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BACKWARD EULER SCHEMES FOR THE KELVIN-VOIGT VISCOELASTIC FLUID FLOW MODEL

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Abstract. In this paper, we discuss the backward Euler method along with its linearized version for the Kelvin-Voigt viscoelastic fluid flow model with non zero forcing function, which is either independent of time or in $\mathbf{L}^\infty(\mathbf{L^2})$. After deriving some bounds for the semidiscrete scheme, *a priori* estimates in Dirichlet norm for the fully discrete scheme are obtained, which are valid uniformly in time using a combination of discrete Gronwall's lemma and Stolz-Cesaro's classical result for sequences. Moreover, an existence of a discrete global attractor for the discrete problem is established. Further, optimal *a priori* error estimates are derived, whose bounds may depend exponentially in time. Under uniqueness condition, these estimates are shown to be uniform in time. Even when $\mathbf{f} = 0$, the present result improves upon earlier result of Bajpai *et al.* (IJNAM,10 (2013),pp.481-507) in the sense that error bounds in this article depend on $1/\sqrt{\kappa}$ as against $1/\kappa^r$, $r \geq 1$. Finally, numerical experiments are conducted which confirm our theoretical findings.

Key words. Viscoelastic fluids, Kelvin-Voigt model, a *priori* bounds, backward Euler method, discrete attractor, optimal error estimates, linearized backward Euler scheme, numerical experiments.

1. Introduction

Let Ω be a bounded convex polygonal or polyhedron domain in $\mathbb{R}^d$ $(d = 2 \text{ or } 3)$ with boundary $\partial\Omega$. Consider the following system of equations described by the Kelvin-Voigt viscoelastic fluid flow model (see, [20]): Find a pair $(\mathbf{u}, p)$ such that

$$(1)\qquad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \kappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(x,t),\ \ x\in\Omega,\ t>0$$

with incompressibility condition

$$(2)\qquad \nabla\cdot\mathbf{u} = 0,\ \ x\in\Omega,\ t>0,$$

initial and boundary conditions

$$(3)\qquad \mathbf{u}(x,0) = \mathbf{u}_0 \ \text{ in } \Omega, \qquad \mathbf{u} = 0 \ \text{ on } \partial\Omega,\ t\geq 0.$$

Here, $\mathbf{u} = \mathbf{u}(x,t)$ denotes the velocity vector, $p = p(x,t)$ is the pressure, $\nu > 0$ represents the kinematic coefficient of viscosity and κ is the retardation in time parameter. For some applications, we refer to [5],[6], [7] and references, therein.

Now, we quickly recall some theoretical developments on the Kelvin-Voigt model. Based on proof techniques of Ladyzenskaya [17] for establishing the wellposedness of the Navier Stokes system, Oskolkov [19, 20] has proved an existence of a global unique 'almost' classical solution in finite time interval for the initial and boundary value problem (1)-(3). Investigations on existence and uniqueness results for all time $t > 0$ have been further continued by him and his collaborators under various conditions on the forcing function $\mathbf{f}$, see [22] and [23].

For earlier results on numerical methods applied to the problem (1)-(3), we refer to [1] and [21]. Under the assumption that the solution is asymptotically stable

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as $t \to \infty$, Oskolkov [21] has proved convergence of spectral Galerkin approximations to the problem (1)-(3) in semi time axis $t \geq 0$. Later on, Pani *et al.* [26] have employed a variant of nonlinear semidiscrete spectral Galerkin method and derived optimal error estimates. Recently, Bajpai *et al.* [1] have applied finite element methods to discretize spatial variables and have established optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$ as well as $L^\infty(\mathbf{H}^1)$-norms and for the pressure term in $L^\infty(L^2)$- norm of the Kelvin-Voigt model with zero forcing function. It is, further, shown that both exact solution and semidiscrete solution decay exponentially in time. Moreover, the error estimates have similar exponential decay property. Subsequently, Bajpai *et al.* [2] have analyzed both first order backward Euler and second order backward difference schemes for the completely discretization of the problem (1)-(2), when the forcing function $\mathbf{f} = 0$. Firstly, an existence result is shown for the discrete nonlinear problem using a variant of Brouwer fixed point argument and optimal error estimates which reflect exponential decay property are proved. Note that their error bounds contain term like $\frac{1}{\kappa^r}$, where $r \geq 1$. For related articles on Navier-Stokes equations, see [11] and on Oldroyd model, refer to [9]-[10], [12], [24]-[27], [29]-[32].

When the non-zero forcing function $\mathbf{f} \in L^\infty(\mathbf{L}^2)$, which is crucial in the study of dynamical system, Pany *et al.* [27] have applied semidiscrete finite element method for the problem (1)-(3) and have proved the existence of a global attractor. New regularity results for the exact solution are established which are valid both uniformly in time as $t \mapsto \infty$ and in κ as $\kappa \mapsto 0$. With the help of Sobolev-Stokes projection introduced in [1], *a priori* optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$ as well as $L^\infty(\mathbf{H}^1)$-norms and for the pressure term in $L^\infty(L^2)$-norm are derived. Under uniqueness assumption, it is shown that error bounds are valid uniformly in time. When $\kappa = O(h^{2\delta})$, $\delta > 0$ small, where h is the spatial discretization parameter, it is, further, established that quasi-optimal error estimates are valid for small κ. Moreover, this articles concludes with several numerical experiments, which are based on backward Euler method with out error analysis. In continuation to the investigation in [27] on semidiscrete problem, in this article, a backward Euler method along with its linearized version for the time discretization is analyzed. *A priori* bounds for the discrete solution, specially in the Dirichlet norm are established using a combination of discrete Gronwall's lemma and Stolz-Cesaro theorem (see, pp 85-87 of [18]) for sequences, which can be thought of a discrete version of the L'Hospital's rule. It is, further, shown that the discrete problem has a global discrete attractor and then optimal error estimates are derived. More precisely, the following estimates are obtained

$$\|\mathbf{u}_h(t_n) - \mathbf{U}^n\| \leq Ck,$$

and

$$\|(p_h(t_n) - P^n)\| \leq \frac{C}{\sqrt{\kappa}}\, k,$$

where the pair $(\mathbf{U}^n, P^n)$ is the fully discrete solution of the backward Euler method and the pair $(\mathbf{u}_h(t_n), p_h(t_n))$ is the semi-discrete solution at time level t_n. Since constants in these error bounds depend on e^{Ct}, these results as in the Navier-Stokes case are valid locally. But under the uniqueness assumption, it is, further, shown that error estimates are valid uniformly in time. Then, using the contribution of semi-discrete error estimates from [27], we, finally, obtain error estimates for the complete discrete scheme.

This paper is organized as follows. Section **2** deals with some assumptions and discusses the weak formulation. In Section **3**, some *a priori* bounds for semidiscrete approximations are obtained which are to be used in our subsequent sections. Section **4** focuses on the backward Euler method and establishes the existence and uniqueness result for the discrete problem. It is, further, shown that discrete solution is bounded in Dirichlet norm and the discrete problem has a global attractor. In Section **5**, we establish optimal error estimates in the velocity and the pressure for the backward Euler method. Section **6** discusses the linearized backward Euler method. In Section **7**, some numerical experiments are conducted which confirm our theoretical findings.

2. Preliminaries and Weak Formulation

We denote by bold face letters the $\mathbb{R}^d$, $(d = 2, 3)$-valued function spaces such as

$$\mathbf{H}_0^1 = (H_0^1(\Omega))^d, \quad \mathbf{L}^2 = (L^2(\Omega))^d \text{ and } \mathbf{H}^m = (H^m(\Omega))^d,$$

where $H^m(\Omega)$ is the standard Hilbert Sobolev space of order m with norm $\|\cdot\|_m$. Note that $\mathbf{H}_0^1$ is equipped with a norm

$$\|\nabla\mathbf{v}\| = \left(\sum_{i=1}^{d}(\nabla v_i, \nabla v_i)\right)^{1/2}.$$

Now, introduce the following spaces of the vector valued functions:

$$\begin{aligned}\mathbf{J}_1 &= \{\boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla\cdot\boldsymbol{\phi} = 0\},\\ \mathbf{J} = \{\boldsymbol{\phi} \in \mathbf{L}^2 : \nabla\cdot\boldsymbol{\phi} &= 0 \text{ in } \Omega, \ \boldsymbol{\phi}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ holds weakly}\},\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$ and $\boldsymbol{\phi}\cdot\mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$, see [28]. Let $H^m/\mathbb{R}$ be the quotient space consisting of equivalence classes of elements of H^m differing by constants, with norm $\|p\|_{H^m/\mathbb{R}} = \inf_{c\in\mathbb{R}}\|p + c\|_m$. For any Banach space $\mathbf{X}$, let $L^p(0, T; X)$ be the space of measurable $\mathbf{X}$- valued functions ϕ on $(0, T)$ such that

$$\int_0^T \|\phi(t)\|_X^p\, dt < \infty \ \ if \ \ 1 \le p \le \infty,$$

and for $p = \infty$,

$$\operatorname*{ess\,sup}_{0<t<T}\|\phi(t)\|_X < \infty.$$

Let P be the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{J}$.

Throughout this article, we make the following assumptions:
(**A1**). For $\mathbf{g} \in \mathbf{L^2}$, let $\{\mathbf{v} \in \mathbf{J}_1, q \in L^2/\mathbb{R}\}$ be the unique pair of solution to the steady state Stokes problem, see [28],

$$\begin{aligned}-\Delta\mathbf{v} + \nabla q &= \mathbf{g},\\ \nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega, &\quad \mathbf{v}|_{\partial\Omega} = 0\end{aligned}$$

satisfying the following regularity result:

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/\mathbb{R}} \le C\|\mathbf{g}\|. \tag{4}$$

Set

$$-\tilde{\Delta} = -P\Delta : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \to \mathbf{J}$$

as the Stokes operator. Then, the assumption (**A1**) shows

$$\|\mathbf{v}\|_2 \le C\|\tilde{\Delta}\mathbf{v}\| \quad \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2. \tag{5}$$

Note that, the following estimates holds:

$$\|\mathbf{v}\|^2 \leq \lambda_1^{-1}\|\nabla\mathbf{v}\|^2 \quad \forall\mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad \text{and } \|\nabla\mathbf{v}\|^2 \leq \lambda_1^{-1}\|\tilde{\Delta}\mathbf{v}\|^2 \quad \forall\mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2, \tag{6}$$

where λ_1^{-1} is the best possible positive constant depending on the domain Ω in the Poincaré inequality.

