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ON THE ZEROS OF A CLASS OF POLYNOMIALS AND RELATED ANALYTIC FUNCTIONS

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Abstract. In this paper we prove some interesting extensions and generalizations of Enestrom-Kakeya Theorem concerning the location of the zeros of a polynomial in a complex plane. We also obtain some zero-free regions for a class of related analytic functions. Our results not only contain some known results as a special case but also a variety of interesting results can be deduced in a unified way by various choices of the parameters.

Key words: *zeros of a polynomial, bounds, analytic functions, moduli of zeros*

AMS (2010) subject classification: 30C10, 30C15

1 Introduction and Statement of Results

The following well-known result is due to Enestrom and Kakeya[7].

Theorem A. *If $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots\cdots + a_1z + a_0$ is a polynomial of degree n, such that $a_n \geq a_{n-1} \geq \cdots\cdots \geq a_1 \geq a_0 > 0$, then $P(z)$ has no zeros in $|z| < 1$.*

With the help of Theorem *A*, one gets the following equivalent form of Enestrom-Kakeya Theorem by considering the polynomial $z^nP(1/z)$.

Theorem B. *If*

$$P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots\cdots + a_1z + a_0$$

is a polynomial of degree n, such that

$$a_n \geq a_{n-1} \geq \cdots\cdots \geq a_1 \geq a_0; \qquad a_0 > 0,$$

then $P(z)$ has no zeros in $|z| < 1$.

In the literature[1, 4−10], there already exist some extensions and generalizations of Enestrom-Kakeya Theorem. Aziz and Zarger[3] relaxed the hypothesis of Theorem A in several ways and have proved some extensions and generalizations of this result. As a generalization of Enestrom-Kakeya Theorem, they proved:

Theorem C. *If $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots\cdots+a_1z+a_0$ is a polynomial of degree n, such that for some $k\geq 1$*

$$ka_n \geq a_{n-1} \geq \cdots\cdots \geq a_1 \geq a_0 > 0, \tag{1}$$

then $P(z)$ has all its zeros in the disk $|z+k-1|\leq k$.

Remark 1. Since the circle $|z+k-1|\leq k$ is contained in the circle $|z|\leq 2k-1$, it follows from Theorem C that all the zeros of $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$, satisfying (I) lie in the circle.

$$|z|\leq 2k-1. \tag{2}$$

Aziz and Mohammad[2] have studied the zeros of a class of related analytic functions and among other things have obtained.

Theorem D. *Let $f(z)=\sum_{j=0}^{\infty} a_jz^j \neq 0$ be analytic in $|z|\leq t$. If $|\arg a_j|\leq \alpha\leq \pi/2$, $j=0,1,2,\cdots$ and for some finite non-negative integer k,*

$$|a_0|\leq t|a_1|\leq\cdots\leq t^k|a_k|\geq t^{k+1}|a_{k+1}|\geq\cdots,$$

then $f(z)$ does not vanish in

$$|z|\leq \frac{t}{\left(2t^k\left|\frac{a_k}{a_0}\right|-1\right)\cos\alpha+\sin\alpha+\frac{2\sin\alpha}{|a_0|}\,|\sum_{j=0}^{\infty} t^j\,|a_j|}.$$

The aim of this paper is to present some more extensions and generalizations of Enestrom-Kakeya Theorem. We also study the zeros of a class of related analytic functions. We start by presenting the following interesting generalization of Theorem C.

Theorem 1. *If $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots\cdots+a_1z+a_0$ is a polynomial of degree n. If for some real number $\rho\geq 0$, such that*

$$\rho+a_n \geq a_{n-1} \geq \cdots\cdots \geq a_1 \geq a_0 > 0, \tag{3}$$

then $P(z)$ has all its zeros in

$$|z+\frac{\rho}{a_n}|\leq 1+\frac{\rho}{a_n}. \tag{4}$$

Remark 2. Theorem C is a special case of Theorem 1 for the choice of $\rho=(k-1)a_n$, where $k\geq 1$. Applying Theorem 1 to polynomial $P(tz)$ we obtain the following result :

