Analysis in Theory and Applications
Anal. Theory Appl., Vol. **30**, No. 1 (2014), pp. 1-19

DOI: 10.4208/ata.2014.v30.n1.1

# Weighted Integral Means of Mixed Areas and Lengths Under Holomorphic Mappings

Jie Xiao[1,*] and Wen Xu[2]

[1] *Department of Mathematics and Statistics, Memorial University, NL A1C 5S7, Canada*
[2] *Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland*

Received 31 October 2013; Accepted (in revised version) 1 March 2014

Available online 31 March 2014

**Abstract.** This note addresses monotonic growths and logarithmic convexities of the weighted ($(1-t^2)^\alpha dt^2$, $-\infty<\alpha<\infty$, $0<t<1$) integral means $\mathsf{A}_{\alpha,\beta}(f,\cdot)$ and $\mathsf{L}_{\alpha,\beta}(f,\cdot)$ of the mixed area $(\pi r^2)^{-\beta}A(f,r)$ and the mixed length $(2\pi r)^{-\beta}L(f,r)$ ($0\le\beta\le 1$ and $0<r<1$) of $f(r\mathbb{D})$ and $\partial f(r\mathbb{D})$ under a holomorphic map $f$ from the unit disk $\mathbb{D}$ into the finite complex plane $\mathbb{C}$.

**Key Words**: Monotonic growth, logarithmic convexity, mean mixed area, mean mixed length, isoperimetric inequality, holomorphic map, univalent function.

**AMS Subject Classifications**: 32A10, 32A36, 51M25

## 1 Introduction

From now on, $\mathbb{D}$ represents the unit disk in the finite complex plane $\mathbb{C}$, $H(\mathbb{D})$ denotes the space of holomorphic mappings $f:\mathbb{D}\to\mathbb{C}$, and $U(\mathbb{D})$ stands for all univalent functions in $H(\mathbb{D})$. For any real number $\alpha$, positive number $r\in(0,1)$ and the standard area measure $dA$, let

$$dA_\alpha(z)=(1-|z|^2)^\alpha dA(z),\quad r\mathbb{D}=\{z\in\mathbb{D}:|z|<r\},\quad r\mathbb{T}=\{z\in\mathbb{D}:|z|=r\}.$$

In their recent paper [11], Xiao and Zhu have discussed the following area $0<p<\infty$-integral mean of $f\in H(\mathbb{D})$:

$$M_{p,\alpha}(f,r)=\Big[\frac{1}{A_\alpha(r\mathbb{D})}\int_{r\mathbb{D}}|f|^p dA_\alpha\Big]^{\frac{1}{p}},$$

*Corresponding author. *Email addresses:* `jxiao@mun.ca` (J. Xiao), `wen.xu@uef.fi` (W. Xu)

proving that $r\mapsto M_{p,\alpha}(f,r)$ is strictly increasing unless $f$ is a constant, and $\log r\mapsto \log M_{p,\alpha}(f,r)$ is not always convex. This last result suggests such a conjecture that $\log r\mapsto\log M_{p,\alpha}(f,r)$ is convex or concave when $\alpha\le 0$ or $\alpha>0$. But, motivated by [11, Example 10, (ii)] we can choose $p=2$, $\alpha=1$, $f(z)=z+c$ and $c>0$ to verify that the conjecture is not true. At the same time, this negative result was also obtained in Wang-Zhu's manuscript [10]. So far it is unknown whether the conjecture is generally true for $p\neq 2$– see [9] for a recent development.

The foregoing observation has actually inspired the following investigation. Our concentration is the fundamental case $p=1$. To understand this new approach, let us take a look at $M_{1,\alpha}(\cdot,\cdot)$ from a differential geometric viewpoint. Note that

$$M_{1,\alpha}(f',r)=\frac{\int_{r\mathbb{D}}|f'|dA_\alpha}{A_\alpha(r\mathbb{D})}=\frac{\int_0^r\left[(2\pi t)^{-1}\int_{t\mathbb{T}}|f'(z)||dz|\right](1-t^2)^\alpha dt^2}{\int_0^r(1-t^2)^\alpha dt^2}.$$

So, if $f\in U(\mathbb{D})$, then

$$(2\pi t)^{-1}\int_{t\mathbb{T}}|f'(z)||dz|$$

is a kind of mean of the length of $\partial f(t\mathbb{D})$, and hence the square of this mean dominates a sort of mean of the area of $f(t\mathbb{D})$ in the isoperimetric sense:

$$\Phi_A(f,t)=(\pi t^2)^{-1}\int_{t\mathbb{D}}|f'(z)|^2dA(z)\le\left[(2\pi t)^{-1}\int_{t\mathbb{T}}|f'(z)||dz|\right]^2=\left[\Phi_L(f,t)\right]^2.$$

In accordance with the well-known Pólya-Szegö monotone principle [8, Problem 309] (or [2, Proposition 6.1]) and the area Schwarz's lemma in Burckel, Marshall, Minda, Poggi-Corradini and Ransford [2, Theorem 1.9], $\Phi_L(f,\cdot)$ and $\Phi_A(f,\cdot)$ are strictly increasing on $(0,1)$ unless $f(z)=a_1z$ with $a_1\neq 0$. Furthermore, $\log\Phi_L(f,r)$ and $\log\Phi_A(f,r)$, equivalently, $\log L(f,r)$ and $\log A(f,r)$, are convex functions of $\log r$ for $r\in(0,1)$, due to the classical Hardy's convexity and [2, Section 5]. Perhaps, it is worthwhile to mention that if $c>0$ is small enough then the universal cover of $\mathbb{D}$ onto the annulus $\{e^{-c\pi/2}<|z|<e^{c\pi/2}\}$:

$$f(z)=\exp\left[ic\log\left(\frac{1+z}{1-z}\right)\right]$$

enjoys the property that $\log r\mapsto\log A(f,r)$ is not convex; see [2, Example 5.1].

In the above and below, we have used the following convention:

$$\Phi_A(f,r)=\frac{A(f,r)}{\pi r^2}\quad\text{and}\quad\Phi_L(f,r)=\frac{L(f,r)}{2\pi r},$$

where under $r\in(0,1)$ and $f\in H(\mathbb{D})$, $A(f,r)$ and $L(f,r)$ stand respectively for the area of $f(r\mathbb{D})$ (the projection of the Riemannian image of $r\mathbb{D}$ by $f$) and the length of $\partial f(r\mathbb{D})$ (the boundary of the projection of the Riemannian image of $r\mathbb{D}$ by $f$) with respect to the standard Euclidean metric on $\mathbb{C}$. For our purpose, we choose a shortcut notation

$$d\mu_\alpha(t)=(1-t^2)^\alpha dt^2\quad\text{and}\quad\nu_\alpha(t)=\mu_\alpha([0,t]),\quad\forall t\in(0,1),$$

and for $0\le\beta\le 1$ define

$$\Phi_{A,\beta}(f,t)=\frac{A(f,t)}{(\pi t^2)^\beta}\quad\text{and}\quad \Phi_{L,\beta}(f,t)=\frac{L(f,t)}{(2\pi t)^\beta},$$

and then introduce two natural analytic-geometric quantities

$$\mathsf{A}_{\alpha,\beta}(f,r)=\frac{\int_0^r\Phi_{A,\beta}(f,t)d\mu_\alpha(t)}{\int_0^r d\mu_\alpha(t)}\quad\text{and}\quad \mathsf{L}_{\alpha,\beta}(f,r)=\frac{\int_0^r\Phi_{L,\beta}(f,t)d\mu_\alpha(t)}{\int_0^r d\mu_\alpha(t)},$$

which are respectively called the weighted integral means of the mixed area and the mixed length for $f(r\mathbb{D})$ and $\partial f(r\mathbb{D})$.

