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Some Results Concerning Growth of Polynomials

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Abstract. Let $P(z)$ be a polynomial of degree n having no zeros in $|z|<1$, then for every real or complex number β with $|\beta|\leq 1$, and $|z|=1$, $R\geq 1$, it is proved by Dewan et al. [4] that

$$\Big|P(Rz)+\beta\Big(\frac{R+1}{2}\Big)^nP(z)\Big|\leq\frac{1}{2}\Big\{\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big)\max_{|z|=1}|P(z)| -\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|-\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n|\Big)\min_{|z|=1}|P(z)|\Big\}.$$

In this paper we generalize the above inequality for polynomials having no zeros in $|z|<k$, $k\leq 1$. Our results generalize certain well-known polynomial inequalities.

Key Words: Polynomial, inequality, maximum modulus, growth of polynomial.

AMS Subject Classifications: 30A10, 30C10, 30E15

1 Introduction and statement of results

It is well known that if $P(z)$ is a polynomial of degree n, then for $|z|=1$ and $R\geq 1$

$$|P(Rz)|+|Q(Rz)|\leq (R^n+1)\max_{|z|=1}|P(z)|, \tag{1.1}$$

where $Q(z)=z^n\overline{P(1/\overline{z})}$ (see [6]).

On the other hand, concerning the estimate of $|P(z)|$ on the disc $|z|\leq R$, $R\geq 1$, we have, as a simple consequence of the principle of maximum modulus (see also [6]), if $P(z)$ is a polynomial of degree n, then for $R\geq 1$

$$\max_{|z|=R}|P(z)|\leq R^n\max_{|z|=1}|P(z)|. \tag{1.2}$$

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The result is best possible and the equality holds for polynomials having zeros at the origin.

It was shown by Ankeny and Rivlin [1] that if $P(z)$ doe not vanish in $|z|<1$, then the inequality (1.2) can be replaced by

$$\max_{|z|=R}|P(z)|\le\frac{R^n+1}{2}\max_{|z|=1}|P(z)|,\quad R\ge 1. \tag{1.3}$$

The inequality (1.3) is sharp and the equality holds for $P(z)=\alpha z^n+\gamma$, where $|\alpha|=|\gamma|$.

The inequality (1.3) was generalized by Jain [5] who proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then for $|\beta|\le 1$, $R\ge 1$ and $|z|=1$,

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+1}{2}\Big)^nP(z)\Big|\\ \le&\frac{1}{2}\Big\{\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big\}\max_{|z|=1}|P(z)|.\end{aligned} \tag{1.4}$$

Aziz and Dawood [3] used

$$\min_{|z|=1}|P(z)| \tag{1.5}$$

to obtain a refinement of the inequality (1.3) and proved, if $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then for $R\ge 1$

$$\max_{|z|=R}|P(z)|\le\Big(\frac{R^n+1}{2}\Big)\max_{|z|=1}|P(z)|-\Big(\frac{R^n-1}{2}\Big)\min_{|z|=1}|P(z)|. \tag{1.6}$$

The result is best possible and the equality holds for $P(z)=\alpha z^n+\gamma$ with $|\alpha|=|\gamma|$.

As refinement of the inequality (1.4) and generalization of the inequality (1.6), Dewan and Hans [4] have proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then for $|\beta|\le 1$, $R\ge 1$ and $|z|=1$,

$$\begin{aligned}\Big|P(Rz)+\beta\Big(\frac{R+1}{2}\Big)^nP(z)\Big|\le&\frac{1}{2}\Big\{\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big)\max_{|z|=1}|P(z)|\\ &-\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|-\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big)\min_{|z|=1}|P(z)|\Big\}.\end{aligned} \tag{1.7}$$

The result is best possible and the equality holds for $P(z)=\alpha z^n+\gamma$ with $|\alpha|=|\gamma|$.

Whereas if $P(z)$ has all its zeros in $|z|\le 1$, then for any $|\beta|\le 1$, $R\ge 1$ and $|z|=1$,

$$\min_{|z|=1}\Big|P(Rz)+\beta\Big(\frac{R+1}{2}\Big)^nP(z)\Big|\ge\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\min_{|z|=1}|P(z)|. \tag{1.8}$$

The result is best possible and the equality holds for $P(z)=me^{i\alpha}z^n$, $m>0$.

