Anal. Theory Appl.
Vol. 28, No. 1 (2012), 49–57

NOTE ON THE PAPER " AN NEGATIVE ANSWER TO A CONJECTURE ON THE SELF-SIMILAR SETS SATISFYING THE OPEN SET CONDITION"

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Received Aug. 11, 2011

Abstract. In this paper, we present a more simple and much shorter proof for the main result in the paper " An negative answer to a conjecture on the self-similar sets satisfying the open set condition", which was published in the journal Analysis in Theory and Applications in 2009.

Key words: *self-similar set, best covering, natural covering, Hausdorff measure and Hausdorff dimension*

AMS (2010) subject classification: 28A78, 28A80

1 Introduction

It is well known that the theory of Hausdorff measure is the basis of fractal geometry and Hausdorff measure is an important notion in the study of fractals (see [1-2]). But unfortunately, it is usually difficult to calculate the exact value of the Hausdorff measure of fractal sets. Since J.E.Hutchinson [3] first introduced self-similar sets satisfying the open set condition (OSC), many authors have studied this class of fractals and obtained a number of meaningful results (see [1-11]). Among them, Z. Zhou and L. Feng's paper[5] has attracted widespread attention since it was published in 2004. In [5], Z. Zhou and L. Feng thought the reason for the difficulty

Supported in part by National Natural Science Foundation of China (No. 10961003).

in calculating the Hausdorff measures of fractals is neither computational trickiness nor computational capacity, but a lack of full understanding of the essence of the Hausdorff measure. They posed eight open problems and six conjectures on Hausdorff measure of similar sets. Among them is a conjecture as follows:

Conjecture 1.1[5]. *E has a best covering if and only if* $H^s(E) = |E|^s$.

Recently, some authors negatively answered the above-mentioned Conjecture 1.1 (see [8, 9]). In [8], the authors neglected an important condition that a best covering $\{U_i\}_{i\in I}$ must meet, that is, for any $i \in I$, the inequality $|U_i| > 0$ should hold. They constructed a self-similar set which is for the IFS consisting of three functions, found a covering $\beta = \{\{1\}, [0, \frac{3}{9}], \cdots\}$ of E and claimed that it was a best covering of E. In fact, this covering is an best almost covering but not a best one since the diameter of the set $\{1\}$ is 0. In [9], the author constructed another self-similar set which is for the IFS consisting of six functions and claimed that by discussing this similar set, the main result (i.e., Theorem 3.1 in [9]) can provide a negative answer to the conjecture, but unfortunately the proof is too long. In this paper, we will present a more simple and much shorter shirt proof of the result. The objective of this paper is to promote more people to develop interest in this conjecture and its answer.

Some definitions, notations and known results are from references [1-5].

Let d be the standard distance function on R^n, where R^n is Euclidian n-space. Denote $d(x,y)$ by $|x-y|$, $\forall x,y \in R^n$. If U is a nonempty subset of R^n, we define the diameter of U as $|U| = \sup\{|x-y| : x,y \in U\}$. Let δ be a positive number. If $E \subset \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a δ-covering of E.

Let $E \subset R^n$ and $s \geq 0$. For $\delta > 0$, define

$$H^s_\delta(E) = \inf\{\sum_i |U_i|^s : \bigcup_i U_i \supset E, 0 < |U_i| \leq \delta\}.$$

Letting $\delta \to 0$, we call the limit

$$H^s(E) = \lim_{\delta\to 0} H^s_\delta(E)$$

the s-dimensional Hausdorff measure of E. Note that the Hausdorff dimension of E is defined as

$$\dim_H E = \inf\{s \geq 0 : H^s(E) = 0\} = \sup\{s \geq 0 : H^s(E) = \infty\}.$$

An H^s-measurable set $E \subset R^n$ with $0 < H^s(E) < \infty$ is termed an s-set.

