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QUARTIC CONVEX TYPE PIECEWISE POLYNOMIAL

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Abstract. In the present paper we consider quartic piecewise polynomial for approximation to the function $f \in C^2[0,1]$. A convex type condition has been imposed in the partition so that the matrix involved for the computation of pp functions is of lower band. This reduces the computation for constructions of the pp functions for the approximation.

Key words: *piecewise polynomial, interpolation, deficient splines*

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1 Introduction and Notations

Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a mesh, denoted by Δ, of $[0,1]$. We write

$$x_i - x_{i-1} = h_i, \qquad i = 1,2,\cdots,n$$

and π_m the set of all real algebraic polynomials of degree at most $m \geq 1$. When the partition points are equidistant i.e, uniform partition we write $h = h_i$, for i. The class of deficient polynomial splines of degree m with deficiency k, a non-negative integer, $k < m-1$ is defined as

$$S(m_k,\Delta) = \{s(x) : s(x) \in C^{m-k-1}[0,1], s(x) \in \pi_m, x \in [x_{i-1},x_i], \qquad i = 1,2,\cdots n\}.$$

For $k = 0$, i.e., $S(m_0,\triangle) = S(m,\triangle)$ denotes the class of splines of degree m.

The approximation by means of different kind of quartic spline functions has been studied by Marsden[5], Sharma and Tzimbalario[10] and Rana[8,9]. Spline functions specially cubic spline functions have been studied extensively, e.g. Meir and Sharma[6,7], Dikshit[1], Dikshit and Powar[2], and Kumar and Govil[4].

In order to reduce continuity requirements of the spline function, correspondingly restrictions of interpolations were imposed and such splines were termed as Deficient Splines. These functions are also termed as *pp* functions.

Here continuity requirement of the second derivative of splines function is replaced by the following condition:

$$\alpha s(x_{i-1}+m_1h_i)+(1-\alpha)s(x_{i-1}+m_2h_i) = s(x_{i-1}+h_i(\alpha m_1+(1-\alpha)m_2)), \qquad i = 1,2,\cdots,n, \tag{1}$$

where α, m_1 and m_2 are positive numbers such that $0 < m_1 < m_2 < 1$ and $0 < \alpha < 1$. We call this condition as a convexity type condition and denote such class of spline functions by $S(4_1^c,\Delta)$. $S(3_1^c,\Delta)$ has a similar meaning for cubic case which was studied by Kumar and Das in [3].

The spline in the class $S(4_1^c,\Delta)$ involves parameters α, m_1 and m_2. The approximation of the function naturally depends on the parameters, which are selected so that the error is minimum. We also consider deviation of the second derivative of the function and the spline function.

First we consider the existence of the Quartic Deficient Spline Function and prove the following:

Theorem 1. *Let α, m_1 and m_2 be non-negative real numbers such that $m_1+m_2=1$, $\alpha \neq \{0,1\}$ and $m_2 \neq \frac{1}{2}$. Then there exists a unique 1-periodic function of the class $S(4_1^c,\triangle)$ for uniform partition satisfying the interpolatory condition:*

$$f(x_i) = s(x_i), i = 1,2,\cdots,n.$$

provided

$$m_2 \leq \frac{1}{\alpha^2+(\alpha-1)^2}\max\{\alpha^2,(\alpha-1)^2\}$$
$$m_2 \geq \frac{1}{\alpha^2+(\alpha-1)^2}\min\{\alpha^2,(\alpha-1)^2\}.$$

We make use of the following result:

Lemma . (a) *Let*

$$C_n(r,q,p;c_1,c_2) = \begin{bmatrix} q & p & 0 & 0 & . & . & . & c_1 \\ r & q & p & 0 & 0 & . & . & 0 \\ 0 & r & q & p & 0 & 0 & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ c_2 & 0 & 0 & . & . & . & r & q \end{bmatrix},$$

and $C_n = C_n(r,q,p;0,0) = C_n(r,q,p)$. We have

(i) $|C_n| = \frac{\beta_1^{n+1}-\beta_2^{n+1}}{\beta_1-\beta_2}$, *where $\beta_1+\beta_2 = q$ and $\beta_1-\beta_2 = \sqrt{q^2-4pr}$;*

(ii) $|C_n(r,q,p;r,p)| = q|C_{n-1}|-2pr|C_{n-2}\,|+(-1)^{n+1}(p^n+r^n)$,

where $|A|$ denotes the determinant of the matrix A.