(**A2**). There exists a positive constant M_0 such that the initial velocity $\mathbf{u}_0$ and the external force $\mathbf{f}$ satisfy

$$\mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{J}_1, \mathbf{f}, \mathbf{f}_t \in L^\infty(0,\infty;\mathbf{L^2})$$

with

$$\|\mathbf{u}_0\|_2 \leq M_0, \sup_{0<t<\infty} \|\mathbf{f}\|, \|\mathbf{f}_t\|, \|\mathbf{f}_{tt}\|_{-1} \leq M_0.$$

Moreover, set a bilinear form $a(\cdot,\cdot)$ on $\mathbf{H}_0^1 \times \mathbf{H}_0^1$ as

$$a(\mathbf{v},\boldsymbol{\phi}) = (\nabla\mathbf{v},\nabla\boldsymbol{\phi}) \quad \forall\mathbf{v},\ \boldsymbol{\phi} \in \mathbf{H}_0^1, \tag{7}$$

and a trilinear form $b(\cdot,\cdot,\cdot)$ on $\mathbf{H}_0^1 \times \mathbf{H}_0^1 \times \mathbf{H}_0^1$ by

$$b(\mathbf{v},\mathbf{w},\boldsymbol{\phi}) = \frac{1}{2}(\mathbf{v}\cdot\nabla\mathbf{w},\boldsymbol{\phi}) - \frac{1}{2}(\mathbf{v}\cdot\nabla\boldsymbol{\phi},\mathbf{w}) \ \forall\mathbf{v},\mathbf{w},\boldsymbol{\phi} \in \mathbf{H}_0^1. \tag{8}$$

Now, the weak formulation of problem (1)-(3) is to find a pair of functions $(\mathbf{u}(t), p(t)) \in \mathbf{H}_0^1 \times \mathrm{L}^2/\mathbb{R}$ with $\mathbf{u}(0) = \mathbf{u}_0$ such that for all $t > 0$

$$\begin{aligned} (\mathbf{u}_t,\boldsymbol{\phi}) + \kappa(\nabla\mathbf{u}_t,\nabla\boldsymbol{\phi}) + \nu(\nabla\mathbf{u},\nabla\boldsymbol{\phi}) + (\mathbf{u}\cdot\nabla\mathbf{u},\boldsymbol{\phi})& \\ + (p,\nabla\cdot\boldsymbol{\phi}) = (\mathbf{f},\boldsymbol{\phi}) \ \ \forall\boldsymbol{\phi} \in \mathbf{H}_0^1&, \\ (\nabla\cdot\mathbf{u},\chi) = 0 \qquad \forall\chi \in L^2& \end{aligned} \tag{9}$$

Equivalently, find $\mathbf{u}(t) \in \mathbf{J}_1$ such that for $t > 0$

$$\begin{aligned} (\mathbf{u}_t,\boldsymbol{\phi}) + \kappa\, a(\mathbf{u}_t,\boldsymbol{\phi}) + \nu\, a(\mathbf{u},\boldsymbol{\phi}) + b(\mathbf{u},\mathbf{u},\boldsymbol{\phi}) = (\mathbf{f},\boldsymbol{\phi}) \quad \forall\boldsymbol{\phi} \in \mathbf{J}_1, \\ \mathbf{u}(0) = \mathbf{u}_0. \end{aligned} \tag{10}$$

3. Finite Element Approximation

Let $\mathbf{H}_h$ and L_h, $0 < h < 1$ be finite dimensional subspaces of $\mathbf{H}_0^1$ and L^2, respectively, where $h > 0$ is a spatial discretization parameter. Further, let subspace $\mathbf{H}_h$ and L_h satisfy the following approximation properties:

(**B1**). For $\mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2$ and $q \in H^1/\mathbb{R}$, there exist approximations $i_h\mathbf{w} \in \mathbf{J_h}$ and $j_h q \in L_h$ such that

$$\|\mathbf{w} - i_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - i_h\mathbf{w})\| \leq K_0 h^2\|\mathbf{w}\|_2, \quad \|q - j_h q\|_{L^2/\mathbb{R}} \leq K_0 h\|q\|_{H^1/\mathbb{R}}.$$

Now, set the subspace $\mathbf{J}_h$ of $\mathbf{H}_h$ as

$$\mathbf{J}_h = \{\mathbf{v}_h \in \mathbf{H}_h : (\chi_h,\nabla\cdot\mathbf{v}_h) = 0 \quad \forall\chi_h \in L_h\}.$$

The semidiscrete formulation of (9) is to find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$\begin{aligned} (\mathbf{u}_{ht},\boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht},\boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h,\boldsymbol{\phi}_h) + b(\mathbf{u}_h,\mathbf{u}_h,\boldsymbol{\phi}_h)& \\ - (p_h,\nabla\cdot\boldsymbol{\phi}_h) = (\mathbf{f},\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{H}_h&, \\ (\nabla\cdot\mathbf{u}_h,\chi_h) = 0 \quad \forall\chi_h \in L_h.& \end{aligned} \tag{11}$$

Equivalently, seek $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$\begin{aligned} &(\mathbf{u}_{ht},\boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht},\boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h,\boldsymbol{\phi}_h) \\ &= -\, b(\mathbf{u}_h,\mathbf{u}_h,\boldsymbol{\phi}_h) + (\mathbf{f},\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{J}_h. \end{aligned} \tag{12}$$

Once, we compute $\mathbf{u}_h(t) \in J_h$, the approximation $p_h(t) \in L_h$ to the pressure $p(t)$ can be computed out by solving the following system

$$\begin{aligned} (p_h, \nabla\cdot\boldsymbol{\phi}_h) &= (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h, \boldsymbol{\phi}_h) \\ &+ \ b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) + (\mathbf{f}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h. \end{aligned} \tag{13}$$

For solvability of the systems (12) and (13), see [1]. Uniqueness is obtained in the quotient space L_h/N_h with norm given by

$$\|q_h\|_{L^2/N_h} = \inf_{\chi_h \in N_h} \|q_h + \chi_h\|,$$

where

$$N_h = \{q_h \in L_h : (q_h, \nabla\cdot\boldsymbol{\phi}_h) = 0 \ \ \forall \boldsymbol{\phi}_h \in \mathbf{H}_h\}.$$

Moreover, assume that the pair $(\mathbf{H}_h, L_h/N_h)$ satisfies the following uniform inf-sup condition:
(**B2**). For every $q_h \in L_h$, there is a non-trivial function $\boldsymbol{\phi}_h \in \mathbf{H_h}$ and a positive constant K_1, independent of h, such that

$$|(q_h, \nabla\cdot\boldsymbol{\phi}_h)| \geq K_1 \|\nabla\boldsymbol{\phi}_h\| \|q_h\|_{L^2/N_h}.$$

As a consequence of (**B1**), the following properties of the L^2 projection $P_h : \mathbf{L^2} \to \mathbf{J_h}$ hold: For $\boldsymbol{\phi} \in \mathbf{J}_1$, we note that, see ([8], [13]),

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla P_h\boldsymbol{\phi}\| \leq Ch\|\nabla\boldsymbol{\phi}\|, \tag{14}$$

and for $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$,

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - P_h\boldsymbol{\phi})\| \leq Ch^2\|\tilde{\Delta}\boldsymbol{\phi}\|. \tag{15}$$

Set the discrete operator $\Delta_h : \mathbf{H}_h \to \mathbf{H}_h$ via the bilinear form $a(\cdot,\cdot)$ as

$$a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (-\Delta_h \mathbf{v}_h, \boldsymbol{\phi}_h) \quad \forall \mathbf{v}_h, \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{16}$$

Now, the discrete analogue of the Stokes operator $\tilde{\Delta} = P\Delta$ is given as $\tilde{\Delta}_h = P_h\Delta_h$. Using Sobolev embedding theorems with Sobolev inequalities, it is a routine calculation to derive the following lemma, see page 360 of [14].

Lemma 3.1. *There exists a positive constant K such that for all $\boldsymbol{\phi}$, $\boldsymbol{\xi}$, $\chi \in \mathbf{H}_h$, the trilinear form $b(\cdot,\cdot,\cdot)$ satisfies the following:*

$$|b(\boldsymbol{\phi}, \boldsymbol{\xi}, \chi)| \leq K \begin{cases} \|\boldsymbol{\phi}\| \|\nabla\boldsymbol{\xi}\| \|\chi\|^{\frac{1}{2}} \|\tilde{\Delta}_h\chi\|^{\frac{1}{2}}, \\ \|\nabla\boldsymbol{\phi}\|^{1/2} \|\tilde{\Delta}_h\boldsymbol{\phi}\|^{1/2} \|\nabla\boldsymbol{\xi}\| \ \|\chi\|, \\ \|\nabla\boldsymbol{\phi}\| \|\nabla\boldsymbol{\xi}\|^{1/2} \|\tilde{\Delta}_h\boldsymbol{\xi}\|^{1/2} \|\chi\|, \\ \|\boldsymbol{\phi}\|^{\frac{1}{2}} \|\nabla\boldsymbol{\phi}\|^{\frac{1}{2}} \|\nabla\boldsymbol{\xi}\| \|\nabla\chi\|. \end{cases} \tag{17}$$

Moreover, the trilinear form satisfies

$$b(\mathbf{v}_h, \mathbf{w}_h, \mathbf{w}_h) = 0 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h. \tag{18}$$

Examples of subspaces $\mathbf{H_h}$ satisfying assumptions (**B1**) and (**B2**) can be found in [3], [4] and [13].
Now, in a series of lemmas given below we derive *a priori* estimates for the discrete solution $\mathbf{u}_h$ of (12) analogous to those known for the continuous solution $\mathbf{u}$ of (10) (see [27]).

Lemma 3.2. *With* $0 \leq \alpha < \dfrac{\nu\lambda_1}{4(1+\kappa\lambda_1)}$*, and* $\mathbf{u}_{0h} = P_h\mathbf{u}_0$*, let the assumptions* (**A1**)–(**A2**) *hold true. Then, the solution* $\mathbf{u}_h$ *of (12) satisfies*

$$\|\mathbf{u}_h(t)\|^2 + \|\nabla \mathbf{u}_h(t)\|^2 + \kappa\|\tilde{\Delta}_h \mathbf{u}_h(t)\|^2$$
$$+ 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\nabla \mathbf{u}_h(s)\|^2 + \|\tilde{\Delta}_h \mathbf{u}_h(s)\|^2)\, ds \leq C(\nu, \alpha, \lambda_1, M_0) \quad t > 0,$$

where $\beta = (\nu/2) - \alpha(\lambda_1^{-1} + \kappa) \geq \nu/4 > 0$.

