LOCAL AND PARALLEL FINITE ELEMENT ALGORITHM BASED ON MULTILEVEL DISCRETIZATION FOR EIGENVALUE PROBLEMS

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Abstract. In this paper, a local and parallel algorithm based on the multilevel discretization is proposed for solving the eigenvalue problem by the finite element method. With this new scheme, the eigenvalue problem solving in the finest grid is transferred to solutions of the eigenvalue problems on the coarsest mesh and a series of solutions of boundary value problems on each level mesh. Therefore this type of multilevel local and parallel method improves the overall efficiency of solving the eigenvalue problem. Some numerical experiments are presented to validate the efficiency of the new method.

Key words. eigenvalue problem, multigrid, multilevel correction, local and parallel method, finite element method.

1. Introduction

Solving large scale eigenvalue problems becomes a fundamental problem in modern science and engineering society. However, it is always a very difficult task to solve high-dimensional eigenvalue problems which arise from physical and chemistry sciences. Xu and Zhou [28] give a type of two-grid discretization method to improve the efficiency of the solution of eigenvalue problems. By the two-grid method, the solution of eigenvalue problem on a fine mesh is reduced to a solution of eigenvalue problem on a coarse mesh (which depends on the fine mesh) and a solution of the corresponding boundary value problem on the fine mesh [28]. For more details, please read [25, 26]. Combing the two-grid idea and the local and parallel finite element technique [27], a type of local and parallel finite element technique to solve the eigenvalue problems is given in [29] (also see [10]). Recently, a new type of multilevel correction method for solving eigenvalue problems which can be implemented on multilevel grids is proposed in [14]. In the multilevel correction scheme, the solution of eigenvalue problem on the finest mesh can be reduced to a series of solutions of the eigenvalue problem on a very coarse mesh (independent of the finest mesh) and a series of solutions of the boundary value problems on the multilevel meshes. The multilevel correction method gives a way to construct a type of multigrid scheme for the eigenvalue problem [15].

In this paper, we propose a type of multilevel local and parallel scheme to solve the eigenvalue problem based on the combination of the multilevel correction method and the local and parallel technique. The special property of this scheme

is that we can do the local and parallel computing for any level grids and the mesh size of the original coarse triangulation is independent of the finest triangulation. With this new method, the solution of the eigenvalue problem is not going to be more difficult than the solution of the boundary value problems by the local and parallel algorithm since the main part of the computation in the multilevel local and parallel method is solving the boundary value problems.

The standard Galerkin finite element method for eigenvalue problems has been extensively investigated, e.g. Babuška and Osborn [2, 3], Chatelin [8] and references cited therein. There also exists analysis for the local and parallel finite element method for the boundary value problems and eigenvalue problems [10, 21, 22, 27, 28, 29]. Here we adopt some basic results in these papers for our analysis. The corresponding error and computational work estimates of the proposed multilevel local and parallel scheme for the eigenvalue problem will be analyzed. Based on the analysis, the new method can obtain optimal errors with an optimal computational work in each processor.

An outline of this paper goes as follows. In the next section, a basic theory about the local error estimate of the finite element method is introduced. In Section 3, we introduce the finite element method for the eigenvalue problem and the corresponding error estimates. A local and parallel type of one correction step and multilevel correction algorithm will be given in Section 4. The estimate of the computational work for the multilevel local and parallel algorithm is presented in section 5. In Section 6, three numerical examples are presented to validate our theoretical analysis and some concluding remarks are given in the last section.

2. Discretization by the finite element method

In this section, we introduce some notation and error estimates of the finite element approximation of linear elliptic problems. The letter C (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences throughout the paper. For convenience, the symbols $\lesssim, \gtrsim$ and $\approx$ will be used in this paper. That $x_1 \lesssim y_1, x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh sizes (see, e.g., [24]). We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms (see, e.g., [1]). For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace, $\| \cdot \|_{s,\Omega} = \| \cdot \|_{s,2,\Omega}$.

For $G \subset D \subset \Omega$, the notation $G \subset\subset D$ means that $\text{dist}(\partial D \setminus \partial\Omega, \partial G \setminus \partial\Omega) > 0$ (see Figure 1). It is well known that any $w \in H_0^1(\Omega_0)$ can be naturally extended to be a function in $H_0^1(\Omega)$ with zero outside of Ω_0, where $\Omega_0 \subset \Omega$. Thus we will show this fact by the abused notation $H_0^1(\Omega_0) \subset H_0^1(\Omega)$.
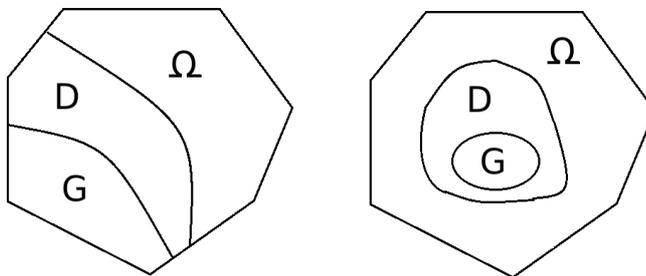
2.1. Finite element space. Now, let us define the finite element space. First we generate a shape-regular decomposition $\mathcal{T}_h(\Omega)$ of the computing domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h(\Omega)$ is denoted by h_K. The mesh size function is denoted by $h(x)$ whose value is the diameter h_K of the element K including x.

For generality, following [27, 29], we shall consider a class of finite element spaces that satisfy certain assumptions. Now we describe such assumptions.

A.0. There exists $\gamma > 1$ such that

$$h_\Omega^\gamma \lesssim h(x), \quad \forall x \in \Omega,$$

where $h_\Omega = \max_{x \in \Omega} h(x)$ is the largest mesh size of $\mathcal{T}_h(\Omega)$.

FIGURE 1. $G \subset\subset D \subset\subset \Omega$.

Based on the triangulation $\mathcal{T}_h(\Omega)$, we define the finite element space $V_h(\Omega)$ as follows

$$V_h(\Omega) = \big\{ v \in C(\bar{\Omega}) : \ v|_K \in \mathcal{P}_k, \ \ \forall K \in \mathcal{T}_h(\Omega) \big\},$$

where $\mathcal{P}_k$ denotes the space of polynomials of degree not greater than a positive integer k. Then we know $V_h(\Omega) \subset H^1(\Omega)$ and define $V_{0h}(\Omega) = V_h(\Omega) \cap H_0^1(\Omega)$. Given $G \subset \Omega$, we define $V_h(G)$ and $\mathcal{T}_h(G)$ to be the restriction of $V_h(\Omega)$ and $\mathcal{T}_h(\Omega)$ to G, respectively, and

$$V_{0h}(G) = \big\{ v \in V_h(\Omega) : \ \mathrm{supp} v \subset\subset G \big\}.$$

For any $G \subset \Omega$ mentioned in this paper, we assume that it aligns with the partition $\mathcal{T}_h(\Omega)$.