(b) *If $C_n(r,q,p;r,p)$ is a non-singular matrix, then the (i,j) entry $\hat{a}_{ij}$ of its inverse matrix is given by*

$$\hat{a}_{ij} = \begin{cases} \dfrac{(-1)^{i-j}\{r^{i-j}|C_{n-(i-j)-1}|+(-1)^n p^{n-(i-j)}|C_{i-j-1}|\}}{|C_n(r,\ q,\ p;\ r,\ p)|}, & j<i, \\ \dfrac{(-1)^{j-i}\{p^{j-i}|C_{n-(j-i)-1}|+(-1)^n r^{n-(j-i)}|C_{j-i-1}|\}}{|C_n(r,\ q,\ p;\ r,\ p)|}, & j>i, \\ \dfrac{|C_{n-1}|}{|C_n(r,\ q,\ p;\ r,\ p)|}, & j=i, \end{cases}$$

where $|C_{-1}| = 0$, $|C_0| = 1$.

The above lemma is contained in [3].

Proof of The Theorem. We write $s_i''(x) = s''(x)$ in $[x_{i-1}, x_i], i = 1, 2, \cdots, n$. We have

$$s''(x)h_i = -M_{i-1}\{2(x_i - x)(x - x_{i-1}) - (x_i - x)^2\} \\ -M_i\{2(x_i - x)(x - x_{i-1}) - (x - x_{i-1})^2\} \\ + \eta_i\{6(x_i - x)(x - x_{i-1})\},$$

where $\eta_i's$ are constants and $s''(x_i) = M_i$. Whence

$$\begin{aligned} s(x)h^2 \;=\; & -M_{i-1}\{\frac{1}{3}(x_i - x)(x - x_{i-1})^3 + \frac{1}{6}(x - x_{i-1})^4 - \frac{1}{12}(x_i - x)^4\} \\ & -M_i\{\frac{1}{3}(x_i - x)(x - x_{i-1})^3 + \frac{1}{12}(x - x_{i-1})^4\} \\ & +\eta_i\{(x_i - x)(x - x_{i-1})^3 + \frac{1}{2}(x - x_{i-1})^4\} + \delta_i(x - x_{i-1})h_i + \gamma_i h_i, \end{aligned} \tag{2}$$

where δ_i and γ_i are again constants. The interpolatory condition gives

$$s_i(x_i) = f(x_i) = -\frac{1}{6}h_i M_{i-1} - \frac{1}{12}M_i h_i + \frac{1}{2_i}\eta_i + \delta_i h_i + \gamma_i h^2, \qquad i = 1, 2, \cdots n. \tag{3}$$

Now we impose continuity requirement on s(x) at x_i, i.e. $s_i(x_i) = s_{i+1}(x_i)$. This yeilds

$$\eta_i = \frac{1}{3}M_i + \frac{1}{3}M_{i-1} - 2\delta_i + 2(\gamma_{i+1} - \gamma_i), \qquad i = 1, 2, \cdots, n. \tag{4}$$

And the continuity of $s'(x)$ at x_i gives

$$\eta_i = -\frac{1}{3}(M_i - M_{i-1}) + (\delta_{i+1} - \delta_i), \qquad i = 1, 2, \cdots, n. \tag{5}$$

We apply the convexity condition, that is, the condition (1), and obtain for $i = 1, 2, \cdots, n$,

$$\frac{A_1}{A_0}M_{i-1} + \frac{A_2}{A_0}M_i = \eta_i, \tag{6}$$

where

$$A_1 = \frac{1}{12}(12\alpha^2 m_2^2 + 3\alpha^2 - 12\alpha^2 m_2 - 12\alpha m_2^2 + 16\alpha m_2 - 5\alpha + 6m_2^2 - 8m_2 + 1),$$
$$A_2 = \frac{1}{12}(-12\alpha^2 m_2 + 12\alpha^2 m_2^2 + 3\alpha^2 - 12\alpha m_2^2 + 8\alpha m_2 - \alpha + 6m_2^2 - 4m_2 - 1)$$

and

$$A_0 = \frac{1}{2}(\alpha^2 - 4\alpha^2 m_2 + 4\alpha^2 m_2^2 - 4\alpha m_2^2 + 4\alpha m_2 - \alpha + 2m_2^2 - 2m_2 - 1).$$

Whence

$$\eta_{i+1} - \eta_i = -\frac{A_1}{A_0}M_{i-1} + (\frac{A_1 - A_2}{A_0})M_i + \frac{A_2}{A_0}M_{i+1}, \qquad i = 1, 2, \cdots, n. \tag{7}$$

