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A MATHEMATICAL PROOF OF A PROBABILISTIC MODEL OF HARDY'S INEQUALITY

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Abstract. In this paper using an argument from [1] , we prove one of the probabilistic version of Hardy's inequality.

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1 Introduction

Hardy's inequality is defined as for a constant $c > 0$, we have

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq c\|f\|_1$$

for all functions $f \in L^1([0,2\pi))$ with $\hat{f}(n) = 0$ for $n < 0$. This inequality is not true for all functions $f \in L^1([0,2\pi))$, which can be seen by letting f to be the Fejér kernel of order N for large enough N.

When McGehee, Pigno and Smith[3] proved the Littlewood conjecture, many questions were asked of how Hardy's inequality can be generalized for all functions $f \in L^1([0,2\pi))$. For instance, one of the expected generalizations is the following:

$$\sum_{n>0} \frac{\hat{f}(n)|}{n} \leq c\|f\|_1 + c\sum_{n>0} \frac{|\hat{f}(-n)|}{n}, \qquad f \in L^1([0,2\pi)),$$

where $c > 0$ is an absolute constant.

In this paper, we prove one version of Hardy's inequality for functions whose Fourier coefficients $\hat{f}(n)$ are random variables on $(0,1)$ for $n > 0$ without conditions on $\hat{f}(n)$ for $n < 0$.

In my proof use a technique that was motivated by Körner[1], who used this technique in a different problem to modify a result of Byrnes (see [1]).

In the sequel, $[0,2\pi)$ denotes the unit circle, $L^1([0,2\pi)$ the space of integrable functions on $[0,2\pi), \mu$ the Lebesgue measure, and B_j the set of integers in the interval $[4^{j-1},4^j)$.

2 Basic Lemmas

In this section, I am going to prove some basic lemmas required for our purpose.

Lemma 2.1. *Let $X_1,X_2,\cdots,X_N$ be independent random variables such that*

$$|X_j| \leq 1 \qquad for\ each \qquad j, 1 \leq j \leq N,$$

and write

$$S_N = X_1 + X_2 + \cdots + X_N.$$

Then, for any $\lambda > 0$,

$$Pr(|S_N - ES_N| \geq \lambda) \leq 4\exp(-\frac{\lambda^2}{100N}).$$

For the proof, see [4, p.398].

The idea of the following proof is due to Köner[1]. The statement of the lemma was observed by Kahane [2] without proof.

Lemma 2.2. *Let (r_k) be a sequence of independent, zero mean random variables defined on the interval $(0,1)$with $|r_k| \leq 1$ for all k. Let*

$$f_n(\theta,t) = \sum_{p=1}^{n} r_p(t)e^{ip\theta} \quad for \quad t \in (0,1) \quad and \quad \theta \in [0,2\pi).$$

Then for $n \geq 27$ and $\lambda \geq 2 \times 2$,

$$\mu(\{t : \sup_{\theta} |f_n(\theta,t)| \geq \lambda\sqrt{n\log n}\}) \leq 4n^{2-\frac{\lambda^2}{400}}.$$

Proof. By applying Lemma 2.1, we find that for fixed $\theta \in [0,2\pi)$,

$$\mu(\{t : \sup_{\theta} |f_n(\theta,t)| \geq \lambda\sqrt{n\log n}\}) \leq 4n^{2-\frac{\lambda^2}{100}}.$$

Let $(\theta_k)_{k=1}^{n^2}$ be a uniform partition of the unit circle. For fixed $t \in (0,1)$ and $\theta_k \in [0,2\pi)$ and for all θ with $|\theta - \theta_k| \leq 2\pi/n^2$, we have

$$|f_n(\theta,t) - f_n(\theta_k,t)| \leq \sum_{p=1}^{n} |r_p(t)||e^{ip\theta} - e^{ip\theta_k}| \leq 2\sum_{p=1}^{n} \frac{2\pi}{n^2}p = \frac{2\pi(n+1)}{n}.$$

