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CONVERGENCE ANALYSES OF CRANK-NICOLSON ORTHOGONAL SPLINE COLLOCATION METHODS FOR LINEAR PARABOLIC PROBLEMS IN TWO SPACE VARIABLES

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Abstract. The Crank-Nicolson (CN) orthogonal spline collocation method and its alternating direction implicit (ADI) counterpart are considered for the approximate solution of a class of linear parabolic problems in two space variables. It is proved that both methods are second order accurate in time and of optimal order in certain H^j norms in space. Also, L^∞ estimates in space are derived.

Key words. parabolic problems, orthogonal spline collocation, Crank-Nicolson method, alternating direction implicit method, optimal global error estimates.

1. Introduction

Consider the initial/boundary value problem comprising

$$\frac{\partial u}{\partial t} + Lu = f(x,y,t), \quad (x,y,t) \in \Omega_T \equiv \Omega \times (0,T], \tag{1}$$

the initial condition,

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega, \tag{2}$$

and the Dirichlet boundary condition,

$$u(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times (0,T], \tag{3}$$

where $\Omega = (0,1) \times (0,1)$ with boundary $\partial\Omega$. Here, $u_0(x,y)$ and $f(x,y,t)$ are given functions in their respective domains of definition, and $L = L_1 + L_2$ with

$$L_1 u = -a_1(x,y,t)u_{xx} + b_1(x,y,t)u_x + c(x,y,t)u, \tag{4}$$

$$L_2 u = -a_2(x,y,t)u_{yy} + b_2(x,y,t)u_y, \tag{5}$$

where

$$0 \le a_{\min} \le a_1(x,y,t), \ \ a_2(x,y,t) \le a_{\max}, \quad (x,y,t) \in \Omega_T.$$

For the approximate solution of this problem, we examine the Crank–Nicolson orthogonal spline collocation (OSC) scheme. In this method, OSC with C^1 piecewise polynomials of arbitrary degree $r \ge 3$ in each space variable is used for the spatial discretization and the resulting system of ordinary differential equations in the time variable is discretized using the trapezoid rule. We also consider an alternating direction implicit (ADI) version of this method. These methods are not new but, to the best of the authors' knowledge, a comprehensive convergence analysis of them has not yet appeared in the literature. Numerical experiments reported in the literature exhibit the expected second order accuracy in time and optimal order error estimates in various norms in space at each time step. In [9], the first convergence analysis of the Crank-Nicolson OSC method was presented for semilinear problems of the form (1)–(3); that is, problems in which the function f depends on the solution u. An algebraically linear form of this method, commonly known as

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the extrapolated Crank-Nicolson method, was also considered. Both methods were proved to be second order accurate in time and of optimal accuracy in the L^2 norm in space. In the 1980s, ADI OSC methods were developed by various authors and used to solve practical problems modeled by parabolic equations in several space variables; see [2] for an overview of these methods. The first analysis of an ADI OSC method was given by Fernandes and Fairweather [7], who considered the ADI Crank-Nicolson OSC method for the heat equation in two space variables. Optimal L^2 and H^1 error estimates in space and second order accuracy in time were derived. An optimal H^2 error estimate for this method is a consequence of analysis given in [11].

In [3], Bialecki and Fernandes considered an ADI Crank-Nicolson OSC for (1)–(3) for the case in which $b_1 = b_2 = c = 0$ and $r = 3$. They proved that the scheme is second order accurate in time and third order accurate in space in a norm which is stronger than the L^2 norm but weaker than the H^1 norm. In [4], the ADI Crank-Nicolson OSC was considered for (1)–(3) with $r = 3$ and proved to be second order in time and of optimal accuracy in space in the H^1 norm. The authors state that the analysis can be easily extended to the case $r > 3$. For an overview of these methods and ADI OSC methods for other equations, see [8].

The primary purpose of this paper is to provide a complete convergence analysis of the Crank-Nicolson OSC method and the ADI Crank-Nicolson OSC method for the approximate solution of (1)–(3). Specifically, we prove that each method is second order accurate in time and of optimal accuracy in H^j norms in space. Moreover, some L^∞ estimates in space are obtained. An outline of the paper is as follows. In section 2, we introduce standard notation and basic lemmas used in the formulation and analysis of OSC methods. In section 3, the Crank-Nicolson OSC scheme is described and the optimal error estimates are derived. The ADI Crank-Nicolson OSC scheme is considered in section 4, and concluding remarks are presented in section 5.

2. Preliminaries and Basic Results

Set $I = (0, 1)$, and let $\delta_x = \{x_i\}_{i=0}^{N_x}$ and $\delta_y = \{y_j\}_{j=0}^{N_y}$ be two partitions of $\bar{I}$ such that

$$0 = x_0 < x_1 < \cdots < x_{N_x-1} < x_{N_x} = 1, \qquad 0 = y_0 < y_1 < \cdots < y_{N_y-1} < y_{N_y} = 1.$$

Assume that the partition $\delta = \delta_x \otimes \delta_y$ of Ω is quasi-uniform. For $1 \le i \le N_x,\ 1 \le j \le N_y$, let

$$I_i^x = (x_{i-1}, x_i), \quad I_j^y = (y_{j-1}, y_j), \quad I_{ij} = I_i^x \times I_j^y,$$
$$h_i^x = x_i - x_{i-1}, \quad h_j^y = y_j - y_{j-1},$$

and set

$$h = \max(\max_i h_i^x, \max_j h_j^y).$$

Let $\mathcal{M}_r(\delta_x)$ and $\mathcal{M}_r^0(\delta_x)$ be the spaces of piecewise polynomials of degree $\le r$ with $r \ge 3$ defined by

$$\begin{aligned} \mathcal{M}_r(\delta_x) &= \{v | v \in C^1[0,1], v|_{I_i^x} \in P_r,\ 1 \le i \le N_x\}, \\ \mathcal{M}_r^0(\delta_x) &= \{v | v \in \mathcal{M}_r(\delta_x),\ v(0) = v(1) = 0\}, \end{aligned}$$

where P_r denotes the set of polynomials of degree $\le r$. The spaces $\mathcal{M}_r(\delta_y)$ and $\mathcal{M}_r^0(\delta_y)$ are defined similarly. Set

$$\mathcal{M}_r^0(\delta) = \mathcal{M}_r^0(\delta_x) \otimes \mathcal{M}_r^0(\delta_y).$$

Let $\{\lambda_k\}_{k=1}^{r-1}$ and $\{\omega_k\}_{k=1}^{r-1}$ be the nodes and weights, respectively, of the $(r-1)$-point Gauss quadrature rule on I. Let

$$\mathcal{G}_x = \{\xi_{i,k}^x\}_{i,k=1}^{N_x,r-1} \quad \text{and} \quad \mathcal{G}_y = \{\xi_{j,l}^y\}_{j,l=1}^{N_y,r-1}$$

be the sets of Gauss points in the x- and y-directions, respectively, where

$$\xi_{i,k}^x = x_{i-1} + h_i^x \lambda_k, \qquad \xi_{j,l}^y = y_{j-1} + h_j^y \lambda_l.$$

