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RECURSIVE REPRODUCING KERNELS HILBERT SPACES USING THE THEORY OF POWER KERNELS

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Abstract. The main objective of this work is to decompose orthogonally the reproducing kernels Hilbert space using any conditionally positive definite kernels into smaller ones by introducing the theory of power kernels, and to show how to do this decomposition recursively. It may be used to split large interpolation problems into smaller ones with different kernels which are related to the original kernels. To reach this objective, we will reconstruct the reproducing kernels Hilbert space for the normalized and the extended kernels and give the recursive algorithm of this decomposition.

Key words: *Hilber space, reproducing kernel, interpolant, power function and Conditionally positive kernel*

AMS (2010) subject classification: 46E22, 46B70, 41A05, 47B34, 41D65, 65D05, 65D15

1 Introduction

The abstract theory of Reproducing Kernels Hilbert Space (RKHS) has been developed over a number of years outside of different domains in Physics, Mathematics and/or Chemistry such as the study of conformal mappings[1], integral equations[2], and partial differential equations[3]. The RKHS method has been used for a variety of applications, especially in data interpolation and smoothing[4−7]. The RKHS method provides a rigorous and effective framework for smooth multivariate interpolation of arbitrarily scattered data and for accurate approximation of general multidimensional functions using conditionally/unconditionally positive kernels. Smooth global multi-dimensional reproducing kernels have been successfully used in other contexts for multivariate interpolation, e.g., in computer aided geometric design [8,9] and to solve differential equations by collocation[10]. These reproducing kernels usually are simple and easily to compute in closed forms[10,11]. The reproducing property imparts a rich physically based structure in the associated Hilbert space that possesses many important properties (e.g., the uniqueness and positive definiteness of the reproducing kernel which are important for its practical utility).

The association of a Hilbert space to each conditionally positive definite function go back to the analysis of Madych[13]. The practical advantage of all of this is that all useful conditionally positive definite functions, which were constructed without any relation to an Hilbert space, can be investigated thoroughly within their native space, once the latter is defined and characterized. RKHS, in the conditionally positive definite case, turns out to be a Hilbert space plus a finite-dimensional space[12,17,19].

Section 2 will summarize the recent work in the construction of RKHS (will be called native space) for the conditionally positive kernels Φ and also introduce the power kernels and its native space [15]. Section 3 will present the construction of RKHS for the normalized kernel[18]. Section 4 will introduce an extended kernel Φ_P of the normalized kernel that have the same RKHS. We will show the condition where the interpolation to Φ_P does coincide with the one associated to Φ. Section 5 is the core of this work. The main idea is to decompose large interpolation problems into smaller ones using the theory of power kernels and its RKHS. The orthogonal decomposition of the original native Hilbert space, involving the native space of the power kernel which is proven in our previous work[15]. We will show how to do this orthogonal decomposition of RKHS recursively. It turns out to be used to split large interpolation problems into smaller ones with different kernels which is related to the original kernels Φ.

2 Native Space for the Power Kernels

The interpolation, of scattered data $(x_i, f_i) \in \mathbf{R}$ for pairwise points of discrete set $X = \{x_1, \cdots, x_N\}$ and real valued data $f(x_1), \cdots, f(x_N)$, uses a symmetric multivariate function $\Phi : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ for all $x, y \in \mathbf{R}^d$ and the Q-dimensional space $\mathbf{P}_m^d$ of polynomials p_k on $\mathbf{R}^d$ of degree m, to construct the interpolant:

$$s(x) = \sum_{j=1}^{N} \alpha_j \Phi(x, x_j) + \sum_{k=1}^{Q} \beta_k p_k(x) \quad \text{where} \quad x \in \mathbf{R}^d, \tag{2.1}$$

where α_i and β_i are real numbers, via the system

$$\begin{cases} \displaystyle\sum_{j=1}^{N} \alpha_j \Phi(x, x_j) + \sum_{k=1}^{Q} \beta_k p_k(x) = f_i, \\ \displaystyle\sum_{j=1}^{N} \alpha_j p_k(x_j) = 0, \end{cases} \tag{2.2}$$

For a finite-dimensional subspace of continuous real-valued functions P on Ω and the coefficients α satisfying (2.2), we define a suitable pre-Hilbert space $L_P(\Omega)$ of functionals by

$$L_P(\Omega) := \{\lambda_{\alpha,X} \mid \lambda_{\alpha,X}(f) := \sum_{j=1}^{N} \alpha_j f(x_j),\ \lambda_{\alpha,X}(P) = \{0\}\}, \tag{2.3}$$

equipped with the bilinear form

$$(\lambda_{\alpha,X}, \lambda_{\beta,Y})_\Phi = \sum_{i=1}^{N} \sum_{j=1}^{M} \alpha_i \beta_j \Phi(x_i, y_j), \tag{2.4}$$

which is an inner product on $L_P(\Omega)$.