In this paper, we obtain further generalizations of the inequalities (1.7) and (1.8). More precisely, we prove

Theorem 1.1. *$P(z)$ is a polynomial of degree n, having all its zeros in $|z|\le k$, $k\le 1$, then for every real or complex number β with $|\beta|\le 1$, $R\ge 1$ and $|z|=1$,*

$$\min_{|z|=1}\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n P(z)\Big|\ge k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\min_{|z|=k}|P(z)|. \tag{1.9}$$

The result is best possible and the equality holds for

$$P(z)=a\Big(\frac{z}{k}\Big)^n.$$

If we take $k=1$ in Theorem 1.1, then the inequality (1.9) reduces to the inequality (1.8).
If we take $\beta=0$ in Theorem 1.1, we have the following interesting result:

Corollary 1.1. If $P(z)$ is a polynomial of degree n, having all its zeros in $|z|\le k$, $k\le 1$, then for $R\ge 1$

$$k^n\min_{|z|=R}|P(z)|\ge R^n\min_{|z|=k}|P(z)|. \tag{1.10}$$

The result is best possible and the equality holds for $P(z)=a(z/k)^n$.

We next generalize the inequality (1.7) by using Theorem 1.1, more precisely

Theorem 1.2. *If $P(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\le 1$, then for every real or complex number β with $|\beta|\le 1$, $R\ge 1$ and $|z|=1$ we have*

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n P(z)\Big|\\ \le&\frac{1}{2}\Big\{\Big(k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\Big)\max_{|z|=k}|P(z)|\\ &-\Big(k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|-\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\Big)\min_{|z|=k}|P(z)|\Big\}.\end{aligned} \tag{1.11}$$

The inequality (1.11) *is sharp and the equality holds for $P(z)=\alpha z^n+\gamma k^n$ with $|\alpha|=|\gamma|$.*

If we take $k=1$ in Theorem 1.2, then the inequality (1.11) reduces to (1.7).
If we take $\beta=0$ in Theorem 1.2, then we get a generalization of the inequality (1.6).

Corollary 1.2. If $P(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\le 1$, then for $R\ge 1$

$$\max_{|z|=R}|P(z)|\le\Big(\frac{R^n+k^n}{2k^n}\Big)\max_{|z|=k}|P(z)|\Big(\frac{R^n-k^n}{2k^n}\Big)\min_{|z|=k}|P(z)|. \tag{1.12}$$

The inequality (1.12) is sharp and the equality holds for $P(z)=\alpha z^n+\gamma k^n$ with $|\alpha|=|\gamma|$.

2 Lemmas

For the proof of our theorems, we need the following lemmas. The first lemma is due to Aziz [2].

Lemma 2.1. *If $P(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z|\le k$, $k\le 1$, then for $R\ge 1$*

$$|P(Rz)|\ge\Big(\frac{R+k}{1+k}\Big)^n|P(z)|,\quad |z|=1. \tag{2.1}$$

Lemma 2.2. *Let $F(z)$ be a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, and $P(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|P(z)|\le|F(z)|$ for $|z|=k$, $k\le 1$, then for any β with $|\beta|\le 1$ and $|z|=1$, $R\ge 1$ we have*

$$\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|\le\Big|F(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nF(z)\Big|. \tag{2.2}$$

Proof. From Rouche's Theorem, it is obvious that for α with $|\alpha|<1$, $F(z)+\alpha P(z)$ has as many zeros in $|z|<k$ as $F(z)$ and so has all of its zeros in $|z|<k$. On the other hand by the inequality $|P(z)|\le|F(z)|$ for $|z|=k$, any zero of $F(z)$ that lies on $|z|=k$, is the zero of $P(z)$. Therefore $F(z)+\alpha P(z)$ has all its zeros in $|z|\le k$. On applying Lemma 2.1, we get for α with $|\alpha|<1$ and $|z|=1$, $R\ge 1$,

$$|F(Rz)+\alpha P(Rz)|\ge\Big(\frac{R+k}{1+k}\Big)^n|F(z)+\alpha P(z)|.$$

Therefore, for any β with $|\beta|<1$, we have

$$\Big(F(Rz)+\alpha P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)\Big)^n(F(z)+\alpha P(z))\neq 0,$$

i.e.,

$$T(z)=F(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nF(z)+\alpha\Big(P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big)\neq 0, \tag{2.3}$$

where $|z|=1$.

