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Results About Parabolic-Like Mappings

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Abstract. In this paper we present the most important definitions and results of the theory of parabolic-like mappings, and we will give an example. The proofs of the results can be found in [2,4] and [3].

Key Words: Polynomials, rational maps, entire and meromorphic functions, renormalization, Holomorphic families of dynamical systems, the Mandelbrot set, bifurcations.

AMS Subject Classifications: 37F10, 37F25, 37F45

1 Introduction

Complex Dynamics is concerned with the study of iteration of holomorphic maps on a Riemann surface. Let $z\in\widehat{\mathbb{C}}$, and let f be a holomorphic map on $\widehat{\mathbb{C}}$, the *orbit* of z under f is the sequence $\{z,f(z),f^2(z),\cdots,\}$ (where f^n means f composed to itself n-times). The main activity in Holomorphic dynamics is the study of the asymptotic behaviour of such orbits and the resulting classification of points in $\widehat{\mathbb{C}}$. The *Fatou set* is the set of points z such that the family (f^n) is equicontinuous near z; the dynamics is chaotic on the complementary *Julia set* (see [5]). An important special case is given by polynomial maps of $\widehat{\mathbb{C}}$. In the polynomial case the Julia set of a map f is the boundary of the basin of the superattracting fixed point at infinity. In this situation is useful to define the *filled Julia set* to be the complement of the basin of attraction of infinity. A classical theorem by Fatou in 1918 asserts that the filled Julia set is connected if and only if it contains all the finite critical points (those points where the derivative vanishes) of f (see [5]). This result motivates the consideration of the *connectedness locus* within a given family of maps, for example the family of polynomial of some degree d. The Mandelbrot set M is the connectedness locus of the quadratic family $z\mapsto z^2+c$.

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In 1985, Adrien Douady and John Hamal Hubbard published a groundbreaking paper entitled *On the dynamics of polynomial-like mappings* (see [1]). A polynomial-like mapping is a proper holomorphic map $f: U' \to U$, where $U', U \approx \mathbb{D}$, and $\overline{U'} \subset U$. The filled Julia set is defined in the polynomial-like case in the same fashion as for polynomials: the set of points which do not escape the domain. A polynomial-like map of degree d is determined up to holomorphic conjugacy by its internal and external classes, that is, the (conjugacy classes of) the restrictions to the filled Julia set and its complement. In particular an external map is a degree d real-analytic orientation preserving and strictly expanding self-covering of the unit circle. A central result of this theory gives a verifiable sufficient condition for the connectedness locus of an analytic family of quadratic polynomial-like maps to be homeomorphic to M. This theory provides a language and firm foundation for the formulation and resolution of numerous problems concerning renormalization, for example, the celebrated Branner-Hubbard description of the locus of cubic polynomials with one escaping critical point.

It has long been clear, from both heuristic considerations and numerical experimentations, that many of the parameter space consequences of the Douady-Hubbard theory should have appropriate analogues in parabolic settings. To understand and study such families we extend in [2] and [3] the polynomial-like theory to a class of parabolic mappings which we called parabolic-like mappings. A parabolic-like mapping is an object similar to a polynomial-like mapping, but with an external map with a parabolic fixed point (see [2]). In this paper we will state the most important definitions and results of the theory of parabolic-like mappings, and we will give an example.

The paper is organized as follows: in Section 2 we will remember some facts about polynomials dynamics on the Riemann Sphere and polynomial-like theory. In Section 3 we will give an idea of what kind of maps we are interested in. In Sections 4 and 5 we will give the main definitions and results of the parabolic-like mappings theory, and in Section 6 we will give an example.

2 Polynomials and polynomial-like mappings

Let P be a polynomial on the Riemann Sphere. Then P has a superattracting fixed point at infinity. Let $A_\infty(P) := \{z | P^n(z) \to \infty \text{ as } n \to \infty\}$ denotes the basin of attraction of infinity. Then $A_\infty(P)$ is a completely invariant Fatou component for the polynomial P. Define the filled Julia set as the complement of the basin of attraction of infinity: $K_P := \widehat{\mathbb{C}} \setminus A_\infty(P)$. The Julia set for the polynomial P is the common boundary between the basin of attraction of infinity and the filled Julia set: $J_P = \partial K_P = \partial A_\infty(P)$.

