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ON THE GENERALIZED RESOLVENT OF LINEAR PENCILS IN BANACH SPACES

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Abstract. Utilizing the stability characterizations of generalized inverses of linear operator, we investigate the existence of generalized resolvent of linear pencils in Banach spaces. Some practical criterions for the existence of generalized resolvents of the linear pencil $\lambda \to T - \lambda S$ are provided and an explicit expression of the generalized resolvent is also given. As applications, the characterization for the Moore-Penrose inverse of the linear pencil to be its generalized resolvent and the existence of the generalized resolvents of linear pencils of finite rank operators, Fredholm operators and semi-Fredholm operators are also considered. The results obtained in this paper extend and improve many results in this area.

Key words: *generalized inverse, generalized resolvent, linear pencils, Moore-Penrose inverse, Fredholm operator, semi-Fredholm operator*

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1 Introduction and Preliminaries

Let X and Y be two Banach spaces. Let $B(X,Y)$ denote the Banach space of all bounded linear operators from X into Y. We write $B(X)$ as $B(X,X)$. The identity operator will be denoted by I. For any $T \in B(X,Y)$, we denote by $N(T)$ and $R(T)$ the null space and the range of T, respectively.

The resolvent set $\rho(T)$ of $T \in B(X)$ is, by definition, the set of all complex number $\lambda \in \mathbf{C}$ such that $T - \lambda I$ is invertible in $B(X)$. And its resolvent $R(T,\lambda) = (T - \lambda I)^{-1}$ is an analytic

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function on $\rho(T)$ since it satisfies the resolvent identity

$$R(T,\lambda)-R(T,\mu)=(\lambda-\mu)R(T,\lambda)R(\mu), \qquad \forall\lambda,\mu\in\rho(T).$$

The spectrum $\sigma(T)$ is the complement of $\rho(T)$ in $\mathbf{C}$. As we all know, the spectral theory plays a fundamental role in functional analysis. If the operator $T-\lambda I$ has a generalized inverse, we can consider the generalized resolvent and generalized spectrum. Some properties of the classical spectrum $\sigma(T)$ remain true in the case of the generalized one[1−4]. Recall that an operator $S\in B(Y,X)$ is said to be an inner inverse of $T\in B(X,Y)$ if $TST=T$ and an outer inverse if $STS=S$. If S is both an inner inverse and outer inverse of T, then S is called a generalized inverse of T[5]. We always write the generalized inverse of T by T^+. If T has a bounded generalized inverse T^+, then TT^+ and T^+T are projectors with

$$R(TT^+)=R(T),\quad R(T^+T)=R(T^+),\quad N(T^+T)=N(T),\quad N(TT^+)=N(T^+)$$
$$X=N(T)\oplus R(T^+),\qquad Y=N(T^+)\oplus R(T).$$

Let is recall the concept of generalized resolvent. Let $T\in B(X)$ and U be an open set in the complex plane. The function

$$U\ni\lambda\to R_g(T,\lambda)\in B(X)$$

is said to be a generalized resolvent of $T-\lambda I$ on U if

(1) $(T-\lambda I)R_g(T,\lambda)(T-\lambda I)=T-\lambda I$ for all $\lambda\in U$;

(2) $R_g(T,\lambda)(T-\lambda I)R_g(T,\lambda)=R_g(T,\lambda)$ for all $\lambda\in U$;

(3) $R_g(T,\lambda)-R_g(T,\mu)=(\lambda-\mu)R_g(T,\lambda)R_g(T,\mu)$ for all λ and μ in U.