Lemma 2.3. *There exists a set $\subset (0,1)$ of measure 1 such that whenever $t \in B$ there exists an index k_t with the property that*

$$\sup_{\theta} |g_j(\theta,t)| \geq 60\sqrt{j4^{-j}}, \qquad \forall j \geq k_t.$$

Proof. Let

$$M_k = \bigcup_{j=k}^{\infty} A_j \qquad \text{also} \qquad M = \bigcap_{k=1}^{\infty} M_k.$$

Thus,

$$\mu(M) = \mu\left(\bigcap_{k=1}^{\infty} M_k\right) \leq \mu(M_k)$$

for all $k \geq 1$, i.e.,

$$\mu(M) \leq \mu(\left(\bigcup_{j=k}^{\infty} A_j\right) \leq \sum_{j=k}^{\infty} \mu(A_j)$$

for all $k \geq 1$. As

$$\mu(A_j) \leq 8 \times 4^{-j/4} \quad \text{and} \quad \sum 4^{-j/4} < \infty,$$

hence

$$\mu(M) \leq \sum_{j=k}^{\infty} \mu(A_j) \to 0 \quad \text{as} \quad k \to \infty.$$

Thus, $\mu(M) = 0$. Putting $B = M^C$, the lemma is proved.

3 Main Result

In this section, we prove the probabilistic version of Hardy's inequality, which is main contribution in this paper.

Thus, for fixed t and θ_k and for all θ such that $|\theta - \theta_k| \leq 2\pi/n^2$, we have

$$|f_n(\theta,t)| \leq \frac{2\pi(n+1)}{n} + |f_n(\theta_k,t)|,$$

and consequently

$$\sup_{|\theta-\theta_k| \leq \pi/n^2} |f_n(\theta,t)| \leq \frac{2\pi(n+1)}{n} + |f_n(\theta_k,t)|.$$

But on a set (of t) of measure $\geq 1 - 4n^{2-\frac{\lambda^2}{100}}$ we have for each θ_k

$$|f_n(\theta_k,t) \leq \lambda\sqrt{n\log n}.$$

Therefore, for any particular θ_k we have on a set (of t) of measure $\geq 1-4n^{2-\frac{\lambda^2}{100}}$,

$$\sup_{\theta-\theta_k|\leq 2\pi/n^2} |f_n(\theta,t)| \leq \frac{2\pi(n+1)}{n} + \lambda\sqrt{n\log n}.$$

Since the set

$$\left\{t : \sup_{\theta} |f_n(\theta,t)| \geq \frac{2\pi(n+1)}{n} + \lambda\sqrt{n\log n}\right\}$$

is contained in the set

$$\bigcup_{k=1}^{n^2} \left\{t : \sup_{|\theta-\theta_k|\leq 2\pi/n^2} |f_n(\theta,t)| \leq \frac{2\pi(n+1)}{n} + \lambda\sqrt{n\log n}\right\},$$

we must have

$$\mu\left(\left\{t : \sup_{\theta} |f_n(\theta,t)| \geq \frac{2\pi(n+1)}{n} + \lambda\sqrt{n\log n}\right\}\right) \leq \sum_{p=1}^{n^2} 4n^{-\frac{\lambda^2}{100}} = 4n^{2-\frac{\lambda^2}{100}}.$$

If $\lambda \geq \sqrt{2}$ and $n \geq 27$, we have

$$\frac{2\pi(n+1)}{n} \leq \lambda\sqrt{n\log n},$$

hence it follows that

$$\mu(\{t : \sup_{\theta} |f_n(\theta,t)| \geq 2\lambda\sqrt{n\log n}\}) \leq 4n^{2-\frac{\lambda^2}{100}}.$$

On replacing 2λ by λ

$$\mu(\{t : \sup_{\theta} |f_n(\theta,t)| \leq \lambda\sqrt{n\log n}\}) \leq 4n^{2-\frac{\lambda^2}{400}}.$$

whenever $\lambda \geq 2\sqrt{2}$ and $n \geq 27$.