In this note, we consider two fundamental properties: monotonic growths and logarithmic convexities of both $\mathsf{A}_{\alpha,\beta}(f,r)$ and $\mathsf{L}_{\alpha,\beta}(f,r)$, thereby giving two applications: (i) if $r\mapsto\Phi_L(f,r)$ is monotone increasing on $(0,1)$, then so is the isoperimetry-induced function:

$$r\mapsto\frac{\int_0^r\big[\Phi_{L,1}(f,t)\big]^2d\mu_\alpha(t)}{\int_0^r d\mu_\alpha(t)}\ge\mathsf{A}_{\alpha,1}(f,r);$$

(ii) the log-convexity for $\mathsf{L}_{\alpha,1}(f,r)$ essentially settles the above-mentioned conjecture. The non-trivial details (results and their proofs) are arranged in the forthcoming two sections.

# 2 Monotonic growth

In this section, we deal with the monotonic growths of $\mathsf{A}_{\alpha,\beta}(f,r)$ and $\mathsf{L}_{\alpha,\beta}(f,r)$, along with their associated Schwarz type lemmas. In what follows, $\mathbb{N}$ is used as the set of all natural numbers.

## 2.1 Two lemmas

The following two preliminary results are needed.

**Lemma 2.1** (see [5]). *Let $f\in H(\mathbb{D})$ be of the form $f(z)=a_0+\sum_{k=n}^\infty a_kz^k$ with $n\in\mathbb{N}$. Then:*

*(i) $\pi r^{2n}\Big[\frac{|f^{(n)}(0)|}{n!}\Big]^2\le A(f,r),\quad\forall r\in(0,1)$.*

*(ii) $2\pi r^n\Big[\frac{|f^{(n)}(0)|}{n!}\Big]\le L(f,r),\quad\forall r\in(0,1)$.*

*Moreover, equality in (i) or (ii) holds if and only if $f(z)=a_0+a_nz^n$.*

*Proof.* This may be viewed as the higher order Schwarz type lemma for area and length. See also the proofs of Theorems 1 and 2 in [5], and their immediate remarks on equalities. Here it is worth noticing three matters: (a) $f^{(n)}(0)/n!$ is just $a_n$; (b) [4, Corollary 3] presents a different argument for the area case; (c) $L(f,r)$ is greater than or equal to the length $l(r,f)$ of the outer boundary of $f(r\mathbb{D})$ (defined in [5]) which is not less than the length $l^\#(r,f)$ of the exact outer boundary of $f(r\mathbb{D})$ (introduced in [12]). □

**Lemma 2.2.** *Let* $0\le\beta\le 1$.

*(i) If* $f\in H(\mathbb{D})$, *then* $r\mapsto\Phi_{A,\beta}(f,r)$ *is strictly increasing on* $(0,1)$ *unless*

$$f=\begin{cases} \textit{constant}, & \textit{when } \beta<1,\\ \textit{linear map}, & \textit{when } \beta=1.\end{cases}$$

*(ii) If* $f\in U(\mathbb{D})$ *or* $f(z)=a_0+a_nz^n$ *with* $n\in\mathbb{N}$, *then* $r\mapsto\Phi_{L,\beta}(f,r)$ *is strictly increasing on* $(0,1)$ *unless*

$$f=\begin{cases} \textit{constant}, & \textit{when } \beta<1,\\ \textit{linear map}, & \textit{when } \beta=1.\end{cases}$$

*Proof.* It is enough to handle $\beta<1$ since the case $\beta=1$ has been treated in [2, Theorem 1.9 and Proposition 6.1]. The monotonic growths in (i) and (ii) follow from

$$\Phi_{A,\beta}(f,r)=(\pi r^2)^{1-\beta}\Phi_{A,1}(f,r) \quad\text{and}\quad L(f,r)=(2\pi r)^{1-\beta}\Phi_{L,1}(f,r).$$

To see the strictness, we consider two cases.

(i) Suppose that $\Phi_{A,\beta}(f,\cdot)$ is not strictly increasing. Then there are $r_1,r_2\in(0,1)$ such that $r_1<r_2$, and $\Phi_{A,\beta}(f,\cdot)$ is a constant on $[r_1,r_2]$. Hence

$$\frac{d}{dr}\Phi_{A,\beta}(f,r)=0,\quad \forall r\in[r_1,r_2].$$

Equivalently,

$$2\beta A(f,r)=r\frac{d}{dr}A(f,r),\quad \forall r\in[r_1,r_2].$$

But, according to [2, (4.2)],

$$2A(f,r)\le r\frac{d}{dr}A(f,r),\quad \forall r\in(0,1).$$

Since $\beta<1$, we get $A(f,r)=0$ for all $r\in[r_1,r_2]$, whence finding that $f$ is constant.

(ii) Now assume that $\Phi_{L,\beta}(f,\cdot)$ is not strictly increasing. There are $r_3,r_4\in(0,1)$ such that $r_3<r_4$ and

$$0=\frac{d}{dr}\Phi_{L,\beta}(f,r)=(2\pi r)^{-\beta}\Big[\frac{d}{dr}L(f,r)-\frac{\beta}{r}L(f,r)\Big]=0,\quad \forall r\in[r_3,r_4].$$

If $f\in U(\mathbb{D})$, then

$$L(f,r)=\int_{r\mathbb{T}}|f'(z)||dz|$$

and hence one has the following "first variation formula"

$$\frac{d}{dr}L(f,r)=\int_0^{2\pi}|f'(re^{i\theta})|d\theta+r\frac{d}{dr}\int_0^{2\pi}|f'(re^{i\theta})|d\theta,\quad \forall r\in[r_3,r_4].$$

The previous three equations yield

$$0=(1-\beta)\int_0^{2\pi}|f'(re^{i\theta})|d\theta+r\frac{d}{dr}\int_0^{2\pi}|f'(re^{i\theta})|d\theta,\quad \forall r\in[r_3,r_4],$$

and so

$$\int_0^{2\pi}|f'(re^{i\theta})|d\theta=0,\quad \forall r\in[r_3,r_4].$$

This ensures that $f$ is a constant, contradicting $f\in U(\mathbb{D})$. Therefore, $f(z)$ is of the form $a_0+a_nz^n$. But, since $L(z^n,r)=2\pi r^n$ is strictly increasing, $f$ must be constant. □

## 2.2 Monotonic growth of $\mathsf{A}_{\alpha,\beta}(f,\cdot)$

This aspect is essentially motivated by the following Schwarz type lemma.

**Proposition 2.1.** Let $-\infty<\alpha<\infty$, $0\le\beta\le 1$, and $f\in H(\mathbb{D})$ be of the form $f(z)=a_0+\sum_{k=n}^{\infty}a_kz^k$ with $n\in\mathbb{N}$. Then

$$\pi^{1-\beta}\Big[\frac{|f^{(n)}(0)|}{n!}\Big]^2\le \mathsf{A}_{\alpha,\beta}(f,r)\Big[\frac{\nu_\alpha(r)}{\int_0^r t^{2(n-\beta)}d\mu_\alpha(t)}\Big],\quad \forall r\in(0,1),$$

with equality if and only if $f(z)=a_0+a_nz^n$.