Let $\mathcal{L}_{\Phi,P}(\Omega) = \mathrm{clos}_{(.,.)_\Phi} L_P(\Omega)$ be the completion form of the pre-Hilbert space $L_P(\Omega)$. This space is a space of functionals and we don't know if it is acting on functions. In order to verify this property, $\mathcal{L}_{\Phi,P}(\Omega)$ must be a Hilbert space. For this, we define a functional for all fixed P-unisolvent subset $Z = \{z_1, z_2, \cdots, z_Q\}$ of points on Ω:

$$\Xi_{(x)}(f) = f(x) - \pi_P(f)(x) \text{ with } \pi_P(f)(x) = \sum_{k=1}^{Q} p_k(x) f(z_k), \tag{2.5}$$

and we define also a map $R_{\Phi,\Omega}$:

$$R_{\Phi,\Omega} : L_P(\Omega) \to R_{\Phi,\Omega}\big(L_P(\Omega)\big)\ ,\ R_{\Phi,\Omega}\big(\lambda_{\alpha,X}\big)(x) = \Big(\lambda_{\alpha,X}, \Xi_{(x)}\Big)_\Phi, \tag{2.6}$$

which is injective on $L_P(\Omega)$. Then, we can interpret $R_{\Phi,\Omega}(L_P(\Omega))$ as a space of functions vanishing in Z and we can define the inner product on the space $R_{\Phi,\Omega}(L_P(\Omega))$ by

$$\Big(R_{\Phi,\Omega}(\lambda_{\alpha,X}), R_{\Phi,\Omega}(\lambda_{\beta,Y})\Big)_\Phi := (\lambda_{\alpha,X}, \lambda_{\beta,Y})_\Phi \quad \text{for all} \quad \lambda_{\alpha,X},\ \lambda_{\beta,Y} \in L_P(\Omega), \tag{2.7}$$

which turns the space $R_{\Phi,\Omega}(L_P(\Omega))$ into a pre-Hilbert space. This application is isometric to $L_P(\Omega)$ via $R_{\Phi,\Omega}$. For simplicity, in both spaces we use the same notion for the inner product as is defined in 4.

Hence, the completions of the pre-Hilbert space $R_{\Phi,\Omega}(L_P(\Omega))$ with respect to the inner products $(.,.)_\Phi$ will be denoted by $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ which is the continuous extension of $R_{\Phi,\Omega}$, and the new inner products are also denoted by $(.,.)_\Phi$ for simplicity. Thus, the completed space $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ is a space of functions satisfying the following property:

$$f \in \mathcal{L}_{\Phi,P}(\Omega) \Rightarrow \mathcal{R}_{\Phi,\Omega}(f)(x) = (f, \Xi_{(x)})_\Phi \quad \text{for all} \quad x \in \Omega. \tag{2.8}$$

Now we are able to define the native space of a conditionally positive definite kernel Φ.

Definition 2.1 The RKHS to a symmetric kernel Φ, which is conditionally positive definite on Ω with respect to P is defined by [15]:

$$\mathcal{N}_\Phi(\Omega) = \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega)) + P,$$

and is equipped with the semi-inner product:

$$\begin{cases} (f,g)_{\mathcal{N}_\Phi(\Omega)} := (\mathcal{R}^{-1}_{\Phi,\Omega}(f-\pi_P(f)), \mathcal{R}^{-1}_{\Phi,\Omega}(g-\pi_P(g)))_\Phi, & \text{for all} \quad f,g \in \mathcal{N}_\Phi(\Omega), \\ (p,.)_{\mathcal{N}_\Phi(\Omega)} := 0 \quad \text{for all} \quad p \in P. & \end{cases}$$

Since $\|f\|_\Phi$ is undefined for $f \in \mathcal{N}_\Phi(\Omega)$, then we characterize the norms $\|.\|_\Phi$ and $\|\|_{\mathcal{N}_\Phi(\Omega)}$ with the following property

$$f - \pi_P(f) \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega)) \quad \text{and} \quad \|f\|_{\mathcal{N}_\Phi(\Omega)} = \|f - \pi_P(f)\|_\Phi. \tag{2.9}$$

Theorem 2.2. *For a fixed finite set X satisfying the system* (2.2)*, if we define the space* $S_{Xm\Psi} := \pi_X(\mathcal{N}_\Phi(\Omega))$ *which is spanned by all interpolants to functions* $f \in \mathcal{N}_\Phi(\Omega)$ *by functions from P and translates of* Φ*, then* $\mathcal{N}_\Phi(\Omega)$ *is an orthogonal decomposition with respect to* $(.,.)_{\mathcal{N}_\Phi(\Omega)}$. *ie:*

$$\mathcal{N}_\Phi(\Omega) = S_{X,\Phi} + \{f \in \mathcal{N}_\Phi(\Omega) \;:\; f(X) = \{0\}\}. \tag{2.10}$$