Let

$$\mathcal{G}_r = \{\xi = (\xi^x, \xi^y) | \xi^x \in \mathcal{G}_x, \xi^y \in \mathcal{G}_y\}$$

be the set of Gauss points in Ω. For V and W defined on $\mathcal{G}_r$, let $\langle V, W\rangle$ and $\|V\|_{\mathcal{M}_r}$ be defined by

$$\langle V, W\rangle = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} h_i^x h_j^y \sum_{k=1}^{r-1}\sum_{l=1}^{r-1} \omega_k \omega_l (VW)(\xi_{i,k}^x, \xi_{j,l}^y), \tag{6}$$

and

$$\|V\|_{\mathcal{M}_r} = \langle V, V\rangle^{\frac{1}{2}}. \tag{7}$$

Let $L^p(\Omega)$, $p = 2, \infty$, denote the Banach space with norm $\|\cdot\|_{L^p}$, where

$$\|v\|_{L^2} = \left(\int_\Omega |v|^2 dxdy\right)^{\frac{1}{2}}, \qquad \|v\|_{L^\infty} = \sup_\Omega |v|.$$

With m a nonnegative integer, let $H^m(\Omega)$ denote the Sobolev space with norm

$$\|v\|_{H^m} = \left(\sum_{\ell=0}^{m} |v|_{H^\ell}^2\right)^{\frac{1}{2}},$$

where

$$|v|_{H^l} = \left(\sum_{i+j=l} \left\|\frac{\partial^{i+j} v}{\partial x^i \partial y^j}\right\|_{L^2}^2\right)^{\frac{1}{2}}.$$

Suppose $\{t_n\}_{n=0}^{M}$ is a uniform partition of $[0, T]$ such that $t_n = nk$, $k = T/M$, and introduce the notation:

$$V^n(\,\cdot\,) = V(\,\cdot\,, t_n), \qquad 0 \le n \le M,$$

$$\partial_t V^n = \frac{V^n - V^{n-1}}{k}, \qquad 1 \le n \le M,$$

$$V^{n-\frac{1}{2}} = \frac{1}{2}\left[V^n + V^{n-1}\right], \qquad 1 \le n \le M.$$

Throughout the paper, we denote by C a generic positive constant that is independent of h and Δt and is not necessarily the same on each occurrence. Next we present several lemmas required in the convergence analyses.

Lemma 2.1. *If $U, V \in \mathcal{M}_0^r(\delta)$, then the following hold:*

$$\langle -\Delta U, V\rangle = \langle U, -\Delta V\rangle, \tag{8}$$

[7, Eq. (3.4)]*;*

$$\langle -\Delta U, U\rangle \ \ge\ C \|\nabla U\|^2 \ \ge\ 0, \tag{9}$$

[7, Eq. (3.5)]*;*

$$(10)\ |\langle \Delta U, V\rangle| \le C\,\|\nabla U\|\,\|\nabla V\|, \quad -\langle \Delta U, V\rangle \le C\left[\|\nabla U\|^2 + \|\nabla V\|^2\right];$$

see, the proof of Lemma 3.3 in [7];

$$\|V\|_{H^2} \le C\|\Delta V\|_{\mathcal{M}_r}, \tag{11}$$

[1, Eq. (3.20)].

In the convergence analysis, we use the elliptic projection $W : [0,T] \to \mathcal{M}_r^0(\delta)$ defined by

$$\langle L(u-W), v\rangle = 0 \quad \forall v \in \mathcal{M}_r^0(\delta), \quad t \in [0,T]. \tag{12}$$

As in [1], it can be shown that for a given u, (12) has a unique solution $W \in \mathcal{M}_r^0(\delta)$. Moreover, the following estimates hold.

Lemma 2.2. *With W defined as in (12), there exists a positive constant C such that*

$$\left\|\frac{\partial^i}{\partial t^i}(u-W)\right\|_{H^j} \le Ch^{r+1-j}\left\|\frac{\partial^i u}{\partial t^i}\right\|_{H^{r+3-j}}, \ j=0,1,2, \ i=0,1,2, \ t\in[0,T]. \tag{13}$$

In the subsequent convergence analyses, we use the following result; for a proof, see [3, Lemma 3.2].

Lemma 2.3. *If $L = L_1 + L_2$, then*

$$\langle L(t)v, Z\rangle = A_0(t;v,Z) + A_1(t;v,Z) \quad \forall v, Z \in \mathcal{M}_r^0(\delta), \quad t \in (0,T], \tag{14}$$

where

$$A_i(t;\cdot,\cdot), \quad i=0,1, \quad t\in(0,T]$$

are bilinear forms on $\mathcal{M}_r^0(\delta)\times\mathcal{M}_r^0(\delta)$ for each $t\in(0,T]$ satisfying:

(i) $A_0(t;Z,v) = A_0(t;v,Z), \quad \forall v, Z \in \mathcal{M}_r^0(\delta), \quad t\in(0,T]$;

(ii) *there exist positive constants $a_{\min}$ and $a_{\max}$ such that*

$$a_{\min}\langle -\Delta v, v\rangle \le A_0(t;v,v) \le a_{\max}\langle -\Delta v, v\rangle, \quad \forall v \in \mathcal{M}_r^0(\delta), \quad t\in(0,T];$$

(iii) *there exists a positive constant C such that*

$$|A_0(t_1;v,v) - A_0(t_2;v,v)| \le C|t_1-t_2|\,\langle -\Delta v, v\rangle, \quad \forall v \in \mathcal{M}_r^0(\delta), \quad t\in(0,T];$$

(iv) *there exists a positive constant C such that*

$$A_1(t_1;v,Z) \le C\,\chi\ \langle -\Delta v, v\rangle^{\frac{1}{2}}\,\|Z\|_{\mathcal{M}_r}, \quad \forall v, Z \in \mathcal{M}_r^0(\delta), t\in(0,T],$$

where

$$|a_i(x,y,t_1) - a_i(x,,y,t_2)| \le K|t_1-t_2|, \quad i=1,2, \quad (x,y)\in\Omega, \quad t_1,t_2\in(0,T],$$

and

$$\chi = \max_{1\le l\le 5}\left(\left\|\frac{\partial^l a_1}{\partial x^l}\right\|_{C(\overline{\Omega}_T)}, \left\|\frac{\partial^l a_2}{\partial y^l}\right\|_{C(\overline{\Omega}_T})\right) + \|b_1\|_{C(\overline{\Omega}_T)} + \|b_2\|_{C(\overline{\Omega}_T)} + \|c\|_{C(\overline{\Omega}_T)}.$$

In the following, we make repeated use of Young's inequality,

$$de \le \varepsilon d^2 + \frac{1}{4\varepsilon}e^2, \quad d,e\in\mathcal{R}, \quad \varepsilon > 0. \tag{15}$$

3. The Crank-Nicolson OSC Method

The Crank-Nicolson OSC method for approximating the solution of (1)–(3) consists in finding $U_h : [0,T] \to \mathcal{M}_r^0(\delta)$ such that

$$\partial_t U_h^n(\xi) + L^{n-\frac{1}{2}} U_h^{n-\frac{1}{2}}(\xi) = f^{n-\frac{1}{2}}(\xi), \quad \xi \in \mathcal{G}_r, \tag{16}$$

where $L^{n-\frac{1}{2}}$ and $f^{n-\frac{1}{2}}$ denote the operator $L(t)$ and the function $f(t)$, respectively, evaluated at $t = t_{n-\frac{1}{2}}$. The quantity $U_h^0 \in \mathcal{M}_r^0(\delta)$ is determined from the initial condition (2) as described in the following. For the error analysis, we rewrite (16) in the equivalent form

$$\langle \partial_t U_h^n, v\rangle + \left\langle L^{n-\frac{1}{2}} U_h^{n-\frac{1}{2}}, v\right\rangle = \left\langle f^{n-\frac{1}{2}}, v\right\rangle, \quad \forall\, v \in \mathcal{M}_r^0. \tag{17}$$

We now derive optimal order error estimates in the $H^\ell(\Omega)$, $\ell = 0,1,2$, norms on each time level.