Now we review the self-similar s-sets satisfying the open set condition. Let $D \subset R^n$ be closed. A map $S : D \to D$ is called a contracting similarity, if there is a number c with $0 < c < 1$ such that

$$|S(x) - S(y)| = c|x-y|, \qquad \forall\, x,y \in D$$

where c is called the similar ratio. It was proved by Hutchinson (see [3]) that given $m \geq 2$ and contracting similarities $S_i : D \to D (i = 1, 2, \cdots, m)$ with similarity ratios c_i there exists a unique nonempty compact set $E \subset R^n$ satisfying

$$E = \bigcup_{i=1}^{m} S_i(E).$$

The set E is called the self-similar s-set for the iterated function system (IFS) $\{S_1, ..., S_m\}$, here we assume that there is a bounded nonempty open set V such that

$$\bigcup_{i=1}^{m} S_i(V) \subset V,$$

and

$$S_i(V) \cap S_j(V) = \emptyset \ \ (i \neq j, j = 1, 2, \cdots, m)$$

which is often referred to as the open set condition (OSC). In this case, we know that $0 < H^s(E) < \infty$ and so E is an s-set. Denote by J_k the set of all k-sequences $(i_1, \cdots, i_k)$, where $1 \leq i_1, \cdots, i_k \leq m, k \geq 1$ and put $E_{i_1 \cdots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(E)$, which is referred to as k-contracting-copy of E. Obviously, $\forall (i_1, \cdots, i_k) \in J_k$, we have

$$|E_{i_1 \cdots i_k}|^s = |S_{i_1} \circ \cdots \circ S_{i_k}(E)|^s = c_{i_1} \cdots c_{i_k} |E|^s.$$

It is not hard to see that for each $k \geq 1$,

$$E = \bigcup_{J_k} E_{i_1 \cdots i_k}.$$

Definition 1.1[5−7]. Let E be an s-set in R^n. A sequence $\{U_i\}$ of subsets in R^n is called an H^s-a.e. covering of E, if there is an H^s-measurable subset E_0 in R^n with $E_0 \subset E$ and $H^s(E_0) = 0$ such that $\{U_i\}$ is a covering of $E - E_0$ (i.e., $E - E_0 \subset \bigcup_i U_i$). A sequence $\{U_i\}$ of subsets in R^n is called a best H^s-a.e. covering of E if it is an H^s-a.e. covering of E such that

$$H^s(E) = \sum_i |U_i|^s.$$

Note that a best H^s-a.e. covering may be alternatively called a best almost covering or an optimal almost covering (see [10]).

A family $\{U_i\}_{i \geq 1}$of subsets in R^n is called a best covering of E (see[5]), if it is a covering of E with $|U_i| > 0 (\forall i \geq 1)$ such that

$$H^s(E) = \sum_{i \geq 1} |U_i|^s.$$

2 A More Simple and Much Shorter Proof for the Answer to Zhou and Feng's Conjecture on Self-similar Sets

In this section, we present a more simple and much shorter proof for the answer to Zhou and Feng's conjecture (Conjecture 1.1).

Lemma 2.1[11]**.** *Let $E \subset R^n$ be a self-similar set for the IFS $\{S_1, \cdots, S_m\}$ satisfying OSC with $c_1 = \cdots = c_m = c$, and $s = \dim_H E$ (i.e., $s = \log_{\frac{1}{c}} m$). If there are positive integers k_0, $N_0 (1 \le N_0 \le m^{k_0})$ and a k-copy group of E consisting of N_0 c^{k_0}-Es, which is denoted by $B_{N_0}^{k_0}$, such that*

$$\frac{m^{k_0}}{N_0}|B_{N_0}^{k_0}|^s \le \frac{m^k}{N}|B_N^k|^s. \tag{2.1}$$

for all integer $k \ge 0$, N $(1 \le N \le m^k)$ and each B_N^k, where B_N^k denotes the union of any N k-copies of E. Then the Hausdorff measure of E can be computed by the formula

$$H^s(E) = \frac{m^{k_0}}{N_0}|B_{N_0}^{k_0}|^s.$$

Lemma 2.2. *Let $n \ge 2$, and $s = \log_{21} 6$. Then the following inequalities hold:*

(i) $(3n)^{\frac{1}{s}} - \frac{4}{7} \ge (3n - \frac{1}{2})^{\frac{1}{s}}$.