From (3) and (4) we find that

$$\gamma_{i+1} = h_i f_i - \frac{1}{12}M_i. \tag{8}$$

From (3) we see that

$$\begin{aligned}(f_{i+1}-f_i)h_i &= -\frac{1}{12}M_i-\frac{1}{12}M_{i+1}+\frac{1}{6}M_{i-1}+\frac{1}{2}(\eta_{i+1}-\eta_i)\\ &\quad +(\delta_{i+1}-\delta_i)+(\gamma_{i+1}-\gamma_i).\end{aligned}$$

Using (5) - (9) and the fact that $M_0=M_n$, we obtain the following equations for $i=1,2,\cdots,n$:

$$\frac{12A_0(f_{i+1}-2f_i+f_{i-1})h_i}{2m_2-1}=rM_{i-1}+qM_i+pM_{i+1}, \tag{10}$$

where

$$\begin{aligned}&r=(2\alpha^2-2\alpha+1)m_2-\alpha^2+2\alpha-1\\ &q=\frac{8(2\alpha-1-2\alpha^2)m_2+8(2\alpha^2-2\alpha+1)m_2^2+4\alpha(\alpha-1)-1}{2m_2-1}\\ &p=(2\alpha^2-2\alpha+1)m_2-\alpha^2.\end{aligned}$$

The coefficient matrix of the equation (10) is $A=C_n(r,q,p;r,p)$. It can be seen that the coefficient r of M_{i-1} is non-negative for

$$m_2\geq\frac{(\alpha-1)^2}{\alpha^2+(\alpha-1)^2}. \tag{11}$$

Further, we see that the numerator of the coefficient of q of M_i is zero for

$$m_2=\frac{1}{2}+\frac{1}{4}\frac{\sqrt{12\alpha^2-12\alpha+6}}{2\alpha^2-2\alpha+1}>1 \quad \text{for} \quad 0\leq\alpha\leq 1 \tag{12}$$

and

$$m_2=\frac{1}{2}-\frac{1}{4}\frac{\sqrt{12\alpha^2-12\alpha+6}}{2\alpha^2-2\alpha+1}<0 \quad \text{for} \quad 0\leq\alpha\leq 1. \tag{13}$$

For $m_2=0$ we find that the numerator of the coefficient q of M_i is

$$4\alpha(\alpha-1)-1<0.$$

Thus, the numerator of q has negative sign for $0\leq m_2\leq 1$.

The coefficient p of M_{i+1} is non-negative for

$$m_2\geq\frac{\alpha^2}{\alpha^2+(\alpha-1)^2}. \tag{14}$$

We have

$$\max\{\alpha^2,(1-\alpha)^2\}=(1-\alpha)^2 \quad \text{or} \quad \alpha^2 \quad \text{if} \quad \alpha<\frac{1}{2} \quad \text{or} \quad \alpha>\frac{1}{2}.$$

Thus for $\alpha<\frac{1}{2}$ and $m_2\geq\frac{(\alpha-1)^2}{\alpha^2+(\alpha-1)^2}>\frac{1}{2}$ or for $\alpha>\frac{1}{2}$ and $m_2\geq\frac{\alpha^2}{\alpha^2+(\alpha-1)^2}>\frac{1}{2}$.
We see that for this case

$$\begin{aligned}D_1 &= -\text{Coeff. of} \quad M_i-\text{Coeff. of} \quad M_{i-1}-\text{Coeff. of} \quad M_{i+1}\\ &= \frac{6[(4\alpha^2-4\alpha+2)m_2(1-m_2)+\alpha(1-\alpha)]}{2m_2-1}>0.\end{aligned} \tag{15}$$

That is the matrix is diagonally dominant with diagonal element of coefficient of M_i for

$$m_2 \geq \frac{1}{\alpha^2+(\alpha-1)^2}\max\{\alpha^2,(1-\alpha)^2\}.$$

Now $\min\{\alpha^2,(\alpha-1)^2\}=(\alpha-1)^2$ or α^2 according as $\alpha>\frac{1}{2}$ or $\alpha<\frac{1}{2}$. We have in either case $m_2<\frac{1}{2}$. Again for $m_2 \leq \frac{1}{\alpha^2+(\alpha-1)^2}\min\{\alpha^2,(\alpha-1)^2\}$.

In this case we have

$$\begin{aligned} D_2 &= \text{Coeff. of } \quad M_i + \text{Coeff. of } \quad M_{i-1} + \text{Coeff. of } \quad M_{i+1} \\ &= \frac{6[(4\alpha^2-4\alpha+2)m_2(1-m_2)+\alpha(1-\alpha)]}{1-2m_2} > 0 \end{aligned} \qquad (15')$$

that is, in this case also the matrix is also diagonally dominant with the diagonal element M_i.