2.1 Polynomials-like mappings

we said in the Introduction, the notion of polynomial-like mappings was introduced by Douady and Hubbard in the landmark paper *On the dynamics of Polynomial-like mappings* (see [1]).

Definition 2.1. A *polynomial-like map* of degree $d \geq 2$ is a triple (f,U,U') where U, U' are open sets of $\mathbb{C}$ isomorphic to discs with $\overline{U'} \subset U$ and $f: U' \to U$ is a proper holomorphic map of degree d.

The filled Julia set and the Julia set are defined for polynomial-like maps in the same fashion as for polynomials.

Definition 2.2. Let $f: U' \to U$ be a polynomial-like map. The *filled Julia set* of f is defined as the set of points in U' that never leave U' under iteration, i.e.,

$$K_f = \{z \in U' | f^n(z) \in U' \ \forall n \geq 0\}.$$

As for polynomials, we define the Julia set of f as

$$J_f := \partial K_f.$$

Any polynomial-like map (f,U',U) of degree d is associated with an *external map* h_f of the same degree d, which describes the dynamics of the polynomial-like map outside the filled Julia set. We will give the construction of an external map for polynomial-like maps in the case K_f is connected. For the case K_f not connected, we refer to [1].

Let (f,U',U) be a polynomial-like map of degree d with connected filled Julia set K_f. Let

$$\alpha: U \setminus K_f \to W = \{z | 1 < |z| < R\},$$

(where $\log R$ is the modulus of $U \setminus K_f$) be an isomorphism such that $|\alpha(z)| \to 1$ as $z \to K_f$. Write $W' = \alpha(U' \setminus K_f)$ and define the map:

$$h^+ := \alpha \circ f \circ \alpha^{-1} : W' \to W.$$

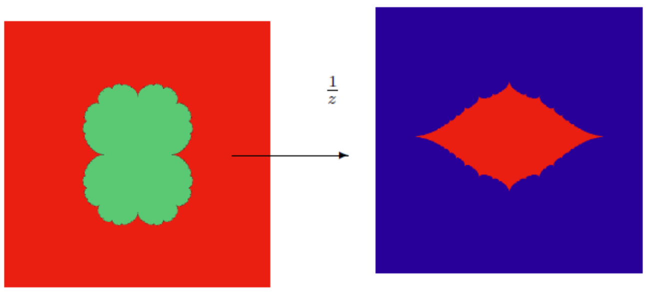
Since the filled Julia set is connected, it contains all the critical points of f, then $f: U' \setminus K_f \to U \setminus K_f$ is a holomorphic degree d covering map, and therefore the map h^+ is a holomorphic degree d covering. Let $\tau(z) = 1/\bar{z}$ be the reflection with respect to the unit circle, and set $W_- = \tau(W)$, $W'_- = \tau(W')$, $\widetilde{W} = W \cup \mathbb{S}^1 \cup W_-$ and $\widetilde{W'} = W' \cup \mathbb{S}^1 \cup W'_-$. By the strong reflection principle with respect to the unit circle we can extend analytically the map $h^+: W' \to W$ to $h: \widetilde{W'} \to \widetilde{W}$. The map h is strictly expanding, indeed $h: \widetilde{W'} \to \widetilde{W}$ is a degree d covering map, and $h^{-1}: \widetilde{W} \to \widetilde{W'} \subsetneq \widetilde{W}$ is strongly contracting for the Poincare metric on $\widetilde{W}$. Let h_f be the restriction of h to the unit circle. Then the map $h_f: \mathbb{S}^1 \to \mathbb{S}^1$ is an *external map* of f (an external map of a polynomial-like map is defined up to real-analytic conjugacy).