3.1. L^2 convergence analysis.

We first derive an optimal $L^2(\Omega)$ error estimate.

Theorem 3.1. *Let U_h^n be the solution of (16) and let U_h^0 be such that*

$$\|u_0 - U_h^0\| \le C h^{r+1}. \tag{18}$$

Then there exists a positive constant C such that, for $J = 1,2,\ldots,M$,

$$\|u(t_J) - U_h^J\| \le C(k^2 + h^{r+1}). \tag{19}$$

Proof. We set $u^n - U_h^n = (u^n - W^n) - (U_h^n - W^n) \equiv \eta^n - \theta^n$. Since estimates of η^n are known from Lemma 2.2, it is sufficient to estimate θ^n.

From (1) and (12) at $t = t_{n-\frac{1}{2}}$, we obtain

$$\begin{aligned}\langle \partial_t\theta^n, v\rangle + \left\langle L^{n-\frac{1}{2}}\theta^{n-\frac{1}{2}}, v\right\rangle &= \left\langle u_t(t_{n-\frac{1}{2}}) - \partial_t u^n, v\right\rangle + \langle \partial_t \eta^n, v\rangle \\ &\quad + \left\langle L^{n-\frac{1}{2}}(W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}}), v\right\rangle,\end{aligned} \tag{20}$$

or

$$\langle \partial_t\theta^n, v\rangle + \left\langle L^{n-\frac{1}{2}}\theta^{n-\frac{1}{2}}, v\right\rangle = \langle \partial_t\eta^n, v\rangle + \left\langle \sigma^{n-\frac{1}{2}}, v\right\rangle \tag{21}$$

where $\sigma = \sigma_1 + \sigma_2$ with

$$\sigma_1^{n-\frac{1}{2}} = u_t(t_{n-\frac{1}{2}}) - \partial_t u^n, \qquad \sigma_2^{n-\frac{1}{2}} = L^{n-\frac{1}{2}}(W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}}). \tag{22}$$

Setting $v = \theta^{n-\frac{1}{2}}$ in (21), we obtain

$$\frac{1}{2}\partial_t\|\theta^n\|_{\mathcal{M}_r} + \left\langle L^{n-\frac{1}{2}}\theta^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}\right\rangle = \left\langle \partial_t\eta^n, \theta^{n-\frac{1}{2}}\right\rangle + \left\langle \sigma^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}\right\rangle. \tag{23}$$

Following (14), we rewrite the second term on the left hand side of (23) as

$$\left\langle L^{n-\frac{1}{2}}\theta^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}\right\rangle = A_0(t_{n-\frac{1}{2}}; \theta^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}) + A_1(t_{n-\frac{1}{2}}; \theta^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}). \tag{24}$$

From properties (ii) and (iv) of Lemma 2.3, we obtain

$$\begin{aligned}\left\langle L^{n-\frac{1}{2}}\theta^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}\right\rangle &\ge a_{\min}\left\langle -\Delta\theta^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}\right\rangle - C\chi\left\langle -\Delta\theta^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}\right\rangle^{\frac{1}{2}}\|\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r} \\ &\ge a_{\min}\|\nabla\theta^{n-\frac{1}{2}}\|^2 - C\chi\|\nabla\theta^{n-\frac{1}{2}}\ \|\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r} \\ &\ge \frac{a_{\min}}{2}\|\nabla\theta^{n-\frac{1}{2}}\|^2 - C\|\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r}^2.\end{aligned}$$

Substituting (25) in (23), multiplying by $2k$, and then summing from $n=1$ to m, we obtain, after using the Cauchy-Schwarz inequality and simplifying the resulting expression,

$$\begin{aligned}
&\|\theta^m\|^2_{\mathcal{M}_r} + Ck\sum_{n=1}^{m}\|\nabla\theta^{n-\frac{1}{2}}\|^2 \\
&\quad\le C\left[\|\theta^0\|^2_{\mathcal{M}_r} + 2k\sum_{n=1}^{m}\left(\|\partial_t\eta^n\|^2_{\mathcal{M}_r} + \|\sigma_1^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} + \|\sigma_2^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\right) + \|\theta^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\right] \\
&\qquad + Ck\sum_{n=1}^{m}\|\theta^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} \\
&\quad\le C\left[\|\theta^0\|^2_{\mathcal{M}_r} + 2k\sum_{n=1}^{m}\left(\|\partial_t\eta^n\|^2_{\mathcal{M}_r} + \|\sigma_1^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} + \|\sigma_2^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\right)\right] + Ck\sum_{n=0}^{m}\|\theta^n\|^2_{\mathcal{M}_r},
\end{aligned}$$

since

$$\|\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r} \le \frac{1}{2}\left(\|\theta^n\|_{\mathcal{M}_r} + \left\|\theta^{n-1}\right\|_{\mathcal{M}_r}\right).$$

On using Gronwall's Lemma, we obtain, for k sufficiently small,

$$\begin{aligned}
&\|\theta^m\|_{\mathcal{M}_r} + Ck\sum_{n=1}^{m}\|\nabla\theta^{n-\frac{1}{2}}\| \\
&\quad\le C\|\theta^0\|_{\mathcal{M}_r} + Ck\sum_{n=1}^{m}\left(\|\partial_t\eta^n\|_{\mathcal{M}_r} + \|\sigma_1^{n-\frac{1}{2}}\|_{\mathcal{M}_r} + \|\sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r}\right).
\end{aligned}$$

From [7, Eq. (4.18)], we note that

$$\sum_{n=1}^{m}\|\partial_t\eta^n\|_{\mathcal{M}_r} \le Ch^{r+1}\|u_t\|_{L^2(H^{r+3})}. \tag{25}$$

To estimate $\sigma_1^{n-\frac{1}{2}}$, we use Taylor's theorem to obtain

$$k\sum_{n=1}^{m}\|\sigma_1^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} \le Ck^4\left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{L^\infty(L^\infty(\Omega))}. \tag{26}$$

For the term $\sigma_2^{n-\frac{1}{2}}$, we obtain, on using Taylor's theorem and the boundedness of the coefficients,

$$\begin{aligned}
\|\sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r} &= \|L^{n-\frac{1}{2}}(W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}})\|_{\mathcal{M}_r} \\
&\le \|W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}}\|_{W^{2,\infty}} \le k^2\|W_{tt}\|_{W^{2,\infty}}.
\end{aligned} \tag{27}$$

Since

$$\|W_{tt}\|_{W^{2,\infty}(I_{ij})} \le \|(W-u)_{tt}\|_{W^{2,\infty}(I_{ij})} + \|u_{tt}\|_{W^{2,\infty}(I_{ij})},$$

it follows on using [7, Lemma 3.2] and Lemma 2.2 with $i=2, j=2$ that

$$k\sum_{n=1}^{m}\|\sigma_2^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} \le Ck^4. \tag{28}$$

Since

$$\|\theta^0\| \le \|u_0 - W^0\| + \|u_0 - U_h^0\|,$$

we obtain on using Lemma 2.2 with $i=0, j=0$ and $t=0$, and (18)

$$\|\theta^0\| \le Ch^{r+1}. \tag{29}$$

Substituting (25), (26), (28) and (29) in (25) yields

$$\|\theta^m\| \le C(k^2 + h^{r+1}), \tag{30}$$

as required. Moreover,

$$\left(k \sum_{n=1}^{m} \|\nabla\theta^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\right)^{\frac{1}{2}} \le C(k^2 + h^{r+1}). \tag{31}$$

The proof is completed by using the triangle inequality, (30) and the estimate in Lemma 2.2 with $i = 0$. □

Remark 3.1. We can choose U_h^0 as the piecewise Hermite interpolant of u_0 without degrading the accuracy.