As we know, the finite element space V_h satisfy the following proposition (see, e.g., [7, 9, 27, 29]).

Proposition 2.1. *(Fractional Norm) For any $G \subset \Omega$, we have*

$$(1) \qquad \inf_{v \in V_{0h}(G)} \|w - v\|_{1,G} \lesssim \|w\|_{1/2, \partial G}, \quad \forall w \in V_h(\Omega).$$

2.2. A linear elliptic problem. In this subsection, we repeat some basic properties of a second order elliptic boundary value problem and its finite element discretization, which will be used in this paper. The following results is presented in [21, 22, 27, 29].

We consider the homogeneous boundary value problem

$$(2) \qquad \begin{cases} Lu &= f, \ \ \text{in } \Omega, \\ u &= 0, \ \ \text{on } \partial\Omega. \end{cases}$$

Here the linear second order elliptic operator $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ is define as

$$Lu = -\mathrm{div}(A\nabla u),$$

where $A = (a_{ij})_{1 \le i,j \le d} \in \mathbb{R}^{d \times d}$ is uniformly positive definite symmetric on Ω with $a_{ij} \in W^{1,\infty}(\Omega)$. The weak form for (2) is as follows:

Find $u \equiv L^{-1}f \in H_0^1(\Omega)$ such that

$$(3) \qquad a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega),$$

where $(\cdot, \cdot)$ is the standard inner-product of $L^2(\Omega)$ and

$$a(u, v) = (A\nabla u, \nabla v).$$

Uniform positive definiteness of the coeffcent matrices implies

$$\|w\|_{1,\Omega}^2 \lesssim a(w, w), \quad \forall w \in H_0^1(\Omega).$$

We assume (c.f. [12]) that the following regularity estimate holds for the solution of (2) or (3)

$$\|u\|_{1+\alpha,\Omega} \lesssim \|f\|_{-1+\alpha,\Omega}$$

for some $\alpha \in (0,1]$ depending on Ω and the coefficient of L.

For the analysis, we define the Galerkin-Projection operator $P_h : H_0^1(\Omega) \to V_{0h}(\Omega)$ by

(4) $$a(u - P_h u, v) = 0, \quad \forall v \in V_{0h}(\Omega)$$

and apparently

(5) $$\|P_h u\|_{1,\Omega} \lesssim \|u\|_{1,\Omega}, \quad \forall u \in H_0^1(\Omega).$$

Based on (5), the global priori error estimate can be obtained from the approximate properties of the finite dimensional subspace $V_{0h}(\Omega)$ (cf. [7, 9]). For the following analysis, we introduce the following quantity:

(6) $$\rho_\Omega(h) = \sup_{f \in L^2(\Omega), \|f\|_{0,\Omega}=1} \inf_{v \in V_{0h}(\Omega)} \|L^{-1}f - v\|_{1,\Omega}.$$

The following results can be found in [3, 7, 9, 28, 29].

Proposition 2.2.

$$\|(I - P_h)L^{-1}f\|_{1,\Omega} \lesssim \rho_\Omega(h)\|f\|_{0,\Omega}, \quad \forall f \in L^2(\Omega),$$
$$\|u - P_h u\|_{0,\Omega} \lesssim \rho_\Omega(h)\|u - P_h u\|_{1,\Omega}, \quad \forall u \in H_0^1(\Omega).$$

Now, we state an important and useful result about the local error estimates [21, 22, 29] which will be used in the following.

Proposition 2.3. *Suppose that $f \in H^{-1}(\Omega)$ and $G \subset\subset \Omega_0 \subset \Omega$. If Assumptions A.0 holds and $w \in V_h(\Omega_0)$ satisfies*

$$a(w, v) = (f, v), \quad \forall v \in V_{0h}(\Omega_0).$$

Then we have the following estimate

$$\|w\|_{1,G} \lesssim \|w\|_{0,\Omega_0} + \|f\|_{-1,\Omega_0}.$$

3. Error estimates for eigenvalue problems

In this section, we introduce the concerned eigenvalue problem and the corresponding finite element discretization.

In this paper, we consider the following eigenvalue problem:

Find $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ such that $b(u, u) = 1$ and

(7) $$a(u, v) = \lambda b(u, v), \quad \forall v \in H_0^1(\Omega),$$

where

$$b(u, u) = (u, u).$$

For the eigenvalue λ, there exists the following Rayleigh quotient expression (see, e.g., [2, 3, 28])

$$\lambda = \frac{a(u, u)}{b(u, u)}.$$

From [3, 8], we know the eigenvalue problem (7) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \cdots, u_k, \cdots,$$

where $b(u_i, u_j) = \delta_{ij}$. In the sequence $\{\lambda_j\}$, the λ_j are repeated according to their multiplicity.

Then we can define the discrete approximation for the exact eigenpair (λ, u) of (7) based on the finite element space as:

Find $(\lambda_h, u_h) \in \mathbb{R} \times V_{0h}(\Omega)$ such that $b(u_h, u_h) = 1$ and

$$(8) \qquad a(u_h, v_h) \quad = \quad \lambda_h b(u_h, v_h), \quad \forall v_h \in V_{0h}(\Omega).$$

From (8), we know the following Rayleigh quotient expression for λ_h holds (see, e.g., [2, 3, 28])

$$\lambda_h \quad = \quad \frac{a(u_h, u_h)}{b(u_h, u_h)}.$$

Similarly, we know from [3, 8] the eigenvalue problem (8) has eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{k,h} \leq \cdots \leq \lambda_{N_h,h},$$

and the corresponding eigenfunctions

$$u_{1,h}, u_{2,h}, \cdots, u_{k,h}, \cdots, u_{N_h,h},$$

where $b(u_{i,h}, u_{j,h}) = \delta_{ij}, 1 \leq i, j \leq N_h$ (N_h is the dimension of the finite element space $V_{0h}(\Omega)$).

From the minimum-maximum principle (see, e.g., [2, 3]), the following upper bound result holds

$$\lambda_i \leq \lambda_{i,h}, \quad i = 1, 2, \cdots, N_h.$$

Let $M(\lambda_i)$ denote the unit set in the eigenspace corresponding to the eigenvalue λ_i which is defined by

$$M(\lambda_i) \quad = \quad \big\{ w \in H_0^1(\Omega) : w \text{ is an eigenvalue of (7) corresponding to } \lambda_i$$

$$(9) \qquad\qquad \text{and } \|w\|_b = 1 \big\},$$

where $\|w\|_b = \sqrt{b(w, w)}$. Then we define

$$(10) \qquad\qquad \delta_h(\lambda_i) = \sup_{w \in M(\lambda_i)} \inf_{v \in V_{0h}(\Omega)} \|w - v\|_1.$$

For the eigenpair approximations by the finite element method, there exist the following error estimates.