Corollary 1. *Let* $P(z) = \sum_{j=0}^{\infty} a^j \mid z^j \mid \neq 0$ *be a polynomial of degree n. If for some real numbers* $\rho \geq 0$ *and* $t > 0$*, such that*

$$\rho = t^n a_n \geq t^{n-1} a_{n-1} \geq \cdots \geq t a_1 \geq a_0 \geq 0$$

then all zeros of $P(z)$ *lie in*

$$\mid z + \frac{\rho}{t^{n-1} a_{n-1}} \mid \leq t + \frac{\rho}{t^{n-1} a_n}.$$

Taking $\rho = a_{n-1} - a_n \geq 0$ in Theorem 1, we immediately get the following result:

Corollary 2. *Let* $P(z) \sum_{j=0}^{\infty} a^j \mid z^j \mid \neq 0$ *be a polynomial of degree n such that* $a_n \leq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$*, then* $P(z)$ *has all its zeros in*

$$\mid z - 1 + \frac{a_{n-1}}{a_n} \mid \leq \frac{a_{n-1}}{a_n}.$$

Next, we prove the following results:

Theorem 2. *If*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots\cdots + a_1 z + a_0$$

is a polynomial of degree n such that

$$a_n \leq a_{n-1} \geq \cdots \geq a_1 \geq a_{\lambda+1} \geq a_\lambda; \quad a_\lambda \leq a_{\lambda-1} \leq \cdots \leq a_0; \quad a_0 > 0,$$

then all zeros of $P(z)$ *lie in the disk*

$$\mid z \mid \leq 1 + \frac{2(a_0 - a_\lambda)}{a_n}. \tag{6}$$

For $\lambda = 0$, Theorem 2 reduces to Theorem 1.

The following result immediately follows by applying Theorem 2 to the polynomial $P(tz)$ where t is some positive real number.

Corollary 3. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots\cdots + a_1 z + a_0,$$

be a polynomial of degree n such that

$$t^n a_n \geq t^{n-1} a_{n-1} \geq \cdots \geq t^{\lambda+1} a_{\lambda+1} \geq t^\lambda a_\lambda \geq a_\lambda; \quad t^\lambda a_\lambda \leq \cdots \leq a_0,$$

then all zeros of $P(z)$ *lie in the disk*

$$|z| \leq t \left\{ 1 + \frac{2(a_0 - t^\lambda a_\lambda)}{t^n a_n} \right\}.$$

Corollary 3 for $\lambda = n$ with the help of Theorem B applied to polynomial $P(tz)$ yields the following interesting result:

Corollary 4. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots\cdots + a_1 z + a_0,$$

be a polynomial of degree n such that

$$t^n a_n \le t^{n-1} a_{n-1} \le \cdots \le t a_1 \le a_0; \qquad a_0 \ge 0,$$

then all the zeros of $P(z)$ *lie in the ring shaped region*

$$t \le |z| \le t\left\{\frac{2a_0}{t^n a_n} - 1\right\}.$$

Now we shall present the following interesting generalization of Theorem A analogous to (2).

Theorem 3. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots\cdots + a_1 z + a_0,$$

be a polynomial of degree n, if for some $k \ge 1$,

$$k a_\lambda \le a_{\lambda+1} \ge \cdots \ge a_1 \ge a_0 \ge 0, \text{ and } a_n \ge a_{n-1} \ge \cdots \ge a_\lambda, \tag{7}$$

then all the zeros of $P(z)$ *lie in the region*

$$|z| \le 1 + 2(k-1)\frac{a_\lambda}{a_n}. \tag{8}$$

For $\lambda = n$, *we get Theorem C and for* $k = 1$, *it reduces to Enestrom - Kakeya Theorem.*

Remark 3. Theorem 3 is applicable to situations where Enestrom-Kakeya Theorem provides no information. To see this consider the polynomial

$$P(z) = 3z^5 + 3z^4 + z^3 + 2z^2 + 2z + 2.$$

Here Enestrom-Kakeya Theorem is not applicable, but according to Theorem 3 all the zeros of $P(z)$ lie in the disk

$$|z| \le 1 + \frac{2(2-1)}{3} = \frac{5}{3},$$

which is much better than the bound obtained by the Cauchy's classical Theorem [7,Theorem 27.2].