(ii) $(3n)^{\frac{1}{s}} - 4 \ge (3n-1)^{\frac{1}{s}}$.

(iii) $(3n)^{\frac{1}{s}} - \frac{32}{7} \ge (3n - \frac{3}{2})^{\frac{1}{s}}$.

(iv) $(3n)^{\frac{1}{s}} - 8 \ge (3n-2)^{\frac{1}{s}}$.

(v) $(3n)^{\frac{1}{s}} - \frac{60}{7} \ge (3n - \frac{5}{2})^{\frac{1}{s}}$.

Proof. (i) Set

$$f(x) = (3x)^{\frac{1}{s}} - \frac{4}{7} - (3x - \frac{1}{2})^{\frac{1}{s}}.$$

Since $s = \log_{21} 6$, it follows that we have

$$f(2) = 6^{\frac{1}{s}} - \frac{4}{7} - (6 - \frac{1}{2})^{\frac{1}{s}} > 0,$$

and

$$f'(x) = (3^{\frac{1}{s}}\frac{1}{s}x^{\frac{1}{s}-1} - \frac{1}{s}(x - \frac{1}{4})^{\frac{1}{s}-1}) > 0, \qquad \forall x \ge 2.$$

Therefore it follows that (i) holds.

(ii)-(v) The proofs of (ii)-(v) are similar to that of (i), so we omit them here.

Corollary 2.1. *Let $n \ge 2$, and $s = \log_{21} 6$. Then the following inequalities hold:*

(i) $[\frac{(3n)^{\frac{1}{s}}}{9 \cdot 21^k} - \frac{4}{3 \cdot 21^{k+1}}]^s \ge \frac{3}{9^s} \cdot \frac{6n-1}{6^{k+1}}$.

(ii) $[\frac{(3n)^{\frac{1}{s}}}{9 \cdot 21^k} - \frac{28}{3 \cdot 21^{k+1}}]^s \ge \frac{3}{9^s} \cdot \frac{6n-2}{6^{k+1}}$.

(iii) $[\frac{(3n)^{\frac{1}{s}}}{9 \cdot 21^k} - \frac{32}{3 \cdot 21^{k+1}}]^s \ge \frac{3}{9^s} \cdot \frac{6n-3}{6^{k+1}}$.

(iv) $[\frac{(3n)^{\frac{1}{s}}}{9\cdot 21^k}-\frac{56}{3\cdot 21^{k+1}}]^s\geq \frac{3}{9^s}\cdot\frac{6n-4}{6^{k+1}}$.

(v) $[\frac{(3n)^{\frac{1}{s}}}{9\cdot 21^k}-\frac{60}{3\cdot 21^{k+1}}]^s\geq \frac{3}{9^s}\cdot\frac{6n-5}{6^{k+1}}$.

Proof. (i) Consider

$$\begin{aligned}[\frac{(3n)^{\frac{1}{s}}}{9\cdot 21^k}-\frac{4}{3\cdot 21^{k+1}}]^s\geq \frac{3}{9^s}\cdot\frac{6n-1}{6^{k+1}} &\iff \frac{21(3n)^{\frac{1}{s}}-12}{9\cdot 21^{k+1}}\geq\frac{3^{\frac{1}{s}}}{9}\cdot\frac{(6n-1)^{\frac{1}{s}}}{21^{k+1}}\\ &\iff 21(3n)^{\frac{1}{s}}-12\geq[3(6n-1)]^{\frac{1}{s}}\\ &\iff 21(3n)^{\frac{1}{s}}-12\geq 21(3n-\frac{1}{2})^{\frac{1}{s}}\\ &\iff (3n)^{\frac{1}{s}}-\frac{4}{7}\geq(3n-\frac{1}{2})^{\frac{1}{s}}.\end{aligned}$$

Therefore it follows from Lemma 2.2 (i) that (i) holds.

(ii)-(v) The proofs of (ii)-(v) are similar to that of (i), so we omit them here.