By replacing the external map of a degree d polynomial-like map with the external map of a degree d polynomial (namely $z \to z^d$) Douady and Hubbard proved the following result.

Theorem 2.1 (Straightening theorem)**.** *Let $f: U' \to U$ be a polynomial-like map of degree d. Then f is hybrid conjugate to a polynomial P of the same degree. Moreover, if K_f is connected, then P is unique (up to conjugation by an affine map).*

3 About external maps

The external map of a degree d polynomial-like map is a degree d real-analytic orientation preserving and strictly expanding self-covering of the unit circle. In this paper we are interested in (restrictions of) maps with external class with a parabolic fixed point. That is, (restrictions of) maps with an attracting petal outside the filled Julia set of the restriction we consider. For example, consider the map $f_1(z)=z^2+1/4$. This map has a parabolic fixed point at $z=1/2$, which basin of attraction resides in the filled Julia set. Outside the filled Julia set there is the basin of attraction of infinity (as for any polynomial), so the external map of $f_1(z)$ is hyperbolic, and the map $f_1(z)$ presents (trivial) polynomial-like restrictions. On the other hand, by interchanging the roles of the filled Julia set and the closure of the basin of attraction of infinity for f_1 we can find a map with a parabolic external class. More precisely, we conjugate $f_1(z)$ by $\iota(z)=1/z$, obtaining the map $f_2(z)=4z^2/(4+z^2)$, and we define as filled Julia set for f_2 the closure of the basin of attraction of the superattracting fixed point $z=0$. Therefore, the parabolic basin of attraction of $z=2$ resides now outside the filled Julia set. This gives rise to an the external class with a parabolic fixed point, and appropriate restrictions of the map f_2 belong to the class of maps we are interested in.



4 Parabolic-like mappings

Let us now introduce the definition of parabolic-like mapping.

Definition 4.1 (Parabolic-Like Maps)**.** A parabolic-like map of degree d is a 4-tuple (f,U',U,γ), where

1. U', U are open subsets of $\mathbb{C}$, with U' U and $U\cup U'$ isomorphic to a disc, and U' not contained in U,
2. $f:U'\to U$ is a proper holomorphic map of degree d with a parabolic fixed point at $z=z_0$ of multiplier 1,

3. $\gamma:[-1,1]\to\overline{U}$, $\gamma(0)=z_0$ is an arc, forward invariant under f, C^1 on $[-1,0]$ and on $[0,1]$, and such that

$$f(\gamma(t))=\gamma(dt),\quad \forall -\frac{1}{d}\le t\le \frac{1}{d},$$
$$\gamma\Big(\Big[\frac{1}{d},1\Big)\cup\Big(-1,-\frac{1}{d}\Big]\Big)\subseteq U\backslash U',\quad \gamma(\pm 1)\in\partial U.$$

It resides in repelling petal(s) of z_0 and it divides U',U into Ω', Δ' and Ω, Δ respectively, such that $\Omega'\subset\subset U$ (and $\Omega'\subset\Omega$), $f:\Delta'\to\Delta$ is an isomorphism (see Fig. 1) and Δ' contains at least one attracting fixed petal of z_0. We call the arc γ a dividing arc.

Let (f,U',U,γ) be a parabolic-like map. We define the *filled Julia set* K_f of f as the set of points in U' that never leave $(\Omega'\cup\gamma_\pm(0))$ under iteration:

$$K_f:=\{z\in U'|\forall n\ge 0,\ f^n(z)\in\Omega'\cup\gamma_\pm(0)\}.$$

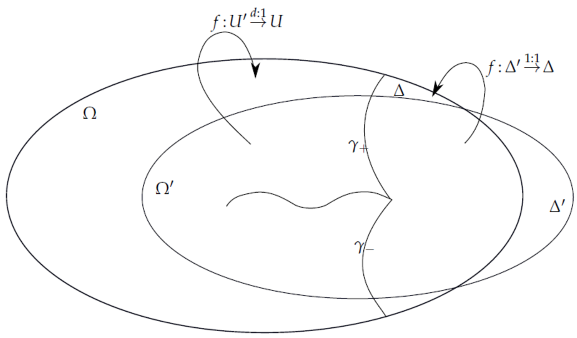


Figure 1: On a parabolic-like map (f,U',U,γ) the arc γ divides U' and U into Ω', Δ' and Ω, Δ respectively. These sets are such that Ω' is compactly contained in U, $\Omega'\subset\Omega$, $f:\Delta'\to\Delta$ is an isomorphism and Δ' contains at least one attracting fixed petal of the parabolic fixed point.