3.2. H^1 convergence analysis. In the following theorem, we derive an optimal $H^1(\Omega)$ estimate of the error.

Theorem 3.2. *Let U_h^n be the solution of the Crank-Nicolson OSC scheme (16) and let U_h^0 be such that*

$$\|u_0 - U_h^0\|_{H^1} \le Ch^r. \tag{32}$$

Then there exists a positive constant C such that, for $J = 1, 2, \ldots, M$, and k sufficiently small,

$$\|u(t_J) - U_h^J\|_{H^1} \le C(k^2 + h^r). \tag{33}$$

Proof. With $\upsilon = \partial_t\theta^n$ in (21), we obtain

$$\|\partial_t\theta^n\|^2 + \left\langle L^{n-\frac{1}{2}}\theta^{n-\frac{1}{2}}, \partial_t\theta^n \right\rangle = \langle \partial_t\eta^n, \partial_t\theta^n\rangle + < \sigma^{n-\frac{1}{2}}, \partial_t\theta^n > . \tag{34}$$

Following (14),

$$\left\langle L^{n-\frac{1}{2}}\theta^{n-\frac{1}{2}}, \partial_t\theta^n \right\rangle = A_0(t_{n-\frac{1}{2}}; \theta^{n-\frac{1}{2}}, \partial_t\theta^n) + A_1(t_{n-\frac{1}{2}}; \theta^{n-\frac{1}{2}}, \partial_t\theta^n). \tag{35}$$

Now

$$\begin{aligned}
A_0(t_{n-\frac{1}{2}}; \theta^{n-\frac{1}{2}}, \partial_t\theta^n) &= A_0(t_{n-\frac{1}{2}}; (\theta^n + \theta^{n-1})/2, (\theta^n - \theta^{n-1})/k) \\
&= \frac{1}{2k}\left[A_0(t_{n-\frac{1}{2}}; \theta^n + \theta^{n-1}, \theta^n - \theta^{n-1})\right] \\
&= \frac{1}{2k}\Big\{\left[A_0(t_{n-\frac{1}{2}}; \theta^n, \theta^n) - A_0(t_{n-\frac{3}{2}}; \theta^{n-1}, \theta^{n-1})\right] \\
&\quad - \left[A_0(t_{n-\frac{1}{2}}; \theta^{n-1}, \theta^{n-1}) - A_0(t_{n-\frac{3}{2}}; \theta^{n-1}, \theta^{n-1})\right]\Big\} \\
&= \frac{1}{2}\partial_t\left(A_0(t_{n-\frac{1}{2}}; \theta^n, \theta^n)\right) - \frac{1}{2}(\partial_t A_0)(t_{n-\frac{1}{2}}; \theta^{n-1}, \theta^{n-1}).
\end{aligned}$$

Thus, (35) becomes

$$\begin{aligned}
\left\langle L^{n-\frac{1}{2}}\theta^{n-\frac{1}{2}}, \partial_t\theta^n \right\rangle &= \frac{1}{2}\partial_t\left(A_0(t_{n-\frac{1}{2}}; \theta^n, \theta^n)\right) - \frac{1}{2}(\partial_t A_0)(t_{n-\frac{1}{2}}; \theta^{n-1}, \theta^{n-1}) \\
&\quad + A_1(t_{n-\frac{1}{2}}; \theta^{n-\frac{1}{2}}, \partial_t\theta^n).
\end{aligned} \tag{36}$$

On substituting (36) in (34), multiplying the resulting equation by $2k$, and summing the resulting equation from $n=2$ to m, we obtain

$$(37)\quad 2k\sum_{n=2}^{m}\|\partial_t\theta^n\|^2_{\mathcal{M}_r}+A_0(t_{m-\frac{1}{2}};\theta^m,\theta^m)$$
$$\leq\ A_0(t_{\frac{1}{2}};\theta^1,\theta^1)+2k\sum_{n=2}^{m}(\partial_tA_0)(t_{n-\frac{1}{2}};\theta^{n-1},\theta^{n-1})-2k\sum_{n=2}^{m}A_1(t_{n-\frac{1}{2}};\theta^{n-\frac{1}{2}},\partial_t\theta^n)$$
$$+\ Ck\sum_{n=2}^{m}\left(\|\partial_t\eta^n\|_{\mathcal{M}_r}+\|\sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r}+\|\sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r}\right)\|\partial_t\theta^n\|_{\mathcal{M}_r}.$$

For $n=1$ in (21), we set $v=\partial_t\theta^1$ giving

$$(38)\quad \left\langle\partial_t\theta^1,\partial_t\theta^1\right\rangle+\left\langle L^{\frac{1}{2}}\theta^{\frac{1}{2}},\partial_t\theta^1\right\rangle = \left\langle\partial_t\eta^1,\partial_t\theta^1\right\rangle+\left\langle\sigma^{\frac{1}{2}},\partial_t\theta^1\right\rangle$$
$$\leq \left(\|\partial_t\eta^1\|_{\mathcal{M}_r}+\|\sigma^{\frac{1}{2}}\|_{\mathcal{M}_r}\right)\|\partial_t\theta^1\|_{\mathcal{M}_r}.$$

Note that

$$(39)\quad \left\langle L^{\frac{1}{2}}\theta^{\frac{1}{2}},\partial_t\theta^1\right\rangle=\frac{1}{2k}\left\langle L^{\frac{1}{2}}(\theta^1+\theta^0),\theta^1-\theta^0)\right\rangle$$
$$=\ \frac{1}{2k}A_0(t_{\frac{1}{2}};\theta^1+\theta^0,\theta^1-\theta^0)+\frac{1}{2k}A_1(t_{\frac{1}{2}};\theta^1+\theta^0,\theta^1-\theta^0)$$
$$=\ \frac{1}{2k}\left[A_0(t_{\frac{1}{2}};\theta^1,\theta^1)-A_0(t_{\frac{1}{2}};\theta^0,\theta^0)\right]+\frac{1}{2k}A_1(t_{\frac{1}{2}};\theta^1+\theta^0,\theta^1-\theta^0).$$

Thus, on substituting (39) in (38), multiplying the resulting expression by $2k$, we obtain

$$(40)\quad 2k\|\partial_t\theta^1\|^2_{\mathcal{M}_r}+A_0(t_{\frac{1}{2}};\theta^1,\theta^1)\leq A_0(t_{\frac{1}{2}};\theta^0,\theta^0)-A_1(t_{\frac{1}{2}};\theta^1+\theta^0,\theta^1-\theta^0)$$
$$+2k\left(\|\partial_t\eta^1\|_{\mathcal{M}_r}+\|\sigma_1^{\frac{1}{2}}\|_{\mathcal{M}_r}+\|\sigma_2^{\frac{1}{2}}\|_{\mathcal{M}_r}\right)\|\partial_t\theta^1\|_{\mathcal{M}_r}.$$