Example 2.1[9]. Let $E\subset R$ be the self-similar set yielded by IFS $\{f_1,f_2,f_3,f_4,f_5,f_6\}$ as follows:

$$f_1(x)=\frac{x}{21},f_2(x)=\frac{x}{21}+\frac{4}{63},f_3(x)=\frac{x}{21}+\frac{8}{28},$$

$$f_4(x)=\frac{x}{21}+\frac{32}{63},f_5(x)=\frac{x}{21}+\frac{56}{63},f_6(x)=\frac{x}{21}+\frac{60}{63}.$$

Let $s=\log_{21}6$, then we have

Theorem 2.1 The exact value of the Hausdorff measure of E is

$$H^s(E)=\frac{3}{9^s}=\frac{3}{9^{\log_{21}6}}.$$

Proof. Now we use Lemma 2.1 and Corollary 2.1 to prove Theorem 2.1. Take $m=6$. Similarly, let $B_{N_0}^{k_0}$ denote the union of the given N_0 $(\frac{1}{21})^{k_0}$-Es, where k_0 is any positive integer and $1\leq N_0\leq 6^{k_0}$. We will check that all conditions of Lemma 2.1 are satisfied. On the one hand, take $B_{N_0}^{k_0}=f_1(E)\cup f_2(E)$. So, we see here

$$N_0=2,k_0=1$$

and

$$\frac{m^{k_0}}{N_0}|B_{N_0}^{k_0}|^s=\frac{6^1}{2}\cdot(\frac{1}{9})^s=\frac{3}{9^{\log_{21}6}}.$$

On the other hand, according to (2.1) in Lemma 2.1, we will have to prove the inequality

$$|B_N^k|^s\geq\frac{3N}{9^{\log_{21}6}\cdot 6^k}=\frac{3}{9^s}\cdot\frac{N}{6^k}. \tag{2.2}$$

holds for all positive integers k and N ($1 \le N \le 6^k$), where B_N^k denotes any k-copy group of E consisting of N c^k-Es. Now we show (2.2) holds by induction. Firstly, when $k=1$, we see that

Case 1 If $N=1$, then

$$|B_1^1|^s \ge (\frac{1}{21})^s \ge \frac{3}{9^s}\cdot\frac{1}{6^1};$$

Case 2 If $N=2$, then

$$|B_2^1|^s \ge (\frac{7}{63})^s \ge \frac{3}{9^s}\cdot\frac{2}{6^1};$$

Case 3 If $N=3$, then

$$|B_3^1|^s \ge (\frac{31}{63})^s \ge \frac{3}{9^s}\cdot\frac{3}{6^1};$$

Case 4 If $N=4$, then

$$|B_4^1|^s \ge (\frac{35}{63})^s \ge \frac{3}{9^s}\cdot\frac{4}{6^1};$$

Case 5 If $N=5$, then

$$|B_5^1|^s \ge (\frac{59}{63})^s \ge \frac{3}{9^s}\cdot\frac{5}{6^1};$$

Case 6 If $N=6$, then

$$|B_6^1|^s \ge 1 \ge \frac{3}{9^s}\cdot\frac{6}{6^1};$$

Secondly, suppose (2.2) holds for some positive integer k, that is, for any $1 \le N \le 6^k$, we get

$$|B_N^k|^s \ge \frac{3N}{9^{\log_{21}6}\cdot 6^k} = \frac{3}{9^s}\cdot\frac{N}{6^k}. \tag{2.3}$$

Then for any $1 \le N \le 6^{k+1}$, we must show that

$$|B_N^{k+1}|^s \ge \frac{3N}{9^{\log_{21}6}\cdot 6^{k+1}} = \frac{3}{9^s}\cdot\frac{N}{6^{k+1}}, \tag{2.4}$$

where B_N^{k+1} denotes any $(k+1)$-copy group of E consisting of N c^{k+1}-Es. In order to prove (2.4), we will make discussion by two steps.

Step 1 Consider the cases of $N=1,2,3,4,5,6$, respectively.