Proposition 3.1. *([2, Lemma 3.7, (3.28b,3.29b)], [3, P. 699] and [8])*
(i) For any eigenfunction approximation $u_{i,h}$ of (8) $(i = 1, 2, \cdots, N_h)$, there is an eigenfunction u_i of (7) corresponding to λ_i such that $\|u_i\|_b = 1$ and

$$\|u_i - u_{i,h}\|_{1,\Omega} \quad \leq \quad C_i \delta_h(\lambda_i).$$

Furthermore,

$$\|u_i - u_{i,h}\|_{0,\Omega} \quad \leq \quad C_i \rho_\Omega(h) \delta_h(\lambda_i).$$

(ii) For each eigenvalue, we have

$$\lambda_i \leq \lambda_{i,h} \leq \lambda_i + C_i \delta_h^2(\lambda_i).$$

Here and hereafter C_i is some constant depending on i but independent of the mesh size h.

To analyze our method, we introduce the error representation of eigenvalue by the Rayleigh quotient formula which comes from [2, 3, 13, 28].

Proposition 3.2. *Assume (λ, u) is the true solution of the eigenvalue problem (7) and $0 \neq \psi \in H_0^1(\Omega)$. Let us define*

$$\widehat{\lambda} = \frac{a(\psi, \psi)}{b(\psi, \psi)}.$$

Then we have

$$\widehat{\lambda} - \lambda \quad = \quad \frac{a(u - \psi, u - \psi)}{b(\psi, \psi)} - \lambda \frac{b(u - \psi, u - \psi)}{b(\psi, \psi)}.$$

4. Multilevel local and parallel algorithms

In this section, we present a new multilevel parallel algorithm to solve the eigenvalue problem based on the combination of the local and parallel finite element technique and the multilevel correction method. First, we introduce an one correction step with the local and parallel finite element scheme and then present a parallel multilevel method for the eigevalue problem.

For the description of the numerical scheme, we need to define some notation. Given an coarsest triangulation $\mathcal{T}_H(\Omega)$, we first divide the domain Ω into a number of disjoint subdomains $D_1, \cdots, D_m$ such that $\bigcup_{j=1}^m \bar{D}_j = \bar{\Omega}$, $D_i \cap D_j = \emptyset$, then enlarge and reduce each D_j to obtain Ω_j and G_j respectly, which both align with $\mathcal{T}_H(\Omega)$. We get the sequence of subdomains $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$ for $i = 1, \cdots, m$ and $G_{m+1} = \Omega \setminus (\cup_{j=1}^m \bar{G}_j)$ (see Figure 2).
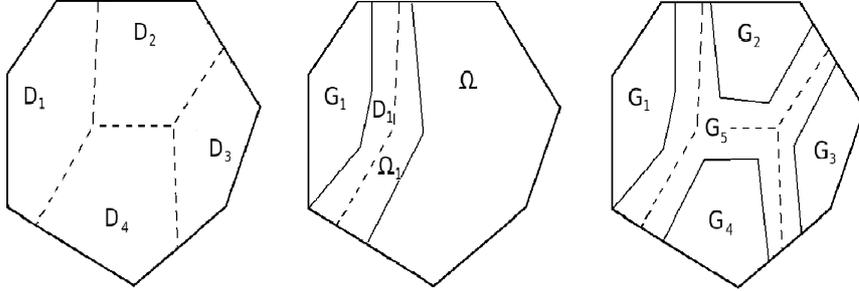

FIGURE 2. the left – $\bigcup_{j=1}^4 \bar{D}_j = \bar{\Omega}$; the middle – $G_1 \subset\subset D_1 \subset \Omega_1$; the right – $G_5 = \Omega \setminus (\cup_{j=1}^4 \bar{G}_j)$.

In this paper we assume the domain decomposition satisfies the following property

$$(11) \qquad \sum_{j=1}^m \|v\|_{\ell,\Omega_j}^2 \lesssim \|v\|_{\ell,\Omega}^2$$

for any $v \in H^\ell(\Omega)$ with $\ell = 0, \ 1$.

4.1. One correction step. First, we present the one correction step to improve the accuracy of the given eigenvalue and eigenfunction approximation. This correction method contains solving an auxiliary boundary value problem in the finer finite element space on each subdomain and an eigenvalue problem on the coarsest finite element space.

For simplicity of notation, we set $(\lambda, u) = (\lambda_i, u_i)$ $(i = 1, 2, \cdots, k, \cdots)$ and $(\lambda_h, u_h) = (\lambda_{i,h}, u_{i,h})$ $(i = 1, 2, \cdots, N_h)$ to denote an eigenpair and its corresponding approximation of problems (7) and (8), respectively. For the clear understanding, we only describe the algorithm for the simple eigenvalue case. The corresponding algorithm for the multiple eigenvalue case can be given in the similar way as in [23].

In order to do the correction step, we build original coarsest finite element space $V_{0H}(\Omega)$ on the background mesh $\mathcal{T}_H(\Omega)$. This coarsest finite element space $V_{0H}(\Omega)$ will be used as the background space in our algorithm.

Assume we have obtained an eigenpair approximation $(\lambda_{h_k}, u_{h_k}) \in \mathbb{R} \times V_{0h_k}(\Omega)$. The one correction step will improve the accuracy of the current eigenpair approximation (λ_{h_k}, u_{h_k}). Let $V_{0h_{k+1}}(\Omega)$ ba a finer finite element space such that $V_{0h_k}(\Omega) \subset V_{0h_{k+1}}(\Omega)$. Here we assume the finite element spaces $V_{0h_k}(\Omega)$ and $V_{0h_{k+1}}(\Omega)$ are consistent with the domain decomposition and $V_{0H}(\Omega) \subset V_{0h_k}(\Omega)$. Based on this finer finite element space $V_{0h_{k+1}}(\Omega)$, we define the following one correction step.

Algorithm 4.1. *One Correction Step*

We have a given eigenpair approximation $(\lambda_{h_k}, u_{h_k}) \in \mathbb{R} \times V_{0h_k}(\Omega)$.

(1) *Define the following auxiliary boundary value problem:*

For each $j = 1, 2, \cdots, m$, find $e_{h_{k+1}}^j \in V_{0h_{k+1}}(\Omega_j)$ such that

$$(12) \quad a(e_{h_{k+1}}^j, v_{h_{k+1}}) = \lambda_{h_k} b(u_{h_k}, v_{h_{k+1}}) - a(u_{h_k}, v_{h_{k+1}}), \quad \forall v_{h_{k+1}} \in V_{0h_{k+1}}(\Omega_j).$$

Set $\widetilde{u}_{h_{k+1}}^j = u_{h_k} + e_{h_{k+1}}^j \in V_{h_{k+1}}(\Omega_j)$.

