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NUMERICAL SHOOTING METHODS FOR OPTIMAL BOUNDARY CONTROL AND EXACT BOUNDARY CONTROL OF 1-D WAVE EQUATIONS

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Abstract. Numerical solutions of optimal Dirichlet boundary control problems for linear and semilinear wave equations are studied. The optimal control problem is reformulated as a system of equations (an optimality system) that consists of an initial value problem for the underlying (linear or semilinear) wave equation and a terminal value problem for the adjoint wave equation. The discretized optimality system is solved by a shooting method. The convergence properties of the numerical shooting method in the context of exact controllability are illustrated through computational experiments. In particular, in the case of the linear wave equation, convergent approximations are obtained for both smooth minimum L^2-norm Dirichlet control and generic, non-smooth minimum L^2-norm Dirichlet controls.

Key words. Controllability, optimal control, wave equation, shooting method, finite difference method.

1. Introduction

In this chapter we consider an optimal boundary control approach for solving the exact boundary control problem for one-dimensional linear or semilinear wave equations defined on a time interval $(0,T)$ and spatial interval $(0,X)$. The exact boundary control problem we consider is to seek a boundary control $\mathbf{g}=(g_L,g_R)\in \mathbf{L^2(0,T)}\subset[L^2(0,T)]^2$ and a corresponding state u such that the following system of equations hold:

$$
(1)\qquad \begin{cases} u_{tt}-u_{xx}+f(u)=V \quad \text{in } Q\equiv(0,T)\times(0,X)\,,\\ u|_{t=0}=u_0 \quad\text{and}\quad u_t|_{t=0}=u_1 \quad \text{in } (0,X)\,,\\ u|_{t=T}=W \quad\text{and}\quad u_t|_{t=T}=Z \quad \text{in } (0,X)\,,\\ u|_{x=0}=g_L \text{ and } u|_{x=1}=g_R \quad \text{in } (0,T)\,, \end{cases}
$$

where u_0 and u_1 are given initial conditions defined on $(0,X)$, $W\in L^2(0,X)$ and $Z\in H^{-1}(0,X)$ are prescribed terminal conditions, V is a given function defined on $(0,T)\times(0,X)$, f is a given function defined on $\mathbb{R}$, and $\mathbf{g}=(g_L,g_R)\in[L^2(0,T)]^2$ is the boundary control.

It is well known (see, e.g., [15, 16, 18, 19]) that when $f=0$ (i.e., the equation is linear) and T is sufficiently large, the exact controllability problem (1) admits at least one state-control solution pair $(u,\mathbf{g})$; furthermore, the exact controller $\mathbf{g}$

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having minimum boundary L^2 norm is unique. Exact boundary controllability for semilinear wave equations have also been established for certain asymptotically linear or superlinear f; see, e.g., [4, 23, 24].

For the exact boundary controllability problem associated with the linear wave equation there are basically two classes of computational methods in the literature. The first class is HUM-based methods; see, e.g., [6, 9, 15, 17, 22]. The approximate solutions obtained by the HUM-based methods in general do not seem to converge (even in a weak sense) to the exact solutions as the temporal and spatial grid sizes tend to zero. Methods of regularization including Tychonoff regularization and filtering that result in convergent approximations were introduced in those papers on HUM-based methods. The second class of computational methods for boundary controllability of the linear wave equation was those based on the method proposed in [8]. One solves a discrete optimization problem that involves the minimization of the discrete boundary L^2 norm subject to the undetermined linear system of equations formed by the discretization of the wave equation and the initial and terminal conditions. This approach was implemented in [12]. The computational results demonstrated the convergence of the discrete solutions when the exact minimum boundary L^2 norm solution is smooth. In the generic case of a non-smooth exact minimum boundary L^2 norm solution the computational results of [12] exhibited at least a weak L^2 convergence of the discrete solutions.

Although there are well-known theoretical results concerning boundary controllability of *semilinear* wave equations (see, e.g., [4, 23, 24]), little seems to exist in the literature about computational methods for such problems.

In this chapter we attempt to solve the exact controllability problems by an optimal control approach. Precisely, we consider the following optimal control problem: minimize the cost functional

$$\begin{aligned}\mathcal{J}_0(u,\mathbf{g}) =& \frac{\sigma}{2}\int_0^1 |u(T,x)-W(x)|^2\,dx + \frac{\tau}{2}\int_0^1 |u_t(T,x)-Z(x)|^2\,dx \\ &+ \frac{1}{2}\int_0^1 (|g_L|^2+|g_R|^2)\,dt\end{aligned} \tag{2}$$

subject to

$$\begin{cases} u_{tt} - u_{xx} + f(u) = V \quad \text{in } Q \equiv (0,T)\times(0,1) \\ u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1 \quad \text{in } (0,1) \\ u|_{x=0} = g_L \text{ and } u|_{x=1} = g_R \quad \text{in } (0,T)\,. \end{cases} \tag{3}$$

The optimal control problem is converted into an optimality system of equations and this optimality system of equations will be solved by a shooting method.

The optimal control approach of this chapter provides an alternative method to the two classes of methods mentioned in the foregoing for solving the exact controllability problem for the linear wave equations; it also offers a systematic procedure for solving exact controllability problems for the semilinear wave equations. The computational solutions of this chapter obtained by an optimal control approach exhibit behaviors similar to those of the solutions obtained in [12]. Note that an optimal solution exists even when the equation is not exactly controllable. Note also that the solution methods in the literature for optimal control of PDEs can be utilized. and that there are certain intrinsic parallelisms to the algorithms studied in this chapter.

The shooting algorithms for solving the optimal control problem will be described for the slightly more general functional

$$
\begin{aligned}
\mathcal{J}(u,\mathbf{g}) =& \frac{\alpha}{2}\int_0^T\int_0^1 |u-U|^2\,dx\,dt + \frac{\sigma}{2}\int_0^1 |u(T,x)-W(x)|^2\,dx \\
&+ \frac{\tau}{2}\int_0^1 |u_t(T,x)-Z(x)|^2\,dx + \frac{1}{2}\int_0^1 (|g_L|^2+|g_R|^2)\,dt
\end{aligned}
\tag{4}
$$

where the term involving $(u-U)$ reflects our desire to match the candidate state u with a given U in the entire domain Q. Our computational experiments of the proposed numerical methods will be performed exclusively for the case of $\alpha = 0$.

The rest of this chapter is organized as follows. In Section 2 we establish the equivalence between the limit of optimal solutions and the minimum boundary L^2 norm exact controller; this justifies the use of the optimal control approach for solving the exact control problem. In Section 3 we formally derive the optimality system of equations for the optimal control problem and discuss the shooting algorithm for solving the optimality system. In Section 4 We state the discrete version of the shooting algorithm for solving the discrete optimality system. Finally in Sections 5 and 6 we present computations of certain concrete controllability problems by the shooting method for solving optimal control problems.

2. The solution of the exact controllability problem as the limit of optimal control solutions

In this section we establish the equivalence between the limit of optimal solutions and the minimum boundary L^2 norm exact controller. We will show that if $\alpha = 0$, $\sigma \to \infty$ and $\tau \to \infty$, then the corresponding optimal solution $(\widehat{u}_{\sigma\tau}, \widehat{\mathbf{g}}_{\sigma\tau})$ converges weakly to the minimum boundary L^2 norm solution of the exact boundary controllability problem (1). The same is also true in the discrete case.

Theorem 2.1. *Assume that the exact boundary controllability problem* (1) *admits a unique minimum boundary L^2 norm solution $(u_{\rm ex}, \mathbf{g}_{\rm ex})$. Assume that for every $(\alpha,\sigma,\tau) \in \{0\}\times\mathbb{R}_+\times\mathbb{R}_+$ (where $\mathbb{R}_+$ is the set of all positive real numbers,) there exists a solution $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$ to the optimal control problem* (17)*. Then*

$$
\|\mathbf{g}_{\sigma\tau}\|_{L^2(\Sigma)} \le \|\mathbf{g}_{\rm ex}\|_{L^2(\Sigma)} \qquad \forall\,(\alpha,\sigma,\tau)\in\{0\}\times\mathbb{R}_+\times\mathbb{R}_+\,. \tag{5}
$$

Assume, in addition, that for a sequence $\{(\sigma_n,\tau_n)\}$ satisfying $\sigma_n\to\infty$ and $\tau_n\to\infty$,

$$
u_{\sigma_n\tau_n} \rightharpoonup \overline{u} \text{ in } L^2(Q) \quad\text{and}\quad f(u_{\sigma_n\tau_n}) \rightharpoonup f(\overline{u}) \text{ in } L^2(0,T;[H^2(\Omega)\cap H_0^1(\Omega)]^*)\,. \tag{6}
$$

Then

$$
\mathbf{g}_{\sigma_n\tau_n} \rightharpoonup \mathbf{g}_{\rm ex} \text{ in } [L^2(0,T)]^2 \text{ and } u_{\sigma_n\tau_n} \rightharpoonup u_{\rm ex} \text{ in } L^2(Q) \text{ as } n\to\infty\,. \tag{7}
$$

Furthermore, if (6) *holds for every sequence $\{(\sigma_n,\tau_n)\}$ satisfying $\sigma_n\to\infty$ and $\tau_n\to\infty$, then*

$$
\mathbf{g}_{\sigma\tau} \rightharpoonup \mathbf{g}_{\rm ex} \text{ in } [L^2(0,T)]^2 \text{ and } u_{\sigma\tau} \rightharpoonup u_{\rm ex} \text{ in } L^2(Q) \text{ as } \sigma,\tau\to\infty\,. \tag{8}
$$

Proof. Since $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$ is an optimal solution, we have that

$$
\begin{aligned}
&\frac{\sigma}{2}\|u_{\sigma\tau}(T)-W\|^2_{L^2(0,1)} + \frac{\tau}{2}\|\partial_t u_{\sigma\tau}(T)-Z\|^2_{H^{-1}(0,1)} + \frac{1}{2}\|\mathbf{g}_{\sigma\tau}\|^2_{L^2(\Sigma)} \\
&= \mathcal{J}(u_{\sigma\tau},\mathbf{g}_{\sigma\tau}) \le \mathcal{J}(u_{\rm ex},\mathbf{g}_{\rm ex}) = \frac{1}{2}\|\mathbf{g}_{\rm ex}\|^2_{L^2(\Sigma)}
\end{aligned}
$$

so that (5) holds,

$$
u_{\sigma\tau}|_{t=T} \to W \text{ in } L^2(0,X) \text{ and } (\partial_t u_{\sigma\tau})|_{t=T} \to Z \text{ in } H^{-1}(0,X) \text{ as } \sigma,\tau\to\infty\,. \tag{9}
$$