Adding (40) to (37), we arrive at

$$(41)\quad 2k\sum_{n=1}^{m}\|\partial_t\theta^n\|^2_{\mathcal{M}_r}+A_0(t_{m-\frac{1}{2}};\theta^m,\theta^m)+A_0(t_{\frac{1}{2}};\theta^1,\theta^1)$$
$$\leq\ A_0(t_{\frac{1}{2}};\theta^0,\theta^0)+2k\sum_{n=2}^{m}(\partial_tA_0)(t_{n-\frac{1}{2}};\theta^{n-1},\theta^{n-1})-2k\sum_{n=1}^{m}A_1(t_{n-\frac{1}{2}};\theta^{n-\frac{1}{2}},\partial_t\theta^n)$$
$$+Ck\sum_{n=1}^{m}\left(\|\partial_t\eta^n\|_{\mathcal{M}_r}+\|\sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r}+\|\sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r}\right)\|\partial_t\theta^n\|_{\mathcal{M}_r}.$$

For the second term on the right hand side, we have

$$(42)\quad A_0(t_{m-\frac{1}{2}};\theta^m,\theta^m)\geq a_{\min}\|\nabla\theta^m\|^2$$

on using Lemma 2.3(ii) and (9). Similarly

$$(43)\quad A_0(t_{\frac{1}{2}};\theta^1,\theta^1)\geq a_{\min}\|\nabla\theta^1\|^2\geq 0.$$

Also,

$$(44)\quad A_0(t_{\frac{1}{2}};\theta^0,\theta^0)\leq a_{\max}\|\nabla\theta^0\|^2$$

on using Lemma 2.3(ii). Then

$$(45)\quad (\partial_t A_0)(t_{n-\frac{1}{2}};\theta^{n-1},\theta^{n-1}) \;\le\; \frac{1}{k}|A_0(t_{n-\frac{1}{2}};\theta^{n-1},\theta^{n-1})-A_0(t_{n-\frac{3}{2}};\theta^{n-1},\theta^{n-1})| \;\le\; C\frac{1}{k}k\|\nabla\theta^{n-1}\|^2 = C\|\nabla\theta^{n-1}\|^2,$$

on using Lemma 2.3(iii). From Lemma 2.3(iv) and (10),

$$(46)\qquad A_1(t_{n-\frac{1}{2}};\theta^{n-\frac{1}{2}},\partial_t\theta^n) \le C\|\nabla\theta^{n-\frac{1}{2}}\|\ \|\partial_t\theta^n\|_{\mathcal{M}_r}.$$

With (42)–(46) in (41), we arrive at

$$(47)\quad 2k\sum_{n=2}^{m}\|\partial_t\theta^n\|^2 + a_{\min}\|\nabla\theta^m\|^2 \le a_{\max}\|\nabla\theta^0\|^2_{\mathcal{M}_r} + C\Bigg[k\sum_{n=1}^{m}\Big(\|\nabla\theta^{n-\frac{1}{2}}\|\ \|\partial_t\theta^n\|_{\mathcal{M}_r}\Big) + k\sum_{n=2}^{m}\Big(\|\partial_t\eta^n\|_{\mathcal{M}_r} + \|\sigma_1^{n-\frac{1}{2}}\|_{\mathcal{M}_r} + \|\sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r}\Big)\|\partial_t\theta^n\|_{\mathcal{M}_r}\Bigg].$$

Using $ab \le a^2 + \frac{1}{4}b^2$ for the second and third term on the right hand side and simplifying, we obtain

$$(48)\quad k\sum_{n=1}^{m}\|\partial_t\theta^n\|^2_{\mathcal{M}_r} + (a_{\min}-Ck)\|\nabla\theta^m\|^2 \le C\Bigg[\|\nabla\theta^0\|^2_{\mathcal{M}_r} + k\sum_{n=1}^{m}\|\nabla\theta^n\|^2 + k\sum_{n=1}^{m}\Big(\|\partial_t\eta^n\|^2_{\mathcal{M}_r} + \|\sigma_1^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} + \|\sigma_2^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\Big)\Bigg].$$

Choose k small so that $(a_{\min} - Ck) > 0$, and hence,

$$\|\nabla\theta^m\|^2 \le C\Bigg[\|\nabla\theta^0\|^2 + k\sum_{n=1}^{m-1}\|\nabla\theta^n\|^2 + k\sum_{n=1}^{m}\Big(\|\partial_t\eta^n\|^2_{\mathcal{M}_r} + \|\sigma_1^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} + \|\sigma_2^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\Big)\Bigg].$$

Using Gronwall's Lemma, we obtain

$$(49)\quad \|\nabla\theta^m\|^2 \le C\Bigg[\left\|\nabla\theta^0\right\|^2 + k\sum_{n=1}^{m-1}\Big(\|\partial_t\eta^n\|^2_{\mathcal{M}_r} + \|\sigma_1^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} + \|\sigma_2^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\Big)\Bigg].$$

Substitute (25), (26) and (28) in (49) then apply the triangle inequality with the estimate (13) for $i = 0$ and $j = 1$ to complete the proof. □

Remark 3.2. If we choose U_h^0 as the elliptic projection W^0 of u_0 defined in (12), then $\theta^0 = 0$. Hence, from (49), we obtain a superconvergence result for $\|\nabla\theta\|$. From Sobolev's inequality, we obtain, since $\theta^m \in \mathcal{M}_r^0$,

$$\|\theta^m\| \le C\ \log\left(\frac{1}{h}\right)\|\nabla\theta^m\|,$$

which together with the triangle inequality yields the L^∞ estimate,

$$\|u(t_J) - U_h^J\|_{L^\infty} \leq C \ \log\left(\frac{1}{h}\right) \left(k^2 + h^{r+1}\right), \quad J = 1, \ldots, M,$$

provided the necessary maximum norm estimate for η, namely

$$\|\eta^J\|_{L^\infty} \leq C\ h^{r+1}, \quad J = 1, \ldots, M,$$

is available.

3.3. H^2 convergence analysis and a superconvergence result. In the following theorem, we derive an $H^2(\Omega)$ estimate of the error.

Theorem 3.3. *Let U_h^n be the solution of the Crank-Nicolson OSC scheme (16) and let $U_h^0 = W^0$. Then there exists a positive constant C such that, for $J = 1, 2, \ldots, M$, and k sufficiently small,*

$$\|u(t_J) - \hat{U}_h^J\|_{H^2} \leq C(k^2 + h^{r-1}), \tag{50}$$

where

$$\hat{U}_h^J = \frac{1}{2}\left(U_h^{J+\frac{1}{2}} + U_h^{J-\frac{1}{2}}\right) = \frac{1}{4}\left(U_h^{J-1} + 2U_h^J + U_h^{J+1}\right).$$

Proof. From (16) and the elliptic projection (12),

$$L^{n-\frac{1}{2}} W(\xi, t_{n-\frac{1}{2}}) = L^{n-\frac{1}{2}} u(\xi, t_{n-\frac{1}{2}}), \quad \xi \in \mathcal{G}, \tag{51}$$

we find that

$$\partial_t \theta^n(\xi) + L^{n-\frac{1}{2}} \theta^{n-\frac{1}{2}}(\xi) = \partial_t \eta^n(\xi) + \sigma^{n-\frac{1}{2}}(\xi), \quad \xi \in \mathcal{G}. \tag{52}$$

Using properties of L in Lemma 2.3, we note

$$\|L^{n-\frac{1}{2}} \theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r} \geq a_{\min} \|\Delta \theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r} - C\|\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r}. \tag{53}$$

Hence, using (52) and (53), we find that

$$\|\Delta \theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r} \leq C\left(\|\partial_t \theta^n\|_{\mathcal{M}_r} + \|\partial_t \eta^n\|_{\mathcal{M}_r} + \|\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r} + \|\sigma^{n-\frac{1}{2}}\|_{\mathcal{M}_r}\right). \tag{54}$$

Since an estimate of $\|\partial_t \eta^n\|_{\mathcal{M}_r}$ is given in (25) and $\|\theta^n\|$ can be estimated from (30), it remains to bound $\|\partial_t \theta^n\|_{\mathcal{M}_r}$.