In particular, if $s_{f,X,\Phi}$ *interpolates f on X using* Φ*, then:*

$$f - s_{f,X,\Phi} \perp_{\mathcal{N}_\Phi(\Omega)} S_{X,\Phi} \quad \text{and} \quad \|f\|^2_{\mathcal{N}_\Phi(\Omega)} = \|f - s_{f,X,\Phi}\|^2_{\mathcal{N}_\Phi(\Omega)} + \|s_{f,X,\Phi}\|^2_{\mathcal{N}_\Phi(\Omega)}.$$

The power kernel $K_X(.,.)$ of Φ with respect to $X = \{x_1, \cdots, x_N\}$ for all $x,y \in \Omega$ is given by

$$K_X(x,y) = \Phi(x,y) - \sum_{i=1}^N u_i^X(x)\Phi(x_i,y) - \sum_{j=1}^N u_j^X(y)\Phi(x,x_j) + \sum_{i=1}^N\sum_{j=1}^N u_i^X(x)u_j^X(y)\Phi(x_i,x_j), \tag{2.11}$$

where u_j are the Lagrange functions satisfying the property[16]: $u_i(x_j) = \delta_{ij}$ for $1 \le i,j \le N$.

The native space for the symmetric unconditionally positive definite function K_X on $\Omega \setminus X$ is given by

$$\mathcal{N}_{K_X}(\Omega \setminus X) = \{f \in \mathcal{N}_\Phi(\Omega) \;:\; f(X) = \{0\}\}, \tag{2.12}$$

equipped with the inner product $(.,.)_{\mathcal{N}_{K_X}}$ and has the reproducing property

$$f(y) = (f, K_X(y,.))_{K_X},$$

For all $f \in \mathcal{N}_{K_X}(\Omega \setminus X)$, the norm $\|\cdot\|_{K_X}$ of $\mathcal{N}_{K_X}(\Omega \setminus X)$ is defined by

$$\|f\|^2_{K_X} = \|g_Y - g_X\|^2_\Phi, \tag{2.13}$$

where g_X and g_Y are in the space spanned by $\Phi(\cdot,x_k)$ and $\Phi(\cdot,y_l)$ respectively

$$g_X(x) \in \text{Span}\{\Phi(\cdot,x_k)\} \quad \text{and} \quad g_X(x) \in \text{Span}\{\Phi(\cdot,y_l)\}. \tag{2.14}$$

Note that if Φ is an positive definite kernel, we have

$$\|f\|^2_{K_X} = \|f\|^2_{\Phi} \quad \text{for all} \quad f \in \mathcal{N}_{K_X}(\Omega \setminus X). \tag{2.15}$$

3 RKHS for the Normalized Kernel

For all $x,y \in \Omega$, we define the normalized kernel function $h\ :\ \Omega\times\Omega \to \mathbf{R}$ by [reference]:

$$h(x,y) = \Big(Id - \pi_P\Big)^x \Big(Id - \pi_P\Big)^y \Phi(x,y) = (\Xi_{(x)}, \Xi_{(y)})_{\Phi}. \tag{3.1}$$

Substituting (2.5) in (3.1), the normalized kernel will be expressed as

$$\begin{aligned} h(x,y) &= \Phi(x,y) - \sum_{j=1}^{Q} p_j(x)\Phi(z_j,y) - \sum_{k=1}^{Q} p_k(y)\Phi(x,z_k) \\ &\quad + \sum_{j=1}^{Q}\sum_{k=1}^{Q} p_j(x)p_k(y)\Phi(z_j,z_k), \end{aligned} \tag{3.2}$$

which is unconditionally positive on $\Omega \setminus Z$.

Note that for all $1 \le k \le Q$ we have $h(.,z_k) = 0$ because $p_1,\cdots,p_Q$ is a Lagrange basis for P with respect to the points $z_1,\cdots,z_Q$.

Using (3.2), we can re-write (2.5) in a simpler form:

$$f(x) = (\pi_P(f))(x) + \Big(h(x,.),f\Big)_{\mathcal{N}_\Phi(\Omega)} \text{ for all } x \in \Omega \quad \text{and} \quad f \in \mathcal{N}_\Phi(\Omega). \tag{3.3}$$

For all function $f \in \mathcal{N}_\Phi(\Omega)$, the equation (3.3) permits the reproduction property in the sens of the following theorem:

Theorem 3.1. *The bilinear form $(.,.)_{\mathcal{N}_\Phi}$ defines an inner product on the Hilbert space*

$$\mathcal{M}_\Phi = \mathcal{N}_\Phi \cap \{f \in \mathcal{N}_\Phi(\Omega) : f(z_k) = 0,\ \ 1 \le k \le Q\}$$

which has the function $h(.,.)$ as reproducing kernel.