If $N=1$, then

$$|B_1^{k+1}|^s \ge (\frac{1}{21^{k+1}})^s \ge \frac{3}{9^s}\cdot\frac{1}{6^{k+1}};$$

If $N=2$, then

$$|B_2^{k+1}|^s \ge (\frac{7}{63^{k+1}})^s \ge (\frac{7}{3\cdot 21^{k+1}})^s \ge \frac{3}{9^s}\cdot\frac{2}{6^{k+1}};$$

If $N=3$, then

$$|B_3^{k+1}|^s \ge (\frac{31}{63^{k+1}})^s \ge (\frac{31}{3\cdot 21^{k+1}})^s \ge \frac{3}{9^s}\cdot\frac{3}{6^{k+1}};$$

If $N=4$, then

$$|B_4^{k+1}|^s \ge (\frac{35}{63^{k+1}})^s \ge (\frac{35}{3\cdot 21^{k+1}})^s \ge \frac{3}{9^s}\cdot\frac{4}{6^{k+1}};$$

If $N=5$, then

$$|B_5^{k+1}|^s \geq (\frac{59}{63^{k+1}})^s \geq (\frac{59}{3\cdot 21^{k+1}})^s \geq \frac{3}{9^s}\cdot\frac{5}{6^{k+1}};$$

If $N=6$, then

$$|B_6^{k+1}|^s \geq 1 \geq \frac{3}{9^s}\cdot\frac{6}{6^{k+1}};$$

Thus (2.3) holds for the case of $N=1,2,3,4,5,6$.

Step 2 Consider the case that N is any of the other positive integers with $1\leq N\leq 6^{k+1}$.

If $N=6n$, where n is any positive integer with $1\leq n\leq 6^k$. Then, by the geometrical structure of E and (2.3), we see

$$|B_N^{k+1}|^s \geq |B_{6n}^{k+1}|^s \geq |B_n^k|^s \geq \frac{3}{9^s}\cdot\frac{n}{6^k}\cdot\frac{6}{6} = \frac{3}{9^s}\cdot\frac{N}{6^{k+1}};$$

If $N=6n-1$, where n is any positive integer with $2\leq n\leq 6^k$. Then, by the geometrical structure of E, Corollary 2.1(i) and (2.3), we see

$$\begin{aligned}|B_N^{k+1}|^s &\geq (|B_n^k|-\frac{4}{3\cdot 21^{k+1}})^s \geq [(\frac{3n}{9^s\cdot 6^k})^{\frac{1}{s}}-\frac{4}{3\cdot 21^{k+1}}]^s\\ &\geq [\frac{(3n)^{\frac{1}{s}}}{9\cdot 21^k}-\frac{4}{3\cdot 21^{k+1}}]^s\\ &\geq \frac{3}{9^s}\cdot\frac{6n-1}{6^{k+1}} \geq \frac{3}{9^s}\cdot\frac{N}{6^{k+1}};\end{aligned}$$

If $N=6n-2$, where n is any positive integer with $2\leq n\leq 4^k$. Then, by the geometrical structure of E, Corollary 2.1(ii) and (2.3), we see

$$\begin{aligned}|B_N^{k+1}|^s &\geq (|B_n^k|-\frac{28}{3\cdot 21^{k+1}})^s \geq [(\frac{3n}{9^s\cdot 6^k})^{\frac{1}{s}}-\frac{28}{3\cdot 21^{k+1}}]^s\\ &\geq [\frac{(3n)^{\frac{1}{s}}}{9\cdot 21^k}-\frac{28}{3\cdot 21^{k+1}}]^s\\ &\geq \frac{3}{9^s}\cdot\frac{6n-2}{6^{k+1}} \geq \frac{3}{9^s}\cdot\frac{N}{6^{k+1}};\end{aligned}$$