Let $\{(\sigma_n, \tau_n)\}$ be the sequence in (6). Estimate (5) implies that a subsequence of $\{(\sigma_n, \tau_n)\}$, denoted by the same, satisfies

$$\mathbf{g}_{\sigma_n\tau_n} \rightharpoonup \overline{\mathbf{g}} \text{ in } [L^2(0,T)]^2 \quad \text{and} \quad \|\overline{\mathbf{g}}\|_{\mathbf{L}^2(0,T)} \le \|\mathbf{g}_{\rm ex}\|_{\mathbf{L}^2(0,T)} \,. \tag{10}$$

$(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$ satisfies the initial value problem in the weak form:

$$\begin{aligned}
&\int_0^T\int_0^X u_{\sigma\tau}(v_{tt} - v_{xx})\,dx\,dt + \int_0^T\int_0^X [f(u_{\sigma\tau}) - V]v\,dx\,dt \\
&+ \int_0^T \mathbf{g}_{\sigma\tau}|_{x=X}(\partial_x v)|_{x=X}\,dt - \int_0^T \mathbf{g}_{\sigma\tau}|_{x=0}(\partial_x v)|_{x=0}\,dt \\
&+ \int_0^X (v\partial_t u_{\sigma\tau})|_{t=T}\,dx - \int_0^X v|_{t=0}u_1\,dx - \int_0^X (u_{\sigma\tau}\partial_t v)|_{t=T}\,dx \\
&+ \int_0^X (u_0\partial_t v)|_{t=0}\,dx = 0 \qquad \forall\, v \in C^2([0,T]; H^2\cap H^1_0(0,X))
\end{aligned} \tag{11}$$

where $\mathbf{g}_{\sigma\tau}|_{x=0}$ denotes the first component of $\mathbf{g}_{\sigma\tau}$ and $\mathbf{g}_{\sigma\tau}|_{x=X}$ the second component of $\mathbf{g}_{\sigma\tau}$. Passing to the limit in (11) as $\sigma,\tau\to\infty$ and using relations (9) and (10) we obtain:

$$\begin{aligned}
&\int_0^T\int_0^X \overline{u}(v_{tt} - v_{xx})\,dx\,dt + \int_0^T\int_0^X [f(\overline{u}) - V]v\,dx\,dt + \int_0^T \overline{g}_R(\partial_x v)|_{x=X}\,dt \\
&- \int_0^T \overline{g}_L(\partial_x v)|_{x=0}\,dt + \int_0^X v|_{t=T}Z(x)\,dx - \int_0^X v|_{t=0}u_1\,dx - \int_0^X W(x)(\partial_t v)|_{t=T}\,dx \\
&+ \int_0^X (u_0\partial_t v)|_{t=0}\,dx = 0 \qquad \forall\, v \in C^2([0,T]; H^2\cap H^1_0(0,X))\,.
\end{aligned}$$

The last relation and (10) imply that $(\overline{u}, \overline{\mathbf{g}})$ is a minimum boundary L^2 norm solution to the exact control problem (1). Hence, $\overline{u} = u_{\rm ex}$ and $\overline{\mathbf{g}} = \mathbf{g}_{\rm ex}$ so that (7) and (8) follows from (6) and (10). □

Remark 2.2. *If the wave equation is linear, i.e., $f = 0$, then assumption (6) is redundant and (8) is guaranteed to hold. Indeed, (11) implies the boundedness of $\{\|u_{\sigma\tau}\|_{L^2(Q)}\}$ which in turn yields (6). The uniqueness of a solution for the linear wave equation implies (6) holds for an arbitrary sequence $\{(\sigma_n,\tau_n)\}$.*

Theorem 2.3. *Assume that*

: i) *for every $(\alpha,\sigma,\tau)\in\{0\}\times\mathbb{R}_+\times\mathbb{R}_+$ there exists a solution $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$ to the optimal control problem (17);*

: ii) *the limit terminal conditions hold:*

$$u_{\sigma\tau}|_{t=T} \to W \text{ in } L^2(0,X) \text{ and } (\partial_t u_{\sigma\tau})|_{t=T} \to Z \text{ in } H^{-1}(0,X) \text{ as } \sigma,\tau\to\infty\,; \tag{12}$$

: iii) *the optimal solution $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$ satisfies the weak limit conditions as $\sigma,\tau\to\infty$:*

$$\mathbf{g}_{\sigma\tau} \rightharpoonup \overline{\mathbf{g}} \text{ in } \mathbf{L}^2(0,T)\,, \quad u_{\sigma\tau} \rightharpoonup \overline{u} \text{ in } L^2(Q)\,, \tag{13}$$

and

$$f(u_{\sigma\tau}) \rightharpoonup f(\overline{u}) \text{ in } L^2(0,T;[H^2(\Omega)\cap H^1_0(\Omega)]^*) \tag{14}$$

for some $\overline{\mathbf{g}}\in\mathbf{L}^2(0,T)$ and $\overline{u}\in L^2(Q)$.

Then $(\overline{u}, \overline{\mathbf{g}})$ is a solution to the exact boundary controllability problem (1) *with $\overline{\mathbf{g}}$ satisfying the minimum boundary L^2 norm property. Furthermore, if the solution to* (1) *admits a unique solution $(u_{\rm ex}, \mathbf{g}_{\rm ex})$, then*

$$\mathbf{g}_{\sigma\tau} \rightharpoonup \mathbf{g}_{\rm ex} \textit{ in } [L^2(0,T)]^2 \textit{ and } u_{\sigma\tau} \rightharpoonup u_{\rm ex} \textit{ in } L^2(Q) \textit{ as } \sigma,\tau \to \infty\,. \tag{15}$$

Proof. $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$ satisfies (11). Passing to the limit in that equation as $\sigma, \tau \to \infty$ and using relations (12), (13) and (14) we obtain:

$$\begin{aligned}&\int_0^T\int_0^X \overline{u}(v_{tt}-v_{xx})\,dx\,dt + \int_0^T\int_0^X [f(\overline{u})-V]v\,dx\,dt + \int_0^T \overline{g}_R(\partial_x v)|_{x=X}\,dt\\ &- \int_0^T \overline{g}_L(\partial_x v)|_{x=0}\,dt + \int_0^X v|_{t=T}Z(x)\,dx - \int_0^X v|_{t=0}u_1\,dx - \int_0^X W(x)(\partial_t v)|_{t=T}\,dx\\ &+ \int_0^X (u_0\partial_t v)|_{t=0}\,dx = 0 \qquad \forall\, v \in C^2([0,T]; H^2\cap H_0^1(0,X))\,.\end{aligned}$$

This implies that $(\overline{u}, \overline{\mathbf{g}})$ is a solution to the exact boundary controllability problem (1).

To prove that $\overline{\mathbf{g}}$ satisfies the minimum boundary L^2 norm property, we proceeds as follows. Let $(u_{\rm ex}, \mathbf{g})$ denotes a exact minimum boundary L^2 norm solution to the controllability problem (1). Since $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$ is an optimal solution, we have that

$$\begin{aligned}&\frac{\sigma}{2}\|u_{\sigma\tau}-W\|^2_{L^2(0,X)} + \frac{\tau}{2}\|\partial_t u_{\sigma\tau} - Z\|^2_{H^{-1}(0,X)} + \frac{1}{2}\|\mathbf{g}_{\sigma\tau}\|^2_{\mathbf{L}^2(0,T)}\\ &= \mathcal{J}(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau}) \le \mathcal{J}(u_{\rm ex}, \mathbf{g}_{\rm ex}) = \frac{1}{2}\|\mathbf{g}_{\rm ex}\|^2_{\mathbf{L}^2(0,T)}\end{aligned}$$

so that

$$\|\mathbf{g}_{\sigma\tau}\|_{\mathbf{L}^2(0,T)} \le \|\mathbf{g}_{\rm ex}\|_{\mathbf{L}^2(0,T)}\,.$$

Passing to the limit in the last estimate we obtain

$$\|\overline{\mathbf{g}}\|_{\mathbf{L}^2(0,T)} \le \|\mathbf{g}_{\rm ex}\|_{\mathbf{L}^2(0,T)}\,. \tag{16}$$

Hence we conclude that $(\overline{u}, \overline{\mathbf{g}})$ is a minimum boundary L^2 norm solution to the exact boundary controllability problem (1).

Furthermore, if the exact controllability problem (1) admits a unique minimum boundary L^2 norm solution $(u_{\rm ex}, \mathbf{g}_{\rm ex})$, then $(\overline{u}, \overline{\mathbf{g}}) = (u_{\rm ex}, \mathbf{g}_{\rm ex})$ and (15) follows from assumption (13). □

Remark 2.4. *If the wave equation is linear, i.e., $f = 0$, then assumptions i) and* (14) *are redundant.*

Remark 2.5. *Assumptions ii) and iii) hold if $\mathbf{g}_{\sigma\tau}$ and $u_{\sigma\tau}$ converges pointwise as $\sigma, \tau \to \infty$.*

Remark 2.6. *A practical implication of Theorem 2.3 is that one can prove the exact controllability for semilinear wave equations by examining the behavior of a sequence of optimal solutions (recall that exact controllability was proved only for some special classes of semilinear wave equations.) If we have found a sequence of optimal control solutions $\{(u_{\sigma_n\tau_n}, \mathbf{g}_{\sigma_n\tau_n})\}$ where $\sigma_n, \tau_n \to \infty$ and this sequence appears to satisfy the convergence assumptions ii) and iii), then we can confidently conclude that the underlying semilinear wave equation is exactly controllable and the optimal solution $(u_{\sigma_n\tau_n}, \mathbf{g}_{\sigma_n\tau_n})$ when n is large provides a good approximation to the minimum boundary L^2 norm exact controller $(u_{\rm ex}, \mathbf{g}_{\rm ex})$.*

3. An optimality system of equations and a continuous shooting method

Under suitable assumptions on f and through the use of Lagrange multiplier rules, the optimal control problem

$$\text{minimize (4) with respect to the control } \mathbf{g} \text{ subject to (3)} \tag{17}$$

may be converted into the following system of equations from which an optimal solution may be determined:

$$\begin{aligned}
&u_{tt} - u_{xx} + f(u) = V \qquad \text{in } (0,T)\times(0,X)\,,\\
&u|_{x=0} = g_L\,, \quad u|_{x=1} = g_R\,, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x)\,,\\
&\xi_{tt} - \xi_{xx} + f'(u)\xi = -\alpha(u-U) \qquad \text{in } (0,T)\times(0,X)\,,\\
&\xi|_{x=0} = 0, \qquad \xi|_{x=1} = 0\,,\\
&\xi(T,x) = -\tau A^{-1}(u_t(T,x) - Z(x)) \qquad \xi_t(T,x) = -\sigma(u(T,x) - W(x))\,,\\
&g_L = -\xi_x|_{x=0}\,, \qquad \text{and} \qquad g_R = \xi_x|_{x=1}\,,
\end{aligned}$$

where the elliptic operator $A : H_0^1(0,X) \to H^{-1}(0,X)$ is defined by $Av = v_{xx}$ for all $v \in H_0^1(0,X)$. By eliminating g_L and g_R in the system we arrive at the optimality system

$$\begin{aligned}
&u_{tt} - u_{xx} + f(u) = V \qquad \text{in } (0,T)\times(0,X)\,,\\
&u|_{x=0} = -\xi_x|_{x=0}\,, \qquad u|_{x=1} = \xi_x|_{x=1}\,,\\
&u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x)\,,\\
&\xi_{tt} - \xi_{xx} + f'(u)\xi = -\alpha(u-U) \qquad \text{in } (0,T)\times(0,X)\,,\\
&\xi|_{x=0} = 0, \qquad \xi|_{x=1} = 0\,,\\
&\xi(T,x) = -\tau A^{-1}(u_t(T,x) - Z(x))\,,\ \xi_t(T,x) = -\sigma(u(T,x) - W(x))\,.
\end{aligned} \tag{18}$$

Derivations and justifications of optimality systems are discussed in [13] for the linear case and in [14] for the semilinear case.