Proof. The space $\mathcal{M}_\Phi$ coincides with $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ due to equation (2.9). Thus it is a Hilbert space under $(.,.)_\Phi$ which is isometric via $\mathcal{R}_{\Phi,\Omega}$ to $\mathcal{L}_{\Phi,P}(\Omega)$. But for all $g \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ we get from (2.9)

$$g(x) = (g,h(x,\cdot))_\Phi = (g,h(x,\cdot))_{\mathcal{N}_\Phi(\Omega)}$$

holds for all $x \in \Omega$.

The inner products $(.,.)_\Phi$ and $(.,.)_{\mathcal{N}_\Phi(\Omega)}$ coincides on $\mathcal{M}_\Phi$. This implies [14]:

Theorem 3.2. *The space $\mathcal{M}_\Phi$ is the native space of the unconditionally positive kernel $h(.,.)$ on $\Omega \setminus Z$.*

4 RKHS for the Extended Kernels of an CPD Kernels

In the previous section, we started with a CPD-kernel Φ on Ω with respect to P and we constructed a RKHS $\mathcal{N}_\Phi(\Omega)$ for Φ on which the normalized kernel function $h(.,.)$ is a generalized reproducing kernel. The native space for Φ, however, was not a Hilbert space, because it carried only a semi-inner product. The new kernel had a native Hilbert space, but on $\Omega \setminus Z$, where Z was a unisolvent set for P. This calls for a new kernel that we will denote Φ_P, now unconditionally positive definite on all of Ω, such that the native space $\mathcal{N}_\Phi(\Omega)$ of Φ coincides as a vector space with the native space of Φ_P, which is now carrying an inner product that is closely related to the previous semi-inner product $(.,.)_\Phi$ defined in (2.4).

Under the assumptions made so far, we can pick a fixed unisolvent set $Z = \{z_1, \cdots, z_Q\}$ for P and a Lagrange basis $p_1, \cdots, p_Q$ of P. Then, for all functions $f, g \in \mathcal{N}_\Phi(\Omega)$ and all $x, y \in \Omega$, we define:

$$\Big(f,g\Big)_P = \sum_{k=1}^{Q} f(z_k)g(z_k) \quad \text{and} \quad \Phi_P(x,y) = h(x,y) + \sum_{l=1}^{Q} p_l(x)p_l(y). \tag{4.1}$$

At this point, one is tempted to use Φ instead of h in the above definition of the kernel, in order to avoid the point set Z to enter into the kernel. However, it turns out that one has difficulties proving positive definiteness in that case.

For all $g, f \in \mathcal{N}_\Phi(\Omega)$, we define the inner product $(\cdot,\cdot)_{\Phi_P}$ by

$$(f,g)_{\Phi_P} = \Big(f,g\Big)_{\mathcal{N}_\Phi(\Omega)} + \Big(f,g\Big)_P.$$

The bilinear form $(.,.)_{\Phi_P}$ is positive definite because for all $f \in \mathcal{N}_\Phi(\Omega)$ with $(f,f)_{\Phi_P} = 0$, we have

$$0 = (f,f)_{\Phi_P} = (f,f)_{\mathcal{N}_\Phi(\Omega)} + \sum_{k=1}^{Q} |f(z_k)|^2.$$

But then $(f,f)_{\mathcal{N}_\Phi(\Omega)} = 0$, $f \in P$, and $f(z_k) = 0$ hence $f = 0$. Thus $(.,.)_{\Phi_P}$ is positive definite. This implies

Theorem 4.1. *The native space $\mathcal{N}_\Phi(\Omega)$ to a conditionally positive definite kernel Φ on Ω with respect to P carries the inner product $(\cdot,\cdot)_{\Phi_P}$ that is given by*

$$(f,g)_{\Phi_P} = \Big(f - \pi_P(f), g - \pi_P(g)\Big)_\Phi + \Big(\pi_P(f), \pi_P(g)\Big)_P,$$

which turns the decomposition of the native space $\mathcal{N}_\Phi(\Omega)$ of Φ in definition 2.1 *into an orthogonal decomposition. Furthermore, the bilinear forms $(.,.)_\Phi$, $(.,.)_{\Phi_P}$, and $(.,.)_{\mathcal{N}_\Phi(\Omega)}$ coincide on $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$.*

Theorem 4.2. *The native space $\mathcal{N}_\Phi(\Omega)$ to a conditionally positive definite kernel Φ with respect to a finite-dimensional subspace P is a Hilbert space with the extended kernel Φ_P from (4.1) as reproducing kernel, if the inner product $(.,.)_{\Phi_P}$ from Theorem* 4.1 *is used.*