The first two conditions say that $R_g(T,\lambda)$ is a generalized inverse of $T-\lambda I$ for each $\lambda\in U$, while the third one is an analogue of the classical resolvent identity. We also refer to it as the generalized resolvent identity, which plays an important role in the spectrum since it assures that $R_g(T,\lambda)$ is locally analytic. The generalized resolvents has been widely used in many fields such as spectrum theory and theory of Fredholm operators[1−4, 6]. According to M. A. Shubin[7], there exists a continuous generalized inverse function (satisfying (1) and (2) but not possibly (3)) meromorphic in the Fredholm domain $\rho_\phi(T)=\{\lambda\in C: T-\lambda I \text{ is Fredholm}\}$. And he points out that it remains an open problem whether or not this can be done while also satisfying (3), i.e., it is not known whether generalized resolvents always exist. Many authors have been interested in the existence problem for the generalized resolvents of linear operators in [1-4,6]. In [2], C. Badea and M. Mbekhta proved that $T-\lambda I$ has an analytic generalized resolvent in a neighborhood of 0 if and only if T has a generalized inverse and $N(T)\subset R(T^m),\forall m\in N$. It is worth mentioning that the condition $N(T)\subset R(T^m)$ is not easy to be verified and its geometric significance is vague. In [3], C. Badea and M. Mbekhta introduced the concept of linear pencil and its generalized resolvents.

Let $T,S \in B(X,Y)$ and U be an open set in the complex plane. The function $U \ni \lambda \to G(\lambda) \in B(Y,X)$ is called a generalized resolvent on U of the linear pencil $\lambda \to T - \lambda S$ if

(1) $(T-\lambda S)G(\lambda)(T-\lambda S) = T - \lambda S$ for all $\lambda \in U$;

(2) $G(\lambda)(T-\lambda S)G(\lambda) = G(\lambda)$ for all $\lambda \in U$;

(3) $G(\lambda) - G(\mu) = (\lambda - \mu)G(\lambda)SG(\mu)$ for all λ and μ in U.

If $X = Y$ and $S = I$, then the two concepts are coincident. We would like to remark that this is not just the formal extension since $N(T) \subset R(T^m)$ does not hold in general if $X \neq Y$. The linear pencils are very useful in the study of stability radius of Fredholm operators[3]. Utilizing the reduced minimum modulus and the gap between closed subspaces, C. Badea and M. Mbekhta proved the following theorem:

Theorem 1.1.[3] *Let X and Y be two Banach spaces. Let $T,S \in B(X,Y)$ and $U \subset C$ be an open set. There exists a generalized resolvent for $\lambda \to T - \lambda S$ on U if and only if the linear pencil $\lambda \to T - \lambda S$ has fixed complements on U, i.e., there exist two closed subspaces E and F of X and Y such that for all $\lambda \in U$,*

$$X = N(T-\lambda S) \oplus E, \qquad Y = R(T-\lambda S) \oplus F.$$

In this paper, we utilize the stability characterizations of generalized inverses of linear operators to investigate the existence of generalized resolvents of linear pencil $\lambda \to T - \lambda S$ in Banach spaces. Some practical criterions for the existence of generalized resolvents of the linear pencils are provided and an explicit expression of the generalized resolvent is also given. As applications, the characterization for the Moore-Penrose inverse of the linear pencil to be its generalized resolvent and the existence of the generalized resolvents of linear pencils of finite rank operators, Fredholm operators and semi-Fredholm operators are also considered. The results obtained in this paper extend and improve many results in this area.

2 Main Results

We start our investigation with the following lemmas, which are preparation for the proofs of our main results.

Lemma 2.1.[8–10] *Let $T \in B(X,Y)$ with two bounded generalized inverses T^+ and $T^{\oplus}$. Then there exists a $\delta > 0$ such that the following statements are equivalent*:

(1) $R(\overline{T}) \cap N(T^+) = \{0\}$;

(2) $R(\overline{T}) \cap N(T^{\oplus}) = \{0\}$,

where $\overline{T} \in B(X,Y)$ satisfies $\|\overline{T} - T\| < \delta$.