The computational algorithm we propose in this chapter is a shooting method for solving the optimality system of equations. The basic idea for a shooting method is to convert the solution of a initial-terminal value problem into that of a purely initial value problem (IVP); see, e.g., [2] for a discussion of shooting methods for systems of ordinary differential equations. The IVP corresponding to the optimality system (18) is described by

$$\begin{aligned}
&u_{tt} - u_{xx} + f(u) = V \qquad \text{in } (0,T)\times(0,X)\,,\\
&u|_{\partial\Omega} = \frac{\partial\xi}{\partial\nu}\Big|_{\partial\Omega}, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x)\,;\\
&\xi_{tt} - \xi_{xx} + f'(u)\xi = -\alpha(u-U) \qquad \text{in } (0,T)\times(0,X)\,,\\
&\xi|_{\partial\Omega} = 0, \qquad \xi(0,x) = \omega(x), \qquad \xi_t(0,x) = \theta(x)\,,
\end{aligned} \tag{19}$$

with unknown initial values ω and θ. Then the goal is to choose ω and θ such that the solution (u,ξ) of the IVP (19) satisfies the terminal conditions

$$\begin{aligned}
&F_1(\omega,\theta) \equiv \partial_{xx}\xi(T,x) + \tau(u_t(T,x) - Z(x)) = 0\,,\\
&F_2(\omega,\theta) \equiv \xi_t(T,x) + \sigma(u(T,x) - W(x)) = 0\,.
\end{aligned} \tag{20}$$

A shooting method for solving (18) can be described by the following iterations:

choose initial guesses ω and θ;
for $iter = 1, 2, \cdots, max_iter$

solve for (u,ξ) from the IVP (19)
update ω and θ.

A criterion for updating (ω,θ) can be derived from the terminal conditions (20). A method for solving the nonlinear system (20) (as a system for the unknowns ω and θ) will yield an updating formula; for instance, the well-known Newton's method may be invoked.

choose initial guesses ω and θ;
for $iter = 1, 2, \cdots, max_iter$
solve for (u,ξ) from the IVP (19)
update ω and θ:
$(\omega^{\text{new}},\theta^{\text{new}}) = (\omega,\theta) - [F'(\omega,\theta)]^{-1}F(\omega,\theta)$;
if $F(\omega^{\text{new}},\theta^{\text{new}}) = \mathbf{0}$, stop; otherwise, set $(\omega,\theta) = (\omega^{\text{new}},\theta^{\text{new}})$.

A discussion of Newton's method for an infinite dimensional nonlinear system can be found in many functional analysis textbooks, and for the suitable assumption convergence of Newton iteration for the optimality system is guaranteed.

4. The discrete shooting method

The shooting method described in Section (3) must be implemented discretely. We discretize the spatial interval $[0,1]$ into $0 = x_0 < x_1 < x_2 < \cdots < x_I < x_{I+1} = 1$ with a uniform spacing $h = 1/(I+1)$ and we divide the time horizon $[0,T]$ into $0 = t_1 < t_2 < t_3 < \cdots < t_N = T$ with a uniform time stepping $\delta = T/(N-1)$. We use the explicit, central difference scheme to approximate the initial value problem (19):

$$
\begin{aligned}
&u_i^1 = (u_0)_i,\;\; u_i^2 = (u_0)_i + \delta(u_1)_i\,,\;\; \xi_i^1 = \omega_i,\;\; \xi_i^2 = \xi_i^1 + \delta\theta_i\,,\;\; i = 1,2,\cdots,I\,;\\
&u_i^{n+1} = -u_i^{n-1} + \lambda u_{i-1}^n + 2(1-\lambda)u_i^n + \lambda u_{i+1}^n\\
&\qquad\qquad - \delta^2 f(u_i^n) + \delta^2 V(t_n,x_i), \quad i = 1,2,\cdots,I\,,\\
&\xi_i^{n+1} = -\xi_i^{n-1} + \lambda\xi_{i-1}^n + 2(1-\lambda)\xi_i^n + \lambda\xi_{i+1}^n\\
&\qquad\qquad - \delta^2 f'(u_i^n)\xi_i^n + \delta^2\alpha(u_i^n - U(t_n,x_i)), \quad i = 1,2,\cdots,I\\
&u_0^{n+1} = -\frac{\xi_1^{n+1} - \xi_0^{n+1}}{h}, \qquad u_{I+1}^{n+1} = -\frac{\xi_{I+1}^{n+1} - \xi_I^{n+1}}{h}.
\end{aligned} \tag{21}
$$

where $\lambda = (\delta/h)^2$ (we also use the convention that $\xi_0^n = \xi_{I+1}^n = 0$.) The gist of a discrete shooting method is to regard the discrete terminal conditions

$$
\begin{aligned}
&F_{2i-1} \equiv \frac{\xi_i^N - \xi_i^{N-1}}{\delta} + \sigma(u_i^N - W_i) = 0\,,\\
&F_{2i} \equiv \frac{\xi_{i-1}^N - 2\xi_i^N + \xi_{i+1}^N}{h^2} + \tau\Big(\frac{u_i^N - u_i^{N-1}}{\delta} - Z_i\Big) = 0\,, \qquad i = 1,2,\cdots,I
\end{aligned} \tag{22}
$$

as a system of equations for the unknown initial condition $\omega_1,\theta_1,\omega_2,\theta_2,\cdots,\omega_m,\theta_m$.

Similar to the continuous case, the discrete shooting method consists of the following iterations:

choose discrete initial guesses $\{\omega_i\}_{i=1}^I$ and $\{\theta\}_{i=1}^I$;
for $iter = 1, 2, \cdots, max_iter$
solve for $\{(u_i^n,\xi_i^n)\}_{i=1,n=1}^{i=I,n=N}$ from the discrete IVP (21)
update $\{\omega_i\}_{i=1}^I$ and $\{\theta_i\}_{i=1}^I$.

The initial conditions $\{\omega_i\}_{i=1}^I$ and $\{\theta_i\}_{i=1}^I$ are updated by Newton's method applied to the discrete nonlinear system (22). This requires the calculations of

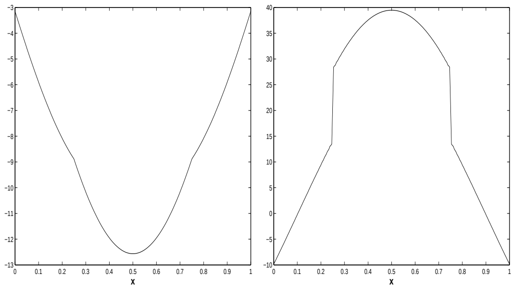

FIGURE 1. left - u_0 , right - u_1 given in (24). $h = 1/256$.

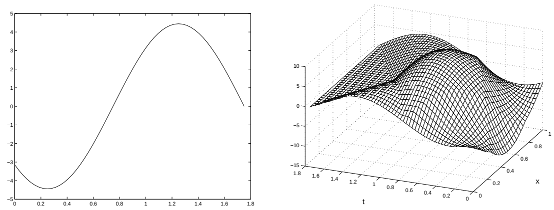

FIGURE 2. left - exact control , right - exact u. with initial data (24). $h = 1/32$.

TABLE 1. Results of computational experiments for the minimum $L^2(\Sigma)$-norm case for the examples with initial data (24).

h		1/16	1/32	1/64	1/128	1/256	1/512	1/1024
$\|g^h\|_{L^2(\Sigma)}$	$\lambda = 1$	5.9339	6.0294	6.0825	6.1103	6.1244	6.1316	6.1352
$\|g^h\|_{L^2(\Sigma)}$	$\lambda = 7/8$	5.9682	6.0468	6.0917	6.1454	6.1262	6.1325	6.1356
$\frac{\|g^h - g\|_{L^2(\Sigma)}}{\|g\|_{L^2(\Sigma)}}$	$\lambda = 1$	6.93%	3.35%	1.63%	0.79%	0.37%	0.18%	0.09%
$\frac{\|g^h - g\|_{L^2(\Sigma)}}{\|g\|_{L^2(\Sigma)}}$	$\lambda = 7/8$	7.53%	4.26%	2.88%	10.15%	0.35%	0.17%	0.08%

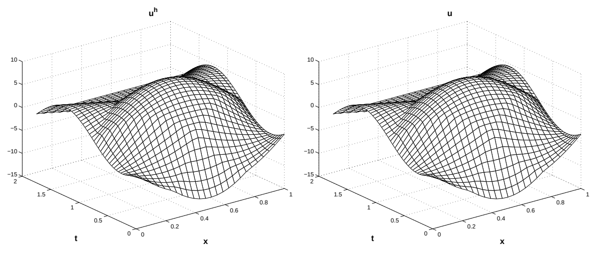

FIGURE 3. left - approximate control u^h , right - exact u with initial data (24). $h = 1/256$. $\lambda = 1$.

partial derivatives. By denoting

$$q_{ij}^n = q_{ij}^n(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I) = \frac{\partial u_i^n}{\partial \omega_j},\ r_{ij}^n = r_{ij}^n(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I) = \frac{\partial u_i^n}{\partial \theta_j},$$

$$\rho_{ij}^n = \rho_{ij}^n(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I) = \frac{\partial \xi_i^n}{\partial \omega_j}\ ,\ \tau_{ij}^n = \tau_{ij}^n(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I) = \frac{\partial \xi_i^n}{\partial \theta_j}\ ,$$