(2) *Construct $\widetilde{u}_{h_{k+1}} \in V_{0h_{k+1}}(\Omega)$ such that $\widetilde{u}_{h_{k+1}} = \widetilde{u}_{h_{k+1}}^j$ in G_j $(j = 1, \cdots, m)$ and $\widetilde{u}_{h_{k+1}} = \widetilde{u}_{h_{k+1}}^{m+1}$ in G_{m+1} with $\widetilde{u}_{h_{k+1}}^{m+1}$ being defined by solving the following problem:*

Find $\widetilde{u}_{h_{k+1}}^{m+1} \in V_{h_{k+1}}(G_{m+1})$ such that $\widetilde{u}_{h_{k+1}}^{m+1}|_{\partial G_j \cap \partial G_{m+1}} = \widetilde{u}_{h_{k+1}}^j$ $(j = 1, \cdots, m)$ and

$$(13) \quad a(\widetilde{u}_{h_{k+1}}^{m+1}, v_{h_{k+1}}) = \lambda_{h_k} b(u_{h_k}, v_{h_{k+1}}), \quad \forall v_{h_{k+1}} \in V_{0h_{k+1}}(G_{m+1}).$$

(3) *Define a new finite element space $V_{H,h_{k+1}} = V_{0H}(\Omega) + \text{span}\{\widetilde{u}_{h_{k+1}}\}$ and solve the following eigenvalue problem:*

Find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathbb{R} \times V_{H,h_{k+1}}$ such that $b(u_{h_{k+1}}, u_{h_{k+1}}) = 1$ and

$$(14) \quad a(u_{h_{k+1}}, v_{H,h_{k+1}}) = \lambda_{h_{k+1}} b(u_{h_{k+1}}, v_{H,h_{k+1}}), \quad \forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}.$$

Summarize the above three steps into

$$(\lambda_{h_{k+1}}, u_{h_{k+1}}) = Correction(V_{0H}(\Omega), \lambda_{h_k}, u_{h_k}, V_{0h_{k+1}}(\Omega)),$$

where λ_{h_k} and u_{h_k} are the given eigenvalue and eigenfunction approximation, respectively.

Theorem 4.1. *Assume the current eigenpair approximation $(\lambda_{h_k}, u_{h_k}) \in \mathbb{R} \times V_{0h_k}(\Omega)$ has the following error estimates*

$$(15) \qquad\qquad \|u - u_{h_k}\|_{1,\Omega} \lesssim \varepsilon_{h_k}(\lambda),$$

$$(16) \qquad\qquad \|u - u_{h_k}\|_{0,\Omega} \lesssim \rho_\Omega(H)\varepsilon_{h_k}(\lambda),$$

$$(17) \qquad\qquad |\lambda - \lambda_{h_k}| \lesssim \varepsilon_{h_k}^2(\lambda).$$

Then after one step correction, the resultant approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathbb{R} \times V_{0h_{k+1}}(\Omega)$ has the following error estimates

$$(18) \qquad\qquad \|u - u_{h_{k+1}}\|_{1,\Omega} \lesssim \varepsilon_{h_{k+1}}(\lambda),$$

$$(19) \qquad \|u - u_{h_{k+1}}\|_{0,\Omega} \;\lesssim\; \rho_\Omega(H)\varepsilon_{h_{k+1}}(\lambda),$$

$$(20) \qquad |\lambda - \lambda_{h_{k+1}}| \;\lesssim\; \varepsilon_{h_{k+1}}^2(\lambda),$$

where $\varepsilon_{h_{k+1}}(\lambda) := \rho_\Omega(H)\varepsilon_{h_k}(\lambda) + \varepsilon_{h_k}^2(\lambda) + \delta_{h_{k+1}}(\lambda)$.

Proof. We focus on estimating $\|u - \widetilde{u}_{h_{k+1}}\|_{1,\Omega}$. First, we have

$$(21) \qquad \|u - \widetilde{u}_{h_{k+1}}\|_{1,\Omega} \le \|u - P_{h_{k+1}}u\|_{1,\Omega} + \|\widetilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,\Omega},$$

and

$$(22) \quad \|\widetilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,\Omega}^2 \lesssim \sum_{j=1}^{m} |\widetilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u|_{1,G_j}^2 + |\widetilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u|_{1,G_{m+1}}^2.$$

From problems (4), (7) and (12), the following equation holds

$$a(\widetilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u, v) = b(\lambda_{h_k}u_{h_k} - \lambda u, v), \quad \forall v \in V_{0h_{k+1}}(\Omega_j),$$

for $j = 1, 2, \cdots, m$. According to Proposition 2.3

$$\|\widetilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u\|_{1,G_j} \lesssim \|\widetilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u\|_{0,\Omega_j} + \|\lambda_{h_k}u_{h_k} - \lambda u\|_{-1,\Omega_j}$$

$$(23) \quad \lesssim \quad \|\widetilde{u}_{h_{k+1}}^j - u_{h_k}\|_{0,\Omega_j} + \|u_{h_k} - P_{h_{k+1}}u\|_{0,\Omega_j} + \|\lambda_{h_k}u_{h_k} - \lambda u\|_{0,\Omega_j}.$$

We will estimate the first term, i.e. $\|e_{h_{k+1}}^j\|_{0,\Omega_j}$ by using the Aubin-Nitsche duality argument.

Given any $\phi \in L^2(\Omega_j)$, there exists $w^j \in H_0^1(\Omega_j)$ such that

$$a(v, w^j) \;=\; b(v, \phi), \quad \forall v \in H_0^1(\Omega_j).$$

Let $w_{h_{k+1}}^j \in V_{0h_{k+1}}(\Omega_j)$ and $w_H^j \in V_{0H}(\Omega_j)$ satisfying

$$a(v_{h_{k+1}}, w_{h_{k+1}}^j) \;=\; a(v_{h_{k+1}}, w^j), \quad \forall v_{h_{k+1}} \in V_{0h_{k+1}}(\Omega_j),$$

$$a(v_H, w_H^j) \;=\; a(v_H, w^j), \qquad \forall v_H \in V_{0H}(\Omega_j).$$

Then the following equations hold

$$b(\widetilde{u}_{h_{k+1}}^j - u_{h_k}, \phi) = a(\widetilde{u}_{h_{k+1}}^j - u_{h_k}, w_{h_{k+1}}^j)$$

$$= \quad b(\lambda_{h_k}u_{h_k}, w_{h_{k+1}}^j) - a(u_{h_k}, w_{h_{k+1}}^j)$$

$$= \quad b(\lambda_{h_k}u_{h_k} - \lambda u, w_{h_{k+1}}^j) + a(P_{h_{k+1}}u - u_{h_k}, w_{h_{k+1}}^j)$$

$$= \quad b(\lambda_{h_k}u_{h_k} - \lambda u, w_{h_{k+1}}^j - w_H^j) + b(\lambda_{h_k}u_{h_k} - \lambda u, w_H^j)$$

$$\qquad + a(P_{h_{k+1}}u - u_{h_k}, w_{h_{k+1}}^j)$$

$$(24) \qquad = \quad b(\lambda_{h_k}u_{h_k} - \lambda u, w_{h_{k+1}}^j - w_H^j) + a(P_{h_{k+1}}u - u_{h_k}, w_{h_{k+1}}^j - w_H^j),$$

where $V_{0H}(\Omega) \subset V_{0h_k}(\Omega)$ and (4), (7), (8), (12) are used in the last equation.