Lemma 2.2.[8,11,12] *Let $T \in B(X,Y)$ with a bounded generalized inverse T^+ and $\overline{T} \in B(X,Y)$ with $\|T^+\|\|\overline{T} - T\| < 1$. Then the following statements are equivalent*:

(1) $B = T^+[I + (\overline{T} - T)T^+]^{-1} = [I + T^+(\overline{T} - T)]^{-1}T^+$ *is a generalized inverse of $\overline{T}$*;

(2) $R(\overline{T}) \cap N(T^+) = \{0\}$;
(3) $Y = R(\overline{T}) \oplus N(T^+)$;
(4) $X = N(\overline{T}) \oplus R(T^+)$.

The following theorem not only provides a practical criterion for the existence of generalized resolvents for the linear pencils, but also gives an explicit expression of the generalized resolvent.

Theorem 2.1. *Let X, Y be two Banach spaces and $T,S \in B(X,Y)$.*

(1) *If the linear pencil $\lambda \to T - \lambda S$ has an analytic generalized resolvent on a neighborhood of 0, then for any generalized inverse T^+ of T, there exists a neighborhood $U(0)$ of 0 such that*

$$R(T-\lambda S) \cap N(T^+) = \{0\}, \qquad \forall\, \lambda \in U(0);$$

(2) *If T has a generalized inverse T^+ and there exists a neighborhood U of 0 such that*

$$R(T-\lambda S) \cap N(T^+) = \{0\}, \qquad \forall\, \lambda \in U,$$

then the linear pencil $\lambda \to T - \lambda S$ has an analytic generalized resolvent on a neighborhood of 0. In fact,

$$G(\lambda) = T^+(I - \lambda ST^+)^{-1} : \, Y \to X$$

is a generalized resolvent of $\lambda \to T - \lambda S$ on a neighborhood of 0.

Proof. (1) If the linear pencil $\lambda \to T - \lambda S$ has an analytic generalized resolvent $G(\lambda)$ on a neighborhood $V(0)$ of zero, then $R(T-\lambda S) \cap N(G(\lambda)) = \{0\}$ and $G(0)$ is a generalized inverse of T, we denote it by $T^{\oplus}$. From the generalized resolvent identity:

$$G(\lambda) - G(\mu) = (\lambda - \mu)G(\lambda)SG(\mu), \qquad \forall \lambda, \mu \in V(0).$$

We get

$$N(G(\lambda)) = N(G(\mu)) \quad \text{and} \quad R(G(\lambda)) = R(G(\mu)), \qquad \forall \lambda, \mu \in V(0).$$

Particularly, $N(G(\lambda)) = N(G(0)) = N(T^{\oplus})$, $\forall \lambda \in V(0)$. Hence for all $\lambda \in V(0)$,

$$R(T-\lambda S) \cap N(T^{\oplus}) = R(T-\lambda S) \cap N(G(\lambda)) = \{0\}.$$

By Lemma 2.1, we can get for any generalized inverse T^+ of T, there exists a neighborhood $U(0)$ of 0 such that

$$R(T-\lambda S) \cap N(T^+) = \{0\}, \qquad \forall\, \lambda \in U(0).$$

(2) If T has a generalized inverse T^+ and there exists a neighborhood U of 0 such that for all $\lambda \in U$, $R(T-\lambda S) \cap N(T^+) = \{0\}$. Without loss of any generality, we can assume $|\lambda| < \|ST^+\|^{-1}, \forall\, \lambda \in U$. Then by Theorem 2.2, for each $\lambda \in U$,

$$G(\lambda) = T^+(I - \lambda ST^+)^{-1} : \, Y \to X$$

is a generalized inverse of $T-\lambda S$ with $R(G(\lambda))=R(T^+)$ and $N(G(\lambda))=N(T^+)$. In the following, we shall show that $G(\lambda)$ is a generalized resolvent of the linear pencil $\lambda\to T-\lambda S$ on U. To this aim, it suffices to prove the generalized resolvent identity, i.e., for all $\lambda,\mu\in U$,

$$G(\lambda)-G(\mu)=(\lambda-\mu)G(\lambda)SG(\mu).$$

Set

$$P(\lambda)=(T-\lambda S)G(\lambda)\quad \text{and}\quad Q(\lambda)=G(\lambda)(T-\lambda S),$$