Proof. Set $\hat{\mathbf{u}}_h(t) = e^{\alpha t}\mathbf{u}_h(t)$ for some $\alpha \geq 0$, then (12) becomes

$$(\hat{\mathbf{u}}_{ht}, \boldsymbol{\phi}_h) - \alpha(\hat{\mathbf{u}}_h, \boldsymbol{\phi}_h) + \kappa(\nabla\hat{\mathbf{u}}_{ht}, \nabla\boldsymbol{\phi}_h) - \kappa\alpha(\nabla\hat{\mathbf{u}}_h, \nabla\boldsymbol{\phi}_h)$$
$$+ \nu(\nabla\hat{\mathbf{u}}_h, \nabla\boldsymbol{\phi}_h) + e^{-\alpha t}b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \boldsymbol{\phi}_h) = (\hat{\mathbf{f}}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{19}$$

With $\boldsymbol{\phi}_h = \hat{\mathbf{u}}_h$ in (19), a use of (18) with (6) yields

$$\frac{1}{2}\frac{d}{dt}(\|\hat{\mathbf{u}}_h\|^2 + \kappa\|\nabla\hat{\mathbf{u}}_h\|^2) + \big(\nu - \alpha\big(\kappa + \lambda_1^{-1}\big)\big)\|\nabla\hat{\mathbf{u}}_h\|^2 \leq (\hat{\mathbf{f}}, \hat{\mathbf{u}}_h)$$
$$\leq \|\hat{\mathbf{f}}\|_{\mathbf{H}^{-1}}\,\|\nabla\hat{\mathbf{u}})\| \leq \frac{\nu}{2}\|\nabla\hat{\mathbf{u}}\|^2 + \frac{1}{2\nu}\|\hat{\mathbf{f}}\|^2_{\mathbf{H}^{-1}}. \tag{20}$$

Employing kick-back arguments and multiply the resulting inequality by 2, then, integrate the resulting one with respect to time to arrive at

$$\|\mathbf{u}_h\|^2 + \kappa\|\nabla\mathbf{u}_h\|^2 + 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_h(s)\|^2 ds$$
$$\leq e^{-2\alpha t}(\|\mathbf{u}_0\|^2 + \kappa\|\nabla\mathbf{u}_0\|^2) + \frac{1 - e^{-2\alpha t}}{\nu\alpha}\|f\|^2_{\mathbf{L}^\infty(\mathbf{H}^{-1})}$$
$$\leq K_1, \tag{21}$$

where K_1 depends on ν, α, λ_1 and M_0. Moreover, after dropping the first two terms on the left hand side of (21), take limit as $t \to \infty$. Then, a use of L'Hospital's rule yields

$$\limsup_{t\to\infty} \|\nabla\mathbf{u}_h\| \leq \frac{1}{\nu}\|\mathbf{f}\|_{\mathbf{L}^\infty(\mathbf{0},\infty;\mathbf{H}^{-1})}. \tag{22}$$

Now, apply the discrete Stokes operator $\tilde{\Delta}_h$ in (19) to rewrite it as

$$(\hat{\mathbf{u}}_{ht}, \boldsymbol{\phi}_h) - \alpha(\hat{\mathbf{u}}_h, \boldsymbol{\phi}_h) - \kappa(\tilde{\Delta}_h\hat{\mathbf{u}}_{ht}, \boldsymbol{\phi}_h) + \kappa\alpha(\tilde{\Delta}_h\hat{\mathbf{u}}_h, \boldsymbol{\phi}_h)$$
$$- \nu(\tilde{\Delta}_h\hat{\mathbf{u}}_h, \boldsymbol{\phi}_h) = -e^{-\alpha t}b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \boldsymbol{\phi}_h) + (\hat{\mathbf{f}}, \boldsymbol{\phi}). \tag{23}$$

Choose $\boldsymbol{\phi}_h = -\tilde{\Delta}_h\hat{\mathbf{u}}_h$ in (23) and apply $-(\hat{\mathbf{u}}_{ht}, \tilde{\Delta}_h\hat{\mathbf{u}}_h) = \frac{1}{2}\frac{d}{dt}\|\nabla\hat{\mathbf{u}}_h\|^2$ to find that

$$\frac{d}{dt}(\|\nabla\hat{\mathbf{u}}_h\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2) + 2(\nu - \kappa\alpha)\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2 - 2\alpha\|\nabla\hat{\mathbf{u}}_h\|^2$$
$$= 2e^{-\alpha t}b(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h, \tilde{\Delta}_h\hat{\mathbf{u}}_h) + 2(\hat{\mathbf{f}}, -\tilde{\Delta}_h\hat{\mathbf{u}}_h) = I_1 + I_2(say). \tag{24}$$

For I_1, a use of generalized Hölder's inequality yields

$$|I_1| \leq e^{-\alpha t}\|\hat{\mathbf{u}}_h\|_{L^4}\,\|\nabla\hat{\mathbf{u}}_h\|_{L^4}\,\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|. \tag{25}$$

For $d = 2$, substitute Ladyzhenskaya's inequality:

$$\|\hat{\mathbf{u}}_h\|_{L^4} \leq C\,\|\hat{\mathbf{u}}_h\|^{\frac{1}{2}}\,\|\nabla\hat{\mathbf{u}}_h\|^{\frac{1}{2}} \quad \text{and} \quad \|\nabla\hat{\mathbf{u}}_h\|_{L^4} \leq \|\nabla\hat{\mathbf{u}}_h\|^{\frac{1}{2}}\,\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^{\frac{1}{2}}.$$

in (25) and use the Young's inequality with $p = 4$, $q = \frac{4}{3}$, $\epsilon = \frac{2\nu}{9}$ to obtain

$$|I_1| \le Ce^{-\alpha t}\|\hat{\mathbf{u}}_h\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{u}}_h\|\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^{\frac{3}{2}}$$

$$\le C\left(\frac{1}{\nu}\right)^3 e^{2\alpha t}\|\mathbf{u}_h\|^2\|\nabla\mathbf{u}_h\|^4 + \frac{\nu}{6}\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2. \tag{26}$$

For I_2, apply the Cauchy-Schwarz inequality with the Young's inequality to arrive at

$$|I_2| = |(\hat{\mathbf{f}}, -\tilde{\Delta}_h\hat{\mathbf{u}}_h)| \le \|\hat{\mathbf{f}}\|\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\| \le \frac{\nu}{3}\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2 + \frac{3}{2\nu}\|\hat{\mathbf{f}}\|^2. \tag{27}$$

Substitute (26) and (27) in (24) to obtain

$$\|\nabla\hat{\mathbf{u}}_h(t)\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{u}}_h(t)\|^2 + \beta\int_0^t e^{2\alpha s}\|\tilde{\Delta}_h\mathbf{u}_h(s)\|^2\,ds \le (\|\nabla\mathbf{u}_{h0}\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{u}_{h0}\|^2)$$

$$+ C(\nu)\int_0^t \|\hat{\mathbf{f}}(s)\|^2\,ds + C(\nu)\int_0^t \|\mathbf{u}_h(s)\|^2\|\nabla\mathbf{u}_h(s)\|^2\|\nabla\hat{\mathbf{u}}_h(s)\|^2 ds. \tag{28}$$

An application of Gronwall's lemma yields

$$\|\nabla\hat{\mathbf{u}}_h(t)\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{u}}_h(t)\|^2 + \beta\int_0^t e^{2\alpha s}\|\tilde{\Delta}_h\mathbf{u}_h(s)\|^2\,ds$$

$$\le \{(\|\nabla\mathbf{u}_h(0)\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{u}_h(0)\|^2)$$

$$+ C(\nu)\int_0^t \|\hat{\mathbf{f}}(s)\|^2 ds\} \times exp\left(C(\nu)\int_0^t \|\mathbf{u}_h(s)\|^2\|\nabla\mathbf{u}_h(s)\|^2 ds\right). \tag{29}$$

Note that

(30)

$$\|\tilde{\Delta}_h\mathbf{u}_h(0)\| = \|\tilde{\Delta}_h\mathbf{u}_{0h}\| = \sup_{0\ne\mathbf{v}_h\in\mathbf{J}_h}\frac{(\tilde{\Delta}_h\mathbf{u}_h(0), \mathbf{v}_h)}{\|\mathbf{v}_h\|} = -\sup_{0\ne\mathbf{v}_h\in\mathbf{J}_h}\frac{(\nabla\mathbf{u}_h(0), \nabla\mathbf{v}_h)}{\|\mathbf{v}_h\|}.$$

Observe that

$$(\nabla\mathbf{u}_{0h}, \nabla\mathbf{v}_h) = (\nabla(\mathbf{u}_{0h} - \mathbf{u}_0), \nabla\mathbf{v}_h) + (\nabla\mathbf{u}_0, \nabla\mathbf{v}_h)$$

$$\le Ch\|\mathbf{u}_0\|_2\|\nabla\mathbf{v}_h\| + \|\tilde{\Delta}\mathbf{u}_0\|\|\mathbf{v}_h\|. \tag{31}$$

Now, a use of inverse property ($\|\nabla\mathbf{v}_h\| \le Ch^{-1}\|\mathbf{v}_h\| \;\;\forall\;\; \mathbf{v}_h \in J_h$) shows that

$$(\nabla\mathbf{u}_{0h}, \nabla\mathbf{v}_h) \le C\;\|\mathbf{u}_0\|_2\|\mathbf{v}_h\|. \tag{32}$$

On substitution (32) in (30), we obtain

$$\|\tilde{\Delta}_h\mathbf{u}_{0h}\| \le C\|\mathbf{u}_0\|_2. \tag{33}$$

Use **(A2)** with (30) in (29) to find that

$$\|\nabla\mathbf{u}_h(t)\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{u}_h(t)\|^2 + \beta\int_0^t e^{2\alpha s}\|\tilde{\Delta}_h\mathbf{u}_h(s)\|^2\,ds \le C(\nu, \alpha, K_1)$$

$$exp\left(C(\nu)\int_0^t \|\mathbf{u}_h(s)\|^2\|\nabla\mathbf{u}_h(s)\|^2 ds\right). \tag{34}$$

Note that for all finite but fixed $T_0 > 0$ and $0 < t \le T_0$, a use of (21) in (34) yields

$$\|\nabla\mathbf{u}_h(t)\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{u}_h(t)\|^2 + \beta\int_0^t e^{2\alpha s}\|\tilde{\Delta}_h\mathbf{u}_h(s)\|^2\,ds \le C(\nu, \alpha, K_1, T_0). \tag{35}$$

Since the inequality (35) is valid for all finite, but fixed T_0, now a use of (22) yields

$$\limsup_{t\to\infty}\|\nabla\mathbf{u}_h\| \le C,$$

and hence, this leads to the boundedness of $\|\nabla \mathbf{u}_h(t)\|$ for all $t > 0$. Combining (21) with (35) completes the rest of the proof. □

Lemma 3.3. *Under assumptions* (**A1**)-(**A2**)*, there exists a positive constant* $C = C(\nu, \alpha, \lambda_1, M)$ *such that the following holds true for* $0 < \alpha < \dfrac{\nu\lambda_1}{4(1+\lambda_1\kappa)}$ *and for all* $t > 0$

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2 + 2\kappa\|\nabla \mathbf{u}_{ht}(s)\|^2)\, ds + \nu\|\nabla \mathbf{u}_h(t)\|^2 \le C.$$

Proof. Set $\boldsymbol{\phi} = e^{2\alpha t}\mathbf{u}_{ht}$ in (11) and obtain

$$e^{2\alpha t}(\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla \mathbf{u}_{ht}\|^2) + \frac{\nu}{2}e^{2\alpha t}\frac{d}{dt}\|\nabla \mathbf{u}_h\|^2 = e^{2\alpha t}(\mathbf{f}, \mathbf{u}_{ht}) - e^{2\alpha t}(\mathbf{u}_h.\nabla \mathbf{u}_h, \mathbf{u}_{ht}). \tag{36}$$