*Proof.* The inequality follows from Lemma 2.1(i) right away. When $f(z)=a_0+a_nz^n$, the last inequality becomes an equality due to the equality case of Lemma 2.1(i). Conversely, suppose that the last inequality is an equality. If $f$ does not have the form $a_0+a_nz^n$, then the equality in Lemma 2.1(i) is not true, then there are $r_1,r_2\in(0,1)$ such that $r_1<r_2$ and

$$A(f,t)>\pi t^{2n}\Big[\frac{|f^{(n)}(0)|}{n!}\Big]^2,\quad \forall t\in[r_1,r_2].$$

This strict inequality forces that for $r\in[r_1,r_2]$,

$$\begin{aligned}&\pi^{1-\beta}\Big[\frac{|f^{(n)}(0)|}{n!}\Big]^2\int_0^r t^{2(n-\beta)}d\mu_\alpha(t)\\ =&\int_0^r(\pi t^2)^{-\beta}A(f,t)d\mu_\alpha(t)=\Big(\int_0^{r_1}+\int_{r_1}^{r_2}+\int_{r_2}^{r}\Big)(\pi t^2)^{-\beta}A(f,t)d\mu_\alpha(t)\\ >&\pi^{1-\beta}\Big[\frac{|f^{(n)}(0)|}{n!}\Big]^2\int_0^r t^{2(n-\beta)}d\mu_\alpha(t),\end{aligned}$$

a contradiction. Thus $f(z)=a_0+a_nz^n$. □

Based on Proposition 2.1, we find the monotonic growth for $\mathsf{A}_{\alpha,\beta}(\cdot,\cdot)$ as follows.

**Theorem 2.1.** *Let* $-\infty<\alpha<\infty$, $0\le\beta\le 1$, *and* $f\in H(\mathbb{D})$. *Then* $r\mapsto \mathsf{A}_{\alpha,\beta}(f,r)$ *is strictly increasing on* $(0,1)$ *unless*

$$f=\begin{cases} \textit{constant}, & \textit{when } \beta<1,\\ \textit{linear map}, & \textit{when } \beta=1.\end{cases}$$

*Consequently,*

*(i)*

$$\lim_{r\to 0}\mathsf{A}_{\alpha,\beta}(f,r)=\begin{cases} 0, & \textit{when } \beta<1,\\ |f'(0)|^2, & \textit{when } \beta=1.\end{cases}$$

*(ii) If*

$$\Phi_{A,\beta}(f,0):=\lim_{r\to 0}\Phi_{A,\beta}(f,r) \quad \textit{and} \quad \Phi_{A,\beta}(f,1):=\lim_{r\to 1}\Phi_{A,\beta}(f,r)<\infty,$$

*then*

$$0<r<s<1\Rightarrow 0\le \frac{\mathsf{A}_{\alpha,\beta}(f,s)-\mathsf{A}_{\alpha,\beta}(f,r)}{\log\nu_\alpha(s)-\log\nu_\alpha(r)}\le \Phi_{A,\beta}(f,s)-\Phi_{A,\beta}(f,0)$$

*with equality if and only if*

$$f=\begin{cases} \textit{constant}, & \textit{when } \beta<1,\\ \textit{linear map}, & \textit{when } \beta=1.\end{cases}$$

*In particular,* $t\mapsto \mathsf{A}_{\alpha,\beta}(f,t)$ *is Lipschitz with respect to* $\log\nu_\alpha(t)$ *for* $t\in(0,1)$.

*Proof.* Note that $\nu_\alpha(r)=\int_0^r d\mu_\alpha(t)$. So $d\nu_\alpha(r)$, the differential of $\nu_\alpha(r)$ with respect to $r\in(0,1)$, equals $d\mu_\alpha(r)$. By integration by parts we have

$$\Phi_{A,\beta}(f,r)\nu_\alpha(r)-\int_0^r\Phi_{A,\beta}(f,t)d\mu_\alpha(t)=\int_0^r\Big[\frac{d}{dt}\Phi_{A,\beta}(f,t)\Big]\nu_\alpha(t)dt.$$

Differentiating the function $\mathsf{A}_{\alpha,\beta}(f,r)$ with respect to $r$ and using Lemma 2.2(i), we get

$$\begin{aligned}\frac{d}{dr}\mathsf{A}_{\alpha,\beta}(f,r)&=\frac{\Phi_{A,\beta}(f,r)2r(1-r^2)^\alpha\nu_\alpha(r)-\Big[\int_0^r\Phi_{A,\beta}(f,t)d\mu_\alpha(t)\Big]2r(1-r^2)^\alpha}{\nu_\alpha(r)^2}\\ &=\frac{2r(1-r^2)^\alpha\big[\Phi_{A,\beta}(f,t)\nu_\alpha(r)-\int_0^r\Phi_{A,\beta}(f,t)d\mu_\alpha(t)\big]}{\nu_\alpha(r)^2}\\ &=\frac{2r(1-r^2)^\alpha\int_0^r\big[\frac{d}{dt}\Phi_{A,\beta}(f,t)\big]\nu_\alpha(t)dt}{\nu_\alpha(r)^2}\ge 0.\end{aligned}$$

As a result, $r\mapsto \mathsf{A}_{\alpha,\beta}(f,r)$ increases on $(0,1)$.

Next suppose that the just-verified monotonicity is not strict. Then there exist two numbers $r_1,r_2\in(0,1)$ such that $r_1<r_2$ and

$$\mathsf{A}_{\alpha,\beta}(f,r_1)=\mathsf{A}_{\alpha,\beta}(f,r)=\mathsf{A}_{\alpha,\beta}(f,r_2),\quad \forall r\in[r_1,r_2].$$

Consequently,

$$\frac{d}{dr}\mathsf{A}_{\alpha,\beta}(f,r)=0,\quad \forall r\in[r_1,r_2],$$

and so

$$\int_0^r\Big[\frac{d}{dt}\Phi_{A,\beta}(f,t)\Big]\nu_\alpha(t)dt=0,\quad \forall r\in[r_1,r_2].$$

Then we must have

$$\frac{d}{dt}\Phi_{A,\beta}(f,t)=0,\quad \forall t\in(0,r),\quad \text{with } r\in[r_1,r_2],$$

whence getting that if $\beta<1$ then $f$ must be constant or if $\beta=1$ then $f$ must be linear, thanks to the argument for the strictness in Lemma 2.2(i).

It remains to check the rest of Theorem 2.1.

(i) The monotonic growth of $\mathsf{A}_{\alpha,\beta}(f,\cdot)$ ensures the existence of the limit. An application of L'Hôpital's rule gives

$$\lim_{r\to0}\mathsf{A}_{\alpha,\beta}(f,r)=\lim_{r\to0}\Phi_{A,\beta}(f,r)=\begin{cases}0, & \text{when } \beta<1,\\ |f'(0)|^2, & \text{when } \beta=1.\end{cases}$$

(ii) Again, the above monotonicity formula of $\mathsf{A}_{\alpha,\beta}(f,\cdot)$ plus the given condition yields that for $s\in(0,1)$,

$$\sup_{r\in(0,s)}\mathsf{A}_{\alpha,\beta}(f,r)=\mathsf{A}_{\alpha,\beta}(f,s)<\infty.$$

Integrating by parts twice and using the monotonicity of $\Phi_{A,\beta}(f,\cdot)$, we obtain that under $0<r<s<1$,

$$\begin{aligned}
0&\le\mathsf{A}_{\alpha,\beta}(f,s)-\mathsf{A}_{\alpha,\beta}(f,r)=\int_r^s\frac{d}{dt}\mathsf{A}_{\alpha,\beta}(f,t)dt\\
&=\int_r^s\Big(\int_0^t\Big[\frac{d}{d\tau}\Phi_{A,\beta}(f,\tau)\Big]\nu_\alpha(\tau)d\tau\Big)\Big[\frac{d\nu_\alpha(t)}{\nu_\alpha(t)^2}\Big]\\
&=\int_r^s\Big(\nu_\alpha(t)\Phi_{A,\beta}(f,t)-\int_0^t\Phi_{A,\beta}(f,\tau)d\nu_\alpha(\tau)\Big)\Big[\frac{d\nu_\alpha(t)}{\nu_\alpha(t)^2}\Big]\\
&\le\big[\Phi_{A,\beta}(f,s)-\Phi_{A,\beta}(f,0)\big]\int_r^s\frac{d\nu_\alpha(t)}{\nu_\alpha(t)}.
\end{aligned}$$

This gives the desired inequality right away. Furthermore, the above argument plus Lemma 2.2(i) derives the equality case. □

As an immediate consequence of Theorem 2.1, we get a sort of "norm" estimate associated with $\Phi_{A,\beta}(f,\cdot)$.