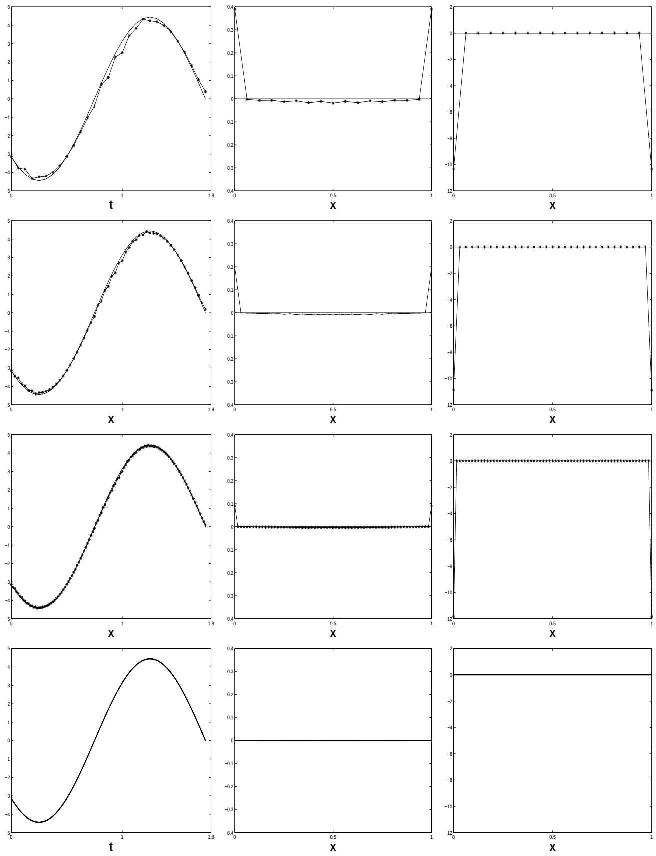

FIGURE 4. left - approximate control g^h and g, middle - approximate u^h and target W, right - approximate u_t^h and target Z with initial data (24). $h = 1/16, 1/32, 1/64, 1/1024$ from top to bottom respectively. $\lambda = 1$.

we obtain the following Newton's iteration formula:

$$\begin{aligned}
&(\omega_1^{\text{new}}, \theta_1^{\text{new}}, \omega_2^{\text{new}}, \theta_2^{\text{new}}, \cdots, \omega_I^{\text{new}}, \theta_I^{\text{new}})^T \\
=&(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I)^T \\
&- [F'(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I)]^{-1} F(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I)
\end{aligned}$$

where the vector F and Jacobian matrix $J = F'$ are defined by

$$F_{2i-1} = \frac{\xi_i^N - \xi_i^{N-1}}{\delta} + \sigma(u_i^N - W_i)\,, \; F_{2i} = \frac{\xi_{i-1}^N - 2\xi_i^N + \xi_{i+1}^N}{h^2} + \tau(\frac{u_i^N - u_i^{N-1}}{\delta} - Z_i)\,,$$

$$\begin{aligned}
J_{2i-1,2j-1} &= \frac{\rho_{ij}^N - \rho_{ij}^{N-1}}{\delta} + \sigma q_{ij}^N\,, \qquad J_{2i-1,2j} = \frac{\tau_{ij}^N - \tau_{ij}^{N-1}}{\delta} + \sigma r_{ij}^N\,, \\
J_{2i,2j-1} &= \frac{\rho_{(i-1)j}^N - 2\rho_{ij}^N + \rho_{(i+1)j}^N}{h^2} + \frac{\tau}{\delta}(q_{ij}^N - q_{ij}^{N-1})\,, \\
J_{2i,2j} &= \frac{\tau_{(i-1)j}^N - 2\tau_{ij}^N + \tau_{(i+1)j}^N}{h^2} + \frac{\tau}{\delta}(r_{ij}^N - r_{ij}^{N-1})\,.
\end{aligned}$$

Moreover, by differentiating (21) with respect to ω_j and θ_j we obtain the equations for determining q_{ij}, r_{ij}, ρ_{ij} and τ_{ij}:

$$
\begin{aligned}
&\begin{matrix} q^1_{ij} = 0, & q^2_{ij} = 0\,, & r^1_{ij} = 0\,, & r^2_{ij} = 0\,, \\ \rho^1_{ij} = \delta_{ij}\,, & \rho^2_{ij} = \delta_{ij}\,, & \tau^1_{ij} = 0\,, & \tau^2_{ij} = \delta\delta_{ij}\,, \end{matrix} \qquad i,j = 1,2,\cdots,I\,;\\
q^{n+1}_{ij} &= -q^{n-1}_{ij} + \lambda q^n_{i-1,j} + 2(1-\lambda)q^n_{ij} + \lambda q^n_{i+1,j}\\
&\qquad - \delta^2 f'(u^n_i)q^n_{ij}, \qquad i,j = 1,2,\cdots,I\,,\\
r^{n+1}_{ij} &= -r^{n-1}_{ij} + \lambda r^n_{i-1,j} + 2(1-\lambda)r^n_{ij} + \lambda r^n_{i+1,j}\\
&\qquad - \delta^2 f'(u^n_i)r^n_{ij}, \qquad i,j = 1,2,\cdots,I\,,\\
\rho^{n+1}_{ij} &= -\rho^{n-1}_{ij} + \lambda \rho^n_{i-1,j} + 2(1-\lambda)\rho^n_{ij} + \lambda \rho^n_{i+1,j}\\
&\quad + \delta^2\alpha q^N_{ij} - \delta^2[f'(u^n_i)]\rho^n_i - \delta^2[f''(u^n_i)q^N_i]\xi^n_i, \qquad i,j = 1,2,\cdots,I\,;\\
\tau^{n+1}_{ij} &= -\tau^{n-1}_{ij} + \lambda \tau^n_{i-1,j} + 2(1-\lambda)\tau^n_{ij} + \lambda \tau^n_{i+1,j}\\
&\quad + \delta^2\alpha r^N_{ij} - \delta^2[f'(u^n_i)]\rho^n_i - \delta^2[f''(u^n_i)r^N_i]\xi^n_i, \qquad i,j = 1,2,\cdots,I\,;
\end{aligned}
\tag{23}
$$

where δ_{ij} is the Chronecker delta. Thus, we have the following Newton's-method-based shooting algorithm:

Algorithm – Newton method based shooting algorithm with central finite difference approximations of the optimality system

choose initial guesses ω_i and θ_i, $i = 1,2,\cdots,I$;
% set initial conditions for u and ξ
for $i = 0,2,\cdots,I+1$
  $u^1_i = (u_0)_i$, $u^2_i = (u_0)_i + \delta(u_1)_i$,
for $i = 1,2,\cdots,I$
  $\xi^1_i = \omega_i$, $\xi^2_i = \xi^1_i + \delta\theta_i$;
% set initial conditions for $q_{ij}, r_{ij}, \rho_{ij}, \tau_{ij}$
for $j = 1,2,\cdots,I$
  for $i = 1,2,\cdots,I$
    $q^1_{ij} = 0,\quad q^2_{ij} = 0,\quad r^1_{ij} = 0,\quad r^2_{ij} = 0,$
    $\rho^1_{ij} = 0,\quad \rho^2_{ij} = 0,\quad \tau^1_{ij} = 0,\quad \tau^2_{ij} = 0;$
    $\rho^1_{jj} = 1,\quad \rho^2_{jj} = 1,\quad \tau^2_{jj} = \delta,$
% Newton iterations
for $m = 1,2,\cdots,M$
  % solve for (u,ξ)
  for $n = 2,3,\cdots,N-1$
    $u^{n+1}_i = -u^{n-1}_i + \lambda u^n_{i-1} + 2(1-\lambda)u^n_i + \lambda u^n_{i+1} - \delta^2 f(u^n_i) +$
$\delta^2 V(t_n,x_i)$,
    $\xi^{n+1}_i = -\xi^{n-1}_i + \lambda\xi^n_{i-1} + 2(1-\lambda)\xi^n_i + \lambda\xi^n_{i+1} - \delta^2 f'(u^n_i)\xi^n_i$
      $+\delta^2\alpha(u^n_i - U(t_n,x_i))$;
  % solve for q,r,ρ,τ
  for $j = 1,2,\cdots,I$
    for $n = 2,3,\cdots,N-1$
      for $i = 2,\cdots,N-1$
        $q^{n+1}_{ij} = -q^{n-1}_{ij} + \lambda q^n_{i-1,j} + 2(1-\lambda)q^n_{ij} + \lambda q^n_{i+1,j}$
          $-\delta^2 f'(u^n_i)q^n_{ij}$,
        $r^{n+1}_{ij} = -r^{n-1}_{ij} + \lambda r^n_{i-1,j} + 2(1-\lambda)r^n_{ij} + \lambda r^n_{i+1,j}$
          $-\delta^2 f'(u^n_i)r^n_{ij}$,

$$\rho_{ij}^{n+1} = -\rho_{ij}^{n-1} + \lambda\rho_{i-1,j}^{n} + 2(1-\lambda)\rho_{ij}^{n} + \lambda\rho_{i+1,j}^{n} + \delta^2\alpha q_{ij}^{n} - \delta^2[f'(u_i^n)]\rho_{ij}^{n} - \delta^2[f''(u_i^n)q_{ij}^{n}]\xi_i^n,$$

$$\tau_{ij}^{n+1} = -\tau_{ij}^{n-1} + \lambda\tau_{i-1,j}^{n} + 2(1-\lambda)\tau_{ij}^{n} + \lambda\tau_{i+1,j}^{n} + \delta^2\alpha r_{ij}^{n} - \delta^2[f'(u_i^n)]\rho_{ij}^{n} - \delta^2[f''(u_i^n)r_{ij}^{n}]\xi_i^n;$$

% (we need to build into the algorithm the following:

$$q_{0j}^{n+1} = -\frac{\rho_{1j}^{n+1}}{h}, r_{0j}^{n} = -\frac{\tau_{1j}^{n+1}}{h}, r_{I+1,j}^{n} = -\frac{\tau_{Ij}^{n+1}}{h}$$

$$q_{I+1,j}^{n} = -\frac{\rho_{Ij}^{n+1}}{h}, \rho_{0j}^{n} = \tau_{0j}^{n} = \rho_{I+1,j}^{n} = \tau_{I+1,j}^{n} = 0.)$$

% evaluate F and F'

for $i = 1, 2, \cdots, I$

$F_{2i-1} = \frac{\xi_i^N - \xi_i^{N-1}}{\delta} + \sigma(u_i^N - W_i)$,

$F_{2i} = \frac{\xi_{i-1}^N - 2\xi_i^N + \xi_{i+1}^N}{h^2} + \tau(\frac{u_i^N - u_i^{N-1}}{\delta} - Z_i)$;

for $j = 1, 2, \cdots, I$

$J_{2i-1,2j-1} = \frac{\rho_{ij}^N - \rho_{ij}^{N-1}}{\delta} + \sigma q_{ij}^N$, $\quad J_{2i-1,2j} = \frac{\tau_{ij}^N - \tau_{ij}^{N-1}}{\delta} + \sigma r_{ij}^N$

$J_{2i,2j-1} = \frac{\rho_{(i-1)j}^N - 2\rho_{ij}^N + \rho_{(i+1)j}^N}{h^2} + \frac{\tau}{\delta}(q_{ij}^N - q_{ij}^{N-1})$,

$J_{2i,2j} = \frac{\tau_{(i-1)j}^N - 2\tau_{ij}^N + \tau_{(i+1)j}^N}{h^2} + \frac{\tau}{\delta}(r_{ij}^N - r_{ij}^{N-1})$;

solve $J\mathbf{c} = -F$ by Gaussian eliminations;

for $i = 1, 2, \cdots, I$

$\omega_i^{\text{new}} = \omega_i + c_{2i-1}$, $\quad \theta_i^{\text{new}} = \theta_i + c_{2i}$;

if $\max_i |\omega_i^{\text{new}} - \omega_i| + \max_i |\theta_i^{\text{new}} - \theta_i| <$ tol, stop;

otherwise, reset $\omega_i = \omega_i^{\text{new}}$ and $\theta_i = \theta_i^{\text{new}}$, $i = 1, 2, \cdots, I$;

As in the continuous case, we have the following convergence result for the shooting algorithm which follows from standard convergence results for Newton's method applied to finite dimensional systems of nonlinear equations.