Finally, we shall present the following result for analytic functions which is a generalization of Theorem D, analogous to Theorem 3:

Theorem 4. *Let*

$$f(z)=\sum_{j=0}^{\infty}a_jz^j\neq 0$$

be analytic in $|z|\leq t$. *If* $|\arg\, a_j|\leq\alpha\leq\pi/2,\ j=0,1,2,\cdots$ *and for some finite non-negative integer* λ *and some* $k,\ 0<k\leq 1$,

$$|a_0|\leq t|a_1|\leq\cdots\leq t^{\lambda}|a_{\lambda}|\geq t^{\lambda+1}|a_{\lambda+1}|\geq\cdots,$$

then $f(z)$ *does not vanish in*

$$|z|\leq\frac{t}{(1-2k)+\left\{\left|\frac{a_\lambda}{a_0}\right|t^{\lambda}\right\}\cos\alpha+\sin\alpha+\frac{2\sin\alpha}{|a_0|}\sum_{j=0}^{\infty}t^j|a_j|}.$$

For $k=1$, it reduces to Theorem D.

2 Proofs of the Theorems

Proof of Theorem 1. Consider

$$\begin{aligned}F(z)&=(1-z)P(z)=(1-z)(a_nz^n+a_{n-1}z^{n-1}+\cdots\cdots+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\cdots+(a_1-a_0)z+a_0.\end{aligned}$$

Therefore, for $|z|>1$, we have

$$\begin{aligned}|F(z)|&=|-a_1z^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\cdots+(a_1-a_0)z+a_0|\\&=|-a_nz^{n+1}-\rho z^n+a_nz^n+(\rho-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\cdots+(a_1-a_0)z+a_0|\\&\geq|a_nz+\rho||z^n|-\{|\rho+a_n-a_{n-1}||z^n|+|a_{n-1}-a_{n-2}||z^{n-1}|\\&\quad+\cdots+|a_1-a_0||z|+|a_0|\}\\&=|z^n|\left[|a_nz+\rho|-\left\{|\rho+a_n-a_{n-1}+|a_{n-1}-a_{n-2}||\frac{1}{|z|}\right.\right.\end{aligned}$$

$$
\begin{aligned}
&+\cdots+|a_1-a_0|\frac{1}{|z^{n-1}|}+|a_0|\frac{1}{|z^n|}\Big\}\Big]\\
> \ & |z^n|\Big[|a_nz+\rho|\\
&-\Big\{(\rho+a_n-a_{n-1})+(a_{n-1}-a_{n-2})+\cdots+(a_1-a_0)+a_0\Big\}\Big]\\
= \ & |z^n|\Big[|a_nz+\rho|-(\rho+a_n)\Big]\\
> \ & 0, \text{ if } \ |a_nz+\rho|>(\rho+a_n).
\end{aligned}
$$

Therefore all the zeros of $F(z)$ whose modulus is greater than 1 lie in

$$\left|z+\frac{\rho}{a_n}\right|\le 1+\frac{\rho}{a_n}.$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the inequality (4). Since the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that all the zeros of $P(z)$ lie in the region.

$$\left|z+\frac{\rho}{a_n}\right|\le 1+\frac{\rho}{a_n},$$

which proves the desired result.

Proof of Theorem 2. Consider

$$
\begin{aligned}
F(z) &= (1-z)P(z)=(1-z)(a_nz^n+a_{n-1}z^{n-1}+\cdots\cdots+a_1z+a_0)\\
&= -a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\cdots+(a_1-a_0)z+a_0.
\end{aligned}
$$