Hence for an appropriate choice of the argument α, one gets

$$\Big|F(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nF(z)\Big|\neq|\alpha|\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|.$$

Therefore we have

$$\Big|F(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nF(z)\Big|\ge\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|, \tag{2.4}$$

where $|z|=1$.

If the inequality (2.3) is not true, then there is a point $z=z_0$ with $|z_0|=1$ such that for $R\geq 1$,

$$\Big|F(Rz_0)+\beta\Big(\frac{R+k}{1+k}\Big)^nF(z_0)\Big|<\Big|P(Rz_0)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z_0)\Big|.$$

We take

$$\alpha=-\frac{F(Rz_0)+\beta\Big(\frac{R+k}{1+k}\Big)^nF(z_0)}{P(Rz_0)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z_0)},$$

then $|\alpha|<1$ and with this choice of α, we have from (2.3), $T(z_0)=0$ for $|z_0|=1$. But this contradicts the fact that $T(z)\neq 0$ for $|z|=1$. For β with $|\beta|=1$, (2.4) follows by continuity. This completes the proof of Lemma 2.2. □

If we take

$$F(z)=\Big(\frac{z}{k}\Big)^n\max_{|z|=k}|P(z)|$$

in Lemma 2.2 we have

Lemma 2.3. *Let $P(z)$ be a polynomial of degree n, then for any $|\beta|\leq 1$, $R\geq 1$, $k\leq 1$ and $|z|=1$ we have*

$$\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|\leq k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\max_{|z|=k}|P(z)|. \tag{2.5}$$

Lemma 2.4. *Let $P(z)$ be a polynomial of degree n, then for any β with $|\beta|\leq 1$, $R\geq 1$ and $|z|=1$ we have*

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|+\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)\Big|\\ \leq&\Big\{k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\Big\}\max_{|z|=k}|P(z)|,\end{aligned} \tag{2.6}$$

where $Q(z)=(z/k)^n\overline{P(k^2/\overline{z})}$ and $k\leq 1$.

Proof. Let $M=\max_{|z|=k}|P(z)|$. For α with $|\alpha|>1$, it follows by Rouche's Theorem that the polynomial $G(z)=P(z)-\alpha M$ has no zeros in $|z|<k$. Correspondingly the polynomial

$$H(z)=\Big(\frac{z}{k}\Big)^n\overline{G(k^2/\overline{z})}$$

has all its zeros in $|z|\leq k$ and $|G(z)|=|H(z)|$ for $|z|=k$. On applying Lemma 2.2, we have for $|\beta|\leq 1$ and $|z|=1$, $R\geq 1$

$$\Big|G(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nG(z)\Big|\leq\Big|H(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nH(z)\Big|.$$

Therefore by the equality

$$H(z)=\Big(\frac{z}{k}\Big)^n\overline{G\Big(\frac{k^2}{\overline{z}}\Big)}=\Big(\frac{z}{k}\Big)^n\overline{P\Big(\frac{k^2}{\overline{z}}\Big)}-\overline{\alpha}\Big(\frac{z}{k}\Big)^nM=Q(z)-\overline{\alpha}\Big(\frac{z}{k}\Big)^nM,$$

i.e.,

$$H(z)=Q(z)-\overline{\alpha}\Big(\frac{z}{k}\Big)^nM.$$

We have

$$\begin{aligned}&\Big|\{P(Rz)-\alpha M\}+\beta\Big(\frac{R+k}{1+k}\Big)^n\{P(z)-\alpha M\}\Big|\\ \le&\Big|\Big\{Q(Rz)-\overline{\alpha}R^n\Big(\frac{z}{k}\Big)^nM\Big\}+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big\{Q(z)-\overline{\alpha}\Big(\frac{z}{k}\Big)^nM\Big\}\Big|.\end{aligned}$$

This implies

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)-\alpha\Big(1+\Big(\frac{R+k}{1+k}\Big)^n\Big)M\Big|\\ \le&\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)-\overline{\alpha}\Big(\frac{z}{k}\Big)^n\Big(R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big)M\Big|.\end{aligned}\tag{2.7}$$

As $|P(z)|=|Q(z)|$ for $|z|=k$, i.e., $M=\max_{|z|=k}|P(z)|=\max_{|z|=k}|Q(z)|$ therefore, by applying Lemma 2.3 for the polynomial $Q(z)$, we have

$$\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)\Big|<|\alpha|k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|M,$$

where $|z|=1$, $|\beta|\le 1$ and $|\alpha|>1$.

