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ABSOLUTE RETRACTIVITY OF SOME SETS TO TWO-VARIABLES MULTIFUNCTIONS

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Abstract. In this paper, by providing some different conditions respect to another works, we shall present two results on absolute retractivity of some sets related to some multifunctions of the form $F: X\times X\to P_{b,cl}(X)$, on complete metric spaces.

Key words: *absolute retract, fixed points set, multifunction*

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1 Introduction

In 1970, Schirmer provided some results about topological properties of the fixed point set of multifunctions[5]. Later, some authors continued this review by providing different conditions [1],[3]. Recently, Sintamarian proved some results on absolute retractivity of the common fixed points set of two multivalued operators[6],[7]. Also, Afshari, Rezapour and Shahzad proved some results about absolute retractivity of the common fixed points set of two multifunctions[4]. In this paper, by providing some different conditions respect to another works, we shall present two results on absolute retractivity of some sets related to some multifunctions of the form $F: X\times X\to P_{b,cl}(X)$. Let X and Y be nonempty sets, $P(Y)$ the set of all nonempty subsets of Y, and $F: X\to P(Y)$ a multifunctions. A mapping $\varphi: X\to Y$ is called a selection of F whenever $\varphi(x)\in Fx$ for all $x\in X$. Throughout the paper, for a topological space X we denote the set of all closed and bounded subsets of X by $P_{b,cl}(X)$ when X is a metric space.

Let (X,d) be a metric space, $B(x_0,r)=\{x\in X: d(x_0,x)<r\}$. For $x\in X$ and $A,B\subseteq X$, set $d(x,A)=\inf_{y\in A} d(x,y)$ and

$$H(A,B)=\max\{\sup_{x\in A} d(x,B),\sup_{y\in B} d(y,A)\}.$$

It is known that, H is a metric on closed bounded subsets of X which is called the Hausdorff metric (for more details see [6] and [7]).

We say that a topological space X is an absolute retract for metric spaces whenever for each metric space Y, $A \in P_{cl}(Y)$ and continuous function $\psi : A \to X$, there exists a continuous function $\varphi : Y \to X$ such that $\varphi|_A = \psi$. Let $\mathcal{M}$ be the set of all metric spaces, $X \in \mathcal{M}$, $\mathcal{D} \in P(\mathcal{M})$ and $F : X \to P_{b,cl}(X)$ a lower semi-continuous multifunction. We say that F has the selection property with respect to $\mathcal{D}$ if for each $Y \in \mathcal{D}$, continuous function $f : Y \to X$ and continuous functional $g : Y \to (0,\infty)$ such that $G(y) := \overline{F(f(y)) \cap B(f(y), g(y))} \neq \emptyset$ for all $y \in Y$, $A \in P_{cl}(Y)$, every continuous selection $\psi : A \to X$ of $G|_A$ admits a continuous extension $\varphi : Y \to X$, which is a selection of G. If $\mathcal{D} = \mathcal{M}$, then we say that F has the selection property and we denote this by $F \in SP(X)$ (for more details see [6] and [7]).

2 Main Results

Theorem 2.1. *Let (X,d) be a complete metric space and absolute retract for metric spaces and $F : X \times X \to P_{b,cl}(X)$ a lower semicontinuous multifunction such that there exist $a_{11}, a_{12}, \cdots, a_{15}, a_{21}, a_{22}, \ldots, a_{25} \in (0,1)$ with $a_{11}+a_{13}+a_{14}+2a_{12} < 1$, $a_{21}+a_{23}+a_{24}+2a_{22} < 1$,*

$$\begin{aligned} H(F(u,v),F(x,y)) \leq\ & a_{11}d(x,u)+a_{12}d(x,F(u,v)) \\ & +a_{13}d(F(x,y),x)+a_{14}d(F(u,v),u)+a_{15}d(u,F(x,y)) \end{aligned}$$

and

$$\begin{aligned} H(F(u,v),F(x,y)) \leq\ & a_{21}d(y,v)+a_{22}d(y,F(u,v)) \\ & +a_{23}d(F(x,y),y)+a_{24}d(F(u,v),v)+a_{25}d(F(x,y),v) \end{aligned}$$

for all $u,v,x,y \in X$. Then the set $B = \{(x,y) : x,y \in F(x,y)\}$ is an absolute retract for metric spaces.