Taking the difference quotient of both sides of (52) with respect to time, we obtain for $n \geq 2$

$$\partial_t^2 \theta^n(\xi) + \partial_t(L^{n-\frac{1}{2}} \theta^{n-\frac{1}{2}})(\xi) = \partial_t^2 \eta^n(\xi) + + \partial_t \sigma^{n-\frac{1}{2}}(\xi), \quad \xi \in \mathcal{G}. \tag{55}$$

Then, (55) can be rewritten as

$$\left\langle \partial_t^2 \theta^n, v\right\rangle + \left\langle L^{n-\frac{1}{2}}(\partial_t \theta^{n-\frac{1}{2}}), v\right\rangle = -\left\langle (\partial_t L^{n-\frac{1}{2}})\theta^{n-\frac{1}{2}}, v\right\rangle + \left\langle \partial_t^2 \eta^n, v\right\rangle + \left\langle \partial_t \sigma^{n-\frac{1}{2}}, v\right\rangle \quad \forall v \in \mathcal{M}_r^0(\delta),$$

and, with $v = \partial_t \theta^{n-\frac{1}{2}}$, we obtain

$$\begin{aligned} &\left\langle \partial_t^2 \theta^n, \partial_t \theta^{n-\frac{1}{2}}\right\rangle + \left\langle L^{n-\frac{1}{2}}(\partial_t \theta^{n-\frac{1}{2}}), \partial_t \theta^{n-\frac{1}{2}}\right\rangle \\ &\quad = -\left\langle (\partial_t L^{n-\frac{1}{2}})\theta^{n-\frac{1}{2}}, \partial_t \theta^{n-\frac{1}{2}}\right\rangle + \left\langle \partial_t^2 \eta^n, \partial_t \theta^{n-\frac{1}{2}}\right\rangle + \left\langle \partial_t \sigma^{n-\frac{1}{2}}, \partial_t \theta^{n-\frac{1}{2}}\right\rangle. \end{aligned} \tag{56}$$

For the terms on the left hand side, we observe that

$$\left\langle \partial_t^2 \theta^n, \partial_t \theta^{n-\frac{1}{2}}\right\rangle = \frac{1}{2} \partial_t \|\partial_t \theta^n\|_{\mathcal{M}_r}^2, \tag{57}$$

and using Lemma 2.3 and following the analysis in (25), we find that

$$\left\langle L^{n-\frac{1}{2}}(\partial_t \theta^{n-\frac{1}{2}}), \partial_t \theta^{n-\frac{1}{2}}\right\rangle \geq \frac{a_{\min}}{2} \|\nabla \partial_t \theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r}^2 - C\|\partial_t \theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r}^2. \tag{58}$$

Furthermore, we can easily estimate the first term on the right hand side of (56) to obtain

$$-\left\langle (\partial_t L^{n-\frac{1}{2}})\theta^{n-\frac{1}{2}}, \partial_t\theta^{n-\frac{1}{2}}\right\rangle \le C\|\nabla\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r}\,\|\nabla\partial_t\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r}, \tag{59}$$

and, for the last two terms, we note that

$$\begin{aligned}\left\langle \partial_t^2\eta^n, \partial_t\theta^{n-\frac{1}{2}}\right\rangle &\le \|\partial_t^2\eta^n\|_{\mathcal{M}_r}\,\|\partial_t\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r},\\ \left\langle \partial_t\sigma^{n-\frac{1}{2}}, \partial_t\theta^{n-\frac{1}{2}}\right\rangle &\le \|\partial_t\sigma^{n-\frac{1}{2}}\|_{\mathcal{M}_r}\,\|\partial_t\theta^{n-\frac{1}{2}}\|_{\mathcal{M}_r}.\end{aligned} \tag{60}$$

On substituting (57)–(60) into (56), we use Young's inequality and then sum the resulting expression from $n=2$ to m to find that

$$\begin{aligned}\|\partial_t\theta^m\|^2_{\mathcal{M}_r} + a_{\min}k\sum_{n=2}^{m}\|\nabla\partial_t\theta^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} &\le \|\partial_t\theta^1\|^2_{\mathcal{M}_r} + Ck\sum_{n=2}^{m}\|\nabla\theta^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\\ +\quad Ck\sum_{n=2}^{m}\left(\|\partial_t^2\eta^n\|^2_{\mathcal{M}_r} + \|\partial_t\sigma^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\right) &+ Ck\sum_{n=2}^{m}\|\partial_t\theta^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}.\end{aligned} \tag{61}$$

We estimate the first term on the right hand side of (61) as follows. For $n=1$, form the discrete inner product of (52) and $\partial_t\theta^1$ to obtain

$$\|\partial_t\theta^1\|^2_{\mathcal{M}_r} + \left\langle L^{\frac{1}{2}}\theta^{\frac{1}{2}}, \partial_t\theta^1\right\rangle = \langle\partial_t\eta^1, \partial_t\theta^1\rangle + \left\langle\partial_t\sigma^{\frac{1}{2}}, \partial_t\theta^1\right\rangle. \tag{62}$$

With $U^0 = W^0$, we obtain

$$\left\langle L^{\frac{1}{2}}\theta^{\frac{1}{2}}, \partial_t\theta^1\right\rangle = \frac{1}{2}\left\langle L^{\frac{1}{2}}\theta^1, \partial_t\theta^1\right\rangle = \frac{1}{2}k\left\langle L^{\frac{1}{2}}\partial_t\theta^1, \partial_t\theta^1\right\rangle, \tag{63}$$

and on substituting in (62), we arrive at

$$\|\partial_t\theta^1\|^2_{\mathcal{M}_r} + k\left\langle L^{\frac{1}{2}}\partial_t\theta^1, \partial_t\theta^1\right\rangle \le C\left(\|\partial_t\eta^1\|^2_{\mathcal{M}_r} + \|\partial_t\sigma^{\frac{1}{2}}\|^2_{\mathcal{M}_r}\right), \tag{64}$$

after multiplying by 2 and using the Cauchy-Schwarz inequality along with Young's inequality. Now, we apply (58) with $n=1$ to the second term on the left hand side of (64) so that

$$(1-Ck)\|\partial_t\theta^1\|^2_{\mathcal{M}_r} + a_{\min}k\|\nabla\partial_t\theta^1\|^2_{\mathcal{M}_r} \le C\left(\|\partial_t\eta^1\|^2_{\mathcal{M}_r} + \|\partial_t\sigma^{\frac{1}{2}}\|^2_{\mathcal{M}_r}\right). \tag{65}$$