A use Sobolev imbedding theorem for the nonlinear term on the right hand side of (36) leads to

$$\begin{aligned} |(\mathbf{u}_h.\nabla \mathbf{u}_h, \mathbf{u}_{ht})| &\le C\|\mathbf{u}_h\|_{\mathbf{L}^4}\ \|\nabla \mathbf{u}_h\|_{\mathbf{L}^4}\ \|\mathbf{u}_{ht}\| \\ &\le C\|\nabla \mathbf{u}_h\|\ \|\tilde{\Delta}_h \mathbf{u}_h\|\ \|\mathbf{u}_{ht}\|. \end{aligned} \tag{37}$$

Substitute (37) in (36) and apply the Young's inequality. Then, integrate the resulting inequality with respect to time from 0 to t and multiply the resulting equation by $e^{-2\alpha t}$ to find that

$$\begin{aligned} e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2 + 2\kappa\|\nabla \mathbf{u}_{ht}(s)\|^2) ds + \nu\|\nabla \mathbf{u}_h(t)\|^2 &\le Ce^{-2\alpha t}\|\nabla \mathbf{u}_{h0}\|^2 \\ + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla \mathbf{u}_h(s)\|^2 ds &+ e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{f}(s)\|^2 ds \\ + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla \mathbf{u}_h(s)\|^2\|\tilde{\Delta}_h \mathbf{u}_h(s)\|^2 ds. \end{aligned} \tag{38}$$

A use of Lemma 3.2 yields to the desired estimate and this completes the rest of the proof. □

Lemma 3.4. *Let the assumptions* (**A1**)-(**A2**) *hold true. Then, there exists a positive constant* $C = C(\nu, \alpha, \lambda_1, M)$ *such that for all* $t > 0$

$$\|\mathbf{u}_{ht}(t)\|^2 + \kappa\|\nabla \mathbf{u}_{ht}(t)\|^2 + \nu e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla \mathbf{u}_{ht}(s)\|^2 ds \le C.$$

Proof. Differentiate (11) with respect to time and obtain

$$\begin{aligned} (\mathbf{u}_{htt}, \boldsymbol{\phi}) + \kappa(\nabla \mathbf{u}_{htt}, \nabla\boldsymbol{\phi}) + \nu(\nabla \mathbf{u}_{ht}, \nabla\boldsymbol{\phi}) &= -(\mathbf{u}_{ht}\cdot\nabla \mathbf{u}_h, \boldsymbol{\phi}) - (\mathbf{u}_h\cdot\nabla \mathbf{u}_{ht}, \boldsymbol{\phi}) \\ &+ (\mathbf{f_t}, \boldsymbol{\phi}) \quad \forall\boldsymbol{\phi}\in \mathbf{J_h}. \end{aligned} \tag{39}$$

Choose $\boldsymbol{\phi} = \mathbf{u}_{ht}$ in (39) with $(\mathbf{u}_h\cdot\nabla \mathbf{u}_{ht}, \mathbf{u}_{ht}) = 0$ to find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla \mathbf{u}_{ht}\|^2) + \nu\|\nabla \mathbf{u}_{ht}\|^2 \\ &= -(\mathbf{u}_{ht}\cdot\nabla \mathbf{u}_h, \mathbf{u}_{ht}) + (\mathbf{f_t}, \mathbf{u_{ht}}). \end{aligned} \tag{40}$$

An application of the Ladyzenskaya's inequality with the Young's inequality (with $p = 8$ and $q = 8/7$) yields

$$\begin{aligned} (\mathbf{u}_{ht}\cdot\nabla \mathbf{u}_h, \mathbf{u}_{ht}) &\le C\|\mathbf{u}_{ht}\|^{1/4}\|\nabla \mathbf{u}_h\|\|\nabla \mathbf{u}_{ht}\|^{7/4} \\ &\le C(\nu)\ \|\nabla \mathbf{u}_h\|^8\ \|\mathbf{u}_{ht}\|^2 + \frac{\nu}{4}\ \|\nabla \mathbf{u}_{ht}\|^2. \end{aligned} \tag{41}$$

A use of the Cauchy-Schwarz inequality with the Young's inequality shows

$$(\mathbf{f}_t, \mathbf{u}_{ht}) \le \|\mathbf{f}_t\| \, \|\mathbf{u}_{ht}\| \le \frac{1}{\sqrt{\lambda_1}}\|\mathbf{f}_t\| \, \|\nabla \mathbf{u}_{ht}\| \le \frac{1}{\lambda_1 \, \nu}\|\mathbf{f}_t\|^2 + \frac{\nu}{4} \, \|\nabla \mathbf{u}_{ht}\|^2. \tag{42}$$

Substitute (41)-(42) in (39) and then, multiply by $e^{2\alpha t}$. An application of *a priori* estimates from Lemmas 3.2 and 3.3 leads to

$$\begin{aligned}\frac{d}{dt}e^{2\alpha t}(\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla \mathbf{u}_{ht}\|^2) + \nu e^{2\alpha t}\|\nabla \mathbf{u}_{ht}\|^2 &\le C(\nu, \lambda_1)e^{2\alpha t}(\|\mathbf{u}_{ht}\|^2 + \|\mathbf{f}_t\|^2) \\ &\quad + 2\alpha e^{2\alpha t}(\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla \mathbf{u}_{ht}\|^2).\end{aligned} \tag{43}$$

Integrate (43) from 0 to t with respect to time to obtain

$$\begin{aligned}&\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla \mathbf{u}_{ht}\|^2 + \nu e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla \mathbf{u}_{ht}(s)\|^2 \, ds \\ \le& e^{-2\alpha t}(\|\mathbf{u}_{ht}(0)\|^2 + \kappa\|\nabla \mathbf{u}_{ht}(0)\|^2) \\ &+ Ce^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2 + \|\mathbf{f}_t(s)\|^2)ds \\ &+ 2\alpha e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{ht}(s)\|^2 + \kappa\|\nabla \mathbf{u}_{ht}(s)\|^2)ds.\end{aligned} \tag{44}$$

From (12), it is observed that

$$\begin{aligned}\|\mathbf{u}_{ht}\|^2 + \kappa\|\nabla \mathbf{u}_{ht}\|^2 &\le C(\|\tilde{\Delta}_h \mathbf{u}_h\|^2 + \|\mathbf{f}\|^2 + \|\mathbf{u}_h\|^2\|\nabla \mathbf{u}_h\|^4) \\ &\le C(\lambda_1)(\|\tilde{\Delta}_h \mathbf{u}_h\|^2 + \|\mathbf{f}\|^2).\end{aligned} \tag{45}$$

and for $t = 0$, it now follows that

$$\begin{aligned}\|\mathbf{u}_{ht}(0)\|^2 + \kappa\|\nabla \mathbf{u}_{ht}(0)\|^2 &\le C(\|\tilde{\Delta}_h \mathbf{u}_h(0)\|^2 + \|\mathbf{f}(0)\|^2 + \|\mathbf{u}_h(0)\|^2\|\nabla \mathbf{u}_h(0)\|^4) \\ &\le C(M_0)\end{aligned} \tag{46}$$

A use of Lemma 3.3, (46) and (45) with **(A2)** in (44) leads to desired estimates and this concludes the proof. □

Lemma 3.5. *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1 + \kappa\lambda_1)}$ *and let the assumptions* (**A1**)–(**A2**) *hold true. Then, there is a positive constant* $C = C(\nu, \alpha, \lambda_1, M_0)$ *such that for all* $t > 0$,

$$\|\mathbf{u}_{htt}(t)\|_{-1,h} + \kappa\|\nabla \mathbf{u}_{htt}\| \le \frac{C}{\sqrt{\kappa}}.$$

Moreover, there holds

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{htt}(s)\|^2_{-1,h} + \kappa\|\mathbf{u}_{htt}(s)\|^2)ds \le C.$$

Proof. Differentiation of (12) with respect to time to find that

$$\begin{aligned}(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + b(\mathbf{u}_{ht}, \mathbf{u}_h, \boldsymbol{\phi}_h) \\ + b(\mathbf{u}_h, \mathbf{u}_{ht}, \boldsymbol{\phi}_h) = (\mathbf{f}_t, \boldsymbol{\phi}) \ \forall \phi_h \in \mathbf{J}_h \ \ t > 0.\end{aligned} \tag{47}$$

Now, (47) is rewritten as

$$\begin{aligned}(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) =& -\kappa a(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) - \nu a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) - b(\mathbf{u}_{ht}, \mathbf{u}_h, \boldsymbol{\phi}_h) \\ &- b(\mathbf{u}_h, \mathbf{u}_{ht}, \boldsymbol{\phi}_h) + (\mathbf{f}_t, \boldsymbol{\phi}_h) \\ &\le (\kappa\|\nabla \mathbf{u}_{htt}\| + \nu\|\nabla \mathbf{u}_h t\| + (\|\nabla \mathbf{u}_{ht}\|\|\nabla \mathbf{u}_h\|) + \|\mathbf{f}_t\|_{-1}) \, \|\nabla \boldsymbol{\phi}_h\|.\end{aligned} \tag{48}$$

Choose $\boldsymbol{\phi} = \mathbf{u}_{htt}$ in (47) and drop the first term from the left hand side to obtain

$$\kappa\|\nabla \mathbf{u}_{htt}\| \leq \Big(\nu\|\nabla \mathbf{u}_{ht}\| + C\|\nabla \mathbf{u}_{ht}\|\|\nabla \mathbf{u}_h\| + \|\mathbf{f}_t\|_{-1}\Big). \tag{49}$$

An application of Lemmas 3.2 and 3.4 in (49) shows

$$\kappa\|\nabla \mathbf{u}_{htt}\| \leq \frac{C}{\sqrt{\kappa}}. \tag{50}$$

From (48), we obtain

$$\begin{aligned} \|\mathbf{u}_{htt}\|_{-1,h} &= \sup_{0\neq\boldsymbol{\phi}_h\in\mathbf{H}_h} \frac{(\mathbf{u}_{htt}, \boldsymbol{\phi})}{\|\nabla\boldsymbol{\phi}_h\|} \\ &\leq \kappa\|\nabla \mathbf{u}_{htt}\| + \|\nabla \mathbf{u}_{ht}\| + C(\|\nabla \mathbf{u}_{ht}\|\|\nabla \mathbf{u}_h\|) + \|\mathbf{f}_t\|_{-1}, \end{aligned} \tag{51}$$

and hence, from *a priori* bounds of Lemmas 3.2 and 3.4, it follows that

$$\|\mathbf{u}_{htt}\|_{-1,h} \leq \frac{1}{\sqrt{\kappa}} C(\lambda_1, \alpha, \nu, M_0). \tag{52}$$

Squaring (50), multiply by $e^{2\alpha t}$ and then integrate from 0 to t. Again multiply the resulting inequality by $e^{2\alpha t}$ to obtain

$$\begin{aligned} &e^{-2\alpha t}\int_0^t e^{2\alpha s}\kappa\|\nabla \mathbf{u}_{htt}\|^2\, ds \\ \leq &e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big(\nu\|\nabla \mathbf{u}_{ht}\|^2 + C\|\nabla \mathbf{u}_{ht}\|^2\|\nabla \mathbf{u}_h\|^2 + \|\mathbf{f}_t\|^2_{-1}\Big)\, ds. \end{aligned} \tag{53}$$