2 Error of Approximation

We denote the error function $s(x)-f(x)$ by $e(x)$. We use the notation $g(x_i)=g_i$. From equation(10), we have

$$re''_{i-1}+qe''_i+pe''_{i+1} = \frac{12A_0(f_{i+1}-2f_i+f_{i-1})h^{-2}}{2m_2-1} - rf''_{i-1}-qf''_i-p\,f''_{i+1}.$$

By Taylor′s theorem we can write

$$(f_{i+1}-2f_i+f_{i-1})h^{-2} = f''(\delta_{i-1})+\frac{1}{2}[f''(\eta_i)-f''(\eta_{i-1})],$$

where $x_i \leq \eta_i \leq x_{i+1}$ and$x_{i-1} \leq \delta_{i-1} \leq x_i$. Using this we get

$$\begin{aligned} &(2m_2-1)(r\,e''_{i-1}+q\,e''_i+p\,e''_{i+1}) \\ &\quad = 12A_0(f''(\delta_{i-1})+\frac{1}{2}[f''(\eta_i)]-f''(\eta_{i-1}))]-(2m_2-1)(r\,f''_{i-1}+q\,f''_i+\ \ p\,f''_{i+1}). \end{aligned}$$

We denote the expression on the right hand side by U$_i$. We have

$$\begin{aligned} U_i &= 6A_0[f''(\delta_{i-1})-f''(\eta_{i-1})]+q(2m_2-1)[f''(\eta_i)-f''_i]+r(2m_2-1)[f''(\delta_{i-1})-f''_{i-1}] \\ &\quad +(R_1-2)f''(\eta_i)+(R_2-4)f''(\delta_{i-1})-(2m_2-1)pf''_{i+1} \\ &= 6A_0[f''(\delta_{i-1})-f''(\eta_{i-1})]+q(2m_2-1)[f''(\eta_i)-f''_i] \\ &\quad +r(2m_2-1)[f''(\delta_{i-1})-f''_{i-1}]+R_1[f''(\eta_i)-f''_{i+1}] \\ &\quad +R_2[f''(\delta_{i-1})-f''_{i+1}]-4f''(\delta_{i-1})-2f''(\eta_i), \end{aligned}$$

where $R_1=6A_0-(2m_2-1)q+2$ and $R_2=6A_0-(2m_2-1)r+4$.

Hence

$$|U_i| \leq [-6A_0+(1-2m_2)q+|r(2m_2-1)|]\omega(f'';h)+(R_1+|R_2|)\omega(f'';2h)+6||f''||,$$

since,

$$|A_0| = -A_0, \quad |q(2m_2-1)| = (1-2m_2)q$$

and

$$|R_1| = R_1, \quad \omega(f,2h) \le 2\omega(f,h).$$

Thus, we have

$$\max|e_i''| = \overline{e} = ||A^{-1}||[-6A_0 + (1-2m_2)q + |r(2m_2-1)|2R_1 + 2|R_2|]\omega(f'';h) + 6||f''||.$$

From (1) and (6), we obtain

$$\begin{aligned} e^{''}(x) &= h^{-2}[\frac{A_1}{A_0}(x_i-x)(x-x_{i-1}) - 2(x_i-x)(x-x_{i-1}) + (x_i-x)^2]e_{i-1}^{''} \\ &\quad +h^{-2}[\frac{A_2}{A_0}(x_i-x)(x-x_{i-1}) - 2(x_i-x)(x-x_{i-1}) + (x-x_{i-1})^2]e_i^{''} \\ &\quad +h^{-2}[\frac{A_1}{A_0}(x_i-x)(x-x_{i-1}) - 2(x_i-x)(x-x_{i-1}) + (x_i-x)^2]f_{i-1}^{''} \\ &\quad +h^{-2}[\frac{A_2}{A_0}(x_i-x)(x-x_{i-1}) - 2(x_i-x)(x-x_{i-1}) + (x-x_{i-1})^2]f_i^{''} - f^{''}(x). \end{aligned}$$