Proof. It is easy to see that $F \in SP(X \times X)$ and $X \times X$ is an absolute retract for metric spaces. Now, put $1 < q < \min\{(a_{11}+a_{13}+a_{14}+2a_{12})^{-1}, (a_{21}+a_{23}+a_{24}+2a_{22})^{-1}\}$ and

$$l := \max\{\frac{a_{11}+a_{12}+a_{13}}{1-(a_{12}+a_{14})}, \frac{a_{21}+a_{22}+a_{23}}{1-(a_{22}+a_{24})}\}.$$

It is not difficult to verify that $ql < 1$. Let Y be a metric space, $A \in P_{cl}(Y)$ and $\psi : A \to B$ a continuous function. Since $X \times X$ is an absolute retract for metric spaces, there exists a continuous function $\varphi_0 : Y \to X \times X$ such that $\varphi_0|_A = \psi$. Let $\varphi_0 = (\varphi_0^1, \varphi_0^2)$. Consider the function

$g_0 : Y \to (0,\infty) \times (0,\infty)$ defined by $g_0 = (g_0^1, g_0^2)$, where g_0^i is defined by

$$g_0^i(y) = \sup\{d(\varphi_0^i(y), z) : z \in F(\varphi_0(y))\} + 1, \qquad i = 1,2$$

for all $y \in Y$. It is easy to see that the function g_0 is continuous. Define

$$G_1{}^1(y) := F(\varphi_0(y)) \cap B(\varphi_0^1(y), g_0^1(y)) = F(\varphi_0(y))$$

for all $y \in Y$. Note that, the function ψ is a continuous selection of the multivalued mapping $A \ni y \vdash F(\varphi_0(y))$. Since $F \in SP(X \times X)$, there exists a continuous function $\varphi_1 : Y \to X \times X$ such that $\varphi_1|_A = \psi$ and $\varphi_1^i(y) \in F(\varphi_0(y))$ (i=1,2), where $\varphi_1 = (\varphi_1^1, \varphi_1^2)$. Thus, we obtain

$$\begin{aligned} d(\varphi_1^1(y), F(\varphi_1(y))) &= d(\varphi_1^1(y), F(\varphi_1^1(y), \varphi_1^2(y))) \\ &\le H(F(\varphi_0(y)), F(\varphi_1(y))) = H(F(\varphi_0^1(y), \varphi_0^2(y)), F(\varphi_1^1(y), \varphi_1^2(y))) \\ &\le a_{11}d(\varphi_0^1(y), \varphi_1^1(y)) + a_{12}d(\varphi_0^1(y), F(\varphi_1^1(y), \varphi_1^2(y))) \\ &\quad + a_{13}d(F(\varphi_0^1(y), \varphi_0^2(y)), \varphi_0^1(y)) + a_{14}d(F(\varphi_1^1(y), \varphi_1^2(y)), \varphi_1^1(y)) \\ &\quad + a_{15}d(\varphi_1^1(y), F(\varphi_0^1(y), \varphi_0^2(y))) \\ &\le a_{11}d(\varphi_0^1(y), \varphi_1^1(y)) + a_{12}d(\varphi_0^1(y), \varphi_1^1(y)) + a_{12}d(\varphi_1^1(y), F(\varphi_1^1)(y), \varphi_1^2(y)) \\ &\quad + a_{13}d(\varphi_0^1(y), \varphi_1^1(y)) + a_{14}d(F(\varphi_1^1(y), \varphi_1^2(y)), \varphi_1^1(y)), \end{aligned}$$

for all $y \in X$. Hence,

$$\begin{aligned} &(1 - a_{12} - a_{14})d(\varphi_1^1(y), F(\varphi_1^1(y), \varphi_1^2(y))) \le (a_{11} + a_{12} + a_{13})d(\varphi_0^1(y), \varphi_1^1(y)) \\ &\le \frac{a_{11} + a_{12} + a_{13}}{1 - (a_{12} + a_{14})} d(\varphi_0^1(y), \varphi_1^1(y)) \le l d(\varphi_0^1(y), \varphi_1^1(y)) < l d(\varphi_0^1(y), \varphi_1^1(y)) + q^{-1}, \end{aligned}$$

for all $y \in X$. Thus, $G_2{}^1(y) := F(\varphi_1(y)) \cap B(\varphi_1{}^1(y), l d(\varphi_0{}^1(y), \varphi_1{}^1(y)) + q^{-1}) \neq \phi$. But since $F \in SP(X \times X)$, there exists a continuous function $\varphi_2{}^1 : Y \to X \times X$ such that $\varphi_2|_A = \psi$ and $\varphi_2{}^1(y) \in \overline{G_2{}^1(y)}$ for all $y \in Y$. Hence, $\varphi_2|_A = \psi$, $\varphi_2{}^1(y) \in F(\varphi_1(y))$ and

$$d(\varphi_1{}^1(y), \varphi_2{}^1(y)) \le l d(\varphi_0{}^1(y), \varphi_1{}^1(y)) + q^{-1}$$

for all $y \in Y$. Now, by using a similar technique we obtain

$$d(\varphi_2{}^1(y), F(\varphi_2(y))) \le l^2 d(\varphi_0{}^1(y), \varphi_1{}^1(y)) + q^{-2}.$$