From Lemmas 3.2-3.4, it follows that

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\kappa\|\nabla \mathbf{u}_{htt}\|^2\, ds \leq C. \tag{54}$$

Similarly for (51), we find that

$$\begin{aligned} &e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_{htt}\|_{-1,h}\, ds \\ \leq &e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big(\kappa^2\|\nabla \mathbf{u}_{htt}\|^2 + \|\nabla \mathbf{u}_{ht}\|^2 + C(\|\nabla \mathbf{u}_{ht}\|^2\|\nabla \mathbf{u}_h\|^2) + \|\mathbf{f}_t\|^2_{-1}\Big)\, ds. \end{aligned} \tag{55}$$

Hence from Lemmas 3.2-3.4 and 54, the rest of the proof follows. This completes the rest of the proof. □

We now recall the following bounds for the semi-discrete method, proved in [27]

Theorem 3.1. *Let conditions* **(A1)-(A2)** *and* **(B1)-(B2)** *be satisfied and let the discrete initial velocity* $\mathbf{u}_{0h} = P_h\mathbf{u}_0$. *Then, there exists a positive constant* $C(\lambda_1, \nu, \alpha, M_0)$ *such that for all* $t > 0$ *and for* $0 \leq \alpha < \dfrac{\nu\lambda_1}{4(1+\lambda_1\kappa)}$,

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\| + h\Big(\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\| + \|(p-p_h)(t)\|\Big) \leq \frac{C}{\sqrt{\kappa}} h^2 e^{Ct}.$$

Moreover, under the assumption of the uniqueness condition, that is

$$\frac{N_0}{\nu}\|\mathbf{f}\|_{L^\infty(\mathbf{H}^{-1})} < 1 \quad \textit{and} \quad N_0 = \sup_{u,v,w\in\mathbf{H}_0^1(\Omega)} \frac{b(u,v,w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|}, \tag{56}$$

the following uniform in time estimate holds

$$\|(\mathbf{u}-\mathbf{u}_h(t))\| + h\|(p-p_h)(t)\| \le \frac{C}{\sqrt{\kappa}}h^2.$$

4. Backward Euler Method

Based on backward Euler scheme, a fully discrete method is analyzed in this section.

Let $k>0$ denote the time step size and $t_n = nk$. For smooth function $\boldsymbol{\phi}$ defined on $[0,T]$, set $\boldsymbol{\phi}^n = \boldsymbol{\phi}(t_n)$ and $\bar{\partial}_t\boldsymbol{\phi}^n = (\boldsymbol{\phi}^n - \boldsymbol{\phi}^{n-1})/k$. Now, the backward Euler scheme applied to (11) is to find $(\mathbf{U}^n, P^n) \in (\mathbf{H}_h, L_h)$ such that for all $n \ge 1$

$$\begin{aligned}(\bar{\partial}_t\mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h)&\\ = -b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) + (P^n, \nabla\cdot\boldsymbol{\phi}_h) + (\mathbf{f}^n, \boldsymbol{\phi}_h) \quad &\forall \boldsymbol{\phi}_h \in \mathbf{H}_h,\\ (\nabla\cdot\mathbf{U}^n, \chi_h) = 0 \qquad &\forall \chi_h \in L_h,\\ \mathbf{U}^0 = P_h\mathbf{u}_0 = \mathbf{u}_{0h}.&\end{aligned} \tag{57}$$

Equivalently, seek $\{\mathbf{U}^n\}_{n\ge1} \in \mathbf{J}_h$ such that

$$\begin{aligned}(\bar{\partial}_t\mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h)&\\ = -b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) + (\mathbf{f}^n, \boldsymbol{\phi}_h) \quad &\forall \boldsymbol{\phi}_h \in \mathbf{J}_h,\\ \mathbf{U}^0 = \mathbf{u}_{0h}.&\end{aligned} \tag{58}$$

Before obtaining *a priori* estimates for the discrete problem (58), we recall the following result for sequences, which is a counter part of the L'Hospital rule. For a proof, see, pp. 85-87 of [18].

Theorem 4.1. *(Stolz-Cesaro Theorem). Let $\{\phi^n\}_{n=0}^{\infty}$ be a sequence of real numbers. Further, let $\{\psi^n\}_{n=0}^{\infty}$ be a strictly monotone and divergent sequence. If*

$$\lim_{n\longrightarrow\infty}\left(\frac{\phi^n - \phi^{n-1}}{\psi^n - \psi^{n-1}}\right) = \ell,$$

then

$$\lim_{n\longrightarrow\infty}\left(\frac{\phi^n}{\psi^n}\right) = \ell$$

holds.

Now, we discuss uniform *a priori* bounds for the discrete solution $\{\mathbf{U}^n\}_{n\ge1}$.

Lemma 4.1. *With $0 \le \alpha < \dfrac{\nu\lambda_1}{4(1+\lambda_1\kappa)}$, choose k_0 so that for $0<k\le k_0$*

$$\left(\frac{\nu k\lambda_1}{4(\kappa\lambda_1+1)} + 1\right) > e^{\alpha k}. \tag{59}$$

Then, the discrete solution $\mathbf{U}^N$, $N\ge1$ of (58) satisfies

$$\begin{aligned}(\|\mathbf{U}^N\|^2 + \kappa\|\nabla\mathbf{U}^N\|^2) + 2\beta_1 e^{-2\alpha t_N}\; k\sum_{n=1}^{N} e^{2\alpha t_n}\|\nabla\mathbf{U}^n\|^2&\\ \le e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2) + \frac{e^{2\alpha k}}{2\alpha\nu}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})},&\end{aligned} \tag{60}$$

where

$$2\beta_1 = \left(\frac{1}{2}e^{-\alpha k}\nu - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\right) \ge \frac{\nu}{2}e^{-\alpha k} \ge \frac{\nu}{2}e^{-2\alpha k} > 0. \tag{61}$$

Moreover, the following estimate holds:

$$\limsup_{N\longrightarrow\infty} \|\nabla U^N\|^2 \leq \frac{1}{\nu^2}\|\mathbf{f}\|^2_{L^\infty(H^{-1})}. \tag{62}$$

Proof. Multiplying (58) by $e^{\alpha t_n}$, choose $\hat{\mathbf{U}}^n = e^{\alpha t_n}\mathbf{U}^n$ to obtain

$$\begin{aligned} e^{\alpha t_n}\Big((\bar{\partial}_t\mathbf{U}^n, \hat{\mathbf{U}}^n) + \kappa a(\bar{\partial}_t\mathbf{U}^n, \hat{\mathbf{U}}^n)\Big) &+ \nu a(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) \\ + e^{-\alpha t_n} b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) &= (\mathbf{f}^n, \hat{\mathbf{U}}^n). \end{aligned} \tag{63}$$

Observe that

$$e^{\alpha t_n}\bar{\partial}_t\mathbf{U}^n = e^{\alpha k}\bar{\partial}_t\hat{\mathbf{U}}^n - \Big(\frac{e^{\alpha k}-1}{k}\Big)\hat{\mathbf{U}}^n. \tag{64}$$

Then, use (64) in (63) and multiply the resulting equation by $e^{-\alpha k}$ to find that

$$\begin{aligned} (\bar{\partial}_t\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) + \kappa a(\bar{\partial}_t\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) - \Big(\frac{1-e^{-\alpha k}}{k}\Big)(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) + e^{-\alpha k}\nu a(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) \\ - \kappa\Big(\frac{1-e^{-\alpha k}}{k}\Big) a(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) + e^{-\alpha t_{n+1}} b(\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) \\ = e^{-\alpha k}(\hat{\mathbf{f}}^n, \hat{\mathbf{U}}^n). \end{aligned} \tag{65}$$

Note that

$$(\bar{\partial}_t\hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) \geq \frac{1}{2k}(\|\hat{\mathbf{U}}^n\|^2 - \|\hat{\mathbf{U}}^{n-1}\|^2) = \frac{1}{2}\bar{\partial}_t\|\hat{\mathbf{U}}^n\|^2. \tag{66}$$

Now using (6) in (66) leads to

$$\begin{aligned} \frac{1}{2}\bar{\partial}_t(\|\hat{\mathbf{U}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^n\|^2) + \Big(e^{-\alpha k}\nu - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{U}}^n\|^2 \\ \leq e^{-\alpha k}\,(\hat{\mathbf{f}}^n, \hat{\mathbf{U}}^n). \end{aligned} \tag{67}$$

The right-hand side of (67) can be estimated as

$$e^{-\alpha k}\,(\hat{\mathbf{f}}^n, \hat{\mathbf{U}}^n) \leq \frac{1}{2}e^{-\alpha k}\nu\|\nabla\hat{\mathbf{U}}^n\|^2 + \frac{1}{2\nu}e^{-\alpha k}\|\hat{\mathbf{f}}^n\|^2_{L^\infty(\mathbf{H}^{-1})}.$$

From (67), we obtain

$$\bar{\partial}_t(\|\hat{\mathbf{U}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^n\|^2) + 2\beta_1\|\nabla\hat{\mathbf{U}}^n\|^2 \leq \frac{e^{-\alpha k}}{\nu}\|\hat{\mathbf{f}}^n\|^2_{L^\infty(\mathbf{H}^{-1})}. \tag{68}$$

Multiplying (67) by k, then sum over $n = 1$ to N to arrive at

$$\begin{aligned} \|\hat{\mathbf{U}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^N\|^2 + 2\beta_1 k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2 \leq \|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2 \\ + \frac{1}{\nu}\,\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}\,e^{-\alpha k}k\sum_{n=1}^{N}e^{2\alpha t_n}. \end{aligned} \tag{69}$$

From geometric series, it follows that

$$\begin{aligned} k\sum_{n=1}^{N}e^{2\alpha t_n} &= e^{2\alpha k}k(e^{2\alpha k}-1)^{-1}e^{2\alpha t_N} \\ &= \frac{k}{(1-e^{-2\alpha k})}e^{2\alpha t_N} \leq \frac{1}{2\alpha}e^{2\alpha k}e^{2\alpha t_N} \end{aligned} \tag{70}$$

for some $k^* \in (0,k)$, On substituting (70) in (69), multiply the resulting form by $e^{-2\alpha t_N}$ and use $\beta_1 \geq (\nu/4)e^{-2\alpha k}$ to obtain

$$\|\mathbf{U}^N\|^2 + \kappa\|\nabla\mathbf{U}^N\|^2 + \frac{\nu}{2}e^{-2\alpha k}ke^{-2\alpha t_N}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2 \leq e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2) + \frac{e^{2\alpha k}}{2\alpha\nu}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}. \tag{71}$$

This complete the first part of the proof. For the remaining part, drop first two terms from (71) and then, to apply Stolz-Cesaro Theorem, that is, Theorem 4.1 to the resulting inequality, we observe that

$$\phi^N = \frac{\nu}{2}e^{-2\alpha k}k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^n\|^2 \quad \text{and} \quad \psi^N = e^{2\alpha t_N}.$$