Combining (24) and the following standard finite element methods error estimates [7, 9]

$$\|w - w_{h_{k+1}}^j\|_{1,\Omega_j} \lesssim \rho_{\Omega_j}(h_{k+1})\|\phi\|_{0,\Omega_j}, \quad \|w - w_H^j\|_{1,\Omega_j} \lesssim \rho_{\Omega_j}(H)\|\phi\|_{0,\Omega_j},$$

we have

$$(25) \quad \|\widetilde{u}_{h_{k+1}}^j - u_{h_k}\|_{0,\Omega_j} \lesssim \rho_{\Omega_j}(H)\big(\|u_{h_k} - P_{h_{k+1}}u\|_{1,\Omega_j} + \|\lambda_{h_k}u_{h_k} - \lambda u\|_{0,\Omega_j}\big).$$

From (23) and (25), for $j = 1, 2 \ldots, m$, we have

$$\|\widetilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u\|_{1,G_j} \lesssim \rho_{\Omega_j}(H)\|u_{h_k} - P_{h_{k+1}}u\|_{1,\Omega_j}$$

$$(26) \qquad\qquad + \|u_{h_k} - P_{h_{k+1}}u\|_{0,\Omega_j} + \|\lambda_{h_k}u_{h_k} - \lambda u\|_{0,\Omega_j}.$$

Now, we estimate $|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u|_{1,G_{m+1}}$. From (4), (7) and (13), we obtain

$$a(\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u, v) = b(\lambda_{h_k}u_{h_k} - \lambda u, v), \quad \forall v \in V_{0h_{k+1}}(G_{m+1}).$$

For any $v \in V_{0h_{k+1}}(G_{m+1})$, the following estimates hold

$$|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u|^2_{1,G_{m+1}}$$
$$\lesssim \ a(\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u, \widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u - v) + b(\lambda_{h_k}u_{h_k} - \lambda u, v)$$
$$\lesssim \ \|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,G_{m+1}} \inf_{\chi \in V_{0h_{k+1}}(G_{m+1})} \|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u - \chi\|_{1,G_{m+1}}$$
$$+ \|\lambda_{h_k}u_{h_k} - \lambda u\|_{-1,G_{m+1}} \big(\|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,G_{m+1}}$$
$$\text{(27)} \qquad + \inf_{\chi \in V_{0h_{k+1}}(G_{m+1})} \|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u - \chi\|_{1,G_{m+1}} \big).$$

Combining (27), the following estimate (by the definition of the norm of $H^{1/2}(\partial G_{m+1})$)

$$\|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1/2,\partial G_{m+1}} \ \lesssim \ \sum_{j=1}^{m} \|\widetilde{u}^{j}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1/2,\partial G_j}$$
$$\lesssim \ \sum_{j=1}^{m} \|\widetilde{u}^{j}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1,G_j},$$

Proposition 2.1, Friedrichs inequality, trace inequality and Cauchy-Schwarz inequality, we have

$$|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u|^2_{1,G_{m+1}}$$
$$\lesssim \ \|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1,G_{m+1}} + \inf_{\chi \in V_{0h_{k+1}}(G_{m+1})} \|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u - \chi\|^2_{1,G_{m+1}}$$
$$+ \|\lambda_{h_k}u_{h_k} - \lambda u\|^2_{-1,G_{m+1}}$$
$$\lesssim \ \|\widetilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{0,\partial G_{m+1}} + \|\widetilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1/2,\partial G_{m+1}}$$
$$+ \|\lambda_{h_k}u_{h_k} - \lambda u\|^2_{0,G_{m+1}}$$
$$\text{(28)} \ \lesssim \ \sum_{j=1}^{m} \|\widetilde{u}^{j}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1,G_j} + \|\lambda_{h_k}u_{h_k} - \lambda u\|^2_{0,G_{m+1}}.$$

Combining (11), (22), (26) and (28) leads to

$$\|\widetilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1,\Omega}$$
$$\lesssim \ \sum_{j=1}^{m} \rho_{\Omega_j}(H)^2 \|u_{h_k} - P_{h_{k+1}}u\|_{1,\Omega_j} + \sum_{j=1}^{m+1} \|u_{h_k} - P_{h_{k+1}}u\|^2_{0,\Omega_j}$$
$$+ \sum_{j=1}^{m} \|\lambda_{h_k}u_{h_k} - \lambda u\|^2_{0,\Omega_j} + \|\lambda_{h_k}u_{h_k} - \lambda u\|^2_{0,G_{m+1}}$$
$$\lesssim \ \rho^2_\Omega(H)\|u_{h_k} - P_{h_{k+1}}u\|^2_{1,\Omega} + \|u_{h_k} - P_{h_{k+1}}u\|^2_{0,\Omega} + \|\lambda_{h_k}u_{h_k} - \lambda u\|^2_{0,\Omega}$$
$$\lesssim \ \rho^2_\Omega(H)\|u_{h_k} - u\|^2_{1,\Omega} + \rho^2_\Omega(H)\|u - P_{h_{k+1}}u\|^2_{1,\Omega} + \|u_{h_k} - u\|^2_{0,\Omega}$$
$$+ \|u - P_{h_{k+1}}u\|^2_{0,\Omega} + |\lambda - \lambda_{h_k}|^2\|u\|^2_{0,\Omega} + \lambda^2\|u_{h_k} - u\|^2_{0,\Omega}.$$

Together with the error estimate of the finite element projection

$$\|u - P_{h_{k+1}}u\|_{1,\Omega} \ \lesssim \ \delta_{h_{k+1}}(\lambda)$$

and (17), (21), we have

$$
\begin{aligned}
\|u - \widetilde{u}_{h_{k+1}}\|_{1,\Omega} \quad &\lesssim \quad \|u - P_{h_{k+1}}u\|_{1,\Omega} + |\lambda - \lambda_{h_k}| + \|u - u_{h_k}\|_{0,\Omega} \\
&\qquad + \rho_\Omega(H)\|u - u_{h_k}\|_{1,\Omega}
\end{aligned}
$$

$$
(29) \qquad\qquad\qquad \lesssim \quad \rho_\Omega(H)\varepsilon_{h_k}(\lambda) + \varepsilon_{h_k}^2(\lambda) + \delta_{h_{k+1}}(\lambda).
$$

We come to estimate the error for the eigenpair solution $(\lambda_{h_{k+1}}, u_{h_{k+1}})$ of problem (14). Based on the error estimate theory of eigenvalue problems by finite element methods (see, e.g., Proposition 3.1 or [3, Theorem 9.1]) and the definition of the space $V_{H,h_{k+1}}$, the following estimates hold

$$
(30) \qquad \|u - u_{h_{k+1}}\|_{1,\Omega} \quad \lesssim \quad \sup_{w \in M(\lambda)} \inf_{v \in V_{H,h_{k+1}}} \|w - v\|_{1,\Omega} \lesssim \|u - \widetilde{u}_{h_{k+1}}\|_{1,\Omega},
$$

and

$$
\|u - u_{h_{k+1}}\|_{0,\Omega} \quad \lesssim \quad \widetilde{\rho}_\Omega(H)\|u - u_{h_{k+1}}\|_{1,\Omega},
$$

where

$$
\widetilde{\rho}_\Omega(H) \quad = \quad \sup_{f \in V, \|f\|_{0,\Omega}=1} \inf_{v \in V_{H,h_{k+1}}} \|L^{-1}f - v\|_{1,\Omega} \le \rho_\Omega(H).
$$

So we obtain the desired result (18), (19) and the estimate (20) can be obtained by Proposition 3.2 and (18).   □

4.2. Multilevel correction process. Now we introduce a type of multilevel local and parallel scheme based on the one correction step defined in Algorithm 4.1. This type of multilevel method can obtain the same optimal error estimate as solving the eigenvalue problem directly in the finest finite element space.