Now by suitable choice of the argument α, we get for $|z|=1$ and $|\beta|\le 1$,

$$\begin{aligned}&\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)-\overline{\alpha}\Big(\frac{z}{k}\Big)^n\Big(R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big)M\Big|\\ =&|\alpha|k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|M-\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)\Big|.\end{aligned}\tag{2.8}$$

Combining (2.7) and (2.8), we have

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|-|\alpha|\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|M\\ \le&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)-\alpha\Big(1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big)M\Big|\\ \le&\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)-\overline{\alpha}\Big(\frac{z}{k}\Big)^n\Big(R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big)M\Big|\\ =&|\alpha|k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|M-\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)\Big|.\end{aligned}$$

i.e.,

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|-|\alpha|\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|M\\ \le&|\alpha|k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|M-\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)\Big|.\end{aligned}$$

This implies

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|+\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)\Big|\\ \le&|\alpha|\Big\{k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\Big\}M.\end{aligned}$$

Making $|\alpha|\to 1$, the lemma follows. □

If we take $\beta=0$ in Lemma 2.4, we have the following generalization of the inequality (1.1).

Corollary 2.1. Let $P(z)$ be a polynomial of degree n, then for any $R\ge 1$ and $|z|=1$ we have

$$|P(Rz)|+|Q(Rz)|\le\frac{R^n+k^n}{k^n}\max_{|z|=k}|P(z)|, \tag{2.9}$$

where $Q(z)=(z/k)^n\overline{P(k^2/\overline{z})}$ and $k\le 1$.

If we take $\beta=0$ in Lemma 2.3, we have the following generalization of the inequality (1.2).

Corollary 2.2. Let $P(z)$ be a polynomial of degree n, then for any $R\ge 1$, $k\le 1$ we have

$$k^n\max_{|z|=R}|P(z)|\le R^n\max_{|z|=k}|P(z)|. \tag{2.10}$$

3 Proof of theorems

Proof of Theorem 1.1: If $P(z)$ has a zero on $|z|=k$, then the inequality (1.9) is trivial. Therefore we assume that $P(z)$ has all its zeros in $|z|<k$. Then $m=\min_{|z|=k}|P(z)|>0$ and for α with $|\alpha|<1$, we have $|\alpha m(z/k)^n|<m\le|P(z)|$, where $|z|=k$. Thereby Rouche's theorem implies that the polynomial $G(z)=P(z)-\alpha m(z/k)^n$ has all its zeros in $|z|<k$. Applying Lemma 2.1, we get for $R\ge 1$, $|\alpha|<1$ and $|z|=1$,

$$\Big|P(Rz)-\alpha mR^n\Big(\frac{z}{k}\Big)^n\Big|\ge\Big(\frac{R+k}{1+k}\Big)^n\Big|P(z)-\alpha m\Big(\frac{z}{k}\Big)^n\Big|.$$

Therefore for $|\beta|<1$ the polynomial

$$P(Rz)-\alpha mR^n\Big(\frac{z}{k}\Big)^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big\{P(z)-\alpha m\Big(\frac{z}{k}\Big)^n\Big\},$$

i.e.,

$$T(z)=\Big\{P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big\}-\alpha m\Big(\frac{z}{k}\Big)^n\Big\{R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big\}$$

will have no zeros on $|z|=1$. As $|\alpha|<1$, we have for $|\beta|<1$

$$\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|\geq\Big|m\Big(\frac{z}{k}\Big)^n\Big\{R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big\}\Big|,$$

i.e.,

$$\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|\geq mk^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|, \tag{3.1}$$

for $|z|=1$.

For β with $|\beta|=1$, (3.1) follows by continuity. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2: Let $m=\min_{|z|=k}|P(z)|$. For α with $|\alpha|<1$, we have $|\alpha m|<m\leq|P(z)|$, where $|z|=k$.