Note that the sequence $\{\psi^n\}$ is monotonically strictly increasing sequence with $\psi^N \longrightarrow \infty$ and $N \longrightarrow \infty$. Hence, an appeal to Stolz-Cesaro Theorem 4.1 with a slightly refined estimate (using middle part of (70)) yields

$$\frac{\nu}{2(1-e^{-2\alpha k})}e^{-2\alpha k}k\limsup_{N\longrightarrow\infty}\|\nabla U^N\|^2 \leq k\frac{1}{\nu(1-e^{-2\alpha k})}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})},$$

and hence, we obtain the desired result. This concludes the proof. □

As in [2], we now appeal to a variant of Brouwers fixed point theorem to prove existence of solution to the discrete problem (58)

Theorem 4.2. *(Brouwer's fixed point theorem)*[16]. *Let $\mathbf{H}$ be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)$ and $\|\cdot\|$. Let $\mathbb{G}:\mathbf{H}\rightarrow\mathbf{H}$ be a continuous function. If there exists $R>0$ such that $(\mathbb{G}(z),z)>0\ \forall z$ with $\|z\|=R$, then there exists $z^*\in\mathbf{H}$ such that $\|z\|\leq R$ and $\mathbb{G}(z^*)=0$.*

Theorem 4.3. *Given $\mathbf{U}^0,\mathbf{U}^1,\mathbf{U}^2,\dots\mathbf{U}^{n-1}$ there exist a unique discrete solution $\mathbf{U}^n$ of (58) for $n\geq 1$.*

Proof. Assuming that $\mathbf{U}^m, m=1,2,\dots n-1$ are known, we need to show the existence of $\mathbf{U}^n$ to the problem (58). Now, define a function $\mathbb{G}:\mathbf{J}_h\rightarrow\mathbf{J}_h$ for a fixed n by

$$\begin{aligned}(\mathbb{G}(\mathbf{w}),\boldsymbol{\phi}_h) &= (\mathbf{w},\boldsymbol{\phi}_h) + \kappa(\nabla\mathbf{w},\nabla\boldsymbol{\phi}_h) + k\nu(\nabla\mathbf{w},\nabla\boldsymbol{\phi}_h)\\ &+ k\ b(\mathbf{w},\mathbf{w},\boldsymbol{\phi}_h) - (\mathbf{U}^{n-1},\boldsymbol{\phi}_h) - \kappa(\nabla\mathbf{U}^{n-1},\nabla\boldsymbol{\phi}_h) - k(\mathbf{f}^n,\boldsymbol{\phi}_h).\end{aligned} \tag{72}$$

Set a norm on $\mathbf{J}_h$ as

$$|||\mathbf{w}||| = (\|\mathbf{w}\|^2 + \kappa\|\nabla\mathbf{w}\|^2)^{\frac{1}{2}}. \tag{73}$$

It is easy to cheek that $\mathbb{G}$ is continuous. Now, after substituting $\boldsymbol{\phi}_h=\mathbf{w}$ in (72), we use (18), (73), the Cauchy-Schwarz's inequality and the Young's inequality to obtain

$$(\mathbb{G}(\mathbf{w}),\mathbf{w}) \geq \Bigg(|||\mathbf{w}||| - |||\mathbf{U}^{n-1}||| - k\|\mathbf{f}^n\|\Bigg)|||\mathbf{w}|||.$$

Choose R such that $|||\mathbf{w}|||=R$ and $\big(R - |||\mathbf{U}^{n-1}||| - k\|\mathbf{f}^n\|\big)>0$ and hence,

$$(\mathbb{G}(\mathbf{w}),\mathbf{w})>0.$$

An appeal to Theorem 4.2 concludes an existence of the discrete solution $\{\mathbf{U}^n\}_{n\geq1}$ of (58).

The part of uniqueness is quite similar to the proof of uniqueness problem in [2], so we skip the proof and this completes the rest of the proof. . □

Remark 4.1. *From the Theorem 4.4, we note that for a given* $\mathbf{U}^{n-1} \in \mathbf{J}_h$*, there exists a unique discrete solution* $\mathbf{U}^n \in \mathbf{J}_h$*. Thus, it defines a map* $S_h^n : \mathbf{J}_h \to \mathbf{J}_h$ *such that* $S_h^n(\mathbf{U}^{n-1}) = U^n$*, which is continuous and globally defined.*

As a consequence of (71), the following result holds on the discrete global attractor.

Theorem 4.4. *There exists a bounded absorbing set*

$$\mathbf{B}_{\rho_0}(\mathbf{0}) : \{(\|\mathbf{U}^N\|^2 + \kappa\|\nabla\mathbf{U}^N\|^2) \leq \rho_0^2\},$$

where ρ_0 *is given by*

$$\rho_0^2 = \frac{e^{2\alpha k}}{\alpha\nu\lambda_1}\|\mathbf{f}\|^2_{L^\infty(L^2)}.$$

Moreover, the discrete problem (58), has a global attractor.

proof. In order to prove the first part of the Theorem 4.4 we now claim that if $\left(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2\right)^{1/2} \in B_{\rho_1}(0)$, there exists $t_{n^*} = n^*k$ depending on $\left(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2\right)^{1/2}$ such that the discrete solution $\left(\|\mathbf{U}^N\|^2 + \kappa\|\nabla\mathbf{U}^N\|^2\right)^{1/2}$ for $t_N \geq t_{n^*}$ lies in $B_{\rho_0}(0)$. To prove this, we observe easily from the estimate (71) that

$$\|\mathbf{U}^N\|^2 + \kappa\|\nabla\mathbf{U}^N\|^2 \leq e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2) + \frac{\rho_0^2}{2}. \tag{74}$$

To complete the first part of the proof, it is enough to claim that

$$e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2) \leq \frac{\rho_0^2}{2}. \tag{75}$$

A use of the fact that $2(a^2+b^2) \geq (a+b)^2$ yields

$$\frac{1}{\rho_0}\|\mathbf{U}^0\| + \kappa\|\nabla\mathbf{U}^0\| \leq e^{\alpha t_N}.$$

That means, there is $t_n^\star = n^\star k \geq \frac{1}{\alpha}\log(\frac{1}{\rho_0}(\|\mathbf{U}^0\| + \kappa\|\nabla\mathbf{U}^0\|))$ such that (75) holds for $\rho_1 > \frac{\rho_0}{2}$ and $t_N \geq t_{n^\star}$ $B_{\rho_1}(0) \subset B_{\rho_0}(0)$. For $\rho_1 < \frac{\rho_0}{2}$ the result trivially holds for any $t_n \geq 0$. Therefore, $B_{\rho_0}(0)$ is an absorbing ball. For a prove of the second part of the Theorem 4.4, we use the Remark 4.1 to infer that S^n possess a global attractor, say $A_{n,k}$, by mimicking the proof of existence of an attractor in the continuous case, see Titi *et al.*[15]. This concludes the rest of the proof. □

Now, we are in a position to sketch a proof of uniform $l^\infty(\mathbf{H}_0^1)$ bound for the discrete solution. As in the Lemma 3.2, we first rewrite the discrete problem (58) using $\tilde{\Delta}_h$ and proceed in a similar manner. Then, an application of discrete Gronwall's Lemma yields estimate, which is valid for all finite $t_N = T > 0$. Now combining with the estimate(62), we complete the proof of the following uniform estimate in $l^\infty(\mathbf{H}_0^1)$-norm.

Lemma 4.2. *(Uniform* $l^\infty(\mathbf{H}_0^1)$ *bounds) With* $0 \leq \alpha < \dfrac{\nu\lambda_1}{4(1+\lambda_1\kappa)}$*, choose* k_0 *so that for* $0 < k \leq k_0$*, the estimate (59) is satisfied. Then, there is a positive constant* K *depending on* $M_0, \nu, \lambda_1, \alpha$ *such that the discrete solution* $\mathbf{U}^N$*,* $N \geq 1$ *of (58) satisfies*

(76)

$$(\|\nabla\mathbf{U}^N\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{U}^N\|^2) + \beta_1 e^{-2\alpha t_N}\; k\sum_{n=1}^{N} e^{2\alpha t_n}\|\tilde{\Delta}_h\mathbf{U}^n\|^2 \leq K(M_0,\nu,\lambda_1,\alpha),$$

where β_1 is as given in (61).

5. Error Analysis of the Backward Euler Method

This section deals with the error estimate of the backward Euler method. Set, for a fixed n, $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}_h(t_n) = \mathbf{U}^n - \mathbf{u}_h^n$. Now, rewrite (12) at $t = t_n$ and subtract it from (58) to obtain

$$(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) = E^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) + \Lambda_h^n(\boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{77}$$

where,

$$\begin{aligned} E^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) &= (\mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n, \boldsymbol{\phi}_h) \\ &= \frac{1}{2k}\int_{t_{n-1}}^{t_n} (t - t_n)(\mathbf{u}_{htt}(t), \boldsymbol{\phi}_h) \quad dt \\ &+ \frac{\kappa}{2k}\int_{t_{n-1}}^{t_n} (t - t_n) a(\mathbf{u}_{htt}(t), \boldsymbol{\phi}_h) \quad dt. \end{aligned} \tag{78}$$

and

$$\begin{aligned} \Lambda_h(\boldsymbol{\phi}_h) &= b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) \\ &= -b(\mathbf{u}_h^n, \mathbf{e}^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^n, \mathbf{e}^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h). \end{aligned} \tag{79}$$

The following theorem provides a bound on the error $\mathbf{e}^n$.