**Corollary 2.1.** Let $-\infty<\alpha<\infty$, $0\le\beta\le 1$ and $f\in H(\mathbb{D})$.
(i) If $-\infty<\alpha\le -1$, then

$$\int_0^1 \Phi_{A,\beta}(f,t)d\mu_\alpha(t)=\sup_{r\in(0,1)}\int_0^r \Phi_{A,\beta}(f,t)d\mu_\alpha(t)<\infty,$$

if and only if $f$ is constant. Moreover, $\sup_{r\in(0,1)}\mathsf{A}_{\alpha,\beta}(f,r)=\Phi_{A,\beta}(f,1)$.
(ii) If $-1<\alpha<\infty$, then

$$\mathsf{A}_{\alpha,\beta}(f,r)\le\mathsf{A}_{\alpha,\beta}(f,1):=\sup_{s\in(0,1)}\mathsf{A}_{\alpha,\beta}(f,s),\quad \forall r\in(0,1),$$

where the inequality becomes an equality for all $r\in(0,1)$ if and only if

$$f=\begin{cases}\text{constant}, & \text{when } \beta<1,\\ \text{linear map}, & \text{when } \beta=1.\end{cases}$$

(iii) The following function $\alpha\mapsto\mathsf{A}_{\alpha,\beta}(f,1)$ is strictly decreasing on $(-1,\infty)$ unless

$$f=\begin{cases}\text{constant}, & \text{when } \beta<1,\\ \text{linear map}, & \text{when } \beta=1.\end{cases}$$

*Proof.* (i) By Theorem 2.1, we have

$$\mathsf{A}_{\alpha,\beta}(f,r)\le\frac{\int_0^s\Phi_{A,\beta}(f,t)d\mu_\alpha(t)}{\nu_\alpha(s)},\quad \forall r\in(0,s).$$

Note that

$$\lim_{s\to1}\nu_\alpha(s)=\infty\quad\text{and}\quad\lim_{s\to1}\int_0^s\Phi_{A,\beta}(f,t)d\mu_\alpha(t)=\int_0^1\Phi_{A,\beta}(f,t)d\mu_\alpha(t).$$

So, the last integral is finite if and only if

$$\Phi_{A,\beta}(f,r)=0,\quad \forall r\in(0,1),$$

equivalently, $A(f,r)=0$ holds for all $r\in(0,1)$, i.e., $f$ is constant.

For the remaining part of (i), we may assume that $f$ is not a constant map. Due to $\lim_{r\to1}\nu_\alpha(r)=\infty$, we obtain

$$\lim_{r\to1}\int_0^r\Phi_{A,\beta}(f,t)d\mu_\alpha(t)=\int_0^1\Phi_{A,\beta}(f,t)d\mu_\alpha(t)=\infty.$$

So, an application of L'Hôpital's rule yields

$$\begin{aligned}\sup_{0<r<1}\mathsf{A}_{\alpha,\beta}(f,r)&=\lim_{r\to 1}\frac{\int_0^r\Phi_{A,\beta}(f,t)d\mu_\alpha(t)}{\nu_\alpha(r)}\\&=\lim_{r\to 1}\frac{\Phi_{A,\beta}(f,r)r(1-r^2)^\alpha}{r(1-r^2)^\alpha}=\Phi_{A,\beta}(f,1).\end{aligned}$$

(ii) Under $-1<\alpha<\infty$, we have

$$\lim_{r\to 1}\nu_\alpha(r)=\nu_\alpha(1)\quad\text{and}\quad\lim_{r\to 1}\int_0^r\Phi_{A,\beta}(f,t)d\mu_\alpha(t)=\int_0^1\Phi_{A,\beta}(f,t)d\mu_\alpha(t).$$

Thus, by Theorem 2.1 it follows that for $r\in(0,1)$,

$$\mathsf{A}_{\alpha,\beta}(f,r)\le\lim_{s\to 1}\mathsf{A}_{\alpha,\beta}(f,s)=\big[\nu_\alpha(1)\big]^{-1}\int_0^1\Phi_{A,\beta}(f,t)d\mu_\alpha(t)=\sup_{s\in(0,1)}\mathsf{A}_{\alpha,\beta}(f,s).$$

The equality case just follows from a straightforward computation and Theorem 2.1.
(iii) Suppose $-1<\alpha_1<\alpha_2<\infty$ and $\mathsf{A}_{\alpha_1,\beta}(f,1)<\infty$, then integrating by parts twice, we obtain

$$\begin{aligned}\mathsf{A}_{\alpha_2,\beta}(f,1)&=\big[\nu_{\alpha_2}(1)\big]^{-1}\int_0^1\Phi_{A,\beta}(f,r)d\mu_{\alpha_2}(r)\\&=\big[\nu_{\alpha_2}(1)\big]^{-1}\int_0^1(1-r^2)^{\alpha_2-\alpha_1}\frac{d}{dr}\Big[\int_0^r\Phi_{A,\beta}(f,t)d\mu_{\alpha_1}(t)\Big]dr\\&=\big[\nu_{\alpha_2}(1)\big]^{-1}\Big[-\int_0^1\Big(\int_0^r\Phi_{A,\beta}(f,t)d\mu_{\alpha_1}(t)\Big)d(1-r^2)^{\alpha_2-\alpha_1}\Big]\\&\le\big[\nu_{\alpha_2}(1)\big]^{-1}\mathsf{A}_{\alpha_1,\beta}(f,1)\int_0^1\nu_{\alpha_1}(r)d\big[-(1-r^2)^{\alpha_2-\alpha_1}\big]\\&=\mathsf{A}_{\alpha_1,\beta}(f,1)\big[\nu_{\alpha_2}(1)\big]^{-1}\Big[\int_0^1(1-r^2)^{\alpha_2-\alpha_1}d\mu_{\alpha_1}(r)\Big]\\&=\mathsf{A}_{\alpha_1,\beta}(f,1),\end{aligned}$$

thereby establishing $\mathsf{A}_{\alpha_2,\beta}(f,1)\le\mathsf{A}_{\alpha_1,\beta}(f,1)$. If this last inequality becomes an equality, then the above argument forces

$$\int_0^r\Phi_{A,\beta}(f,t)d\mu_{\alpha_1}(t)=\mathsf{A}_{\alpha_1,\beta}(f,1)\nu_{\alpha_1}(r),\quad\forall r\in(0,1),$$

whence yielding (via the just-verified (ii))

$$f=\begin{cases}\text{constant}, & \text{when }\beta<1,\\ \text{linear map}, & \text{when }\beta=1.\end{cases}$$

Thus, we complete the proof. □

### 2.3 Monotonic growth of $\mathsf{L}_{\alpha,\beta}(f,\cdot)$

Correspondingly, we first have the following Schwarz type lemma.