Remark 4.1. *The algorithms we propose are well suited for implementations on a parallel computing platform such as a massive cluster of processors. The shooting algorithms of this chapter can be regarded as a generalization of their counterpart for systems of ODE (see, e.g.,* [2]*.) There has been a substantial literature on the parallelization of shooting methods for ODEs* [3, 10, 11]*; these results will be helpful in parallelizing the shooting algorithms of this chapter.*

TABLE 2. Results of computational experiments for the minimum $L^2(\Sigma)$-norm case for Examples I with initial data (26) and for $\lambda = 1, 4/5$.

h		1/32	1/64	1/128	1/256	1/512	1/1024
$\|g^h\|_{L^2(\Sigma)}$	$\lambda = 1$	0.12934	0.12908	0.12906	0.12907	0.12908	0.12909
$\|g^h\|_{L^2(\Sigma)}$	$\lambda = 4/5$	0.15941	0.15269	0.14522	0.14216	0.13907	0.13622

5. Computational experiments for controllability of the linear wave equation

We will apply Algorithm 1 to the special case of $f = 0$, $V = 0$, $W = 0$, $Z = 0$, $\alpha = 0$ and $\sigma, \tau >> 1$. In other words, we will approximate the null controllability

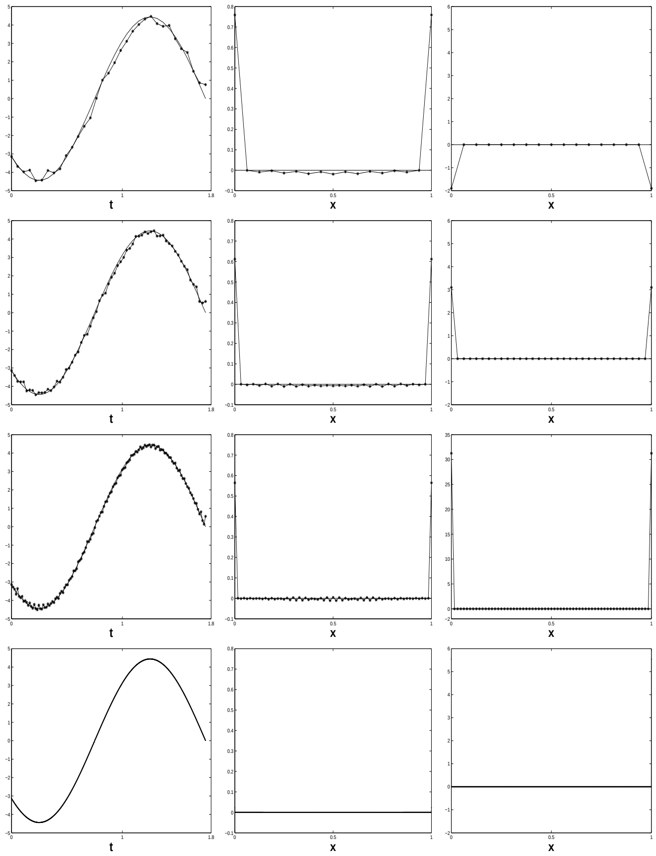

FIGURE 5. left - approximate control g^h and g, middle - approximate u^h and target W, right - approximate u_t^h and target Z with initial data (24). $h = 1/16, 1/32, 1/64, 1/1024$ from top to bottom respectively. $\lambda = 7/8$.

TABLE 3. Results of computational experiments for the minimum $L^2(\Sigma)$-norm case for Examples II with initial data (26) and for $\lambda = 1, 7/8$.

h		1/16	1/32	1/64	1/128	1/256	1/512	1/1024
$\|g^h\|_{L^2(\Sigma)}$	$\lambda = 1$	0.6838	0.6388	0.6162	0.6049	0.5992	0.5963	0.5949
$\|g^h\|_{L^2(\Sigma)}$	$\lambda = 7/8$	0.6734	0.6348	0.6138	0.6039	0.5988	0.5963	0.5949

TABLE 4. Results of computational experiments for the minimum $L^2(\Sigma)$-norm case for for Examples III with initial data (26) and for $\lambda = 1, 7/8$.

h		1/16	1/32	1/64	1/128	1/256	1/512	1/1024
$\|g^h\|_{L^2(\Sigma)}$	$\lambda = 1$	1.4277	1.3187	1.2605	1.2303	1.2149	1.2071	1.2032
$\|g^h\|_{L^2(\Sigma)}$	$\lambda = 7/8$	1.3932	1.3007	1.2493	1.2252	1.2124	1.2065	1.2028

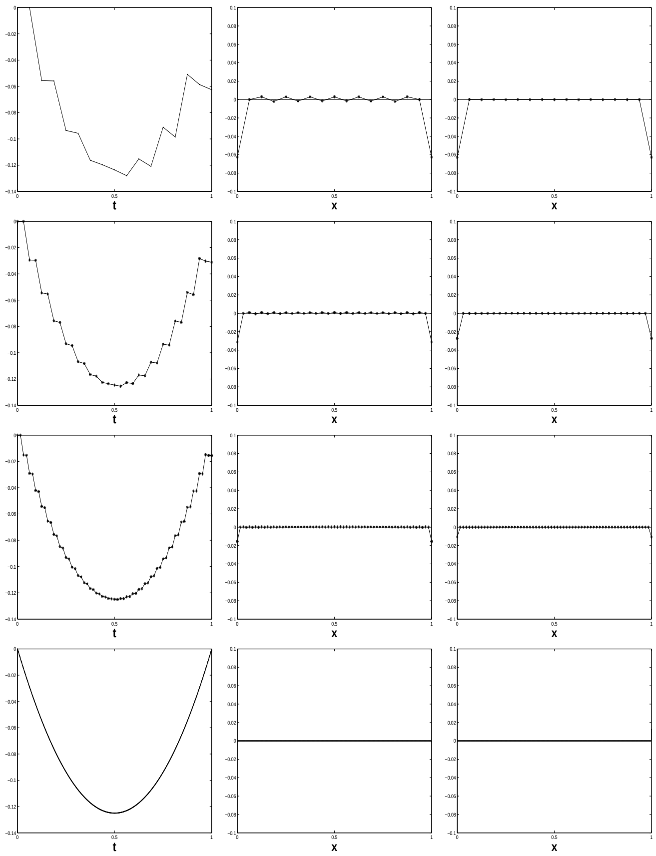

Figure 6. left - approximate control g^h , middle - approximate u^h and target W, right - approximate u_t^h and target Z with initial data (26-I). $h = 1/16, 1/32, 1/64, 1/1024$ from top to bottom respectively. $\lambda = 1$.

Table 5. Results of computational experiments for the minimum $L^2(\Sigma)$-norm case for Examples I, II, III in (27) with initial data I in (26) and for $\lambda = 1$.

	h	1/16	1/32	1/64	1/128
I	$\Vert g^h\Vert_{L^2(\Sigma)}$	0.08084810765	0.08038960736	0.08021218880	0.08013073451
	count	17	16	16	17
II	$\Vert g^h\Vert_{L^2(\Sigma)}$	0.07346047350	0.07314351955	0.07307515741	0.07306230119
	count	8	8	10	12
III	$\Vert g^h\Vert_{L^2(\Sigma)}$	0.07438729446	0.07404916115	0.07397863882	0.07396393744
	count	6	5	5	5

problem for the linear wave equation by optimal control problems. We will test our algorithm with a smooth example (i.e., the continuous minimum boundary L^2 norm controller g and the corresponding state u are smooth) and with three generic examples. It was reported in [12] that the discrete minimum boundary L^2 norm controllers converge strongly to the continuous minimum boundary L^2 norm

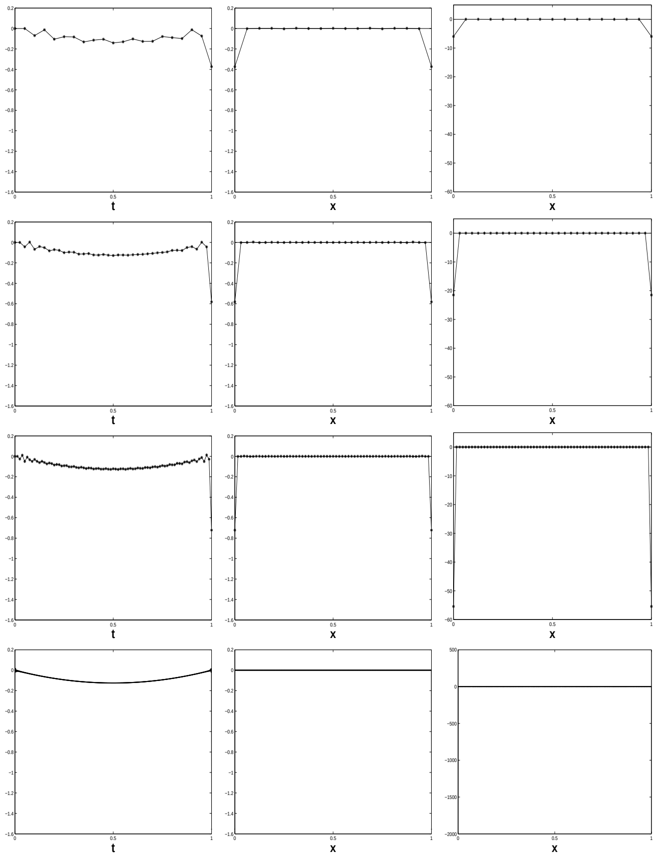

FIGURE 7. left - approximate control g^h , middle - approximate u^h and target W, right - approximate u_t^h and target Z with initial data (26-I). $h = 1/16, 1/32, 1/64, 1/1024$ from top to bottom respectively. $\lambda = 4/5$.

TABLE 6. Results of computational experiments for the minimum $L^2(\Sigma)$-norm case for Examples I, II, III in (27) with initial data II in (26) and for $\lambda = 1$.

	h	1/16	1/32	1/64	1/128
I	$\lVert g^h\rVert_{L^2(\Sigma)}$	0.45499490909	0.43856890841	0.43072624972	0.42688986194
	count	12	15	16	14
II	$\lVert g^h\rVert_{L^2(\Sigma)}$	0.45794379129	0.43786262977	0.42823496745	0.42350319979
	count	12	12	12	17
III	$\lVert g^h\rVert_{L^2(\Sigma)}$	0.45251184147	0.43223256110	0.42248163895	0.41769256787
	count	6	6	6	6

controller for the smooth example and converge weakly in the generic case. The discrete optimal solutions found by Algorithm 1 will exhibit similar behaviors.