Then, for k sufficiently small, (65) gives

$$\|\partial_t\theta^1\|^2_{\mathcal{M}_r} \le C\left(\|\partial_t\eta^1\|^2_{\mathcal{M}_r} + \|\partial_t\sigma^{\frac{1}{2}}\|^2_{\mathcal{M}_r}\right). \tag{66}$$

On substituting (66) into (61), we obtain

$$\begin{aligned}\|\partial_t\theta^m\|^2_{\mathcal{M}_r} \le C\Bigg\{ & k\sum_{n=1}^{m}\|\nabla\theta^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} + k\sum_{n=1}^{m}\left(\|\partial_t^2\eta^n\|^2_{\mathcal{M}_r} + \|\partial_t\sigma^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\right)\\ & +k\sum_{n=2}^{m}\left(\|\partial_t\theta^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r} + \|\partial_t\sigma^{n-\frac{1}{2}}\|^2_{\mathcal{M}_r}\right) + \|\partial_t\eta^1\|^2_{\mathcal{M}_r} + \|\sigma^{\frac{1}{2}}\|^2_{\mathcal{M}_r}\Bigg\}.\end{aligned} \tag{67}$$

Bounds on the first two terms on the right hand side of (67) can be obtained from (31) and (25). Moreover, estimates for $\|\partial_t\eta^1\|^2_{\mathcal{M}_r}$ and $\|\sigma^{\frac{1}{2}}\|^2_{\mathcal{M}_r}$ are derived earlier. Thus, it remains to estimate $\|\partial_t\sigma^{n-\frac{1}{2}}\|^{\frac{1}{2}}_{\mathcal{M}_r}$.

Now $\sigma = \sigma_1 + \sigma_2$ where σ_1 and σ_1 are given by (22). From [10, page 491], we obtain

$$k \sum_{n=1}^{m} \|\partial_t \sigma_1^{n-\frac{1}{2}}\|_{L^\infty}^2 \le Ck^4 \|u_{tttt}\|_{L^2(L^\infty)}^2. \tag{68}$$

For $\sigma_2^{n-\frac{1}{2}}$, note that

$$\begin{aligned}
\partial_t \sigma_2^{n-\frac{1}{2}} &= \partial_t (L^{n-\frac{1}{2}} (W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}})) \\
&= \frac{1}{k} \left(L^{n-\frac{1}{2}} (W(t_{n-\frac{1}{2}}) - L^{n-\frac{3}{2}} (W(t_{n-\frac{3}{2}}) \right) \\
&= L^{n-\frac{3}{2}} \partial_t \left(W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}} \right) + \partial_t (L^{n-\frac{1}{2}}) \left(W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}} \right).
\end{aligned}$$

Then,

$$\begin{aligned}
\|\partial_t \sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r} &\le \|L^{n-\frac{3}{2}} \partial_t (W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}})\|_{\mathcal{M}_r} \\
&\quad + \|\partial_t (L^{n-\frac{1}{2}})(W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}})\|_{\mathcal{M}_r} \\
&\le C \left(\|W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}}\|_{W^{2,\infty}} + k \|\partial_t (W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}})\|_{W^{2,\infty}} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\partial_t (W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}}) = -\frac{k^2}{2} \partial_t (W_{tt}(t_{n-\frac{1}{2}})) \\
&= -\frac{1}{2k} \left(\int_{t_{n-\frac{1}{2}}}^{t_n} (t_n - s)^2 W_{ttt}(s)\, ds - \int_{t_{n-\frac{3}{2}}}^{t_{n-1}} (t_{n-1} - s)^2 W_{ttt}(s)\, ds \right) \\
&\quad + \frac{1}{2k} \left(\int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (s - t_{n-1})^2) W_{ttt}(s)\, ds - \int_{t_{n-2}}^{t_{n-\frac{3}{2}}} (t_{n-1} - s)^2 W_{ttt}(s)\, ds \right),
\end{aligned}$$

since

$$\partial_t W_{tt}(t_{n-\frac{1}{2}}) = \frac{1}{k} \int_{t_{n-2}}^{t_{n-\frac{3}{2}}} W_{ttt}(s)\, ds.$$

Thus,

$$\|\partial_t (W(t_{n-\frac{1}{2}}) - W^{n-\frac{1}{2}})\|_{W^{2,\infty}} \le k^2 \|W_{ttt}\|_{L^\infty(W^{2,\infty})},$$

and, proceeding as in the derivation of (28), we obtain

$$k \sum_{n=2}^{m} \|\partial_t \sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r} \le k^4 \|W_{ttt}\|_{L^\infty(W^{2,\infty})}^2 \le k^4 \|u_{ttt}\|_{L^\infty(W^{2,\infty})}^2. \tag{69}$$

Combining (54), (67) and (69) and using (11), we then obtain

$$\|\theta^{J-\frac{1}{2}}\|_{H^2} \le C(k^2 + h^{r+1}),$$

and, on using the triangle inequality and Lemma 2.2,

$$\|u(t_{J-\frac{1}{2}}) - U_h^{J-\frac{1}{2}}\|_{H^2} \le C(k^2 + h^{r-1}). \tag{70}$$

The desired result, (50), then follows from the triangle inequality. □

Remark 3.3. This theorem yields a superconvergence result for $\|\theta\|_{H^2}$. Using Sobolev's inequality, we obtain

$$\|\theta^{J-\frac{1}{2}}\|_{W^{1,\infty}} \le C \log\left(\frac{1}{h}\right) \|\theta^{J-\frac{1}{2}}\|_{H^2} \le C \log\left(\frac{1}{h}\right) (k^2 + h^{r+1}).$$

Thus, provided the necessary approximation properties of η^J are available, it follows that

$$\|u(t_J) - \hat{U}_h^J\|_{W^{1,\infty}} \le C \log\left(\frac{1}{h}\right)(k^2 + h^r).$$

4. The ADI Crank-Nicolson OSC Method

OSC combined with the Crank-Nicolson and ADI scheme consists in determining $\{U_h^n\}_{n=1}^M \subset \mathcal{M}_r^0(\delta)$ such that

$$\left(\frac{V_h^{n,\frac12} - U_h^{n-1}}{0.5k} + L_1^{n-\frac12} V_h^{n,\frac12} + L_2^{n-\frac12} U_h^{n-1}\right)(\xi) \;=\; f^{n-\frac12}(\xi), \quad \xi \in \mathcal{G}_r \tag{71}$$

$$\left(\frac{U_h^n - V_h^{n,\frac12}}{0.5k} + L_1^{n-\frac12} V_h^{n,\frac12} + L_2^{n-\frac12} U_h^{n}\right)(\xi) \;=\; f^{n-\frac12}(\xi), \quad \xi \in \mathcal{G}_r, \tag{72}$$

where $V_h^{n,\frac12}$ is an auxiliary quantity, and $U_h^0 \in \mathcal{M}_r^0(\delta)$ is to be chosen later. On eliminating $V_h^{n,\frac12}$ from (71)–(72),we obtain

$$\left(\partial_t U_h^n + L^{n-\frac12} U_h^{n-\frac12} + \frac{k^2}{4} L_1^{n-\frac12} L_2^{n-\frac12} \partial_t U_h^n\right)(\xi) = f^{n-\frac12}(\xi), \quad \xi \in \mathcal{G}_r. \tag{73}$$

Equivalently,

$$\langle \partial_t U_h^n, v\rangle + \langle L^{n-\frac12} U_h^{n-\frac12}, v\rangle + \frac{k^2}{4}\langle L_1^{n-\frac12} L_2^{n-\frac12} \partial_t U_h^n, v\rangle = \langle f^{n-\frac12}, v\rangle \quad \forall v \in \mathcal{M}_r^0(\delta). \tag{74}$$

In this section, we briefly sketch the derivation of $H^j, j = 0, 1$, estimates.