If $N=6n-3$, where n is any positive integer with $2\leq n\leq 4^k$. Then, by the geometrical structure of E, Corollary 2.1(iii) and (2.3), we see

$$\begin{aligned}|B_N^{k+1}|^s &\geq (|B_n^k|-\frac{32}{3\cdot 21^{k+1}})^s \geq [(\frac{3n}{9^s\cdot 6^k})^{\frac{1}{s}}-\frac{32}{3\cdot 21^{k+1}}]^s\\ &\geq [\frac{(3n)^{\frac{1}{s}}}{9\cdot 21^k}-\frac{32}{3\cdot 21^{k+1}}]^s\\ &\geq \frac{3}{9^s}\cdot\frac{6n-3}{6^{k+1}} \geq \frac{3}{9^s}\cdot\frac{N}{6^{k+1}};\end{aligned}$$

If $N = 6n-4$, where n is any positive integer with $2 \le n \le 4^k$. Then, by the geometrical structure of E, Corollary 2.1(iv) and (2.3), we see

$$\begin{aligned} |B_N^{k+1}|^s &\ge (|B_n^k| - \frac{56}{3\cdot 21^{k+1}})^s \ge [(\frac{3n}{9^s\cdot 6^k})^{\frac{1}{s}} - \frac{56}{3\cdot 21^{k+1}}]^s \\ &\ge [\frac{(3n)^{\frac{1}{s}}}{9\cdot 21^k} - \frac{56}{3\cdot 21^{k+1}}]^s \\ &\ge \frac{3}{9^s}\cdot\frac{6n-4}{6^{k+1}} \ge \frac{3}{9^s}\cdot\frac{N}{6^{k+1}}; \end{aligned}$$

If $N = 6n-5$, where n is any positive integer with $2 \le n \le 4^k$. Then, by the geometrical structure of E, Corollary 2.1(v) and (2.3), we see

$$\begin{aligned} |B_N^{k+1}|^s &\ge (|B_n^k| - \frac{60}{3\cdot 21^{k+1}})^s \ge [(\frac{3n}{9^s\cdot 6^k})^{\frac{1}{s}} - \frac{60}{3\cdot 21^{k+1}}]^s \\ &\ge [\frac{(3n)^{\frac{1}{s}}}{9\cdot 21^k} - \frac{60}{3\cdot 21^{k+1}}]^s \\ &\ge \frac{3}{9^s}\cdot\frac{6n-5}{6^{k+1}} \ge \frac{3}{9^s}\cdot\frac{N}{6^{k+1}}; \end{aligned}$$

Hence, it follows from Step 1 and Step 2 that (2.4) holds for any $1 \le N \le 6^{k+1}$. So by induction, (2.2) is true for all positive integers k and N $(1 \le N \le 6^k)$. Therefore, it follows from Lemma 2.1 that

$$H^s(E) = \frac{3}{9^{\log_{21} 6}},$$

which completes the proof of Theorem 2.1.

Theorem 2.2[9]. *There exists a self-similar set E satisfying OSC such that the following conditions hold:*

(i) *The natural covering is not a best one, that is, $H^s(E) < |E|^s$;*

(ii) *E has a best covering, that is, there is a sequence $\{U_i\}$ with $|U_i| > 0$ such that $E \subset \bigcup_i U_i$ and $H^s(E) = \sum_i |U_i|^s$.*

Proof. Let E be the self-similar set in Example 2.1. Set $\beta = \{U_1, U_2, U_3\}$, where $U_1 = [0, \frac{1}{9}]$, $U_2 = [\frac{28}{63}, \frac{35}{63}]$, $U_3 = [\frac{56}{63}, 1]$ then it follows from Theorem 2.1 that β is a best covering of E, since $E \subset U_1 \bigcup U_2 \bigcup U_3$ and

$$H^s(E) = \frac{3}{9^{\log_{21} 6}} = 3\cdot(\frac{1}{9})^{\log_{21} 6} = |U_1|^s + |U_2|^s + |U_3|^s,$$

where $s = \log_{21} 6$. However, $H^s(E) < 1 = |E|^s$, which completes the proof of Theorem 2.2.

Remark 2.1. Compared with the main result in [9], the proof of Theorem 2.2 is much simpler and shorter. It is more readable since the method for calculating the exact measure of the self-similar set discussed is simple and elementary.

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