By continuing this process, we obtain a sequence of continuous functions $\{\varphi_n : Y \to X\}_{n \ge 0}$ such that $\varphi_n{}^1|_A = \psi$, $\varphi_n{}^1(y) \in F\varphi_{n-1}(y)$ and

$$d(\varphi_{n-1}{}^1(y), \varphi_n{}^1(y)) \le l^{n-1} d(\varphi_0{}^1(y), \varphi_1{}^1(y)) + q^{-(n-1)}$$

for all $n \geq 1$ and $y \in Y$. Define

$$Y_\lambda := \{y \in Y : d(\varphi_0{}^1(y), \varphi_1{}^1(y)) < \lambda\}$$

for all $\lambda > 0$. Now, we prove that the family $\{Y_\lambda : \lambda > 0\}$ is an open covering of Y. Note that for each $y \in Y$, since $\varphi_1{}^1(y) \in F(\varphi_0(y))$ and $F(\varphi_0(y)) \cap B(\varphi_0^1(y), g_0^1(y)) = F(\varphi_0(y))$, we have $\varphi_1{}^1(y) \in B(\varphi_0^1(y), g_0{}^1(y))$. If $\lambda = g_0(y)$, then $d(\varphi_0{}^1(y), \varphi_1{}^1(y)) < \lambda$. Thus, $y \in Y_\lambda$ and so $Y \subseteq \cup_{\lambda>0}\{y \in Y : d(\varphi_0{}^1(y), \varphi_1{}^1(y)) < \lambda\}$. Since $l < 1$, $q > 1$ and X is complete, the sequence $\{\varphi_n{}^1\}_{n\geq 0}$ converges uniformly on Y_λ for all $\lambda > 0$. Note that,

$$\begin{aligned}
d(\varphi_1{}^2(y), F(\varphi_1(y))) &= d(\varphi_1{}^2(y), F(\varphi_1{}^1(y), \varphi_1{}^2(y))) \\
&\leq H(F(\varphi_0(y)), F(\varphi_1(y))) = H(F(\varphi_0{}^1(y), \varphi_0{}^2(y)), F(\varphi_1{}^1(y), \varphi_1{}^2(y))) \\
&\leq a_{21} d(\varphi_0{}^2(y), \varphi_1{}^2(y)) + a_{22} d(\varphi_0{}^2(y), F(\varphi_1{}^1(y), \varphi_1{}^2(y))) \\
&\quad + a_{23} d(F(\varphi_0{}^1(y), \varphi_0{}^2(y)), \varphi_0{}^2(y)) \\
&\quad + a_{24} d(F(\varphi_1{}^1(y), \varphi_1{}^2(y)), \varphi_1{}^2(y)) + a_{25} d(\varphi_1{}^2(y), F(\varphi_0{}^1(y), \varphi_0{}^2(y))) \\
&\leq a_{21} d(\varphi_0{}^2(y), \varphi_1{}^2(y)) + a_{22} d(\varphi_0{}^2(y), \varphi_1{}^2(y)) + a_{22} d(\varphi_1{}^2(y), F(\varphi_1{}^1(y), \varphi_1{}^2(y))) \\
&\quad + a_{23} d(\varphi_0{}^2(y), \varphi_1{}^2(y)) + a_{24} d(\varphi_1{}^2(y), F(\varphi_1{}^1(y), \varphi_1{}^2(y)))
\end{aligned}$$

for all $y \in Y$. Thus,

$$\begin{aligned}
d(\varphi_1{}^2(y), F(\varphi_1{}^1(y), \varphi_1{}^2(y))) &\leq \frac{a_{21} + a_{22} + a_{23}}{1 - (a_{22} + a_{24})} d(\varphi_0{}^2(y), \varphi_1{}^2(y)) \\
&\leq l d(\varphi_0{}^2(y), \varphi_1{}^2(y)) < l d(\varphi_0{}^2(y), \varphi_1{}^2(y)) + q^{-1}
\end{aligned}$$

for all $y \in Y$. Hence $G_2{}^2(y) := F(\varphi_1(y)) \cap B(\varphi_1{}^2(y), l d(\varphi_0{}^2(y), \varphi_1{}^2(y)) + q^{-1}) \neq \phi$. But since $F \in SP(X \times X)$, there exists a continuous function $\varphi_2 : Y \to X \times X$ such that $\varphi_2|_A = \psi$, $\varphi_2^2(y) \in F(\varphi_1(y))$ and

$$d(\varphi_1{}^2(y), \varphi_2{}^2(y)) \leq l d(\varphi_0{}^2(y), \varphi_1{}^2(y)) + q^{-1}$$