Therefore, for $|z|>1$, using the hypothesis we have

$$
\begin{aligned}
|F(z)| \ \ge \ & |a_n||z^{n+1}|-|z^n|\Big\{|(a_n-a_{n-1})+\left(\frac{a_{n-1}-a_{n-2}}{z}\right)\\
&+\cdots+\left(\frac{a_{\lambda+1}-a_\lambda}{z^{n-\lambda-1}}\right)+\left(\frac{a_\lambda-a_{\lambda-1}}{z^{n-\lambda}}\right)+\cdots+\left(\frac{a_1-a_0}{z^{n-1}}\right)+\left(\frac{a_0}{z^n}\right)\\
\ge \ & |a_n||z|^{n+1}+|z|^n-\Big\{|a_n-a_{n-1}|+\left|\frac{a_{n-1}-a_{n-2}}{z}\right||\\
&+\cdots+\left|\frac{a_{\lambda+1}-a_\lambda}{z^{n-\lambda-1}}\right|+\left|\frac{a_\lambda-a_{\lambda-1}}{z^{n-\lambda}}\right|+\cdots+\left|\frac{a_1-a_0}{z^{n-1}}\right|+\left|\frac{a_0}{z^n}\right|\Big\}\\
\ge \ & |z^n|\{|z||a_n|-(a_n-a_{n-1})+(a_{n-1}-a_{n-2})+\\
&+\cdots+(a_{\lambda+1}-a_\lambda)+(a_\lambda-a_{\lambda-1})+\cdots+(a_1-a_0)+(a_0)\}\\
\ge \ & |z^n|\{|z||a_n|-(a_n+2a_0-2a_\lambda)\}\\
= \ & |a_n||z^n\Big\{|z|-\frac{a_n+2a_0-2a_\lambda}{|a_n|}\Big\}\\
> \ & 0 \ \text{if} \ |z|>\frac{a_n+2a_0-2a_\lambda}{|a_n|}=1-\frac{2(a_\lambda-a_0)}{|a_n|}.
\end{aligned}
$$

Therefore, all the zeros of $F(z)$, whose modulus is greater than 1 lie in

$$|z| \leq 1 - \frac{2(a_\lambda - a_0)}{|a_n|}.$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the inequality (6). Since all the zeros of $P(z)$ are also the zeros of $F(z)$, so it follows that all the zeros of $P(z)$ lie in

$$|z| \leq 1 - \frac{2(a_\lambda - a_0)}{|a_n|}.$$

which completes the proof of the desired result.

Proof of Theorem 3. Consider

$$\begin{aligned} F(z) &= (1-z)P(z) = (1-z)(a_n z^n + a_{n-1} z^{n-1} + \cdots\cdots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_1 - a_0) z + a_0. \end{aligned}$$

Therefore, for $|z| > 1$, using the hypothesis we have

$$\begin{aligned} |F(z)| &\geq |a_n||z^{n+1}| - \{|(a_n - a_{n-1})z^n + \cdots + (a_\lambda - a_{\lambda-1})z^\lambda + \cdots + (a_1 - a_0)z + a_0|\} \\ &\geq |a_n||z^{n+1}| - |z^n| \{|a_n - a_{n-1}| + \left|\frac{a_{n-1} - a_{n-2}}{z}\right| \cdots + \left|\frac{a_{\lambda+1} - a_\lambda}{z^{n-\lambda-1}}\right| + \\ &\quad \cdots + \left|\frac{ka_\lambda - a_\lambda}{z^{n-\lambda}}\right| + \left|\frac{ka_\lambda - a_{\lambda-1}}{z^{n-\lambda}}\right| + \cdots + \left|\frac{a_1 - a_0}{z^{n-1}}\right| + \left|\frac{a_0}{z^n}\right| \\ &\geq |a_n||z^{n+1}| + |z^n| - \left\{ |a_n - a_{n-1}| + \left|\frac{a_{n-1} - a_{n-2}}{z}\right| \right. \\ &\quad \left. + \cdots + \left|\frac{a_{\lambda+1} - a_\lambda}{z^{n-\lambda-1}}\right| + \left|\frac{a_\lambda - a_{\lambda-1}}{z^{n-\lambda}}\right| + \cdots + \left|\frac{a_1 - a_0}{z^{n-1}}\right| + \left|\frac{a_0}{z^n}\right| \right\} \\ &\geq |z^n||a_n| [|z| - \{(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{\lambda+1} - a_\lambda) \\ &\quad (ka_\lambda - a_{\lambda-1}) + (ka_\lambda - a_\lambda) + (a_1 - a_0) + (a_0)\}] \\ &\geq |z^n||a_n| \left\{ |z| - \frac{(a_n + 2(k-1)a_\lambda)}{|a_n|} \right\} \\ &> 0 \text{ if } |z| > \frac{(a_n + 2(k-1)a_\lambda)}{a_n}. \end{aligned}$$