**Proposition 2.2.** Let $-\infty<\alpha<\infty$, $0\le\beta\le1$, and $f\in H(\mathbb{D})$ be of the form $f(z)=a_0+\sum_{k=n}^{\infty}a_kz^k$ with $n\in\mathbb{N}$. Then

$$(2\pi)^{1-\beta}\Big[\frac{|f^{(n)}(0)|}{n!}\Big]\le\mathsf{L}_{\alpha,\beta}(f,r)\Big[\frac{\nu_\alpha(r)}{\int_0^r t^{n-\beta}d\mu_\alpha(t)}\Big],\quad\forall r\in(0,1),$$

with equality when and only when $f=a_0+a_nz^n$.

*Proof.* This follows from Lemma 2.1(ii) and its equality case. □

The coming-up-next monotonicity contains a hypothesis stronger than that for Theorem 2.1.

**Theorem 2.2.** *Let $-\infty<\alpha<\infty$, $0\le\beta\le1$, and $f\in U(\mathbb{D})$ or $f(z)=a_0+a_nz^n$ with $n\in\mathbb{N}$. Then $r\mapsto\mathsf{L}_{\alpha,\beta}(f,r)$ is strictly increasing on $(0,1)$ unless*

$$f=\begin{cases}\text{constant}, & \text{when } \beta<1,\\ \text{linear map}, & \text{when } \beta=1.\end{cases}$$

*Consequently,*

*(i)*

$$\lim_{r\to0}\mathsf{L}_{\alpha,\beta}(f,r)=\begin{cases}0, & \text{when } \beta<1,\\ |f'(0)|, & \text{when } \beta=1.\end{cases}$$

*(ii) If*

$$\Phi_{L,\beta}(f,0):=\lim_{r\to0}\Phi_{L,\beta}(f,r)\quad\text{and}\quad\Phi_{L,\beta}(f,1):=\lim_{r\to1}\Phi_{L,\beta}(f,r)<\infty,$$

*then*

$$0<r<s<1\Rightarrow0\le\frac{\mathsf{L}_{\alpha,\beta}(f,s)-\mathsf{L}_{\alpha,\beta}(f,r)}{\log\nu_\alpha(s)-\log\nu_\alpha(r)}\le\Phi_{L,\beta}(f,s)-\Phi_{L,\beta}(f,0)$$

*with equality if and only if*

$$f=\begin{cases}\text{constant}, & \text{when } \beta<1,\\ \text{linear map}, & \text{when } \beta=1.\end{cases}$$

*In particular, $t\mapsto\mathsf{L}_{\alpha,\beta}(f,t)$ is Lipschitz with respect to $\log\nu_\alpha(t)$ for $t\in(0,1)$.*

*Proof.* Similar to that for Theorem 2.1, but this time by Lemma 2.2(ii). □

Naturally, we can establish the so-called "norm" estimate associated to $\Phi_{L,\beta}(f,\cdot)$.

**Corollary 2.2.** Let $0\le\beta\le1$ and $f\in U(\mathbb{D})$ or $f(z)=a_0+a_nz^n$ with $n\in\mathbb{N}$,
(i) If $-\infty<\alpha\le-1$, then

$$\int_0^1\Phi_{L,\beta}(f,t)d\mu_\alpha(t)=\sup_{r\in(0,1)}\int_0^r\Phi_{L,\beta}(f,t)d\mu_\alpha(t)<\infty$$

if and only if $f$ is constant. Moreover, $\sup_{r\in(0,1)}\mathsf{L}_{\alpha,\beta}(f,r)=\Phi_{L,\beta}(f,1)$.
(ii) If $-1<\alpha<\infty$, then

$$\mathsf{L}_{\alpha,\beta}(f,r)\le\mathsf{L}_{\alpha,\beta}(f,1):=\sup_{s\in(0,1)}\mathsf{L}_{\alpha,\beta}(f,s),\quad\forall r\in(0,1),$$

where the inequality becomes an equality for all $r\in(0,1)$ if and only if

$$f=\begin{cases}\text{constant,} & \text{when }\ \beta<1,\\ \text{linear map,} & \text{when }\ \beta=1.\end{cases}$$

(iii) $\alpha\mapsto\mathsf{L}_{\alpha,\beta}(f,1)$ is strictly decreasing on $(-1,\infty)$ unless

$$f=\begin{cases}\text{constant,} & \text{when }\ \beta<1,\\ \text{linear map,} & \text{when }\ \beta=1.\end{cases}$$

*Proof.* The argument is similar to that for Corollary 2.1, but via Lemma 2.2(ii). □

# 3 Logarithmic convexity

In this section, we treat the convexities of the following two functions: $\log r\mapsto\log\mathsf{A}_{\alpha,\beta}(f,r)$ and $\log r\mapsto\log\mathsf{L}_{\alpha,\beta}(f,r)$ for $r\in(0,1)$.

## 3.1 Two more lemmas

The following are two technical preliminaries.

**Lemma 3.1** (see [10]). *Suppose that $f(x)$ and $\{h_k(x)\}_{k=0}^\infty$ are positive and twice differentiable for $x\in(0,1)$ such that the function $H(x)=\sum_{k=0}^\infty h_k(x)$ is also twice differentiable for $x\in(0,1)$. Then:*
*(i) $\log x\mapsto\log f(x)$ is convex if and only if $\log x\mapsto\log f(x^2)$ is convex.*
*(ii) The function $\log x\mapsto\log f(x)$ is convex if and only if the D-notation of f*

$$D(f(x)):=\frac{f'(x)}{f(x)}+x\Big(\frac{f'(x)}{f(x)}\Big)'\ge0,\quad\forall x\in(0,1).$$

*(iii) If for each k the function $\log x\mapsto\log h_k(x)$ is convex, then $\log x\mapsto\log H(x)$ is also convex.*

**Lemma 3.2.** *Let $f\in H(\mathbb{D})$. Then $f$ belongs to $U(\mathbb{D})$ provided that one of the following two conditions is valid:*
*(i) see* [7] *or* [1, Lemma 2.1]

$$f(0)=f'(0)-1=0 \quad and \quad \Big|\frac{z^2 f'(z)}{f^2(z)}-1\Big|<1, \quad \forall z\in\mathbb{D}.$$

*(ii) see* [6, Theorem 1] *or* [3, Theorem 8.12]

$$\Big|\Big[\frac{f''(z)}{f'(z)}\Big]'-\frac{1}{2}\Big[\frac{f''(z)}{f'(z)}\Big]^2\Big|\le 2(1-|z|^2)^{-2}, \quad \forall z\in\mathbb{D}.$$

### 3.2 Log-convexity for $\mathsf{A}_{\alpha,\beta}(f,\cdot)$

Such a property is given below.