for all $y \in Y$. Thus,

$$d(\varphi_2{}^2(y), F(\varphi_2{}^2(y))) \leq l^2 d(\varphi_0{}^2(y), \varphi_1{}^2(y)) + q^{-2}$$

for all $y \in Y$. Again by continuing this process, we obtain a sequence of continuous functions $\{\varphi_n{}^2 : Y \to X\}_{n\geq 0}$ such that $\varphi_n{}^2|_A = \psi$, $\varphi_n{}^2(y) \in F(\varphi_{n-1}(y))$ and

$$d(\varphi_{n-1}{}^2(y), \varphi_n{}^2(y)) \leq l^{n-1} d(\varphi_0{}^2(y), \varphi_1{}^2(y)) + q^{-(n-1)}$$

for all $n \geq 0$ and $y \in Y$. Now for each $\lambda > 0$, put $Y_\lambda = \{y \in Y : d(\varphi_0{}^2(y), \varphi_1{}^2(y)) < \lambda\}$. Since $l < 1$, $q > 1$ and X is complete, the sequence $\{\varphi_n{}^2\}_{n\geq 1}$ converges uniformly on Y_λ for all $\lambda > 0$.

Let $\varphi^i : Y \to X$ be the pointwise limit of $\{\varphi_n{}^i\}_{n\geq 0}$ (i=1,2). Then, the function φ^i is continuous. Thus $\varphi : Y \to X\times X$ defined by $\varphi(y) = (\varphi^1(y),\varphi^2(y))$ is continuous. Since $\varphi_n|_A = \psi$ for all $n\geq 1$, $\varphi|_A = \psi$. Note that,

$$\varphi_n^1(y)\in F(\varphi_{n-1}(y)),\varphi_n{}^2(y)\in F(\varphi_{n-1}(y))$$

for all $y\in Y$ and $n\geq 1$. If $n\to\infty$, then $\varphi^1(y),\varphi^2(y)\in F(\varphi(y))$. Therefore, the set $\{(x,y) : x,y\in F(x,y)\}$ is an absolute retract for metric spaces.

Also by providing a similar technique, we can prove the following results.

Theorem 2.2. *Let (X,d) be a complete metric space and absolute retract for metric spaces and $F : X\times X \to P_{b,cl}(X)$ a lower semicontinuous multivalued mapping such that there exist $a_{11},\dots,a_{15},a_{21},\dots,a_{25}\in(0,1)$ with $a_{11}+a_{13}+a_{14}+2a_{12}<1$, $a_{21}+a_{23}+a_{24}+2a_{22}<1$,*

$$\begin{aligned}H(F(u,v),F(x,y)) \;\leq\;& a_{11}d(x,u)+a_{12}d(x,F(u,v))\\ &+a_{13}d(F(x,y),x)+a_{14}d(F(u,v),u)+a_{15}d(u,F(x,y))\end{aligned}$$

and

$$\begin{aligned}H(F(x,y),F(u,v)) \;\leq\;& a_{21}d(x,u)+a_{22}d(x,F(u,v))\\ &+a_{23}d(y,F(x,y))+a_{24}d(u,F(u,v))+a_{25}d(v,F(x,y))\end{aligned}$$

for all $u,v,x,y\in X$. Then $\{(x,y) : x\in F(x,y), y\in F(y,x)\}$ is an absolute retract for metric spaces.

Theorem 2.3. *Let (X,d) be a complete metric space and absolute retract for metric spaces and $F_1,F_2 : X\times X\to P_{b,cl}(X)$ lower semicontinuous multifunctions such that there exist $a_{11},\cdots,a_{15},a_{21},\cdots,a_{25}\in(0,1)$ with $a_{11}+a_{13}+a_{14}+2\max\{a_{12},a_{15}\}<1$, $a_{21}+a_{23}+a_{24}+2\max\{a_{22},a_{25}\}<1$,*

$$\begin{aligned}H(F_1(x,y),F_2(u,v)) \;\leq\;& a_{11}d(x,u)+a_{12}d(x,F_2(u,v))\\ &+a_{13}d(F_1(x,y),x)+a_{14}d(F_2(u,v),u)+a_{15}d(u,F_1(x,y))\end{aligned}$$

and

$$\begin{aligned}H(F_1(x,y),F_2(u,v)) \;\leq\;& a_{21}d(y,v)+a_{22}d(y,F_2(u,v))\\ &+a_{23}d(F_1(x,y),y)+a_{24}d(F_2(u,v),v)+a_{25}d(F_1(x,y),v)\end{aligned}$$

for all $u,v,x,y\in X$. Then the set $\{(x,y) : x,y\in F_1(x,y)\cap F_2(x,y)\}$ is an absolute retract for metric spaces.

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