4.1 Example

Let $(C_a(z)=z+az^2+z^3)$, for $a=i$. This map has a parabolic fixed point at $z=0$ with multiplier and parabolic multiplicity 1, a superattracting fixed point s at $z=(-a-\sqrt{a^2-3})/3$, and a critical point c at $z=(-a+\sqrt{a^2-3})/3$ which belongs to the immediate basin of attraction $\mathcal{A}(0)$ of 0. Let $\varphi:\mathcal{A}(0)\to\mathbb{D}$ be the Riemann map with $\varphi(c)=0$ and $\varphi(z)\overset{z\to 0}{\longrightarrow}1$, and let $\psi:\mathbb{D}\to\mathcal{A}(0)$ be its inverse, which extends continuously to $\mathbb{S}^1$. Let $w\in\mathbb{S}^1$ be a $\varphi\circ f\circ\psi$ periodic point in the first quadrant, such that the hyperbolic geodesic $\widetilde{\gamma}\in\mathbb{D}$ connecting w and $\overline{w}$ separates the critical value from the parabolic fixed point of this map. Let U be the Jordan domain bounded by $\widehat{\gamma}=\psi(\widetilde{\gamma})$, union the arcs up to potential level 1 of the external rays landing at $\psi(w)$ and $\psi(\overline{w})$, together with the arc of the level 1 equipotential connecting this two rays around s (see Fig. 2). Let U' be the connected component of $f^{-1}(U)$

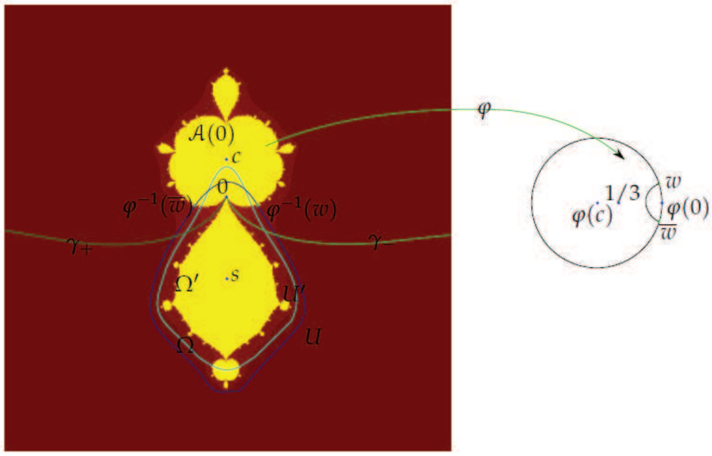


Figure 2: Construction of a degree 2 parabolic-like map from the map $(C_a(z)=z+az^2+z^3)$, for $a=i$. The superattracting fixed point $z=(-a-\sqrt{a^2-3})/3$ is denoted by s, and the critical point $z=(-a+\sqrt{a^2-3})/3$ in the basin of attraction of the parabolic fixed point is denoted by c.

containing 0 and the dividing arcs $\gamma_\pm$ be the fixed external rays landing at the parabolic fixed point 0 and parametrized by potential. Then (f,U',U,γ) is a parabolic-like map of degree 2 (see Fig. 2), and its filled Julia set is the closure of the connected component of the filled Julia set of C_i containing the superattracting fixed point s.