In order to do multilevel local and parallel scheme, we define a sequence of triangulations $\mathcal{T}_{h_k}(\Omega)$ of Ω determined as follows. Suppose $\mathcal{T}_{h_1}(\Omega)$ is obtained from $\mathcal{T}_H(\Omega)$ by the regular refinement and let $\mathcal{T}_{h_k}(\Omega)$ be obtained from $\mathcal{T}_{h_{k-1}}(\Omega)$ via regular refinement (produce β^d congruent elements) such that

$$
h_k \approx \frac{1}{\beta} h_{k-1} \qquad \text{for } k \ge 2.
$$

Based on this sequence of meshes, we construct the corresponding linear finite element spaces such that for each $j = 1, 2, \cdots, m$

$$
V_{0H}(\Omega_j) \subset V_{0h_1}(\Omega_j) \subset V_{0h_2}(\Omega_j) \subset \cdots \subset V_{0h_n}(\Omega_j)
$$

and the following relation of approximation errors holds

$$
(31) \qquad\qquad \delta_{h_k}(\lambda) \approx \frac{1}{\beta}\delta_{h_{k-1}}(\lambda), \quad k = 2, \cdots, n.
$$

Remark 4.1. *The relation (31) is reasonable since we can choose $\delta_{h_k}(\lambda) = h_k$ ($k = 1, \cdots, n$). Always the upper bound of the estimate $\delta_{h_k}(\lambda) \lesssim h_k$ holds. Recently, we also obtain the lower bound $\delta_{h_k}(\lambda) \gtrsim h_k$ (c.f. [16]).*

Algorithm 4.2. *Multilevel Correction Scheme*

(1) *Solve the following eigenvalue problem in $V_{0h_1}(\Omega)$:*
    Find $(\lambda_{h_1}, u_{h_1}) \in \mathbb{R} \times V_{0h_1}(\Omega)$ such that $b(u_{h_1}, u_{h_1}) = 1$ and

$$
a(u_{h_1}, v_{h_1}) = \lambda_{h_1}b(u_{h_1}, v_{h_1}), \qquad \forall v_{h_1} \in V_{0h_1}(\Omega).
$$

(2) *Construct a series of finer finite element spaces $V_{0h_2}(\Omega_j), \cdots, V_{0h_n}(\Omega_j)$ such that $\rho_\Omega(H) \gtrsim \delta_{h_1}(\lambda) \ge \delta_{h_2}(\lambda) \ge \cdots \ge \delta_{h_n}(\lambda)$ and (31) holds.*

(3) *Do $k = 1, \cdots, n-1$*

- *Obtain a new eigenpair approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathbb{R} \times V_{0h_{k+1}}(\Omega)$ by Algorithm 4.1*

$$(\lambda_{h_{k+1}}, u_{h_{k+1}}) = Correction(V_{0H}(\Omega), \lambda_{h_k}, u_{h_k}, V_{0h_{k+1}}(\Omega)).$$

    end Do

Finally, we obtain an eigenpair approximation $(\lambda_{h_n}, u_{h_n}) \in \mathbb{R} \times V_{0h_n}(\Omega)$.

The following theorem states the prior analysis of an eigenpair approximation (λ_{h_n}, u_{h_n}) obtained by Algorithm 4.2.

Theorem 4.2. *After implementing Algorithm 4.2, there exists an eigenfunction $u \in M(\lambda)$ such that the resultant eigenpair approximation (λ_{h_n}, u_{h_n}) has the following error estimate*

$$(32) \qquad \|u - u_{h_n}\|_{1,\Omega} \quad \lesssim \quad \delta_{h_n}(\lambda),$$

$$(33) \qquad \|u - u_{h_n}\|_{0,\Omega} \quad \lesssim \quad \rho_\Omega(H)\delta_{h_n}(\lambda),$$

$$(34) \qquad |\lambda - \lambda_{h_n}| \quad \lesssim \quad \delta_{h_n}^2(\lambda),$$

under the condition $C\beta\rho_\Omega(H) < 1$ for some constant C.

Proof. Based on Proposition 3.1, there exists an eigenfunction $u \in M(\lambda)$ such that

$$(35) \qquad |\lambda - \lambda_{h_1}| \quad \lesssim \quad \delta_{h_1}^2(\lambda),$$

$$(36) \qquad \|u - u_{h_1}\|_{1,\Omega} \quad \lesssim \quad \delta_{h_1}(\lambda),$$

$$(37) \qquad \|u - u_{h_1}\|_{0,\Omega} \quad \lesssim \quad \rho_\Omega(h_1)\delta_{h_1}(\lambda).$$

Let $\varepsilon_{h_1}(\lambda) := \delta_{h_1}(\lambda)$. From (35)-(37) and Theorem 4.1, we have

$$\begin{aligned}
\varepsilon_{h_{k+1}}(\lambda) \quad &\lesssim \quad \rho_\Omega(H)\varepsilon_{h_k}(\lambda) + \varepsilon_{h_k}^2(\lambda) + \delta_{h_{k+1}}(\lambda) \\
&\lesssim \quad \rho_\Omega(H)\varepsilon_{h_k}(\lambda) + \delta_{h_{k+1}}(\lambda), \quad \text{for } 1 \le k \le n-1.
\end{aligned}$$

by a process of induction with the condition $\rho_\Omega(H) \gtrsim \delta_{h_1}(\lambda) \ge \delta_{h_2}(\lambda) \ge \cdots \ge \delta_{h_n}(\lambda)$. Then by recursive relation, we obtain

$$\begin{aligned}
\varepsilon_{h_n}(\lambda) \quad &\lesssim \quad \rho_\Omega(H)\varepsilon_{h_{n-1}}(\lambda) + \delta_{h_n}(\lambda) \\
&\lesssim \quad \rho_\Omega^2(H)\varepsilon_{h_{n-2}}(\lambda) + \rho_\Omega(H)\delta_{h_{n-1}}(\lambda) + \delta_{h_n}(\lambda) \\
(38) \qquad &\lesssim \quad \sum_{k=1}^{n}(\rho_\Omega(H))^{n-k}\delta_{h_k}(\lambda).
\end{aligned}$$