Thus, by letting

$$g_j(\theta,t) = \sum_{n\in B_j} r_n(t)e^{in\theta},$$

where B_j denotes the set of integers in the interval $[4^{j-1}, 4^j)$, we see that

$$\mu\left(\left\{t : \sup_{\theta} |g_j(\theta,t)| \geq 2\lambda\sqrt{j4^{-j}}\right\}\right) \leq 4(\frac{3}{4})^{2-\frac{\lambda^2}{400}}(4^{(2-\lambda^2)\frac{j}{400}}$$

for all $\lambda \geq 2\sqrt{2}$ and $j \geq 4$. By choosing $\lambda = 30$, we see that

$$\mu(A_j) \leq 8\times 4^{-j/4} \qquad \text{for} \qquad j \geq 4,$$

where

$$A_j = \left\{ t : \sup_{\theta} |g_j(\theta,t)| \geq 60\sqrt{j4^{-j}} \right\}.$$

Theorem 3.1. *Let (r_k) be a sequence of independent, zero mean random variables on the interval $(0,1)$, with $|r_k| = 1$ for all k. Then there exists a set $S \subset (0,1)$ of measure 1 such that*

$$\sum_{n=1}^{\infty} \frac{\hat{f}(n)|}{n} \leq c\|f\|_1$$

for all functions $f \in L^1([0,2\pi))$ satisfying the condition

$$\hat{f}(n)\overline{r_n(t)} \geq 0, \qquad \text{for all} \qquad n > 0.$$

Proof. Let $t \in B$ be fixed. It suffices to prove the result for all trigonometric polynomials f with

$$\hat{f}(n)\overline{r_n(t)} \geq 0, \qquad \text{for all} \qquad n > 0.$$

Thus, let f be a trigonometric polynomials with $\hat{f}(n)\overline{r_n(t)} \geq 0$

$$F(\theta) = \sum_{j=1}^{\infty} g_j(\theta,t).$$

It is clear from the definition of g_j that

$$\hat{F}(n) = \frac{r_n(t)}{4^j}, \qquad \text{for} \qquad n > 0,$$

where j is the unique index such that $n \in B_j$. Also, we see that

$$\hat{F}(n) = 0, \qquad \text{for} \qquad n \leq 0.$$

Since $t \in B$, we conclude that

$$\sum_{j=1}^{\infty} \sup_{\theta\in[0,2\pi)} |g_j(\theta,t)| := K < \infty.$$

Therefore, F is a bounded function on the circle with $\|F\|_\infty \leq K$.

Now, we apply a standard duality argument to obtain

$$\begin{aligned} K\|f\|_1 &:= \|F\|_\infty\|f\|_1 \geq \frac{1}{2\pi}\left|\int_0^{2\pi} f(\theta)\overline{F(\theta)}\mathrm{d}\theta\right| = \left|\sum_{n\in\mathbf{Z}} \hat{f}(n)\overline{\hat{F}(n)}\right| \\ &= \left|\hat{f}(0)\overline{\hat{F}(0)} + \sum_{n>0} \hat{f}(n)\overline{\hat{F}(n)}\right|, \\ K\|f\|_1 &= \left|\sum_{n>0} \hat{f}(n)\overline{\hat{F}(n)}\right| - |\hat{f}(0)||\hat{F}(0)|, \end{aligned}$$

hence,

$$2\|f\|_1\|F\|_\infty \geq \left|\sum_{j=1}^{\infty}\sum_{n\in B_j}\hat{f}(n)\overline{\hat{F}(n)}\right| = \left|\sum_{j=1}^{\infty}\sum_{n\in B_j}\hat{f}(n)\frac{\overline{r_n(t)}}{4^j}\right| \geq \frac{1}{4}\sum_{j=1}^{\infty}\sum_{n\in B_j}\frac{|\hat{f}(n)|}{n}.$$

Thus we have proven the above theorem for $c = 8k$.

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