Proof. For proving the reproducing kernel property, we have for all $f \in \mathcal{N}_\Phi(\Omega)$:

$$\begin{aligned}
(f,\Phi_P(x,.))_{\Phi_P} &= (f,\Phi_P(x,.))_{\mathcal{N}_\Phi(\Omega)} + (f,\Phi_P(x,.))_P \\
&= (f,h(x,.))_{\mathcal{N}_\Phi(\Omega)} + 0 + \sum_{k=1}^{Q} f(z_k)h(x,z_k) + \sum_{k=1}^{Q} f(z_k)\sum_{l=1}^{Q} p_l(x)p_l(z_k) \\
&= f(x) - \pi_P\big(f\big)(x) + 0 + \sum_{k=1}^{Q} f(z_k)p_k(x) \\
&= f(x).
\end{aligned}$$

The reproduction property implies that Φ_P is positive semi-definite. To prove positive definiteness, we take a set of points $Y=\{y_1,\cdots,y_M\}\in\Omega\setminus Z$, vectors $\beta=(\beta_1,\cdots,\beta_M)\in\mathbf{R}^M$ and $\gamma=(\gamma_1,\cdots,\gamma_Q)\in\mathbf{R}^Q$. Then we look at the quadratic form:

$$\begin{aligned}
&\sum_{j,k=1}^{M}\beta_j\beta_k\Phi_P(y_j,y_k) + 2\sum_{j=1}^{M}\sum_{k=1}^{Q}\beta_j\gamma_k\Phi_P(y_j,z_k) + \sum_{j,k=1}^{Q}\gamma_j\gamma_k\Phi_P(z_j,z_k) \\
=\;&\sum_{j,k=1}^{M}\beta_j\beta_k h(y_j,y_k) + 0 + 0 + \\
&+\sum_{l=1}^{Q}\sum_{j,k=1}^{M}\beta_j\beta_k p_l(y_j)p_l(y_k) + 2\sum_{l=1}^{Q}\sum_{j=1}^{M}\sum_{k=1}^{Q}\beta_j\gamma_k p_l(y_j)p_l(z_k) \\
&+\sum_{l=1}^{Q}\sum_{j,k=1}^{Q}\gamma_j\gamma_k p_l(z_j)p_l(z_k) \\
=\;&\sum_{j,k=1}^{M}\beta_j\beta_k h(y_j,y_k) + \sum_{l=1}^{Q}\left(\sum_{j=1}^{M}\beta_j p_l(y_j) + \sum_{k=1}^{Q}\gamma_k p_l(z_k)\right)^2 \\
=\;&\sum_{j,k=1}^{M}\beta_j\beta_k h(y_j,y_k) + \sum_{l=1}^{Q}\left(\gamma_l + \sum_{j=1}^{M}\beta_j p_l(y_j)\right)^2,
\end{aligned}$$

and this is nonnegative, since h is positive definite on $\Omega\setminus Z$. If the quadratic form is zero, then we conclude that β vanishes. Finally, γ must also vanish.

Theorem 4.3. *The interpolation associated to Φ_P does coincide with the interpolation associated to Φ, if the data points include the point set Z.*

If the data point set $X = \{x_1, \cdots, x_N\}$ for interpolation includes the point set Z, the interpolant $s_{f,X}$ to some function $f \in \mathcal{N}_\Phi(\Omega)$ on the data X with respect to Φ or Φ_P can be calculated as follows: First calculate $\pi(f)$, this interpolates f on Z. Second interpolate $f - \pi(f)$ in $X \setminus Z$ using the kernels Φ_P or h with no functions from P added, and with no conditions on the coefficients. Call the resulting function s. Then $s + \pi(f) = s_{f,X}$ is the solution.

5 Multistage Recursive RKHS

We are given a set $X = \{x_1, \ldots, x_N\}$ of pairwise distinct points $x_1, \ldots, x_N$ in a set $\Omega \subseteq \mathbf{R}^d$, and a real-valued function f with $f : \Omega \subseteq \mathbf{R}^d \to \mathbf{R}$, $d \geq 1$. We take a conditionally positive definite continuous kernel $\Phi : \Omega \times \Omega \to \mathbf{R}$ with respect to a finite-dimensional subspace P. To avoid complications as in Theorem 4.3, we shall always assume $Z \subseteq X$ in the rest of this paper.

Then we denote the resulting interpolant to f by $s_{f,X,\Phi}$, making the dependence on f, X, and Φ transparent. For all functions $f \in \mathcal{N}_\Phi(\Omega)$, we define the *residual function* or *error function* g_f on Ω by:

$$g_f : x \mapsto f(x) - s_{f,X,\Phi}(x).$$

We now interpolate the function g_f on a new finite set Y of points from $\Omega \setminus X$ using K_X, and denote the interpolant to g_f on Y associated to K_X by s_{g_f,Y,K_X}. We remark that for all $x_j \in X$ with $1 \leq j \leq N$ we have $g_f(x_j) = 0$ and $s_{g_f,Y,K_X}(x_j) = 0$. Then we conclude that for all $x \in X \cup Y$

$$\left(g_f - s_{g_f,Y,K_X}\right)(x) = \left(f - s_{f,X,\Phi} - s_{g_f,Y,K_X}\right)(x) = 0.$$