5.1. An example with known smooth exact solution. A smooth exact solution to the minimum boundary L^2-norm controllability problem was constructed in [12] by using Fourier series in a way similar to that used in [6]. Suppose that $Q =$

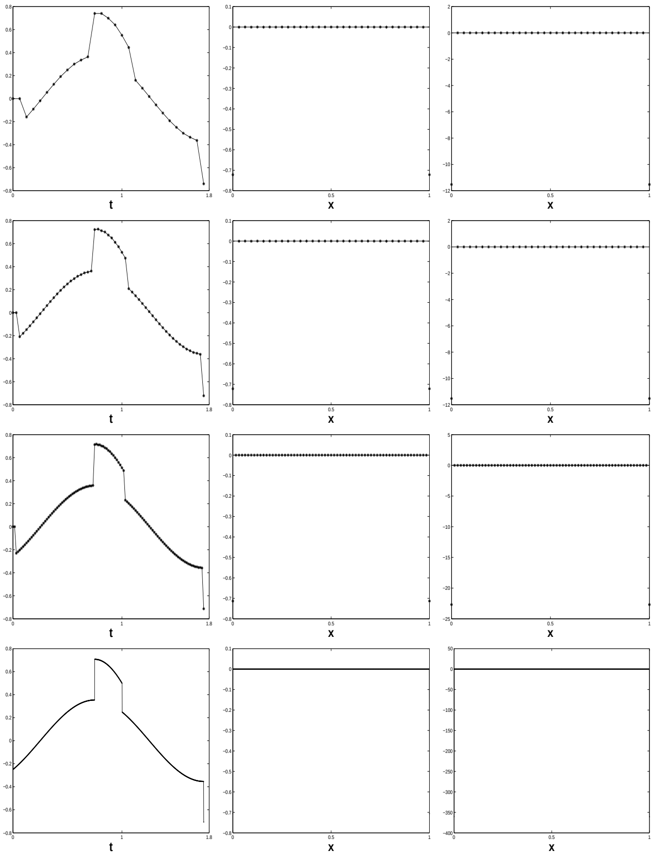

FIGURE 8. left - approximate control g^h , middle - approximate u^h and target W, right - approximate u_t^h and target Z with initial data (26-II). $h = 1/16, 1/32, 1/64, 1/1024$ from top to bottom respectively. $\lambda = 1$.

TABLE 7. Results of computational experiments for the minimum $L^2(\Sigma)$-norm case for Examples I, II, III in (27) with initial data III in (26) and for $\lambda = 1$.

	h	1/16	1/32	1/64	1/128
I	$\|g^h\|_{L^2(\Sigma)}$	0.94846408635	0.86623499117	0.82305989083	0.80084943961
	count	11	11	11	11
II	$\|g^h\|_{L^2(\Sigma)}$	0.99946692390	0.90706619363	0.85837779589	0.83325287647
	count	13	16	18	20
III	$\|g^h\|_{L^2(\Sigma)}$	0.95205362894	0.86225101645	0.81393537311	0.78872084130
	count	6	6	6	6

$(0, 7/4) \times (0, 1)$ and $\Sigma = (0, 7/4) \times \{0, 1\}$. Let $\psi_0(t, x) = -\sqrt{2}\pi \cos \pi(t - \frac{1}{4}) \cos 2\pi x$

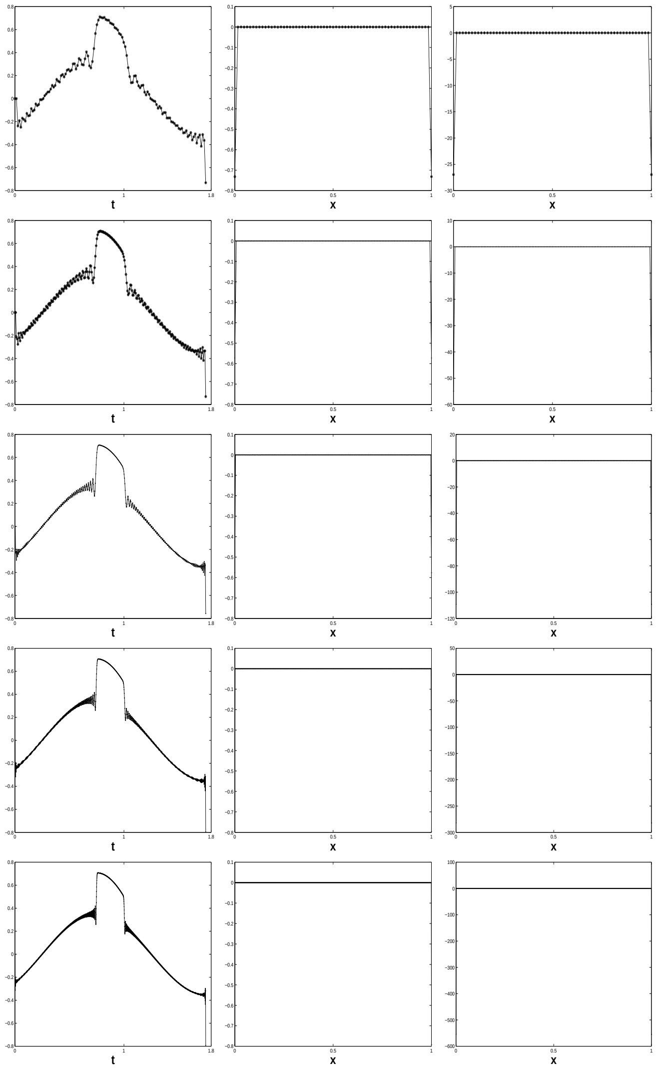

FIGURE 9. left - approximate control g^h , middle - approximate u^h and target W, right - approximate u_t^h and target Z with initial data (26-II). $h = 1/64, 1/128, 1/256, 1/512, 1/1024$ from top to bottom respectively. $\lambda = 7/8$.

and

$$\psi_1(t,x) = \left[2\sqrt{2}\pi(T-t)\sin\pi(t-\frac{1}{4}) - \frac{10}{3}\sqrt{2}\sin\pi(t-T)\right]\sin\pi x$$
$$+ \sum_{p\geq 3 \text{ and } p \text{ odd}} \frac{4\sqrt{2}}{p^2-1}\left[\frac{3p}{p^2-4}\cos\pi(t-\frac{1}{4}) + \sin p\pi(t-T)\right]\sin p\pi x\,.$$

Then, set the initial conditions

$$u_0(x) = \psi_0(0,x) + \psi_1(0,x) \quad \text{and} \quad u_1(x) = \frac{\partial\psi_0}{\partial t}(0,x) + \frac{\partial\psi_1}{\partial t}(0,x)\,. \tag{24}$$

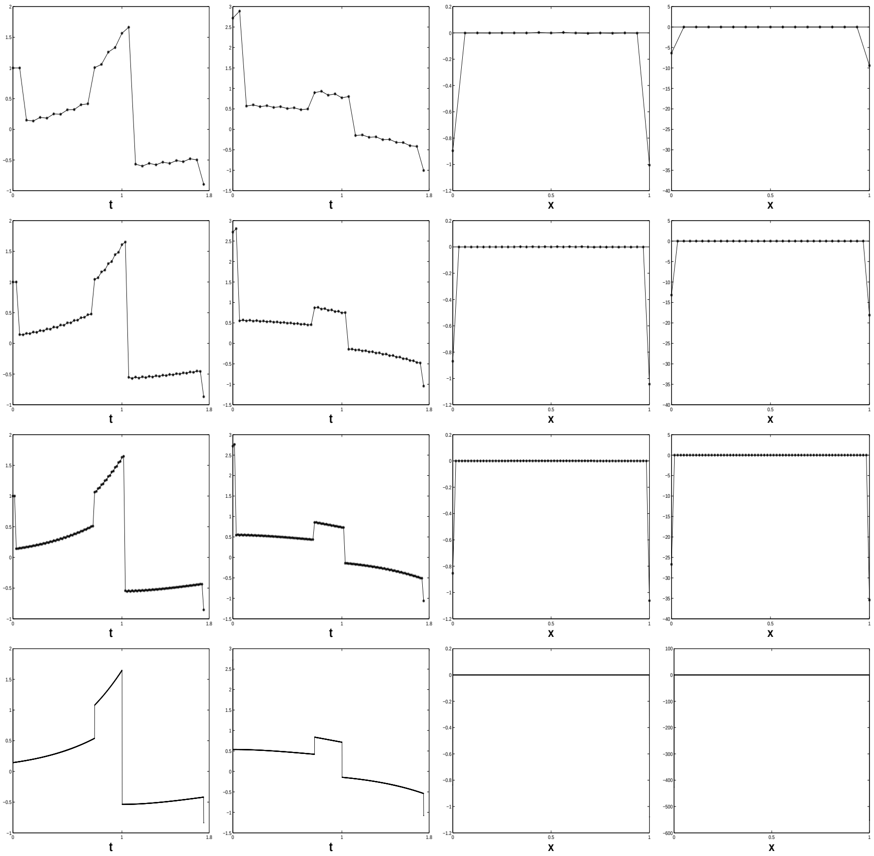

FIGURE 10. g_L, g_R, $u^h(x,T)$, and $u^h_t(x,T)$ from left to right with initial data (26-III). $h = 1/16, 1/32, 1/64, 1/1024$ from top to bottom respectively. $\lambda = 1$.

The computation of u_0 and u_1 involve the summation of infinite trigonometric series. Figure 1 and Figure 2 provides plots of u_0 and u_1, and the exact control and exact solution respectively. Note that initial conditions vanish at the boundary, and due to symmetry, we have $g_L(t) = u(t,0) = u(t,1) = g_R(t)$. i.e the controls at two sides of Q are the same. It is worth noting that u_0 is a Lipschitz continuous function but does not belong to $C^1[0,1]$ and u_1 is a bounded function but does not belong to $C^0[0,1]$. For the initial data (24), it can be shown that $u(t,x) = \psi_0(t,x) + \psi_1(t,x)$ is the exact solution having minimum boundary L^2-norm of the controllability problem given by the first three equations in (1) provided $f = 0$, $V = 0$. Let g be the corresponding exact Dirichlet control given by restricting $u(t,x)$ to the lateral sides Σ. i.e $g(t) = (g_L(t), g_R(t)) = (u(t,0), u(t,1))$, and

$$g_L(t) = g_R(t) = -\sqrt{2}\pi \cos \pi\Big(t - \frac{1}{4}\Big)\,. \tag{25}$$

For future reference, note that $\|g\|_{L^2(\Sigma)} = \sqrt{2\pi^2(\frac{7}{4} + \frac{1}{2\pi})} \approx 6.13882$.