4.2 Model family

Introduced parabolic-like mappings, we want to straighten degree 2 parabolic-like maps to members of a family of maps with a parabolic external class. We consider as model family the family of quadratic rational maps with a parabolic fixed point of multiplier 1 at infinity, normalized by setting the critical points at $+1$ and -1, this is

$$Per_1(1)=\{[P_A] | P_A(z)=z+1/z+A,\ A\in\mathbb{C}\},$$

where $[P_A]$ is the equivalence class of P_A under holomorphic conjugacy, this is $[P_A]=\{P_A,P_{-A}\}$.

Let us clarify what we mean by "external class" for this family of rational maps. Let us remember that, for polynomials, the filled Julia set is defined to be the complement of the basin of attraction of infinity, and such basin is a completely invariant Fatou component. For all the maps in $Per_1(1)$ the parabolic basin of attraction of infinity is a completely invariant Fatou component, call it Λ. So for these maps we can define as filled Julia set the complement of the parabolic basin of attraction of infinity, this is

$$K_A=\widehat{\mathbb{C}}\setminus\Lambda,$$

(this is well defined for every $A\neq 0$, since for every $A\neq 0$, P_A has a unique completely invariant Fatou component Λ. For the map $P_0(z)=z+1/z$ we need to make a choice, and after such choice K_0 is well defined).

Every member of this family has an external class which consists of maps with a parabolic fixed point. We prove in Proposition 4.2 in [2] that every member of this family has as external class the class of $h_2(z)=(3z^2+1)/(3+z^2)$.

4.3 Conjugacies and external maps for parabolic-like mappings

We say that (f,U_1',U_1,γ_1) is a *parabolic-like restriction* of (f,U_2',U_2,γ_2) if $U_1'\subseteq U_2'$ and (f,U_i',U_i,γ_i), $i=1,2$ are parabolic-like maps with the same degree and filled Julia set.

Definition 4.2 (Conjugacy for Parabolic-Like Mappings)**.** We say that the parabolic-like mappings (f,U',U,γ_f) and (g,V',V,γ_g) are *topologically conjugate* if there exist parabolic-like restrictions (f,A',A,γ_f) and (g,B',B,γ_g), and a homeomorphism $\varphi: A\to B$ such that $\varphi(\gamma_{\pm f})=\gamma_{\pm g}$ and

$$\varphi(f(z))=g(\varphi(z)) \quad \text{on } \Omega'_{A_f}\cup\gamma_f.$$

If moreover φ is quasiconformal (and $\bar\partial\varphi=0$ a.e. on K_f), we say that f and g are *quasiconformally* (*hybrid*) conjugate.

On the other side, we say that the parabolic-like mappings (f,U',U,γ_f) and (g,V',V,γ_g) are *holomorphically conjugate* if there exist parabolic-like restrictions (f,A',A,γ_f) and (g,B',B,γ_g), and a conformal map $\varphi: A\to B$ conjugating dynamics on A.

We associate to any degree d parabolic-like map (f,U',U,γ_f) a degree d real analytic orientation preserving map $h_f:\mathbb{S}^1\to\mathbb{S}^1$ with a parabolic fixed point, which we call an *external map* of the parabolic-like map. An external map h_f of a parabolic-like map (f,U',U,γ_f) is unique up to real-analytic conjugacy. The conjugacy class of h_f under real-analytic diffeomorphisms is called the external class of the parabolic-like map, and denoted by $[h_f]$.

The construction of an external map of a parabolic-like map is given in [2]. Anyway, in the connected case such construction follows the construction of an external map for polynomial-like maps, construction which we gave in Section 2. We say that the parabolic-like mappings (f,U',U,γ_f) and (g,V',V,γ_g) are *externally equivalent* if their external maps are conjugate by a real-analytic diffeomorphism, or in other words if their external maps belong to the same external class.