We want to find a relation between the interpolants $s_{f,X,\Phi} + s_{g_f,Y,K_X}$ and $s_{f,X\cup Y,\Phi}$ to f at all points in $X \cup Y$. The uniqueness of the interpolant for data on $X \cup Y$ with centers in $X \cup Y$ proves the following:

Proposition 5.1. *The interpolant of Φ on $X \cup Y$ is given by*

$$s_{f,X\cup Y,\Phi} = s_{f,X,\Phi} + s_{g_f,Y,K_X}.$$

There is a interesting relation between the power function associated to Φ on X and the power function to K_X on Y. To present this relation, we define the power function as

$$P_{X,\Phi} = \sup\{f(x) : f \in \mathcal{N}_\Phi(\Omega), \ \|f\|_{\mathcal{N}_\Phi(\Omega)} \leq 1, \ f(X) = \{0\}\}. \tag{5.1}$$

Proposition 5.2. *If Φ is a positive definite kernel, then the power function $P_{X\cup Y,\Phi}$ is given by:*

$$P_{X\cup Y,\Phi} = P_{Y,K_X}.$$

Proof. We use the equations (5.1) for $P_{X\cup Y,\Phi}$ and for P_{Y,K_X}. Then, we get

$$P_{X\cup Y,\Phi} = \sup\{f(x) : f \in \mathcal{N}_\Phi(\Omega),\ \|f\|_{\mathcal{N}_\Phi(\Omega)} \le 1,\ \ f(X\cup Y) = \{0\}\}$$

and

$$\begin{aligned}
P_{Y,K_X} &= \sup\{f(x) : f \in \mathcal{N}_{K_X},\ \|f\|_{K_X} \le 1,\ \ f(Y) = \{0\}\} && (5.2)\\
&= \sup\{f(x) : f \in \mathcal{N}_{K_X},\ \|f\|_{\mathcal{N}_\Phi(\Omega)} \le 1,\ \ f(Y) = \{0\}\} && (5.3)\\
&= \sup\left\{ (g-\pi_X(g))(x) : \begin{array}{c} g\in\mathcal{N}_\Phi(\Omega),\ \|g-\pi_X(g)\|_\Phi \le 1 \\ (g-\pi_X(g))(Y) = \{0\} \end{array} \right\} && (5.4)\\
&\le \sup\{f(x) : f \in \mathcal{N}_\Phi(\Omega),\ \|f\|_{\mathcal{N}_\Phi(\Omega)} \le 1,\ \ f(X\cup Y) = \{0\}\} && (5.5)\\
&= P_{X\cup Y,\Phi}. && (5.6)
\end{aligned}$$

The inequality sign in (5.5) follows from the fact that every $g-\pi_X(g)$ of (5.4) is some function f (5.5). The other inequality follows directly when we take any f from (5.5) and define $g := f$ with $\pi_X(f) = \pi_X(g) = 0$ in (5.4).

The proposition 5.2 implies that the interpolation error can be bounded as:

$$|f(x) - s_{f,X\cup Y,\Phi}(x)| \le P_{Y,K_X}\|f\|_\Phi \quad \text{for all} \quad f\in\mathcal{N}_\Phi(\Omega). \tag{5.7}$$

Proposition 5.3. *If Φ is an unconditionally positive definite kernel, then for finite sets X, Y with $X\cap Y = \emptyset$ we have*

$$(K_{X,\Phi})_{Y,K_{X,\Phi}} = K_{X\cup Y,\Phi},$$

where we indicated the appropriate "mother" kernels in the notation.

Proof. The native space of the right-hand side is

$$\{f\in\mathcal{N}_\Phi(\Omega)\ :\ f(X\cup Y) = \{0\}\},$$

while the native space for the left-hand side is the same:

$$\left\{f\in\mathcal{N}_{K_{X,\Phi}}\ :\ f(Y) = \{0\}\right\} = \{f\in\mathcal{N}_\Phi\ :\ f(X\cup Y) = \{0\}\}.$$

If f is an element of that space, the two reproduction properties are

$$\begin{aligned}
f(x) &= (f, K_{X\cup Y,\Phi}(x,\cdot))_\Phi \\
f(x) &= (f, (K_{X,\Phi})_{Y,K_{X,\Phi}}(x,\cdot))_{K_{X,\Phi}}
\end{aligned},$$

and since Φ is an unconditionally positive definite kernel, we can use any of the inner products $(.,.)_\Phi$, $(.,.)_{K_{X,\Phi}}$ and $(.,.)_{K_{X\cup Y,\Phi}}$ here. Now by uniqueness of reproducing kernels, the assertion follows first on $\Omega \setminus (X\cup Y)$, but since both kernels vanish on $X\cup Y$, we are done.