We apply our numerical method to this example. Computational experiments were carried out for $h = 1/16$, $1/32$, $1/64$, $1/128$, $1/256$, $1/512$, and $1/1024$ with $\lambda = 1$ and $\lambda = 7/8$ respectively, so that the stability condition is satisfied.

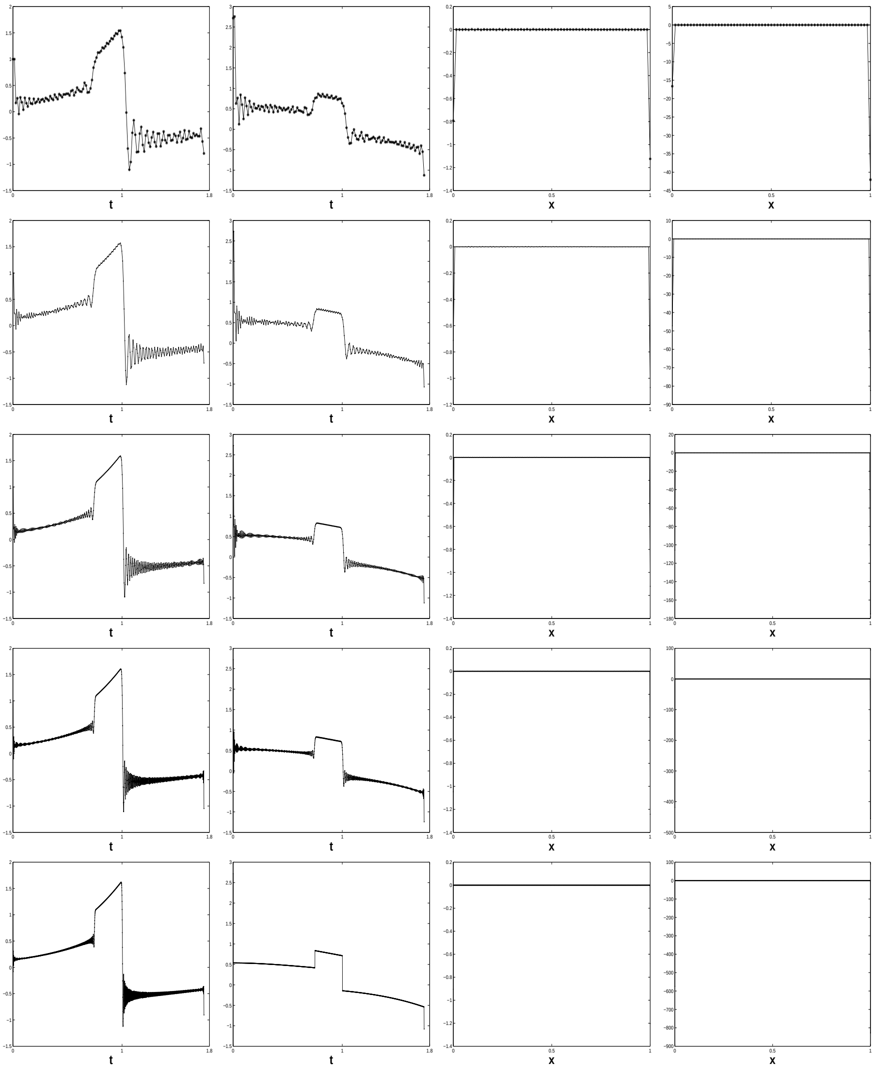

FIGURE 11. g_L, g_R, $u^h(x,T)$, and $u_t^h(x,T)$ from left to right with initial data (26-III). $h = 1/64, 1/128, 1/256, 1/512, 1/1024$ from top to bottom respectively. $\lambda = 7/8$.

The results of our computational experiments are summarized in Table 1, where g^h are the computed approximations of the exact solutions g. All norms were calculated by linearly interpolating the nodal values of g^h. From this table, it seems that g^h converges to g in the $L^2(\Sigma)$-norm at a rate of roughly 1. In order to visualize the convergence of our method as h becomes smaller, in Figure 3 we provide plots of the exact solution u and the corresponding computed discrete solutions u^h for $h = 1/256$ with $\lambda = 1$. Figure 4 and 5 are plots of the exact solution g and the corresponding computed discrete solutions g^h, a given function W and approximate solutions u^h, and a given function Z and approximate solutions u_t^h for $h = 1/16$, $h = 1/32$, $h = 1/64$, and $h = 1/1024$.

It seems that our method produces (pointwise) convergent approximations for both $\lambda = 1$ and $\lambda = 7/8$ *without the need for regularization.* This should be

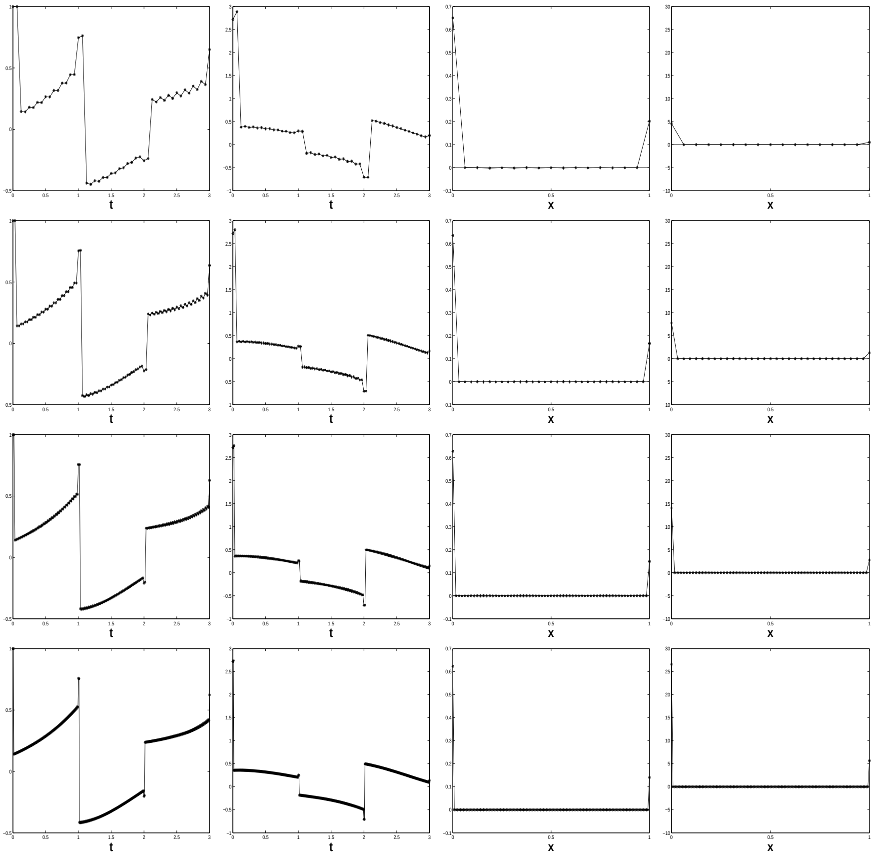

FIGURE 12. g_L, g_R, $u^h(x,T)$, and $u^h_t(x,T)$ from left to right with $f(u) = \sin u$ and initial data (26-III). $h = 1/16, 1/32, 1/64, 1/128$ from top to bottom respectively. $\lambda = 1$.

contrasted with other methods, e.g., that of [6], for which when $\lambda < 1$, regularization was needed in order to obtain convergence. Also, the results obtained by our method behave very similarly to those obtained in [12].

5.2. Generic examples with minimum $L^2(\Sigma)$-norm boundary control. In the example discussed in Section 5.1, the minimum $L^2(\Sigma)$-norm control is very smooth. Using our methods, we obtained good approximations for this example without the need for regularization. However, this is not the generic case. In general, even for smooth initial data, the minimum $L^2(\Sigma)$-norm Dirichlet control for the controllability problem (1) will not be smooth. In this section, we illustrate this point and also examine the performance of our method for the generic case.

We choose $Q = (0,1) \times (0,1)$ in example I and $Q = (0,7/4) \times (0,1)$ in example II, III, and consider three sets of $C^\infty(\overline{\Omega})$ initial data:

$$\begin{array}{rlcl} \text{I.} & u_0(x) = x(x-1) & \text{and} & u_1(x) = 0 \\ \text{II.} & u_0(x) = \sin(\pi x) & \text{and} & u_1(x) = \pi \sin(\pi x) \\ \text{III.} & u_0(x) = e^x & \text{and} & u_1(x) = xe^x. \end{array} \tag{26}$$

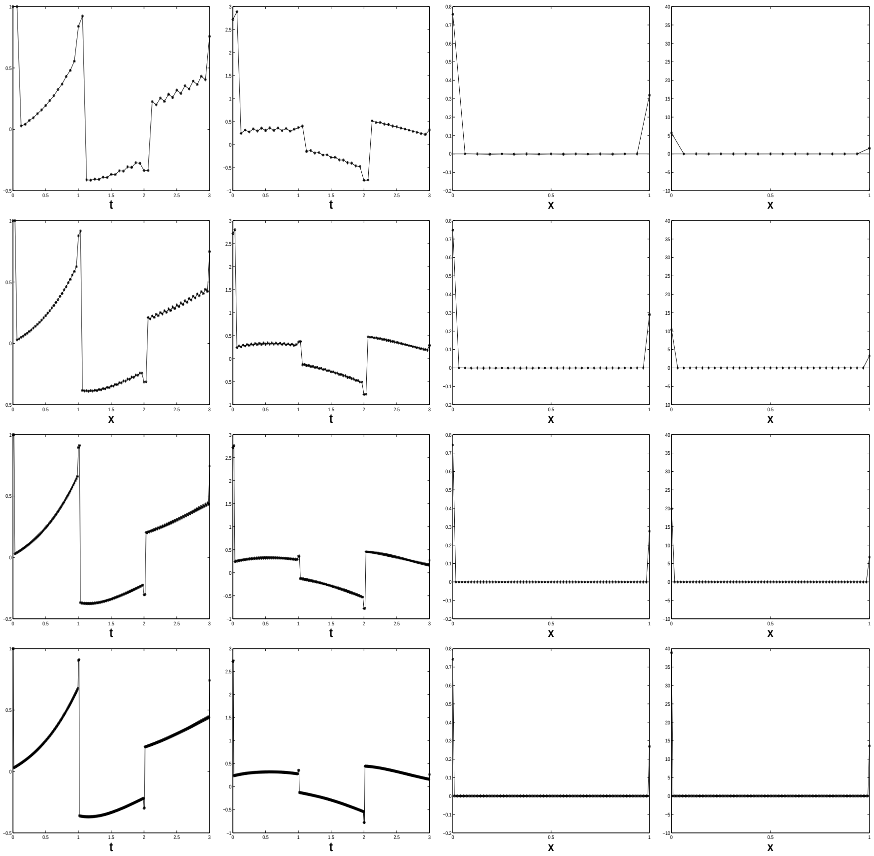

FIGURE 13. g_L, g_R, $u^h(x,T)$, and $u_t^h(x,T)$ from left to right with $f(u) = u^{3/2}$ and initial data (26-III). $h = 1/16, 1/32, 1/64, 1/128$ from top to bottom respectively. $\lambda = 1$.