therefore all the zeros of $F(z)$ whose modulus is greater than 1, lie in the region

$$|z| > 1 + \frac{2(k-1)a_\lambda}{a_n}.$$

But all those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the inequality (8). Since all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence all the zeros of $P(z)$ lie in

$$|z| \leq 1 + \frac{2(k-1)a_\lambda}{a_n},$$

which completes the Proof of Theorem 3

Proof of Theorem 4. It is obvious that $\lim t^j a_j = 0$. Consider

$$F(z) = (z-t)f(z) = -ta_0 + \sum_{j=0}^{\infty}(a_{j-1}ta_j)z^{j-1} - ta_0 + zG(z).$$

Since $|\arg a_j| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \cdots$.

It can be easily verified that

$$|ta_j - a_{j-1}| \leq |ta_j - a_{j-1}|\cos\alpha + (|a_j| + |a_{j-1}|)\sin\alpha.$$

Hence for $|z| = t$, we have

$$\begin{aligned}
|G(z)| &= \Big|\sum_{j=0}^{\infty}(a_{j-1} - ta_j)z^{j-1}\Big| \leq \sum_{j=0}^{\infty}|(a_{j-1} - ta_j)z^{j-1}| \\
&= \sum_{j=0}^{\infty}\big|t|a_j| - |a_{j-1}|\big|t^{j-1}\cos\alpha + \sum_{j=0}^{\infty}(t|a_j| + |a_{j-1}|)t^{j-1}\sin\alpha \\
&\leq \Bigg[\Big(\big|t|a_1| - |a_0|\big| + \sum_{j=2}^{\infty}\big|t|a_j| - |a_{j-1}|\big|t^{j-1}\Big)\cos\alpha \\
&\quad + \sum_{j=1}^{\infty}(t|a_j| + |a_{j-1}|)t^{j-1}\sin\alpha\Bigg] \\
&\leq \Bigg[\Big(\big|t|a_1| - k|a_0| - (1-k)|a_0|\big| \\
&\quad + \sum_{j=2}^{\infty}\big|t|a_j| - |a_{j-1}|\big|t^{j-1}\Big)\cos\alpha + \sum_{j=1}^{\infty}(t|a_j| + |a_{j-1}|)t^{j-1}\sin\alpha\Bigg] \\
&\leq \Bigg[\Big\{(1-2k)|a_0| + t|a_1| + t^2|a_2| + \cdots + t^\lambda|a_\lambda| \\
&\quad - t^{\lambda-1}|a_{\lambda-1}| - t^{\lambda+1}|a_{\lambda+1}| + \cdots\Big\}\cos\alpha + \sin\alpha + 2\sin\alpha\sum_{j=1}^{\infty}|a_j|t^j\Bigg] \\
&= \{(1-2k)|a_0| + 2t^\lambda|a_\lambda|\}\cos\alpha + 2\sin\alpha\sum_{j=1}^{\infty}|a_j|t^j \\
&= |a_0|\left\{(1-2k)|a_0| + 2t^\lambda\left|\frac{a_\lambda}{a_0}\right|\right\}\cos\alpha + 2\sin\alpha\sum_{j=1}^{\infty}|a_j|t^j = |a_0|H \ \text{ say}
\end{aligned}$$

Since $G(0) = 0$, using Schwarz Lemma that $|G(z)| \leq |a_0|M$ for $|z| \leq t$.

From equation (11), it follows that

$$|F(z)| \leq t|a_0| - |z||a_0|M \geq |a_0|(t - M|z|), \ \ for \ \ |z| \leq t,$$

therefore, $|F(z)| > 0$, if

$$|z| > \frac{1}{M}.$$

Consequently $F(z)$,and therefore $f(z)$ does not vanish in $|z| \le \frac{1}{M}$, which is equivalent to the desired result.

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