The appropriate mother kernels $(K_{X,\Phi})_{Y,K_{X,\Phi}}$ implies that there are orthogonal decompositions with respect to the inner product $(.,.)_\Phi$ of $\mathcal{N}_\Phi(\Omega)$ given by:

$$\mathcal{N}_\Phi(\Omega) = S_{X,\Phi} +, \mathcal{N}_{K_{X,\Phi}}(\Omega\setminus X) = S_{X\cup Y,\Phi} + \mathcal{N}_{K_{X\cup Y,\Phi}}(\Omega\setminus (X\cup Y)). \tag{5.8}$$

So far, we have made a step from X to $X\cup Y$. We now want to do a sequence of such steps.

Definition 5.4. We can assume to have a kernel Φ on Ω which is conditionally positive definite with respect to some finite-dimensional space P of functions on Ω, and we want to interpolate a function $f \in \mathcal{N}_\Phi(\Omega)$. We start with a finite set X of data points which contains a P-unisolvent subset Z. The recursion algorithm will be then given by the Table-1.

Table 1. Recursive Algorithm

Start a recursion with	**Do the following for** $j \geq 0$
$j := 0$	$j \geq 0$
$X_0 := X$	$X_{j+1} := X_j \cup Y_j \supset X_j$
$\Omega_0 := \Omega \setminus X$	$\Omega_{j+1} := \Omega \setminus X_{j+1} \subset \Omega_j$
$Y_0 \subset \Omega_0$	$Y_j \subset \Omega_j$
$\Phi_0 := \Phi$	$\Phi_{j+1} := K_{X_j}$ **Positive Definite** on Ω_j
$f_0 := f - s_{f,X_0,\Phi_0}$	$f_{j+1} := f_j - s_{f_j,Y_j,\Phi_{j+1}}$
	$j := j+1$, **Repeat**.

Note that from the second step on we have unconditional positive definiteness, while the first step from $\Phi_0 := \Phi$ to $\Phi_1 := K_X$ has the complications we encountered around equation (2.15).

Theorem 5.5. *The native space $\mathcal{N}_{\Phi_{j+1}}$ of Φ_{j+1} can be expressed as*

$$\mathcal{N}_{\Phi_{j+1}} = \mathcal{N}_{\Phi_1} \cap \{f |\ f(X_j) = \{0\}\}\ \textit{for all}\ j \geq 0.$$

and for all $j \geq 1$ and $f \in \mathcal{N}_\Phi(\Omega)$, using equation (2.15) we get

$$\|f\|^2_{\Phi_j} = \|f\|^2_{\Phi_{j-1}} = \|f\|^2_{\Phi_{j-2}} = \dots = \|f\|^2_{\Phi_1} = \|f\|^2_\Phi + \|s_{g_y - g_X,X,\Phi}\|^2_\Phi. \tag{5.9}$$

with g_Y and g_X from (2.14).

Now we need the recursive power kernel form of our kernels. We conclude

$$\Phi_{j+1} = K_{X_j,\Phi} = K_{Y_{j-1}\cup X_{j-1},\Phi} = \big(K_{X_{j-1},\Phi}\big)_{Y_{j-1},K_{X_{j-1},\Phi}}.$$

Using Proposition 5.3, the last implies

$$\Phi_{j+1} = (\Phi_j)_{Y_{j-1},\Phi_j} \text{ for all } j \geq 1.$$

Note that we can always write the function f_j in the form

$$f_j = f_j - s_{f_j,Y_j,\Phi_{j+1}} + s_{f_j,Y_j,\Phi_{j+1}} = f_{j+1} + s_{f_j,Y_j,\Phi_{j+1}} \text{ for all } j \geq 0. \tag{5.10}$$

The orthogonality property of the interpolation together with (2.15) allows us to write

$$\|f_j\|^2_{\Phi_{j+1}} = \|f_{j+1}\|^2_{\Phi_{j+1}} + \|s_{f_j,Y_j,\Phi_{j+1}}\|^2_{\Phi_{j+1}} = \|f_j\|^2_{\Phi_j} \quad \text{for all } j \geq 0. \tag{5.11}$$

Under the hypotheses of the definition 5.4 and using (5.11), the summation term-to-term gives

$$\|f_1\|^2_{\Phi_1} = \|f_{j+1}\|^2_{\Phi_{j+1}} + \sum_{r=1}^{j} \|s_{f_r,Y_r,\Phi_{r+1}}\|^2_{\Phi_{r+1}} \text{ for all } j \geq 0. \tag{5.12}$$

Now we need the recursive interpolation error form using our kernels. Under the hypotheses of the definition 5.4, we can generalize Proposition 5.1 to get

$$s_{f,X_{j+1},\Phi} = s_{f,X_j,\Phi} + s_{f_j,Y_j,\Phi_{j+1}} \qquad \text{for all } j \geq 0. \tag{5.13}$$