Note that the initial conditions (I), (II) vanish at the boundary and, that due to symmetry, we have that $u(t,0) = u(t,1)$, i.e., the control at the two sides of Q are the same. For the initial conditions (III), we have that $u(t,0) \neq u(t,1)$.

First We examine the case $\lambda = 1$. In Figure 6, 8 and 10, we show the results for the control for several grid sizes ranging from $h = 1/16$ to $h = 1/1024$. The (pointwise) convergence of the approximations is evident. Note that for the initial conditions given in (26), the minimum $L^2(\Sigma)$-norm controls are seemingly piecewise smooth, i.e., they contain jump discontinuities. The pointwise convergence of the approximate control for the case of $\lambda = 1$ is probably a one-dimensional artifact; it is likely due to the fact that both the space and time variables in the wave equation in one dimension can act as time-like variables.

Further details about the computational results for the examples with initial conditions (I) given in (26) with $\lambda = 4/5$ are given in Table 2 and Figure 7. The convergence in $L^2(\Sigma)$ of the approximate minimum $L^2(\Sigma)$-norm controls g^h is evident as is the convergence in $L^2(Q)$ of the approximate solution u^h; the rates of convergence are seemingly first order.

Computational experiments were also carried out for $\lambda = 7/8$ for several values of the grid size ranging from $h = 1/16$ to $h = 1/1024$. The results are summarized

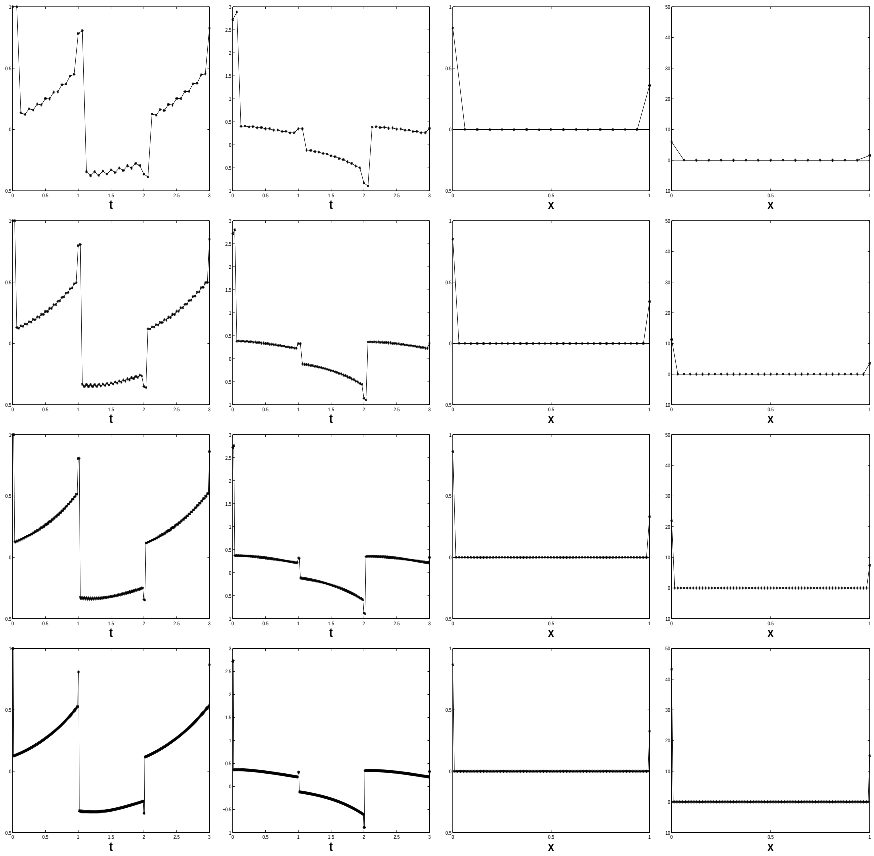

FIGURE 14. g_L, g_R, $u^h(x,T)$, and $u_t^h(x,T)$ from left to right with $f(u) = \ln(u^2+1)$ and initial data (26-III). $h = 1/16, 1/32, 1/64, 1/128$ from top to bottom respectively. $\lambda = 1$.

in Table 3 and 4. In Figures 9 and 11, we respectively provide, for the two sets of initial conditions (II) and (III), plots of the computed discrete solution g^h, u^h, u_t^h for the two different values of λ and for different values of the grid size.

From Figures 9 and 11, we see that the approximate minimum $L^2(\Sigma)$-norm Dirichlet controls obtained with values of $\lambda < 1$ are highly oscillatory. In fact, the frequencies of the oscillations increase with decreasing grid size. However, it seems that the amplitudes of the oscillations do not increase as the grid size decreases. Furthermore, from the results in Table 3 and 4, it seems that for $\lambda < 1$, the approximate controls g^h, although oscillatory in nature and nonconvergent in a pointwise sense, converge in an $L^2(\Sigma)$ sense.

The results of Table 2, 3, 4 and Figures 6, 7, 8, 9, 10, 11 indicate that for the generic case of non-smooth minimum $L^2(\Sigma)$ controls and for general $\lambda < 1$, our method produces convergent (in $L^2(Q)$ and $L^2(\Sigma)$) approximations *without the need of regulatization* but the approximations are not in general convergent in a pointwise sense. Of course, approximations that do not converge in a pointwise sense may be of little practical use, even if they converge in a root mean square sense.

6. Computational experiments for controllability of semilinear wave equations

We will again apply Algorithm 1 to the special case of $V = 0$, $W = 0$, $Z = 0$, $\alpha = 0$ and $\sigma, \tau >> 1$. We will test our algorithm with generic examples. If nonlinear term f satisfies a certain property such as asymptotically linear or superlinear, then the exact control problem of the system 1 can be solvable;see, e.g., [4, 23, 24]. In this section, we examine the performance of our method for the asymptotically linear and superlinear cases.

We choose $Q = (0, 3) \times (0, 1)$ in example I, II, III, and consider three sets of nonlinear term f:

$$\begin{aligned} &\text{I.} && f(u) = \sin u \\ &\text{II.} && f(u) = u^{3/2} \\ &\text{III.} && f(u) = \ln(u^2 + 1). \end{aligned} \tag{27}$$

Note that we choose $T = 3$ for existence of control; see, e.g., [23, 24]. (I) is an example of the asymptotically linear case and (II) is one of the superlinear case. (III) can be considered as either case. In general, we can not expect $g_L = g_R$ due to the nonlinear terms. We test the case $\lambda = 1$. The numerical approximations by Algorithm 1 is convergent in L_2 sense, that is, they have jump discontinuities as well. We will illustrate those through the figures 12, 13, and 14. For the linear cases, the number of iterations of the shooting methods is about 2 or 3, according to the tolerance and the accuracy of the machines we used. However the nonlinear cases are different and we need more iterations than the linear cases. We denote the number of iterations as *count*. It is contained in the tables 5, 6 and 7 with $L^2(\Sigma)$-norm of controls g^h.

References

[1] R. Adams, Sobolev Spaces, Academic, New York, 1975.

[2] U. Ascher, R.M.M. Mattheij and R.D. Russell, Numerical Solution of Boundary Value Problems for Ordinary Differential Equations (1988), Prentice-Hall.

[3] W.H. Enright and K. Chow, Distributed parallel shooting for BVODEs, Proceedings of High Performance Computing 1998, A. Tentner, ed., The Society for Computer Simulation, Boston, 1998, pp. 203-210.

[4] O. Yu. Emanuilov, Boundary controllability of semilinear evolution equations, Russian Math. Surveys., 44 (1989) 183-184.

[5] R. Glowinski and C.-H. Li, On the numerical implementation of the Hilbert Uniqueness Method for the exact boundary controllability of the wave equation, C. R. Adad. Sc., Paris, T. 311, Serie I, 1990, pp. 135-142.

[6] R. Glowinski, C.-H. Li, and J.-L. Lions, A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: description of the numerical methods, Japan J. Appl. Math. 7 (1990) 1-76.

[7] M. Gunzburger, L.S. Hou and L.-L. Ju, A numerical method for exact boundary controllability problems for the wave equation, submitted to Control, Optimization and Calculus of Variations, 2002.

[8] M. Gunzburger and R. Nicolaides, An algorithm for the boundary control of the wave equation, Appl. Math. Lett. 2 (1989) 225-228.

[9] J. A. Infante and E. Zuazua, Boundary observability for the space semi-discretizations of the 1-D wave equation, Mathematical Modelling abd Numerical Analysis, 33 (1999) 407-438.

[10] K. Jackson, A Survey of Parallel Numerical Methods for Initial Value Problems for Ordinary Differential Equations, IEEE Transactions on Magnetics, 27 (1991) 3792-3797.

[11] K. Jackson and S. Norsett, The potential for parallelism in Runge-Kutta methods. Part 1: RK formulas in standard form, SIAM J. Numer. Anal., 32 (1995) 49-82.

[12] L.-L. Ju, M. Gunzburger and L.S. Hou, Approximation of exact boundary controllability problems for the 1-D wave equation by optimization-based methods, Contemp. Math., 330 (2003) 133–153.

[13] J.-L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York, 1971.
[14] J.-L. Lions, Control of Distributed Singular Systems, Bordas, Paris, 1985.
[15] J.-L. Lions, Exact controllability and stabilization and perturbations for distributed systems, SIAM Review, 30 (1988) 1-68.
[16] J.-L. Lions, Controlabilité Exacte, Perturbation et Stabilisation des Systèmes Destribués, Vol. 1 and 2, Masson, Paris, 1988.
[17] M. Negreanu and E. Zuazua, Uniform boundary controllability of a discrete 1-D wave equation, preprint, 2002.
[18] D. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, SIAM Review 20 (1978) 639-739.
[19] D. Russell, A unified boundary controllability theory, Studies in Appl. Math. 52 (1973) 189-211.
[20] D. Russell, Mathematics of Finite-Dimensional Control Systems – Theory and Design, Deckker, New York, 1979.
[21] J. W. Thomas, Numerical Partial Differential Equations – Finite Difference Methods, Springer, New York, 1998.
[22] E. Zuazua, Boundary observability for the finite difference space semi-discretizations of the 2-D wave equation in the square, J. Math. Pures Appl., 78 (1999) 523-563.
[23] E. Zuazua, Exact controllability for semilinear wave equations in one space dimension, Ann. Inst. H. Poincar Anal. Non Linaire., 10 (1993) 109-129.
[24] E. Zuazua, Exact controllability for the semilinear wave equation, J. Math. Pures Appl., 69 (1990) 1-31.

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