**Theorem 3.1.** *Let $0\le\beta\le 1$ and $0<r<1$.*
*(i) If $\alpha\in(-\infty,-3)$, then there exist two maps $f,g\in H(\mathbb{D})$ such that $\log r\mapsto\log\mathsf{A}_{\alpha,\beta}(f,r)$ is not convex and $\log r\mapsto\log\mathsf{A}_{\alpha,\beta}(g,r)$ is not concave.*
*(ii) If $\alpha\in[-3,0]$, then $\log r\mapsto\log\mathsf{A}_{\alpha,1}(a_n z^n,r)$ is convex for $a_n\neq 0$ with $n\in\mathbb{N}$. Consequently,*

$$\log r\mapsto\log\mathsf{A}_{\alpha,1}(f,r)$$

*is convex for all $f\in U(\mathbb{D})$.*
*(iii) If $\alpha\in(0,\infty)$, then $\log r\mapsto\log\mathsf{A}_{\alpha,\beta}(a_n z^n,r)$ is not convex for $a_n\neq 0$ and $n\in\mathbb{N}$.*

*Proof.* The key issue is to check whether or not $\log r\mapsto\log\mathsf{A}_{\alpha,\beta}(z^n,r)$ is convex for $r\in(0,1)$.

To see this, let us borrow some symbols from [10]. For $\lambda\ge 0$ and $0<x<1$, we define

$$f_\lambda(x)=\int_0^x t^\lambda(1-t)^\alpha dt$$

and

$$\Delta(\lambda,x)=\frac{f'_\lambda(x)}{f_\lambda(x)}+x\Big(\frac{f'_\lambda(x)}{f_\lambda(x)}\Big)'-\Big[\frac{f'_0(x)}{f_0(x)}+x\Big(\frac{f'_0(x)}{f_0(x)}\Big)'\Big].$$

Given $n\in\mathbb{N}$. A simple calculation shows $\Phi_{A,\beta}(z^n,t)=\pi^{1-\beta}t^{2(n-\beta)}$, and then a change of variable derives

$$\mathsf{A}_{\alpha,\beta}(z^n,r)=\frac{\int_0^r\Phi_{A,\beta}(z^n,t)d\mu_\alpha(t)}{\nu_\alpha(r)}=\frac{\pi^{1-\beta}\int_0^{r^2}t^{n-\beta}(1-t)^\alpha dt}{\int_0^{r^2}(1-t)^\alpha dt}=\pi^{1-\beta}\Big[\frac{f_{n-\beta}(r^2)}{f_0(r^2)}\Big].$$

In accordance with Lemma 3.1(i)-(ii), it is easy to work out that $\log r\mapsto\log\mathsf{A}_{\alpha,\beta}(z^n,r)$ is convex for $r\in(0,1)$ if and only if $\Delta(n-\beta,x)\ge 0$ for any $x\in(0,1)$.

(i) Under $\alpha\in(-\infty,-3)$, we follow the argument for [10, Proposition 6] to get

$$\lim_{x\to 1}\Delta(\lambda,x)=\frac{\lambda(\alpha+1)(\lambda+2+\alpha)}{(\alpha+2)^2(\alpha+3)}.$$

Choosing

$$f(z)=z^n=\begin{cases} z, & \text{when } \beta<1,\\ z^2, & \text{when } \beta=1,\end{cases}$$

and $\lambda=n-\beta$, we find $\lim_{x\to 1}\Delta(\lambda,x)<0$, whence deriving that $\log r\mapsto\log A_\alpha(f,r)$ is not convex.

In the meantime, picking $n\in\mathbb{N}$ such that $n>\beta-(2+\alpha)$ and putting $g(z)=z^n$, we obtain

$$\lim_{x\to 1}\Delta(n-\beta,x)=\frac{(n-\beta)(\alpha+1)(n-\beta+2+\alpha)}{(\alpha+2)^2(\alpha+3)}>0,$$

whence deriving that $\log r\mapsto\log \mathsf{A}_{\alpha,\beta}(g,r)$ is not concave.

(ii) Under $\alpha\in[-3,0]$, we handle the two situations.

Situation 1: $f\in U(\mathbb{D})$. Upon writing $f(z)=\sum_{n=0}^\infty a_n z^n$, we compute

$$\Phi_{A,1}\big(f(z),t\big)=(\pi t^2)^{-1}A(f,t)=\sum_{n=0}^\infty n|a_n|^2 t^{2(n-1)},$$

and consequently,

$$\mathsf{A}_{\alpha,1}(f,r)=\frac{\sum_{n=0}^\infty n|a_n|^2\int_0^r(\pi t^2)^{-1}A(z^n,t)d\mu_\alpha(t)}{\nu_\alpha(r)}=\sum_{n=0}^\infty n|a_n|^2\mathsf{A}_{\alpha,1}(z^n,r).$$

So, by Lemma 3.1(iii), we see that the convexity of

$$\log r\mapsto\log\mathsf{A}_{\alpha,1}(f,r)\quad\text{under } f\in U(\mathbb{D}),$$

follows from the convexity of

$$\log r\mapsto\log\mathsf{A}_{\alpha,1}(z^n,r)\quad\text{under } n\in\mathbb{N}.$$

So, it remains to verify this last convexity via the coming-up-next consideration.

Situation 2: $f(z)=a_n z^n$ with $a_n\neq 0$. Three cases are required to control.

Case 1: $\alpha=0$. An easy computation shows

$$\mathsf{A}_{0,1}(z^n,r)=n^{-1}r^{2(n-1)}$$

and so $\log r\mapsto\log\mathsf{A}_{0,1}(z^n,r)$ is convex.

Case 2: $-2\le\alpha<0$. Under this condition, we see from the arguments for [10, Propositions 4-5] that

$$\Delta(n-1,x)\ge 0,\quad \forall n-1\ge 0,\quad 0<x<1,$$

and so that $\log r\mapsto\log\mathsf{A}_{\alpha,1}(z^n,r)$ is convex.

Case 3: $-3\le\alpha<-2$. With the assumption, we also get from the arguments for [10, Propositions 4-5] that

$$\Delta(n-1,x)\ge\Delta(-2-\alpha,x)>0,\quad \forall x\in(0,1),\quad n-1\in[-2-\alpha,\infty),$$

and so that $\log r\mapsto\log\mathsf{A}_{\alpha,1}(z^n,r)$ is convex when $n\ge2$. Here it is worth noting that the convexity of $\log r\mapsto\log\mathsf{A}_{\alpha,1}(z,r)=0$ is trivial.

(iii) Under $0<\alpha<\infty$, from the argument for [10, Proposition 6] we know that $\Delta(n-\beta,x)<0$ as $x$ is sufficiently close to 1. Thus $\log r\mapsto\log\mathsf{A}_{\alpha,\beta}(a_nz^n,r)$ is not convex under $a_n\neq0$. □

The following illustrates that the function $\log r\mapsto\log\mathsf{A}_{\alpha,\beta}(f,r)$ is not always concave for $\alpha>0$, $0\le\beta\le1$, and $f\in U(\mathbb{D})$.

**Example 3.1.** Let $\alpha=1$, $\beta\in\{0,1\}$ and $f(z)=z+z^2/2$. Then the function $\log r\mapsto\log\mathsf{A}_{\alpha,\beta}(f,r)$ is neither convex nor concave for $r\in(0,1)$.