then $P(\lambda)$ is the projector from Y onto $R((T-\lambda S))$ with $N(P(\lambda))=N(G(\lambda))=N(T^+)$ and $R(P(\lambda))=R(T-\lambda S)$, and $Q(\lambda)$ is the projector from X onto $R(T^+)$ with $N(Q(\lambda))=N(T-\lambda S)$ and $R(Q(\lambda))=R(G(\lambda))=R(T^+)$. Now we claim

$$P(\lambda)P(\mu)=P(\lambda)\quad \text{and}\quad Q(\lambda)Q(\mu)=Q(\mu),\qquad \forall\lambda,\mu\in U.$$

In fact, for any $y\in Y$, $[P(\lambda)P(\mu)-P(\lambda)]y=-P(\lambda)[(I-P(\mu))y]$. Noting

$$[I-P(\mu)]y\in N(P(\mu))=N(T^+)=N(P(\lambda)),$$

we get $P(\lambda)[(I-P(\mu))y]=0$. Then $P(\lambda)P(\mu)=P(\lambda)$. For any $x\in X$, $Q(\mu)x\in R(G(\mu))=R(T^+)=R(G(\lambda))=R(Q(\lambda))$, we obtain $(I-Q(\lambda))Q(\mu)x=0$. Hence $Q(\lambda)Q(\mu)=Q(\mu)$. Thus

$$\begin{aligned}(\lambda-\mu)G(\lambda)SG(\mu) &= G(\lambda)[(T-\mu S)-(T-\lambda S)]G(\mu)\\ &= G(\lambda)(T-\mu S)G(\mu)-G(\lambda)(T-\lambda S)G(\mu)\\ &= G(\lambda)P(\mu)-Q(\lambda)G(\mu)\\ &= G(\lambda)P(\lambda)P(\mu)-Q(\lambda)Q(\mu)G(\mu)\\ &= G(\lambda)P(\lambda)-Q(\mu)G(\mu)\\ &= G(\lambda)-G(\mu).\end{aligned}$$

Therefore,

$$G(\lambda)=T^+(I-\lambda ST^+)^{-1}:\ Y\to X$$

is a generalized resolvent of $\lambda\to T-\lambda S$ on a neighborhood U of zero. This completes the proof.

From Theorem 2.1 and Lemma 2.2, we can get the following corollary which is a generalization of Theorem 1.1 (i.e., Theorem 3.5 in [3]).

Corollary 2.1. *Let X, Y be two Banach spaces and $T,S\in B(X,Y)$. Then the following statements are equivalent:*

(1) *the linear pencil* $\lambda \to T - \lambda S$ *has an analytic generalized resolvent on a neighborhood of zero;*

(2) T *has a generalized inverse* T^+ *and there exists a neighborhood* U *of* 0 *such that*

$$X = N(T - \lambda S) \oplus R(T^+), \qquad \forall\, \lambda \in U;$$

(3) T *has a generalized inverse* T^+ *and there exists a neighborhood* U *of* 0 *such that*

$$Y = R(T - \lambda S) \oplus N(T^+), \qquad \forall\, \lambda \in U;$$

(4) *for any generalized inverse* T^+ *of* T*, there exists a neighborhood* U *of* 0 *such that*

$$X = N(T - \lambda S) \oplus R(T^+), \qquad \forall\, \lambda \in U;$$

(5) *for any generalized inverse* T^+ *of* T*, there exists a neighborhood* U *of* 0 *such that*

$$Y = R(T - \lambda S) \oplus N(T^+), \qquad \forall\, \lambda \in U;$$

In this case,

$$G(\lambda) = T^+(I - \lambda S T^+)^{-1} : \; Y \to X$$

is a generalized resolvent of $\lambda \to T - \lambda S$ *on a neighborhood of* 0.