Theorem 5.1. *Let $0 \le \alpha < \dfrac{\nu\lambda_1}{4(1+\kappa\lambda_1)}$ and $k_0 > 0$ be such that for $0 < k \le k_0$, (59) is satisfied. Further, let $\mathbf{u}_h(t)$ satisfy (12). Then, there exists a positive constant C, independent of k, such that for $n = 1, 2, \cdots$*

$$\|\mathbf{e}^n\|^2 + \kappa\|\nabla\mathbf{e}^n\|^2 + \beta_1 k e^{-2\alpha t_n}\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}_i\|^2 \le Ce^{C\,t_n}k^2. \tag{80}$$

Proof. Multiply (77) by $e^{\alpha t_n}$ and then, divide the resulting equation by $e^{\alpha k}$ to obtain as in the proof of Lemma 4.1

$$\begin{aligned} (\bar{\partial}_t \hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) - (\frac{1-e^{-\alpha k}}{k})(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) - (\frac{1-e^{-\alpha k}}{k})\kappa a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) \\ + \nu e^{-\alpha k} a(\hat{\mathbf{e}}^n, \boldsymbol{\phi}_h) = e^{-\alpha k} e^{\alpha t_n} E^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) + e^{-\alpha k} e^{\alpha t_n}\Lambda_h^n(\boldsymbol{\phi}_h). \end{aligned} \tag{81}$$

Set $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$ in (81) and use (6) to find that

$$\begin{aligned} \frac{1}{2}\bar{\partial}_t\big(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2\big) + \Big(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\|\nabla\hat{\mathbf{e}}^n\|^2 \\ = e^{-\alpha k} e^{\alpha t_n} E^n(\mathbf{u}_h)(\hat{\mathbf{e}}^n) + e^{-\alpha k} e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n). \end{aligned} \tag{82}$$

Multiply (82) by $2k$ and then, sum over $n = 1$ to N to arrive at

$$\begin{aligned} \|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + 2k\Big(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big)\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2 \\ \le 2ke^{-\alpha k}\sum_{n=1}^{N} e^{\alpha t_n} E^n(\mathbf{u}_h)(\hat{\mathbf{e}}^n) + 2ke^{-\alpha k}\sum_{n=1}^{N} e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n). \end{aligned} \tag{83}$$

To estimate the first term on the right hand side of (83), we observe that

$$\begin{aligned}|2ke^{-\alpha k}\sum_{n=1}^{N}e^{\alpha t_n}E^n(\mathbf{u}_h)(\hat{\mathbf{e}}^n)| \leq& |2ke^{-\alpha k}\sum_{n=1}^{N}e^{\alpha t_n}\frac{1}{2k}\int_{t_{n-1}}^{t_n}(t-t_n)(\mathbf{u}_{htt}(t),\hat{\mathbf{e}}^n)\quad dt| \\ &+ |2ke^{-\alpha k}\sum_{n=1}^{N}e^{\alpha t_n}\frac{1}{2k}\int_{t_{n-1}}^{t_n}(t-t_n)a(\mathbf{u}_{htt}(t),\hat{\mathbf{e}}^n)\quad dt| \\ =& I_1^N + I_2^N. \end{aligned}\tag{84}$$

Using the Cauchy-Schwarz inequality, (6) with the Young's inequality, we estimate I_1^N as:

$$\begin{aligned}|I_1^N| \leq& C(\nu,\lambda_1)ke^{-\alpha k}\sum_{n=1}^{N}e^{2\alpha t_n}\frac{1}{k^2}\left(\int_{t_{n-1}}^{t_n}(t_n-s)\|\mathbf{u}_{htt}(s)\|_{-1,h}ds\right)^2 \\ &+\frac{\nu}{3}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2.\end{aligned}\tag{85}$$

An application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}&e^{2\alpha t_n}\frac{1}{k^2}\left(\int_{t_{n-1}}^{t_n}(t_n-s)\|\mathbf{u}_{htt}(s)\|_{-1,h}ds\right)^2 \\ \leq&\frac{1}{k^2}\left(\int_{t_{n-1}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{htt}(s)\|_{-1,h}^2ds\right)\left(\int_{t_{n-1}}^{t_n}(t_n-s)^2ds\right) \\ =&\frac{k}{3}\int_{t_{n-1}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{htt}(t)\|_{-1,h}^2\,dt,\end{aligned}\tag{86}$$

and hence, using (86) and Lemma (3.5), we find that

$$\begin{aligned}&k\sum_{n=1}^{N}e^{2\alpha t_n}\frac{1}{k^2}\left(\int_{t_{n-1}}^{t_n}(t_n-s)\|\mathbf{u}_{htt}(s)\|_{-1,h}\,ds\right)^2 \\ \leq&\frac{k^2}{3}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{htt}(s)\|_{-1,h}^2\,ds \\ =&\frac{k^2}{3}e^{2\alpha k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}e^{2\alpha t_{n-1}}\|\mathbf{u}_{htt}(s)\|_{-1,h}^2\,ds \\ \leq&\frac{k^2}{3}e^{2\alpha k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}e^{2\alpha s}\|\mathbf{u}_{htt}(s)\|_{-1,h}^2\,ds \\ =&\frac{k^2}{3}e^{2\alpha k}\int_{0}^{t_N}e^{2\alpha s}\|\mathbf{u}_{htt}(s)\|_{-1,h}^2\,ds \\ \leq&C(\nu,\alpha,\lambda_1,M_0)k^2e^{2\alpha t_{N+1}}.\end{aligned}\tag{87}$$

Apply (87) in (85) and obtain

$$|I_1^N| \leq C(\nu,\alpha,\lambda_1,M_0)k^2e^{2\alpha t_n}+\frac{\nu}{3}ke^{-\alpha k}\sum_{n=1}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2.\tag{88}$$

Similarly for $|I_2^N|$, we easily arrive at

$$|I_2^N| \leq C(\nu, \alpha, \lambda_1, M_0)k^2 e^{2\alpha t_n} + \frac{\nu}{3} k e^{-\alpha k} \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2. \tag{89}$$

To estimate the second term on the right hand side of (83), we note from the anti-symmetric property of the trilinear form that

$$e^{\alpha t_n} |\Lambda_h(\hat{\mathbf{e}}^n)| \leq e^{-\alpha t_n} \left| b(\hat{\mathbf{e}}^n, \hat{\mathbf{u}}_h^n, \hat{\mathbf{e}}^n) \right|, \tag{90}$$

and hence, a use of the generalized Holder's inequality with Sobolev's embedding theorem in (90) yields

$$\begin{aligned} e^{\alpha t_n} |\Lambda_h(\hat{\mathbf{e}}^n)| &\leq C e^{-\alpha t_n} \|\hat{\mathbf{e}}^n\|^{\frac{1}{2}} \|\nabla \hat{\mathbf{e}}^n\|^{\frac{1}{2}} \|\nabla \hat{\mathbf{u}}_h^n\| \|\hat{\mathbf{e}}^n\|^{\frac{1}{2}} \|\nabla \hat{\mathbf{e}}^n\|^{\frac{1}{2}} \\ &\leq C \, \|\nabla \mathbf{u}_h^n\| \|\hat{\mathbf{e}}^n\| \|\nabla \hat{\mathbf{e}}^n\|. \end{aligned} \tag{91}$$

Using the Young's inequality, an application of Lemma 4.1 leads to

$$\begin{aligned} k e^{-\alpha k} \sum_{n=1}^{N} e^{\alpha t_n} \left| \Lambda_h(\hat{\mathbf{e}}^n) \right| &\leq C(\nu) \sum_{n=1}^{N-1} k e^{-\alpha k} \|\nabla \mathbf{u}_h^n\|^2 \, \|\hat{\mathbf{e}}^n\|^2 \\ &+ C(\nu, M) k e^{-\alpha k} \, \|\hat{\mathbf{e}}^N\|^2 + \frac{\nu}{3} k e^{-\alpha k} \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2. \end{aligned} \tag{92}$$

A use of (88), (89) and (92) in (83) with $\mathbf{e}^0 = 0$ yields

$$\begin{aligned} \left(\|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 \right) \quad + \beta_1 k \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2 &\leq \frac{C(\nu, \alpha, \lambda_1, M)}{\kappa} e^{2\alpha t_N} k^2 \\ &+ C(\nu) k e^{-\alpha k} \sum_{n=0}^{N-1} \|\nabla \mathbf{u}_h^n\|^2 \|\hat{\mathbf{e}}^n\|^2 \\ &+ C(\nu, M) k e^{-\alpha k} (\|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2). \end{aligned} \tag{93}$$

Now, choose $k_0 > 0$ such that for $0 < k < k_0$, $(1 - C(\nu, M) k e^{-\alpha k}) > 0$ and (59) is satisfied. Then, an application of the discrete Gronwall's Lemma yields

$$\begin{aligned} \|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 + \beta_1 k \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2 \\ \leq C(\nu, \alpha, \lambda_1, M_0) k^2 \; e^{2\alpha t_N} \times \exp\left(k \sum_{n=0}^{N-1} \|\nabla \mathbf{u}_h^n\|^2 \right). \end{aligned} \tag{94}$$

With the help of Lemma 4.1 with $\alpha = 0$, we bound

$$k \sum_{n=0}^{N-1} \|\nabla \mathbf{u}_h^n\|^2 \leq C t_N. \tag{95}$$

Using (95) in (94), we arrive at

$$\|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 + \beta_1 k \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2 \leq C(\nu, \alpha, \lambda_1, M_0) e^{C t_N} k^2. \tag{96}$$

For $0 < k \leq k_0$, the coefficient of the third term on the left-hand side of (96) becomes positive. Dividing (96) by $e^{2\alpha t_N}$, we obtain (80) and this completes the rest of the proof. □

Remark 5.1. *Note from Theorem 5.1 that*

$$\sqrt{\kappa}\|\nabla \mathbf{e}^n\| \leq C\ k \tag{97}$$

Lemma 5.1. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{4(1+\kappa\lambda_1)}$ *and* $k_0 > 0$ *be such that for* $0 < k \leq k_0$, *(59) is satisfied. Further, let* $\mathbf{u}_h(t)$ *satisfy (12). Then, there exists a positive constant* C, *independent of* k, *such that for* $n = 1, 2, \cdots, N$

$$\|\bar{\partial}_t \mathbf{e}^n\|_{-1,h} + \kappa\|\bar{\partial}_t \nabla \mathbf{e}^n\| \leq \frac{C}{\sqrt{\kappa}} e^{ct_N} k. \tag{98}$$

Proof. Choose $\boldsymbol{\phi}_h = \bar{\partial}_t \mathbf{e}^n$ in (77) to find that

$$\kappa\|\bar{\partial}_t \nabla \mathbf{e}^n\|^2 \leq -\nu a(\mathbf{e}^n, \bar{\partial}_t \mathbf{e}^n) + E^n(\mathbf{u}_h^n)(\bar{\partial}_t \mathbf{e}^n) + \Lambda_h(\bar{\partial}_t \mathbf{e}^n). \tag{99}$$

Using (78), (17) and (6), we observe that

$$\begin{aligned} |E^n(\mathbf{u}_h)(\bar{\partial}_t \mathbf{e}^n)| &\leq |\frac{1}{2k}\int_{t_{n-1}}^{t_n} (t-t_n)(\mathbf{u}_{htt}(t), \bar{\partial}_t \mathbf{e}^n) \quad dt| \\ &+ |\frac{\kappa}{2k}\int_{t_{n-1}}^{t_n} (t-t_n) a(\mathbf{u}_{htt}(t), \bar{\partial}_t \mathbf{e}^n) \quad dt|. \end{aligned} \tag{100}$$

A use of the Cauchy-Schwarz inequality with Lemma (3.5) for the first term on the right hand side of the (100) yields

$$\begin{aligned} |\frac{1}{2k}\int_{t_{n-1}}^{t_n} (t-t_n)(\mathbf{u}_{htt}(t), \bar{\partial}_t \hat{\mathbf{e}}^n) \quad dt| &\leq \left(\frac{1}{2k}\int_{t_{n-1}}^{t_n} (t-t_n)\|\mathbf{u}_{htt}(t)\|_{-1,h} dt\right)\|\nabla \bar{\partial}_t \mathbf{e}^n\| \\ &\leq C(\alpha, \nu, \lambda_1, M_0)\frac{1}{\sqrt{\kappa}} k\|\nabla \bar{\partial}_t \mathbf{e}^n\|. \end{aligned} \tag{101}$$