Theorem 5.6. *Under the hypotheses of the definition* 5.4 *and using* (5.13)*, the interpolation error for any function $f \in \mathcal{N}_\Phi(\Omega)$ is given by*

$$f_j = f - s_{f,X_j,\Phi}.$$

Proof. We will prove it by induction. The assertion is true for $j = 0$. If it holds for j, we use (5.13) to get

$$\begin{aligned} f_{j+1} &= f_j - s_{f_j,Y_j,\Phi_{j+1}} \\ &= f - s_{f,X_j,\Phi} - s_{f_j,Y_j,\Phi_{j+1}} \\ &= f - s_{f,X_{j+1},\Phi}. \end{aligned}$$

The relation of the power kernel to the power function implies that we can generalize the result of Proposition 5.2 to get

$$P_{X_{j+1},\Phi_j}(x) = P_{Y_j,\Phi_{j+1}}(x) \quad \text{for all} \quad j \geq 1. \tag{5.14}$$

Proposition 5.2 and (5.14) show

$$P_{X_{j+1},\Phi_j}(x) = P_{X_{j+1},\Phi_1}(x) \quad \text{for all} \quad j \geq 1. \tag{5.15}$$

The last work that we wanr to show is to bound the interpolation error on the set X_i.

Lemma 5.7. *Under the preceding assumptions of the definition 5.4 and for all $x \in \Omega_j$ we have:*

$$|f(x) - s_{f,X_j,\Phi}(x)| \leq P_{X_j,\Phi_1}(x)\|f_{j-1}\|_{\Phi_1}.$$

Proof. We start with the definition of f_j to get

$$|f_j(x)| = |f_{j-1}(x) - s_{f_{j-1},X_{j-1},\Phi_j}(x)| \leq P_{Y_{j-1},\Phi_j}(x)\|f_{j-1}\|_{\Phi_j}.$$

Using (5.14) and (5.15), we get:

$$P_{Y_{j-1},\Phi_j}(x) = P_{X_j,\Phi_{j-1}}(x) = P_{X_j,\Phi_1}(x).$$

Then

$$|f_j(x)| \leq P_{X_j,\Phi_1}(x)\|f_{j-1}\|_{\Phi_1}.$$

Theorem 5.8. Let Φ be a positive definite kernel on a domain Ω, X_i be a set of point satisfied the hypothesis of the assumptions of the definition 5.4, and K_x be the power kernel. Then for any function $f \in NPO$ we have:

$$|f(x) - s_{f,X_j,\Phi}(x)| \leq P_{X_j,K_X}(x)\|f - s_{f,X,\Phi}\|_{\Phi}.$$

Proof. Using Corollary 3.1 until order $j-1$ we get the equation:

$$\|f_{j-1}\|^2_{\Phi_1} = \|f_{j-1}\|^2_{\Phi_{j-1}} = \|f_1\|^2_{\Phi_1} - \sum_{r=1}^{j-2} \|s_{f_r,Y_r,\Phi_{r+1}}\|^2_{\Phi_r},$$

for all $j \geq 2$. We can go one step further, using:

$$\begin{aligned} \|f_0\|^2_{\Phi_1} &= \|f_0 - s_{f_0,Y_0,\Phi_1}\|^2_{\Phi_1} + \|s_{f_0,Y_0,\Phi_1}\|^2_{\Phi_1} \\ &= \|f_1\|^2_{\Phi_1} + \|s_{f_0,Y_0,\Phi_1}\|^2_{\Phi_1} \end{aligned}$$

to get:

$$\|f_{j-1}\|^2_{\Phi_1} = \|f_0\|^2_{\Phi_1} - \sum_{r=0}^{j-2} \|s_{f_r,Y_r,\Phi_{r+1}}\|^2_{\Phi_r} \leq \|f - s_{f,X,\Phi}\|^2_{\Phi_1}.$$

If Φ is unconditionally positive definite, we can replace the norm by $\|.\|_{\Phi}$. Thus the assertion is proven via Lemma 5.7.

Using (5.8) and the hypothesis of of the assumptions of the definition 5.4, one can write down an orthogonal decomposition of the native space related to the above construction as follows:

Theorem 5.9. *The orthogonal decomposition of the native space of order n under the assumptions of the definition 5.4 is given by:*

$$\mathcal{N}_{\Phi}(\Omega) = S_{X_n,\Phi} + \mathcal{N}_{\Phi_{n+1}}(\Omega_n).$$

Conclusion

The theory of power kernels and their native spaces from any conditionally positive definite permits to decompose orthogonally the reproducing kernels Hilbert space of a superior order using any conditionally positive definite kernels into smaller ones. We have shown how to do this decomposition recursively. We also have shown how the power kernels are used to split large interpolation problems into smaller ones with different kernels which are related to the original kernels. This permits to characterize the the interpolant of superior order.

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