Based on the proof in Theorem 4.1, (31) and (38), the final eigenfunction approximation u_{h_n} has the error estimate

$$\begin{aligned}
\|u - u_{h_n}\|_{1,\Omega} \quad &\lesssim \quad \varepsilon_{h_n}(\lambda) \lesssim \sum_{k=1}^{n}(\rho_\Omega(H))^{n-k}\delta_{h_k}(\lambda) \\
&= \quad \sum_{k=1}^{n}\left(\beta\rho_\Omega(H)\right)^{n-k}\delta_{h_n}(\lambda) \lesssim \frac{\delta_{h_n}(\lambda)}{1 - \beta\rho_\Omega(H)} \\
&\lesssim \quad \delta_{h_n}(\lambda).
\end{aligned}$$

The desired result (33) and (34) can also be proved with the similar way in the proof of Theorem 4.1. $\qquad\qquad\square$

5. Work estimate of algorithm

In this section, we turn our attention to the estimate of computational work for Algorithm 4.2. We will show that Algorithm 4.2 makes solving eigenvalue problem need almost the same work as solving the boundary value problem by the local and parallel finite element method.

First, we define the dimension of each level linear finite element space as

$$N_k^j := \dim V_{0h_k}(\Omega_j) \text{ and } N_k := \dim V_{0h_k}(\Omega), \quad k = 1, \cdots, n, \ j = 1, \cdots, m+1.$$

Then we have

$$(39) \qquad N_k^j \approx \left(\frac{1}{\beta}\right)^{d(n-k)} N_n^j \text{ and } N_k^j \approx \frac{N_k}{m}, \quad k = 1, \cdots, n.$$

Theorem 5.1. *Assume the eigenvalue problem solving in the coarsest spaces $V_{0H}(\Omega)$ and $V_{0h_1}(\Omega)$ need work $\mathcal{O}(M_H)$ and $\mathcal{O}(M_{h_1})$, respectively, and the work of solving the boundary value problem in $V_{h_k}(\Omega_j)$ and $V_{h_k}(G_{m+1})$ be $\mathcal{O}(N_k^j)$ and $\mathcal{O}(N_k^{m+1})$, $\forall k = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. Then the tasks involved in Algorithm 4.2 is $\mathcal{O}(N_n/m + M_H \log N_n + M_{h_1})$ for each computer node of the cluster. Furthermore, the tasks in each computer node will be $\mathcal{O}(N_n/m)$ provided $M_H \ll N_n/m$ and $M_{h_1} \leq N_n/m$.*

Proof. Let W_k denote the work in any computer node of the one correction step in the k-th finite element space V_{h_k}. Then with the definition, we have

$$(40) \qquad W_k = \mathcal{O}(N_k/m + M_H) \quad \text{for } k \geq 2.$$

Iterating (40) and using the fact (39), we obtain

$$\text{The total work in any computer node} \leq \sum_{k=1}^{n} W_k$$

$$= \mathcal{O}\left(M_{h_1} + \sum_{k=2}^{n} (N_k/m + M_H)\right)$$

$$= \mathcal{O}\left(\sum_{k=2}^{n} N_k/m + (n-2)M_H + M_{h_1}\right)$$

$$= \mathcal{O}\left(\sum_{k=2}^{n} \left(\frac{1}{\beta}\right)^{d(n-k)} N_n/m + (n-2)M_H + M_{h_1}\right)$$

$$(41) \qquad = \mathcal{O}(N_n/m + M_H \log N_n + M_{h_1}).$$

This is the desired result $\mathcal{O}(N_n/m + M_H \log N_n + M_{h_1})$ and the one $\mathcal{O}(N_n/m)$ can be obtained by the conditions $M_H \ll N_n/m$ and $M_{h_1} \leq N_n/m$. $\qquad\square$

Remark 5.1. *The linear complexity $\mathcal{O}(N_k^j)$ and $\mathcal{O}(N_k^{m+1})$ can be arrived by the so-called multigrid method (see, e.g., [5, 6, 11, 17, 24]).*

6. Numerical result

In this section, we give two numerical examples to illustrate the efficiency of the multilevel correction algorithm (Algorithm 4.2) proposed in this paper.

Example 6.1. *In this example, the eigenvalue problem (7) is solved on the square $\Omega = (-1, 1) \times (-1, 1)$ with $a(u, v) = \int_\Omega \nabla u \cdot \nabla v d\Omega$ and $b(u, v) = \int_\Omega uv d\Omega$.*
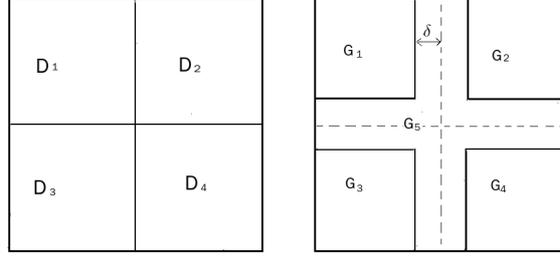
FIGURE 3. $\bigcup_{j=1}^{4} \bar{D}_j = \bar{\Omega}$, $G_5 = \Omega \setminus (\cup_{j=1}^{4} \bar{G}_j)$.

As in Figure 3, we first divide the domain Ω into four disjoint subdomains D_1, $\cdots$, D_4 such that $\bigcup_{j=1}^{4} \bar{D}_j = \bar{\Omega}$, $D_i \cap D_j = \emptyset$, then enlarge each D_j to obtain Ω_j such that $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$ for $i, j = 1, 2, 3, 4$ and

$$G_5 = \Omega \setminus (\cup_{j=1}^{4} \bar{G}_j).$$

TABLE 1. The errors for the first 5 eigenvalue approximations.

DOFS	Errors	1-st order	2-nd order	3-rd order	4-th order	5-th order
41	$|\lambda - \lambda_{h_1}|$	0.628291 -	3.999872 -	5.883511 -	14.020924 -	19.994750 -
137	$|\lambda - \lambda_{h_2}|$	0.167761 1.905019	1.109862 1.849572	1.536843 1.936706	4.242046 1.724749	4.609045 2.117081
497	$|\lambda - \lambda_{h_3}|$	0.042247 1.989463	0.264674 2.068088	0.364833 2.074660	0.912915 2.216207	1.199432 1.942116
1889	$|\lambda - \lambda_{h_4}|$	0.010636 1.989823	0.065304 2.018958	0.090301 2.014416	0.222163 2.038859	0.324016 1.888211
7361	$|\lambda - \lambda_{h_5}|$	0.002668 1.994903	0.016257 2.006052	0.022555 2.001300	0.055572 1.999169	0.088903 1.865761
29057	$|\lambda - \lambda_{h_6}|$	0.000667 1.998504	0.004051 2.004732	0.005641 1.999316	0.014060 1.982763	0.025010 1.829718
115457	$|\lambda - \lambda_{h_7}|$	0.000166 1.999660	0.001011 2.002346	0.001413 1.996823	0.003588 1.970195	0.007459 1.745342
460289	$|\lambda - \lambda_{h_8}|$	0.000041 1.999928	0.000252 1.999725	0.000354 1.994054	0.000924 1.955971	0.002412 1.628473

TABLE 2. The errors for the simple (1-st and 5-th) eigenfunction approximations.