Theorem 4.1. *Let U_h^n be the solution of the ADI Crank-Nicolson OSC scheme (74) and let $U_h^0 = W^0$. Then there exists a positive constant C such that, for $J = 1, 2, \ldots, M$, and k sufficiently small,*

$$\|u(t_J) - U_h^J\|_{H^j} \le C(k^2 + h^{r+j}), \quad j = 0, 1. \tag{75}$$

Proof. As usual, we write $u(t_n) - U_h^n = (u(t_n) - W^n) - (U_h^n - W^n) = \eta^n - \theta^n$. Then, using (1) with (12) at $t = t_{n-\frac12}$ and (74), it follows that

$$\begin{aligned} &\langle \partial_t \theta^n, v\rangle + \langle L^{n-\frac12} \theta^{n-\frac12}, v\rangle + \frac{k^2}{4}\langle L_1^{n-\frac12} L_2^{n-\frac12} \partial_t \theta^n, v\rangle \\ &\quad = \langle \partial_t \eta^n, v\rangle + \langle \partial_t \sigma^{n-\frac12}, v\rangle - \frac{k^2}{4}\langle L_1^{n-\frac12} L_2^{n-\frac12} \partial_t W^n, v\rangle. \end{aligned} \tag{76}$$

Setting $v = \partial_t \theta^n$ in (76), we obtain

$$\begin{aligned} &\|\partial_t \theta^n\|_{\mathcal{M}_r}^2 + \langle L^{n-\frac12} \theta^{n-\frac12}, \partial_t \theta^n\rangle + \frac{k^2}{4}\langle L_1^{n-\frac12} L_2^{n-\frac12} \partial_t \theta^n, \partial_t \theta^n\rangle \\ &\quad = \langle \partial_t \eta^n, \partial_t \theta^n\rangle + \langle \partial_t \sigma^{n-\frac12}, \partial_t \theta^n\rangle - \frac{k^2}{4}\langle L_1^{n-\frac12} L_2^{n-\frac12} \partial_t W^n, \partial_t \theta^n\rangle. \end{aligned} \tag{77}$$

There are only two terms in (77) which are not estimated in the proof of Theorem 3.1, namely,

$$\frac{k^2}{4}\langle L_1^{n-\frac12} L_2^{n-\frac12} \partial_t \theta^n, \partial_t \theta^n\rangle$$

and

$$\frac{k^2}{4}\langle L_1^{n-\frac12} L_2^{n-\frac12} \partial_t W^n, \partial_t \theta^n\rangle.$$

To estimate these terms, we first note that, using [4, Lemma 4.2],

$$\begin{aligned}(78)\qquad \langle L_1^{n-\frac{1}{2}} L_2^{n-\frac{1}{2}} \partial_t\theta^n, \partial_t\theta^n\rangle &\geq -Ck^2\|\partial_t\theta^n\|_{H^1}^2 \\ &= -C\|\theta^n-\theta^{n-1}\|_{H^1}^2 \geq -C(\|\theta^n\|_{H^1}^2+\|\theta^n\|_{H^1}^2).\end{aligned}$$

Since the coefficients of L_1 and L_2 are smooth and $W^n = u^n - \eta^n$, we have

$$\begin{aligned}(79)\ \frac{k^2}{4}&\langle L_1^{n-\frac{1}{2}} L_2^{n-\frac{1}{2}} \partial_t W^n, \partial_t\theta^n\rangle \\ &\leq \frac{k^2}{4}\left(\|L_1^{n-\frac{1}{2}} L_2^{n-\frac{1}{2}}\partial_t\eta^n\|_{\mathcal{M}_r} + \|L_1^{n-\frac{1}{2}} L_2^{n-\frac{1}{2}}\partial_t u^n\|_{\mathcal{M}_r}\right)\|\partial_t\theta^n\| \\ &\leq \frac{k^2}{4}\int_{t_{n-1}}^{t_n}(\|\eta_{txxyy}\|_{\mathcal{M}_r}+\|\eta_{txxy}\|_{\mathcal{M}_r}+\|\eta_t\|_{H^2}+\|u_t\|_{W^{4,\infty}})\|\partial_t\theta_n\|\,ds \\ &\leq Ck^2\|\partial_t\theta_n\|.\end{aligned}$$

On substituting (78) and (79) in (76) with $v = \partial_t\theta^n$ and using Young's inequality, we obtain on following the proof of Theorem 3.2,

$$\begin{aligned}(80)\quad 2k\sum_{n=1}^m\|\partial_t\theta^n\|^2 + (a_{\min}-Ck)\|\nabla\theta^n\|_{\mathcal{M}_r}^2 &\leq C\left(\|\partial_t\theta^0\|^2 + k^4 + k\sum_{n=1}^m\|\nabla\theta^n\|^2\right) \\ &+Ck\sum_{n=1}^m\left(\|\partial_t\eta^n\|_{\mathcal{M}_r}^2 + \|\sigma_1^{n-\frac{1}{2}}\|_{\mathcal{M}_r}^2 + \|\sigma_2^{n-\frac{1}{2}}\|_{\mathcal{M}_r}^2\right);\end{aligned}$$

cf. (48). Using Gronwall's lemma, we obtain, for k sufficiently small,

$$(81)\qquad \|\nabla\theta^m\| \leq C(k^2+h^{r+1}),$$

provided $\theta^0 = 0$; that is, $U_h^0 = W^0$. Thus, if $U_h^0 = W^0$, there exists a positive constant C such that

$$\|u(t_J) - U_h^J\| \leq C(k^2+h^{r+1}),$$

and

$$\|\nabla(u(t_J) - U_h^J)\| \leq C(k^2+h^r).$$

Remark 4.1. As a consequence of (81) and Sobolev's inequality, we obtain

$$\|\theta^J\|_{L^\infty} \leq C\log\left(\frac{1}{h}\right)\|\nabla\theta^J\| \leq C\log\left(\frac{1}{h}\right)(k^2+h^{r+1}),$$

as in Remark 3.2.

5. Concluding Remarks

In this article, we have presented convergence analyses of the Crank-Nicolson OSC method and the ADI Crank-Nicolson OSC method for a class of general linear parabolic initial/boundary value problems on rectangular domains. These analyses yield rates of convergence observed in numerical experiments reported in the literature.

Significant extensions of the ADI Crank-Nicolson OSC method have been formulated. In particular, Bialecki and Fernandes [5] extend the method to a nonlinear parabolic equation in a rectangular polygon with Robin's boundary conditions. To obtain higher accuracy in time, these authors [6] formulated an ADI OSC scheme which is third order in time. Specifically, the standard OSC discretization with $r = 3$ is used for the spatial discretization, but for the time discretization, an ADI backward differentiation formula (BDF) of order three is employed. The results of numerical experiments demonstrate the expected convergence rates in various norms but an L^2 convergence analysis is provided for the heat equation only. The rigorous analysis of these extensions is a topic of future research.

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