Remark 2.1. In [3], C. Badea and M. Mbekhta proved that both $N(T - \lambda S)$ and $R(T - \lambda S)$ have the fixed complement if and only if the linear pencil $\lambda \to T - \lambda S$ has an analytic generalized resolvent on a neighborhood of zero. It should be noted that Corollary 2.1 shows that each one of $N(T - \lambda S)$ and $R(T - \lambda S)$ has the fixed complement if and only if $\lambda \to T - \lambda S$ has an analytic generalized resolvent.

Theorem 2.2. *Let* $T, S \in B(X,Y)$*. Then there is an analytic generalized resolvent for the linear pencil* $\lambda \to T - \lambda S$ *on a neighborhood of zero if and only if there exists a neighborhood* $U(0)$ *of* 0 *such that for all* $\lambda \in U(0)$*,* $T - \lambda S$ *has the generalized inverse* $(T - \lambda S)^+$ *satisfying*

$$\lim_{\lambda \to 0} (T - \lambda S)^+ = T^+.$$

Proof. It suffices to prove the sufficiency. If there exists a neighborhood $U(0)$ of 0 such that for all $\lambda \in U(0)$, $T - \lambda S$ has the generalized inverse $(T - \lambda S)^+$ satisfying $\lim_{\lambda \to 0} (T - \lambda S)^+ = T^+$, then we put

$$P_\lambda = I - (T - \lambda S)^+ (T - \lambda S).$$

Hence $\lim_{\lambda \to 0} P_\lambda = P_0$. Without loss of generality, we can suppose that for all $\lambda \in U(0)$, $\|P_\lambda - P_0\| \cdot \|P_0\| < 1$ and $\lambda \cdot \|T^+\| \cdot \|S\| < 1$. Then

$$\begin{aligned} P_0 R(P_\lambda) &= (I - T^+T) N(T - \lambda S) \\ &= (I - T^+T + \lambda T^+S - \lambda T^+S) N(T - \lambda S) \\ &= [I - T^+(T - \lambda S) - \lambda T^+S] N(T - \lambda S) \\ &= (I - \lambda T^+S) N(T - \lambda S) \end{aligned}$$

and by the Banach Lemma, the operator

$$W = I - P_0 + P_\lambda P_0 = I + (P_\lambda - P_0)P_0$$

is invertible and its inverse $W^{-1} : X \to X$ is bounded. Next, by Theorem 2.1, we only need to show

$$R(T - \lambda S) \cap N(T^+) = \{0\}, \qquad \forall \lambda \in U(0).$$

Take $y \in R(T - \lambda S) \cap N(T^+)$, then $y = (T - \lambda S)x$ and $T^+(T - \lambda S)x = 0$, where $x \in X$. Hence

$$T(I - \lambda T^+S)x \;=\; T[I + T^+(T - \lambda S) - T^+T]x = Tx + T^+(T - \lambda S)x - TT^+Tx = 0$$

which implies

$$(I - \lambda T^+S)x \in N(T) = R(P_0).$$

Therefore

$$\begin{aligned}
(I - \lambda T^+S)x &= P_0(I - \lambda T^+S)x \\
&= P_0WW^{-1}(I - \lambda T^+S)x \\
&= P_0(I - P_0 + P_\lambda P_0)W^{-1}(I - \lambda T^+S)x \\
&= P_0P_\lambda P_0W^{-1}(I - \lambda T^+S)x \\
&\in P_0R(P_\lambda) \\
&= (I - \lambda T^+S)N(T - \lambda S).
\end{aligned}$$

By the invertibility of $(I - \lambda T^+S)$, we get $x \in N(T - \lambda S)$. Thus $y = (T - \lambda S)x = 0$. This completes the proof.

Remark 2.2. According to M. A. Shubin[7], there exists a continuous generalized inverse function but not an analytic generalized resolvent. From Theorem 2.2, we can see that if there exists a continuous generalized inverse function, then we can find a relevant analytic generalized resolvent.

Next we shall give the characterizations for the existence of generalized resolvents of the finite rank operators, Fredholm operators and semi-Fredholm operators. Their proofs come directly from Theorem 3.1 and Theorems in [8,11,13,14], we omit them.