*Proof.* A direct computation shows

$$\Big|\frac{z^2f'(z)}{f^2(z)}-1\Big|=\Big|\frac{z^2(1+z)}{(z+\frac{z^2}{2})^2}-1\Big|=\frac{|z|^2}{|z+2|^2}<1,$$

since

$$|z|<1<2-|z|\le|z+2|,\quad \forall z\in\mathbb{D}.$$

So, $f\in U(\mathbb{D})$ owing to Lemma 3.2(i). By $f'(z)=z+1$ we have

$$A(f,t)=\int_{t\mathbb{D}}|z+1|^2dA(z)=\pi\Big(t^2+\frac{t^4}{2}\Big),$$

plus

$$\int_0^r\Phi_{A,\beta}(f,t)d\mu_1(t)=\begin{cases}\dfrac{\pi}{2}\Big(r^4-\dfrac{r^6}{3}-\dfrac{r^8}{4}\Big), & \text{when }\ \beta=0,\\ r^2-\dfrac{r^4}{4}-\dfrac{r^6}{6}, & \text{when }\ \beta=1.\end{cases}$$

Meanwhile,

$$\nu_1(r)=\int_0^r(1-t^2)dt^2=r^2-\frac{r^4}{2}.$$

So, we get

$$\mathsf{A}_{1,\beta}(f,r)=\begin{cases}\dfrac{\pi(12r^2-4r^4-3r^6)}{12(2-r^2)}, & \text{when }\ \beta=0,\\ \dfrac{12-3r^2-2r^4}{6(2-r^2)}, & \text{when }\ \beta=1,\end{cases}$$

and in turn consider the logarithmic convexities of the following function

$$h_\beta(x)=\begin{cases}\dfrac{12x-4x^2-3x^3}{2-x}, & \text{when } \beta=0,\\ \dfrac{12-3x-2x^2}{2-x}, & \text{when } \beta=1,\end{cases}$$

for $x\in(0,1)$.

Using the so-called D-notation in Lemma 3.1, we have

$$D(h_\beta(x))=\begin{cases}D(12x-4x^2-3x^3)-D(2-x), & \text{when } \beta=0,\\ D(12-3x-2x^2)-D(2-x), & \text{when } \beta=1,\end{cases}$$

for $x\in(0,1)$. By an elementary calculation, we get

$$\begin{cases}D(12x-4x^2-3x^3)=\dfrac{-48-144x+12x^2}{(12-4x-3x^2)^2},\\ D(2-x)=\dfrac{-2}{(2-x)^2},\\ D(12-3x-2x^2)=\dfrac{-36-96x+6x^2}{(12-3x-2x^2)^2}.\end{cases}$$

Consequently,

$$D(h_\beta(x))=\begin{cases}\dfrac{2g_\beta(x)}{(12-4x-3x^2)^2(2-x)^2}, & \text{when } \beta=0,\\ \dfrac{2g_\beta(x)}{(12-3x-2x^2)^2(2-x)^2}, & \text{when } \beta=1,\end{cases}$$

where

$$g_\beta(x)=\begin{cases}48-288x+232x^2-72x^3+15x^4, & \text{when } \beta=0,\\ 72-192x+147x^2-48x^3+7x^4, & \text{when } \beta=1.\end{cases}$$

Now, under $x\in(0,1)$ we find

$$g_0'(x)=-288+464x-216x^2+60x^3 \quad\text{and}\quad g_0''(x)=464-432x+180x^2.$$

Clearly, $g_0''(x)$ is an open-upward parabola with the axis of symmetry $x=6/5>1$. By $g_0''(1)=212>0$ and the monotonicity of $g_0''$ on $(0,1)$, we have $g_0''(x)>0$ for all $x\in(0,1)$. Thus $g_0'$ is increasing on $(0,1)$. The following condition

$$g_0'(0)=-288<0 \quad\text{and}\quad g_0'(1)=20>0$$

yields an $x_1\in(0,1)$ such that $g_0'(x)<0$ for $x\in(0,x_1)$ and $g_0'(x)>0$ for $x\in(x_1,1)$. Since $g_0(0)=48$ and $g_0(1)=-65$, there exists an $x_0\in(0,1)$ such that $g_0(x)>0$ for $x\in(0,x_0)$ and $g_0(x)<0$ for $x\in(x_0,1)$. Thus the function $\log x\mapsto\log h_0(x)$ is neither convex nor concave.

Similarly, under $x\in(0,1)$ we have

$$g_1'(x)=-192+294x-144x^2+28x^3 \quad \text{and} \quad g_1''(x)=294-288x+84x^2.$$

Obviously, $g_1''(x)$ is an open-upward parabola with the axis of symmetry $x=12/7>1$. By $g_1''(1)=90>0$ and the monotonicity of $g_1''$ on $(0,1)$, we have $g_1''(x)>0$ for all $x\in(0,1)$. Thus $g_1'$ is increasing on $(0,1)$. The following condition

$$g_1'(0)=-192<0 \quad \text{and} \quad g_1'(1)=-14<0$$

yields $g_1'(x)<0$ for $x\in(0,1)$. Since $g_1(0)=72$ and $g_1(1)=-14$, there exists an $x_0\in(0,1)$ such that $g_1(x)>0$ for $x\in(0,x_0)$ and $g_1(x)<0$ for $x\in(x_0,1)$. Thus the function $\log x\mapsto\log h_1(x)$ is neither convex nor concave. □

### 3.3 Log-convexity for $\mathsf{L}_{\alpha,\beta}(f,\cdot)$

Analogously, we can establish the expected convexity for the mixed lengths.

**Theorem 3.2.** *Let* $0\le\beta\le1$ *and* $0<r<1$.

*(i) If* $\alpha\in(-\infty,-3)$, *then there exist two maps* $f,g\in H(\mathbb{D})$ *such that* $\log r\mapsto\log\mathsf{L}_{\alpha,\beta}(f,r)$ *is not convex and* $\log r\mapsto\log\mathsf{L}_{\alpha,\beta}(g,r)$ *is not concave.*

*(ii) If* $\alpha\in[-3,0]$, *then* $\log r\mapsto\log\mathsf{L}_{\alpha,1}(a_nz^n,r)$ *is convex for* $a_n\neq0$ *with* $n\in\mathbb{N}$. *Consequently,* $\log r\mapsto\log\mathsf{L}_{\alpha,1}(f,r)$ *is convex for* $f\in U(\mathbb{D})$.

*(iii) If* $\alpha\in(0,\infty)$, *then* $\log r\mapsto\log\mathsf{L}_{\alpha,\beta}(a_nz^n,r)$ *is not convex for* $a_n\neq0$ *and* $n\in\mathbb{N}$.

*Proof.* The argument is similar to that for Theorem 3.1 except using the following statement for $\alpha\in[-3,0]$–If $f\in U(\mathbb{D})$, then there exists $g(z)=\sum_{n=0}^{\infty}b_nz^n$ such that $g$ is the square root of the zero-free derivative $f'$ on $\mathbb{D}$ and $f'(0)=g^2(0)$, and hence

$$\Phi_{L,1}(f,t)=(2\pi t)^{-1}\int_{t\mathbb{T}}|f'(z)||dz|=(2\pi t)^{-1}\int_{t\mathbb{T}}|g(z)|^2|dz|=\sum_{n=0}^{\infty}|b_n|^2t^{2n}.$$

Thus, we complete the proof. □

Our concluding example shows that under $0<\alpha<\infty$ and $0\le\beta\le1$ one cannot get that $\log\mathsf{L}_{\alpha,\beta}(f,r)$ is convex or concave in $\log r$ for all functions $f\in U(\mathbb{D})$.

**Example 3.2.** Let $\alpha=1$, $\beta\in\{0,1\}$ and $f(z)=(z+2)^3$. Then the function $\log r\mapsto\log\mathsf{L}_{\alpha,\beta}(f,r)$ is neither convex nor concave for $r\in(0,1)$.