Eigenfunctions	$\|u - u_{h_1}\|_{1,\Omega}$	$\|u - u_{h_2}\|_{1,\Omega}$	$\|u - u_{h_3}\|_{1,\Omega}$	$\|u - u_{h_4}\|_{1,\Omega}$	$\|u - u_{h_5}\|_{1,\Omega}$
1-st	0.269991	0.135956	0.068195	0.034119	0.017064
Order	$-$	0.989771	0.995402	0.999091	0.999619
4-th	1.025704	0.514925	0.259424	0.129254	0.064645
Order	$-$	0.994180	0.989050	1.005103	0.999598

The sequence of finite element spaces is constructed by using the linear or quadratic element on the nested sequence of triangulations which are produced by the regular refinement with $\beta = 2$ (connecting the midpoints of each edge).

Algorithm 4.2 is applied to solve the eigenvalue problem. If the linear element is used, from Theorem 4.2, we have the following error estimates for eigenpair approximation

$$|\lambda_{h_n} - \lambda| \lesssim h_n^2, \quad \|u_{h_n} - u\|_{1,\Omega} \lesssim h_n$$

which means the multilevel correction method can also obtain the optimal convergence order.

Tables 1 and 2 shows the numerical results for the first five eigenvalues and the 1-st, 4-th eigenfunctions (they are simple) by the linear finite element method in which "Order" means the estimation of the convergence rate calculated from the simulation. It is observed from Tables 1 and 2 that the numerical results confirm the efficiency of the proposed algorithm.

Next we discuss the effectiveness of δ and the coarsest mesh size H to the numerical results by Algorithm 4.2. The variation of approximation error with different δ and H is illustrated in Figure 4, which shows that Algorithm 4.2 obtains the optimal convergence order under the soft requirements $H \leq 0.25$ and $\delta \geq 0.1$ (easy to be met).
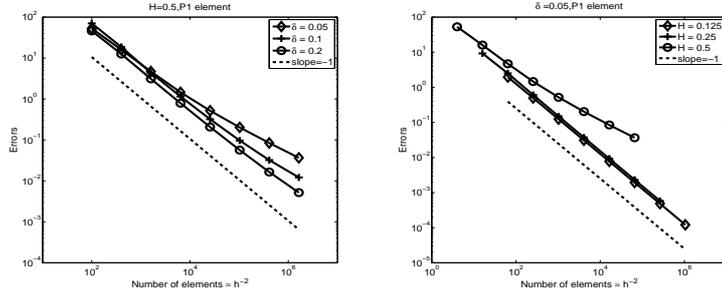

FIGURE 4. The error estimate for the first 6 eigenvalue approximations by the linear element: The left subfigure is for $H = 0.5$ and $\delta = 0.05, 0.1, 0.2$. The right subfigure is for $\delta = 0.05$ and $H = 0.5, 0.25, 0.125$.

Example 6.2. *In the second example, we solve the eigenvalue problem (7) using linear and quadratic element on the square* $\Omega = (-1, 1) \times (-1, 1)$ *with* $a(u, v) = \int_\Omega A\nabla u \cdot \nabla v \mathrm{d}\Omega$, $b(u, v) = \int_\Omega \phi uv \mathrm{d}\Omega$ *and*

$$A = \begin{pmatrix} e^{1+x^2} & e^{xy} \\ e^{xy} & e^{1+y^2} \end{pmatrix} \quad and \quad \phi = (1 + x^2)(1 + y^2).$$

Since the exact eigenvalue is not known, we use the accurate enough approximations $[17.982932, 33.384973, 38.381968, 47.670103, 66.874113, 68.323961]$ by the extrapolation method as the first 6 exact eigenvalues to investigate the errors. The corresponding numerical results for the first 6 eigenvalues by the linear finite element method are shown in Figure 5, from which the efficiency of the proposed algorithm in this paper is confirmed again.
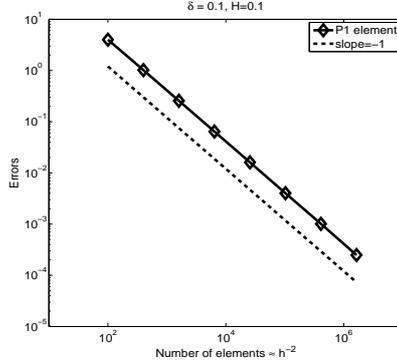
FIGURE 5. The error estimate for the first 6 eigenvalue approximations with $H = 0.1$ and $\delta = 0.1$.

Example 6.3. *In the third example, the eigenvalue problem with discontinious coefficients is considered. We solve the eigenvalue problem (7) with the linear finite element method on the square $\Omega = (-1, 1) \times (-1, 1)$ and $a(u, v) = \int_\Omega A\nabla u \cdot \nabla v d\Omega$, $b(u, v) = \int_\Omega uv d\Omega$ and*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{in } (-1, 0] \times (-1, 0] \cup (0, 1) \times (0, 1),$$

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{in } (0, 1) \times (-1, 0] \cup (-1, 0] \times (0, 1).$$

Similar to the second example, the exact eigenvalues are not known. Then we also use the accurate enough approximations [14.428518, 30.468178, 37.830313, 67.778845, 77.228072, 79.293376] by the extrapolation method as the first 6 exact eigenvalues to investigate the convergence behavior. The corresponding numerical results for the first 6 eigenvalues by the linear finite element method are shown in Figure 6, from which the efficiency of the proposed algorithm in this paper is confirmed again.
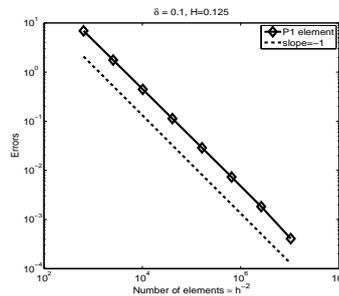

FIGURE 6. The error estimate for the first 6 eigenvalue approximations with $H = 0.125$ and $\delta = 0.1$.

7. Concluding remarks

In this paper, we give a new type of multilevel local and parallel method based on multigrid discretization to solve the eigenvalue problems. The idea here is to use

the multilevel correction method to transform the solution of eigenvalue problem to a series of solutions of the corresponding boundary value problems with the local and parallel method. As stated in the numerical examples, Algorithm 4.2 for simple eigenvalue cases can be extended to the corresponding version for multiple eigenvalue cases. For more information, please refer [23].

Furthermore, the framework here can also be coupled with the adaptive refinement technique. Some articles [4, 18, 19, 20] have contributed to the adaptive finite element methods for eigenvalue problems. Also the ideas can be extended to other types of linear and nonlinear eigenvalue problems. These will be investigated in our future work.

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