Theorem 2.3. *Let $T, S \in B(X,Y)$ and T be a finite rank operator. Then there is an analytic generalized resolvent for the linear pencil $\lambda \to T - \lambda S$ on a neighborhood of zero if and only if there exists a neighborhood U of* 0 *such that*

$$\text{Rank}(T - \lambda S) = \text{Rank } T, \qquad \forall\, \lambda \in U.$$

Theorem 2.4. *Let $T, S \in B(X,Y)$ and T be an Fredholm operator. Then there is an analytic generalized resolvent for the linear pencil $\lambda \to T - \lambda S$ on a neighborhood of zero if and only if* *there exists a neighborhood U of* 0 *such that for all $\lambda \in U$, either*

$$\dim N(T-\lambda S)=\dim N(T) \quad or \quad \mathrm{codim} R(T-\lambda S)=\mathrm{codim} R(T).$$

Remark 2.3. Theorem 2.4 is a generaliztion of Theorem 4.1 in [3].

Theorem 2.5. *Let $T,S\in B(X,Y)$ and T be an semi-Fredholm operator with a generalized inverse. Then there is an analytic generalized resolvent for the linear pencil $\lambda\to T-\lambda S$ on a neighborhood of zero if and only if there exists a neighborhood U of* 0 *such that for all $\lambda\in U$, either*

$$\dim N(T-\lambda S)=\dim N(T)<\infty \quad or \quad \mathrm{codim} R(T-\lambda S)=\mathrm{codim} R(T)<\infty.$$

In the following, we shall give the characterization for the Moore-Penrose inverse of the linear pencil $\lambda\to T-\lambda S$ to be its generalized resolvent. We recall that if the operator $T^\dagger\in B(Y,X)$ satisfies

$$TT^\dagger T=T, \quad T^\dagger TT^\dagger=T^\dagger, \quad (TT^\dagger)^*=TT^\dagger, \quad (T^\dagger T)^*=T^\dagger T,$$

where T^* denotes the adjoint operator of T, then $T^\dagger$ is called to be the Moore-Penrose inverse of T. If $T^\dagger$ is the Moore-Penrose inverse of T, then $T^\dagger T=P^\perp_{R(T^\dagger)}$ and $TT^\dagger=P^\perp_{R(T)}$, where $P^\perp_M$ is the orthogonal projector on M.

Theorem 2.6. *Let X and Y be two Hilbert spaces. Let $T,S\in B(X,Y)$ and $R(T)$ be closed. Then the Moore-Penrose inverse $(T-\lambda S)^\dagger$ of the linear pencil $\lambda\to T-\lambda S$ is its analytic generalized resolvent on a neighborhood of 0 if and only if there exists a neighborhood U of* 0 *such that for all $\lambda\in U$,*

$$N(T-\lambda S)=N(T) \quad and \quad R(T-\lambda S)=R(T).$$

Proof. If the Moore-Penrose inverse $(T-\lambda S)^\dagger$ of the linear pencil $\lambda\to T-\lambda S$ is its analytic generalized resolvent on a neighborhood U of zero, then from the generalized resolvent identity, we get

$$N((T-\lambda S)^\dagger)=N(T^\dagger) \quad \text{and} \quad R((T-\lambda S)^\dagger)=R(T^\dagger), \quad \forall\lambda\in U.$$

Hence

$$N(T-\lambda S)=[R((T-\lambda S)^\dagger)]^\perp=[R(T^\dagger)]^\perp=N(T)$$

and

$$R(T-\lambda S)=[N((T-\lambda S)^\dagger)]^\perp=[N(T^\dagger)]^\perp=R(T).$$

Conversely, if there exists a neighborhood U of 0 such that for all $\lambda\in U$, $N(T-\lambda S)=N(T)$ and $R(T-\lambda S)=R(T)$, then $R(T-\lambda S)\cap N(T^\dagger)=R(T)\cap N(T^\dagger)=\{0\}$. By Theorem 2.1,