*Proof.* Clearly, we have

$$f'(z)=3(z+2)^2 \quad \text{and} \quad f''(z)=6(z+2)$$

as well as the Schwarizian derivative

$$\Big[\frac{f''(z)}{f'(z)}\Big]'-\frac{1}{2}\Big[\frac{f''(z)}{f'(z)}\Big]^2=\frac{-4}{(z+2)^2}.$$

It is easy to see that

$$\sqrt{2}(1-|z|^2)\leq 2-|z|,\quad \forall z\in\mathbb{D}.$$

So,

$$\Big|\Big[\frac{f''(z)}{f'(z)}\Big]'-\frac{1}{2}\Big[\frac{f''(z)}{f'(z)}\Big]^2\Big|=\frac{4}{|z+2|^2}\leq\frac{4}{(2-|z|)^2}\leq\frac{2}{(1-|z|^2)^2}.$$

By Lemma 3.2(ii), $f$ belongs to $U(\mathbb{D})$. Consequently,

$$L(f,t)=\int_0^{2\pi}|f'(te^{i\theta})|td\theta=6\pi t(t^2+4)$$

and

$$\int_0^r\Phi_{L,\beta}(f,t)d\mu_1(t)=\begin{cases}12\pi\Big(\frac{4}{3}r^3-\frac{3}{5}r^5-\frac{1}{7}r^7\Big), & \text{when } \beta=0,\\ 12r^2-\frac{9}{2}r^4-r^6, & \text{when } \beta=1.\end{cases}$$

Note that $\nu_1(r)=r^2-r^4/2$. So,

$$\mathsf{L}_{1,\beta}(f,r)=\begin{cases}\dfrac{24\pi(140r-63r^3-15r^5)}{105(2-r^2)}, & \text{when } \beta=0,\\ \dfrac{24-9r^2-2r^4}{2-r^2}, & \text{when } \beta=1.\end{cases}$$

To gain our conclusion, we only need to consider the logarithmic convexity of the function

$$h_\beta(x)=\begin{cases}\dfrac{140x-63x^3-15x^5}{2-x^2}, & \text{when } \beta=0,\\ \dfrac{24-9x-2x^2}{2-x}, & \text{when } \beta=1.\end{cases}$$

Case 1: $\beta=0$. Applying the definition of $D$-notation, we obtain

$$D(140x-63x^3-15x^5)=\frac{-35280x-33600x^3+3780x^5}{(140-63x^2-15x^4)^2}$$

and

$$D(2-x^2)=\frac{-8x}{(2-x^2)^2},$$

whence reaching

$$D\big(h_0(x)\big)=D(140x-63x^3-15x^5)-D(2-x^2)=\frac{4xg_0(x)}{(140-63x^2-15x^4)^2(2-x^2)^2},$$

where

$$g_0(x)=3920-33600x^2+28098x^4-8400x^6+1395x^8.$$

Obviously,

$$g_0(0)=3920>0 \quad \text{and} \quad g_0(1)=-8587<0.$$

Now letting $s=x^2$, we get

$$g_0(x)=G_0(s)=3920-33600s+28098s^2-8400s^3+1395s^4,$$

and

$$G_0'(s)=-33600+56196s-25200s^2+5580s^3 \quad \text{and} \quad G_0''(s)=56196-50400s+16740s^2.$$

Since the axis of symmetry of $G_0''$ is $s=140/93>1$, $G_0''$ is decreasing on $(0,1)$. Due to $G_0''(1)=22536>0$, we have $G_0''(s)>0$ for all $s\in(0,1)$, i.e., $G_0'(s)$ is increasing on $(0,1)$. By

$$G_0'(0)=-33600<0 \quad \text{and} \quad G_0'(1)=2976>0,$$

we conclude that there exists an $s_0\in(0,1)$ such that $G_0'(s)<0$ for $s\in(0,s_0)$ and $G_0'(s)>0$ for $s\in(s_0,1)$. Then there exists an $x_0\in(0,1)$ such that $g_0(x)$ is decreasing for $x\in(0,x_0)$ and $g_0(x)$ is increasing for $x\in(x_0,1)$. Thus there exists an $x_1\in(0,1)$ such that $g_0(x)>0$ for $x\in(0,x_1)$ and $g_0(x)<0$ for $x\in(x_1,1)$. As a result, we find that $\log r\mapsto\log \mathsf{L}_{\alpha,0}(f,r)$ is neither concave nor convex.

Case 2: $\beta=1$. Again using the $D$-notation, we obtain

$$D(24-9x-2x^2)=\frac{-216-192x+18x^2}{(24-9x-2x^2)^2}$$

and

$$D(2-x)=\frac{-2}{(2-x)^2},$$

whence deriving

$$D\big(h_1(x)\big)=D(24-9x-2x^2)-D(2-x)=\frac{2g_1(x)}{(24-9x-2x^2)^2(2-x)^2},$$

where

$$g_1(x)=144-384x+297x^2-96x^3+13x^4.$$

Now we have

$$g_1'(x)=-384+594x-288x^2+52x^3 \quad \text{and} \quad g_1''(x)=594-576x+156x^2.$$

Since the axis of symmetry of $g_1''(x)$ is $x=24/13>1$, $g_1''(x)$ is decreasing on $(0,1)$. Due to $g_1''(1)=174>0$, we have $g_1''(x)>0$ for all $x\in(0,1)$, i.e., $g_1'(x)$ is increasing on $(0,1)$. By

$$g_1'(0)=-384<0 \quad \text{and} \quad g_1'(1)=-26<0,$$

we conclude that $g_1'(x)<0$ for $x\in(0,1)$. Obviously,

$$g_1(0)=144>0 \quad \text{and} \quad g_1(1)=-26<0.$$

Hence there exists an $x_0\in(0,1)$ such that $g_1(x)>0$ for $x\in(0,x_0)$ and $g_1(x)<0$ for $x\in(x_0,1)$. Consequently, we find that $\log r\mapsto \log \mathsf{L}_{\alpha,\beta=1}(f,r)$ is neither concave nor convex. □

# Acknowledgements

J. Xiao and W. Xu were in part supported by NSERC of Canada and the Finnish Cultural Foundation, respectively.

**References**

[1] M. H. Al-Abbadi and M. Darus, Angular estimates for certain analytic univalent functions, Int. J. Open Problems Complex Anal., 2 (2010), 212–220.
[2] R. B. Burckel, D. E. Marshall, D. Minda, P. Poggi-Corradini and T. J. Ransford, Area, capacity and diameter versions of Schwarz's Lemma, Conform. Geom. Dyn., 12 (2008), 133–152.
[3] P. L. Duren, Univalent Functions, Springer-Verlag, NewYork, 1983.
[4] P. Järvi, On some function-theoretic extremal problems, Complex Variables Theory Appl., 24 (1994), 267–270.
[5] T. H. Macgregor, Length and area estimates for analytic functions, Michigan Math. J., 11 (1964), 317–320.
[6] Z. Nehari, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc., 55 (1949), 545–551.
[7] M. Nunokawa, On some angular estimates of analytic functions, Math. Japan, 41(2) (1995), 447–452.
[8] G. Pólya and G. Szegö, Problems and Theorems in Analysis, I., Springer, 1978.
[9] C. Wang, J. Xiao and K. Zhu, Logarithmic convexity of area integral means for analytic functions II, arXiv:1308.4881v1.
[10] C. Wang and K. Zhu, Logarithmic convexity of area integral means for analytic functions, Math. Scand., to appear.
[11] J. Xiao and K. Zhu, Volume integral means of holomorphic functions, Proc. Amer. Math. Soc., 139 (2011), 1455–1465.
[12] S. Yamashita, Length estimates for holomorphic functions, Proc. Amer. Math. Soc., 81 (1981), 250–252.