Observing that for $x_{i-1} \le x \le x_i$,

$$(x_i-x)(x-x_{i-1}) \le \frac{h^2}{4}, \quad -2(x_i-x)(x-x_{i-1}) + (x_i-x)^2 \le h^2,$$

and

$$-2(x_i-x)(x-x_{i-1}) + (x-x_{i-1})^2 \le h_i,$$

we get

$$\begin{aligned} |e^{''}(x)| &\le (\frac{3}{2}|\frac{A_1}{A_0}|+1)\left(|e_{i-1}^{''}| + |f_{i-1}^{''}|\right) + \left(\frac{3}{2}|\frac{A_2}{A_0}|+1\right)\left(|e_i^{''}| + |f_i^{''}|\right) + |f^{''}(x)| \\ &\le (\frac{3}{2}(|\frac{A_1}{A_0}| + |\frac{A_2}{A_0}|+2))\overline{e} + ||f''||((\frac{3}{2}(|\frac{A_1}{A_0}| + |\frac{A_2}{A_0}|+2))+1). \end{aligned}$$

Since $e(x_{i-1}) = e(x_i) = 0$, we get by Rolle$'s$ theorem a point $x_{i-1}^{'}$ say, where $x_{i-1} \le x_{i-1}^{'} \le x_i$ such that $e^{'}(x_{i-1}') = 0$. Hence

$$|e^{'}(x)| \le \int_{x_{i-1}^{'}}^{x} |e^{''}(x)|\mathrm{d}x.$$

Similarly,

$$|e(x)| \le \int_{x_{i-1}}^{x} |e^{'}(x)|\mathrm{d}x.$$

Hence,

$$\begin{aligned} ||e|| \le h^2 \max||e''(x)|| &\le h^2[(\frac{3}{2}(|\frac{A_1}{A_0}| + |\frac{A_2}{A_0}|+2))([-6A_0 \\ &\quad +(1-2m_2)q + 2R_1 + 2|R_2||r(2m_2-1)|]\omega(f'';h)) \qquad (16) \\ &\quad +||f''||(6 + (\frac{3}{2}(|\frac{A_1}{A_0}| + |\frac{A_2}{A_0}|+2)+1))]. \end{aligned}$$

Thus we have proved the following :

Theorem 2. *The spline function of Theorem 1 approximates $f \in C^2[0,1]$ by the error given by relation* (16).

Remarks. We consider particular cases for illustrations : Let $m_1 = 0.1, m_2 = 0.9$ and $\alpha = 0.25$. The system of equation (10) becomes

$$(f_{i+1} - 2f_i + f_{i-1})h^{-2} = \frac{11}{39}M_i - \frac{2}{39}M_{i+1}.$$

In general from the equation (10) it follows that the coefficient of M_{i-1} is zero for

$$m_2 = \frac{(\alpha-1)^2}{\alpha^2 + (\alpha-1)^2}$$

and the coefficient of M_{i+1} is zero for

$$m_2 = \frac{\alpha^2}{\alpha^2 + (\alpha-1)^2}.$$

Thus the equation (10) can be reduced so that the coefficient matrix A of $M_i's$ becomes two band matrix.

Now we consider the case in more detail when the coefficient of M_{i-1} is zero then the system of equation (10) takes the form

$$-6(f_{i+1} - 2f_i + f_{i-1})h^{-2}(\alpha^2 - \alpha + 1) = q_0 M_i + p_0 M_{i+1},$$

where $q_0 = 2\alpha^2 - 2\alpha - 1$ and $p_0 = 4\alpha^2 - 4\alpha + 1$.

We see that, by the lemma $|C_n| = |A| = q_0^n$. And the inverse of matrix A is

$$\hat{a}_{ij} = \begin{cases} \dfrac{(-1)^{i-j}\left\{(-1)^n p_0^{n-(i-j)} q_0^{i-j-1}\right\}}{q_0^n + (-1)^{n+1} p_0^n}, & j < i; \\ \dfrac{(-1)^{j-i}\left\{p_0^{j-i} q_0^{n-(j-i)-1}\right\}}{q_0^n + (-1)^{n+1} p_0^n}, & j > i; \\ \dfrac{q_0^{n-1}}{q_0^n + (-1)^{n+1} p_0^n}, & j = i. \end{cases}$$

From the above we find that

$$\sum_{j=1}^{n} |\hat{a}_{ij}| \le \frac{p_0(p_0^{n-1} - q_0^{n-1})}{(p_0 - q_0)(q_0^n + (-1)^{n+1} p_0^n)},$$

i.e.,

$$||A^{-1}|| \le \frac{p_0(p_0^{n-1} - q_0^{n-1})}{(p_0 - q_0)(q_0^n + (-1)^{n+1} p_0^n)}.$$

For the general case $||A^{-1}||$ can be obtained from (15) or (15$'$).

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