A *parabolic external map* is a real-analytic orientation preserving and metrically expanding map $h:\mathbb{S}^1\to\mathbb{S}^1$, this is, a degree d real analytic orientation preserving map $h:\mathbb{S}^1\to\mathbb{S}^1$ such that $\exists z_0\in\mathbb{S}^1$ for which $h'(z_0)=1$ and for all $z\in\mathbb{S}^1$, $z\neq z_0$, $|(h)'(z)|>1$. We prove in [4] that an external map of a parabolic-like map is a parabolic external map, and that for a parabolic external map there exists an extension which has all the properties an external map of a parabolic-like map has.

4.4 The Straightening theorem

In [2] we prove the following results:

Proposition 4.1. A degree 2 parabolic-like map is holomorphically conjugate to a member of the family $Per_1(1)$ if and only if its external class is given by the class of h_2.

Theorem 4.1. *Let (f,U',U,γ_f) be a parabolic-like mapping of some degree $d>1$, and $h:\mathbb{S}^1\to\mathbb{S}^1$ be a parabolic external map of the same degree d. Then there exists a parabolic-like mapping (g,V',V,γ_g) which is hybrid equivalent to f and whose external class is $[h]$.*

Proposition 4.2. Let $f:U'\to U$ and $g:V'\to V$ be two parabolic-like mappings of degree d with connected Julia sets. If they are hybrid and externally equivalent, then they are holomorphically equivalent.

Combining these propositions (namely, Proposition 4.1 together with Theorem 4.1 for part 1 and Proposition 4.2 for part 2), we obtain the Straightening Theorem for parabolic-like mappings:

1. Every degree 2 parabolic-like mapping (f,U',U,γ_f) is hybrid equivalent to a member of the family $Per_1(1)$.
2. Moreover, if K_f is connected, this member is unique (up to holomorphic conjugacy).

The idea of the Straightening Theorem is to glue, outside the filled Julia set of a degree 2 parabolic-like map, the basin of attraction of the parabolic fixed point. Fig. 3 shows the filled Julia set of the map $C_a(z)=z^3+az^2+z$, $a=i$, which restricts to a degree 2 parabolic-like map, and Fig. 4 shows the hybrid equivalent member of the family $Per_1(1)$.

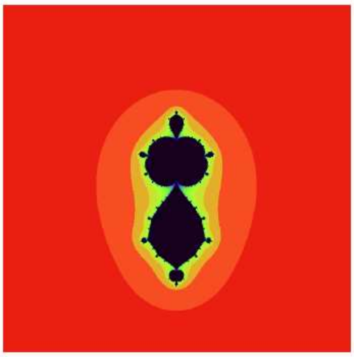


Figure 3: Julia set of the map $C_a(z)=z^3+az^2+z$, $a=i$.

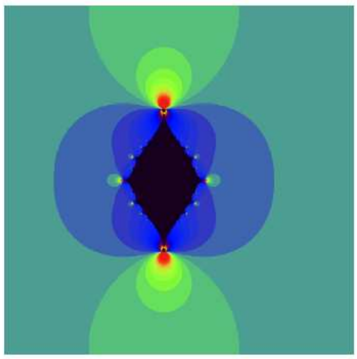


Figure 4: Julia set of the map $P_1(z)=z+1/z+A$, $A=1$.

5 Families of parabolic-like mappings

Definition 5.1. Let $\Lambda \subset \mathbb{C}$, $\Lambda \approx \mathbb{D}$ and let $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ be a family of degree d parabolic-like mappings. Set $\mathbb{U}' = \{(\lambda,z) | z \in U'_\lambda\}$, $\mathbb{U} = \{(\lambda,z) | z \in U_\lambda\}$, $\mathbb{O}' = \{(\lambda,z) | z \in \Omega'_\lambda\}$, and $\mathbb{O} = \{(\lambda,z) | z \in \Omega_\lambda\}$. Then $\mathbf{f}$ is a degree d analytic family of parabolic-like maps if the following conditions are satisfied:

1. $\mathbb{U}'$, $\mathbb{U}$, $\mathbb{O}'$ and $\mathbb{O}$ are domains in $\mathbb{C}^2$.
2. the map $\mathbf{f}: \mathbb{U}' \to \mathbb{U}$ is holomorphic in (λ,z).
3. all the parabolic-like maps in the family have the same number of attracting petals in the filled Julia set.