Therefore by Rouche's theorem the polynomial $G(z)=P(z)-\alpha m$ has no zeros in $|z|\leq k$. Correspondingly the polynomial

$$H(z)=\Big(\frac{z}{k}\Big)^n\overline{G(k^2/\overline{z})}$$

has all its zeros in $|z|\leq k$ and $|G(z)|=|H(z)|$ for $|z|=k$. Therefore, by Lemma 2.2, we have for $|\beta|\leq 1$ and $|z|=1$, $R\geq 1$,

$$\Big|G(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nG(z)\Big|\leq\Big|H(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nH(z)\Big|.$$

Hence by the equality

$$H(z)=\Big(\frac{z}{k}\Big)^n\overline{G\Big(\frac{k^2}{\overline{z}}\Big)}=\Big(\frac{z}{k}\Big)^n\overline{P\Big(\frac{k^2}{\overline{z}}\Big)}-\overline{\alpha}m\Big(\frac{z}{k}\Big)^n=Q(z)-\overline{\alpha}m\Big(\frac{z}{k}\Big)^n$$

satsfies

$$\begin{aligned}&\Big|\{P(Rz)-\alpha m\}+\beta\Big(\frac{R+k}{1+k}\Big)^n\{P(z)-\alpha m\}\Big|\\ \leq&\Big|\Big\{Q(Rz)-\overline{\alpha}R^nm\Big(\frac{z}{k}\Big)^n\Big\}+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big\{Q(z)-\overline{\alpha}m\Big(\frac{z}{k}\Big)^n\Big\}\Big|.\end{aligned}$$

This implies

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nP(z)\Big|-|\alpha|m\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\\ \leq&\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^nQ(z)-\overline{\alpha}m\Big(\frac{z}{k}\Big)^n\Big|R^n+\Big|\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|.\end{aligned} \tag{3.2}$$

As $|P(z)|=|Q(z)|$ for $|z|=k$, i.e., $m=\min_{|z|=k}|P(z)|=\min_{|z|=k}|Q(z)|$. On applying Theorem 1.1 for the polynomial $Q(z)$

$$\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n Q(z)\Big|\geq k^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|m,$$

where $|z|=1$ and $|\beta|\leq 1$.

Now by suitable choice of the argument α, we get for $|z|=1$ and $|\beta|\leq 1$,

$$\begin{aligned}&\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n Q(z)-\overline{\alpha}m\Big(\frac{z}{k}\Big)^n\Big|R^n+\Big|\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\\ =&\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n Q(z)\Big|-|\alpha|mk^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|.\end{aligned} \tag{3.3}$$

Thereby we can rewrite (3.2) as

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n P(z)\Big|-|\alpha|m\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\\ \leq&\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n Q(z)-|\alpha|mk^{-n}\Big|R^n+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|,\end{aligned}$$

i.e.,

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n P(z)\Big|-\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n Q(z)\Big|\\ \leq&-|\alpha|\Big\{k^{-n}\Big|R^n+\beta(\frac{R+k}{1+k}\Big)^n\Big|-\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\Big\}m\end{aligned}$$

for $|z|=1$.

Making $|\alpha|\to 1$, we get for $|z|=1$ and $R\geq 1$,

$$\begin{aligned}&\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n P(z)\Big|-\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n Q(z)\Big|\\ \leq&-\Big\{k^{-n}\Big|R^n+\beta(\frac{R+k}{1+k}\Big)^n\Big|-\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\Big\}m.\end{aligned} \tag{3.4}$$

On the other hand, by Lemma 2.4, we have for $|z|=1$ and $R\geq 1$,

$$\Big|P(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n P(z)\Big|+\Big|Q(Rz)+\beta\Big(\frac{R+k}{1+k}\Big)^n Q(z)\Big| \tag{3.5}$$

$$\leq\Big\{k^{-n}\Big|R^n+\beta(\frac{R+k}{1+k}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+k}{1+k}\Big)^n\Big|\Big\}\max_{|z|=k}|P(z)|. \tag{3.6}$$

Addition of the inequalities (3.4) and (3.5) easily leads to the inequality (1.11) and the theorem follows.

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References

[1] N. C. Ankeny and T. J. Rivlin, On a theorem of S. Bernstein, Pacific J. Math., 5 (1955), 849–852.
[2] A. Aziz, Growth of polynomials whose zeros are within or outside a circle, Bull. Austral. Math. Soc., 35 (1987), 247–256.
[3] Z. Aziz and Q. M. Dawood, Inequalities for a polynomial and its derivative, J. Approx. Theory, 54 (1988), 306–313.
[4] K. K. Dewan and S. Hans, Some polynomial inequalities in the complex domain, Anal. Theory Appl., 26 (2010), 1–6.
[5] V. K. Jain, Generalization of certain well known inequalities for polynomials, Glasnik Matematicki, 32 (1997), 45–51.
[6] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press, New York, 2002.