$$G(\lambda)=T^\dagger(I-\lambda ST^\dagger)^{-1}=(I-\lambda T^\dagger S)^{-1}T^\dagger$$

is an analytic generalized resolvent of linear pencil $\lambda \to T - \lambda S$. Next we shall show that $G(\lambda)$ is also its Moore-Penrose inverse. Indeed, by $N(T-\lambda S) = N(T)$ and $R(T-\lambda S) = R(T)$, then the orthogonal projector $P^{\perp}_{N(T-\lambda S)} = P^{\perp}_{N(T)} = I - T^{\dagger}T$ and $P^{\perp}_{R(T-\lambda S)} = P^{\perp}_{R(T)} = TT^{\dagger}$. Therefore,

$$\begin{aligned} G(\lambda) &= T^{\dagger}(I-\lambda ST^{\dagger})^{-1} = T^{\dagger}TT^{\dagger}(I-\lambda ST^{\dagger})^{-1} = T^{\dagger}T(I-\lambda T^{\dagger}S)^{-1}T^{\dagger} \\ &= T^{\dagger}T(I-\lambda T^{\dagger}S)^{-1}T^{\dagger}TT^{\dagger} = [I-P^{\perp}_{N(T-\lambda S)}](I-\lambda T^{\dagger}S)^{-1}T^{\dagger}P^{\perp}_{R(T-\lambda S)} \\ &= [I-P^{\perp}_{N(T-\lambda S)}]G(\lambda)P^{\perp}_{R(T-\lambda S)}. \end{aligned}$$

So we get

$$\begin{aligned} G(\lambda)(T-\lambda S) &= [I-P^{\perp}_{N(T-\lambda S)}]G(\lambda)P^{\perp}_{R(T-\lambda S)}(T-\lambda S) \\ &= [I-P^{\perp}_{N(T-\lambda S)}]G(\lambda)(T-\lambda S) \\ &= I-P^{\perp}_{N(T-\lambda S)} = T^{\dagger}T \end{aligned}$$

and

$$\begin{aligned} (T-\lambda S)G(\lambda) &= (T-\lambda S)[I-P^{\perp}_{N(T-\lambda S)}]G(\lambda)P^{\perp}_{R(T-\lambda S)} \\ &= (T-\lambda S)G(\lambda)P^{\perp}_{R(T-\lambda S)} = P^{\perp}_{R(T-\lambda S)} = TT^{\dagger}. \end{aligned}$$

Hence $[G(\lambda)(T-\lambda S)]^* = G(\lambda)(T-\lambda S)$ and $[(T-\lambda S)G(\lambda)]^* = (T-\lambda S)G(\lambda)$. Thus $G(\lambda)$ is the Moore-Penrose inverse of $T-\lambda S$. This completes the proof.

Corollary 2.2. *Let X be a Hilbert spaces and $T \in B(X)$, $R(T)$ be closed. Then the Moore-Penrose inverse $(T-\lambda I)^{\dagger}$ of the linear pencil $\lambda \to T-\lambda I$ is the analytic generalized resolvent on a neighborhood of zero if and only if*

$$N(T) = \{0\} \quad \textit{and} \quad R(T) = X.$$

In this case, T is invertible, the Moore-Penrose generalized inverse is the inverse and the generalized resolvent is exactly its classical resolvent.

Proof. If $N(T) = \{0\}$ and $R(T) = X$, by the inverse operator theorem, we can get what we desired. Conversely, if the Moore-Penrose inverse $(T-\lambda I)^{\dagger}$ is the generalized resolvent of $T-\lambda I$ on a neighborhood of zero, then from Theorem 2.6, $N(T-\lambda I) = N(T)$ and $R(T-\lambda I) = R(T)$ hold on a neighborhood of zero. It is easy to be verified that $N(T) = \{0\}$ and $R(T) = X$. This completes the proof.

Remark 2.4. Corollary 2.2 points out that why we use the generalized inverse instead of the Moore-Penrose inverse to define the generalized resolvent. Just the nonuniqueness can give the generalized inverse more significance.

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