Define
$$M_f = \{\lambda | K_\lambda \text{ is connected}\}.$$

5.1 Nice families

An analytic family of parabolic-like mappings is *nice* if there exists a holomorphic motion of the dividing arcs
$$\Phi: \Lambda \times \gamma_{\lambda_0} \to \mathbb{C},$$
and there exists a holomorphic motion of the ranges
$$B: \Lambda \times \partial U_{\lambda_0} \to \mathbb{C},$$
which is a piecewise C^1-diffeomorphism with no cusps in z (for every fixed λ), and $B_\lambda(\gamma_{\lambda_0}(\pm 1)) = \gamma_\lambda(\pm 1)$.

By the Straightening theorem, if $(f_\lambda)_{\lambda\in\Lambda}$ be is a degree 2 nice analytic family of parabolic-like maps, $\forall \lambda \in \Lambda$, f_λ is equivalent to a member in $Per_1(1)$, and if K_λ is connected this member is unique (up to holomorphic conjugacy). The family $Per_1(1)$ is typically parametrized by $B = 1 - A^2$, which is the multiplier of the "free" fixed point $z = -1/A$ of P_A, and the connectedness locus of the family $Per_1(1)$ is called M_1. So we can define a map
$$\begin{aligned} &\chi: M_f \to M_1, \\ &\lambda \to B, \end{aligned}$$
which associates to each λ the multiplier of the fixed point $z = -1/A$ of the member $[P_A]$ hybrid equivalent to f_λ. In [3] we show that we can extend the map χ to a map defined on the whole of Λ, and such that the restriction $\chi_{|\Lambda \setminus M_f}$ is quasiregular. Finally, we prove that:

Theorem 5.1. *Under suitable conditions, the map χ restricts to a ramified covering from the connectedness locus of $(f_\lambda)_{\lambda\in\Lambda}$ to $M_1 \setminus \{1\}$.*

6 Example

Consider the family $C_a(z)=z+az+z^3$, call $\mathcal{C}$ its connectedness locus and define $\Lambda\subset\mathcal{C}$ as the open set bounded by the external rays of angle $1/6$ and $2/6$. Every member C_a of the family $(C_a(z)=z+az+z^3)_{a\in\Lambda}$ presents a degree 2 parabolic-like restriction with no petals in the filled Julia set (see [3]), and we prove in [3] that the family $(C_a(z)=z+az+z^3)_{a\in\Lambda}$ restricts to a degree 2 nice analytic family of parabolic-like mappings. Hence the connectedness locus $\mathcal{C}$ of the family $(C_a(z)=z+az+z^3)$ presents a baby M_1, namely in the open component bounded by the external rays of angle $1/6$ and $2/6$. By symmetry we can repete the argument for Λ being the open set bounded by the external rays of argument $4/6$ and $5/6$. So $\mathcal{C}$ presents two babies M_1, see Figs. 5 and 6.

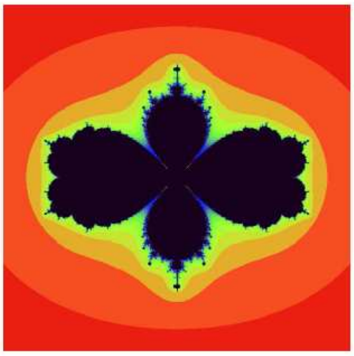


Figure 5: Connectedness locus of the family C_a.

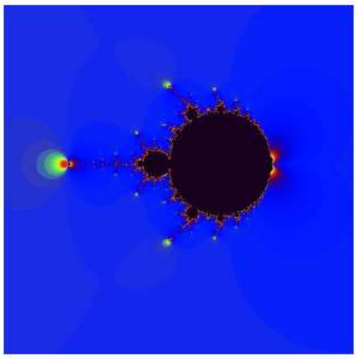


Figure 6: Connectedness locus M_1 of the family $Per_1(1)$.

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