Similarly, the second term of right hand side of (100) is estimated as

$$\begin{aligned} |\frac{\kappa}{2k}\int_{t_{n-1}}^{t_n} (t-t_n) a(\mathbf{u}_{htt}(t), \bar{\partial}_t \mathbf{e}^n) \quad dt| &\leq \left(\frac{\kappa}{2k}\int_{t_{n-1}}^{t_n} (t-t_n)\|\nabla \mathbf{u}_{htt}(t)\|\ dt\right)\|\nabla \bar{\partial}_t \mathbf{e}^n\| \\ &\leq C(\alpha, \nu, \lambda_1, M_0)\frac{1}{\sqrt{\kappa}} k\|\nabla \bar{\partial}_t \mathbf{e}^n\|. \end{aligned} \tag{102}$$

Now, using (79), (17) and (6), it follows that

$$\begin{aligned} |\Lambda_h(\bar{\partial}_t \mathbf{e}^n)| &= |b(\mathbf{u}_h^n, \mathbf{e}^n, \bar{\partial}_t \mathbf{e}^n) + b(\mathbf{e}^n, \mathbf{U}_h^n, \bar{\partial}_t \mathbf{e}^n), \bar{\partial}_t \mathbf{e}^n)| \\ &\leq C(\lambda_1)\Big(\|\nabla \mathbf{u}_h^n\| + \|\nabla \mathbf{U}_h^n\|\Big)\|\nabla \mathbf{e}^n\|\|\nabla \bar{\partial}_t \mathbf{e}^n\|. \end{aligned} \tag{103}$$

With the help of Lemmas 3.2, 4.1 and 4.2, we obtain

$$|\Lambda_h(\bar{\partial}_t \mathbf{e}^n)| \leq C(\nu, \alpha, \lambda_1, M_0)\|\nabla \hat{\mathbf{e}}^n\|\ \|\nabla \bar{\partial}_t \mathbf{e}^n\|. \tag{104}$$

Substitute (104), (102) and (101) in (99) and apply Theorem 5.1 to find that

$$\kappa\|\bar{\partial}_t \nabla \mathbf{e}^n\| \leq \frac{C(\alpha, \nu, \lambda_1, M_0)}{\sqrt{\kappa}} e^{Ct_N} k. \tag{105}$$

Now, (77) is rewritten as

$$(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) = -\kappa a(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) - \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) + E^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) + \Lambda_h^n(\boldsymbol{\phi}_h). \tag{106}$$

A use of (101) with (102) and (105) yields

$$
\begin{aligned}
\|\bar{\partial}_t \mathbf{e}^n\|_{-1,h} &= \sup_{0\neq \boldsymbol{\phi}_h \in \mathbf{H}_h} \frac{(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h)}{\|\nabla \boldsymbol{\phi}_h\|} \\
&\leq \left(\kappa \|\bar{\partial}_t \nabla \mathbf{e}^n\| + \nu \|\nabla \mathbf{e}^n\| + C(\alpha,\nu,\lambda_1,M_0)\frac{1}{\sqrt{\kappa}} e^{Ct_N} k\right) \\
&\leq C(\alpha,\nu,\lambda_1,M_0)\frac{1}{\sqrt{\kappa}} e^{Ct_N} k. \qquad (107)
\end{aligned}
$$

This completes the rest of the proof. □.

Now, we derive error estimate for the pressure term.

Theorem 5.2. *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and* $k_0 > 0$ *be such that for* $0 < k \leq k_0$*, (59) is satisfied. Further, let* $\mathbf{u}_h(t)$ *satisfy (12). Then, there exists a positive constant* C*, independent of* k*, such that for* $n = 1, 2, \cdots, N$

$$
\|P^n - p_h(t_n)\| = \|\boldsymbol{\rho}^n\| \leq \frac{C}{\sqrt{\kappa}} e^{ct_N} k. \qquad (108)
$$

Proof. Consider (11) at $t = t_n$ and subtract it from (57) to obtain

$$
\begin{aligned}
(\boldsymbol{\rho}^n, \nabla\cdot\boldsymbol{\phi}_h) &= (\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) \\
&- E^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) - \Lambda_h(\boldsymbol{\phi}_h).
\end{aligned}
$$

Using the Cauchy-Schwarz's inequality, (6) and (104), we now arrive at

(109)

$$
(\boldsymbol{\rho}^n, \nabla\cdot\boldsymbol{\phi}_h) \leq C(\kappa,\nu,\lambda_1)\left(\|\bar{\partial}_t \nabla \mathbf{e}^n\| + \|\nabla \mathbf{e}^n\| + \frac{C(\nu,\lambda_1,\alpha,M_0)k}{\sqrt{\kappa}}\right)\|\nabla\boldsymbol{\phi}_h\|.
$$

A use of Theorem 5.1, (101), (102) and (104) in (109) would lead us to the desired result, that is,

$$
\|\boldsymbol{\rho}^n\| \leq \frac{C(\nu,\alpha,\lambda_1,M_0)}{\sqrt{\kappa}} k. \qquad (110)
$$

This concludes the proof. □

Theorem 5.3. *In addition to the hypothese of Theorem 5.1, assume that the following uniqueness condition*

$$
2\frac{N_0}{\nu}\|\mathbf{f}\|_{L^\infty(\mathbf{H}^{-1})} < 1 \quad \textit{and} \quad N_0 = \sup_{u,v,w \in \mathbf{H}_0^1(\Omega)} \frac{b(u,v,w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|} \qquad (111)
$$

holds. Then, there is a positive constant C *which is valid uniformly in time such that for* $n > 0$

$$
\|\mathbf{e}^n\|^2 + \kappa\|\nabla \mathbf{e}^n\|^2 \leq C\, k^2.
$$

Proof. In (82), we need to estimate the last term, that is, the non-linear term on the right hand side using the uniqueness condition. Now rewrite (79) as

$$
\begin{aligned}
\Lambda_h(\boldsymbol{\phi}_h) &= b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) \\
&= -b(\mathbf{U}^n, \mathbf{e}^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^n, \hat{\mathbf{u}_h}^n, \boldsymbol{\phi}_h), \qquad (112)
\end{aligned}
$$

and hence, it easily follows that

$$
e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}^n)| \leq N_0\|\nabla \mathbf{u}_h^n\|\,\|\nabla \hat{\mathbf{e}}^n\|^2. \qquad (113)
$$

We note from (22) that for large N^* , that is, for all $n \geq N^*$

$$\|\nabla \mathbf{u}_h^n\| \leq \frac{1}{\nu}\|\mathbf{f}\|_{L^\infty(\mathbf{H}^{-1})}. \tag{114}$$

Then, as in the proof of Theorem 5.1, after multiplying by $2k$, sum up from $n = N^*+1$ to N to obtain On substitution (93) and for $n \geq N^*$, (93) is rewrite as

$$\begin{aligned}
&\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + 2\beta_1 k \sum_{n=N^*+1}^{N} \|\nabla\hat{\mathbf{e}}^n\|^2 \\
\leq & \frac{C(\nu,\alpha,\lambda_1,M_0)}{\kappa} e^{2\alpha t_n} k^2 + e^{2\alpha N^*} \ \|\mathbf{e}^{N^*}\|^2 \\
& + ke^{-\alpha k} \sum_{n=N^*+1}^{N} \frac{N_0}{\nu}\|\mathbf{f}\|_{L^\infty(\mathbf{H}^{-1})} \ \|\nabla\hat{\mathbf{e}}^n\|^2.
\end{aligned} \tag{115}$$

As $\beta \geq e^{-\alpha k}\nu/4$, we arrive at

$$\begin{aligned}
&\|\hat{\mathbf{e}}^N\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^N\|^2 + ke^{-\alpha k}\left(\frac{\nu}{2} - \frac{N_0}{\nu}\|\mathbf{f}\|_{L^\infty(\mathbf{H}^{-1})}\right) \sum_{n=N^*+1}^{N} \|\nabla\hat{\mathbf{e}}^n\|^2 \\
&\leq C(\nu,\alpha,\lambda_1,M_0) e^{2\alpha t_n} k^2 + e^{2\alpha N^*} \ \|\mathbf{e}^{N^*}\|^2.
\end{aligned} \tag{116}$$

Now a use of uniqueness condition and choose $k_0 > 0$ such that for $0 < k < k_0$, $\left(\nu - 2\frac{N_0}{\nu^2} \ \|\mathbf{f}\|_{L^\infty(\mathbf{H}^{-1})}\right) > 0$. Multiplying $e^{2\alpha t_n}$ in (116), a use of Theorem 5.1 for the last term on the right hand side completes the rest of the proof. $\square$.
Now a use of Theorems 3.1, 5.1 and (110) completes the proof of the following Theorem.

Theorem 5.4. *Under the assumptions of Theorems 3.1 and 5.3, the following holds true:*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \leq C(\frac{h^2}{\sqrt{\kappa}} + k)$$

and

$$\|\nabla(\mathbf{u}(t_n - \mathbf{U}^n)\| + \|(p(t_n) - P^n)\| \leq \frac{C}{\sqrt{\kappa}}(h+k),$$

where C depends on e^{Ct_n} and under uniqueness condition (111), C is positive constant which is valid uniformly in time $t > 0$.

6. Linearized Backward Euler Method

Since the backward Euler method applied to (11) yields a system of non linear equations at each time level $t = t_n$, the system has to be solved using Newton type iterative methods and hence, it may be computationally expensive. Therefore, in this section we introduce a linearized version of this method, which gives rise a system of linear equations at each time step. Thus, the linearized backward Euler method is to find a sequence of functions $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{H}_h$ and $\{P^n\}_{n\geq 1} \in L_h$ satisfying for $\boldsymbol{\phi}_h \in H_h$

$$\begin{aligned}
&(\bar{\partial}_t\mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t\mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) \\
&\qquad\qquad = (P^n, \nabla\cdot\boldsymbol{\phi}_h) + (\mathbf{f}^n, \boldsymbol{\phi}_h), \\
&(\nabla\cdot\mathbf{U}^n, \chi_h) = 0 \qquad \forall \chi_h \in L_h, \\
&\mathbf{U}^0 = \mathbf{u}_{0h}.
\end{aligned} \tag{117}$$

Equivalently, we seek $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{J}_h$ such that

$$(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) = (\mathbf{f}^n, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \qquad \mathbf{U}^0 = \mathbf{u}_{0h}. \tag{118}$$

Compared to (11), the linearized backward Euler method (118) differs only in the nonlinear term, we proceeds along the same lines of proof of the Theorem 5.1 and only discuss the difference in the analysis. The equation in error $\mathbf{e}^n$ reads as: find $\mathbf{e}^n \in \mathbf{J}_h$ such that

$$(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \kappa a(\bar{\partial}_t \mathbf{e}^n, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) = E^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{119}$$

where $E^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) = (\mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n, \boldsymbol{\phi}_h)$ and $\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h)$.

Following argument in the proof of Theorem 5.1, we easily find that

$$\begin{aligned} \|\hat{\mathbf{e}}^N\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^N\|^2 + 2\Big(\nu e^{-\alpha k} - \big(\frac{1-e^{-\alpha k}}{k}\big)\big(\kappa + \frac{1}{\lambda_1}\big)\Big) k \sum_{n=1}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2 \\ \leq 2k e^{-\alpha k} \sum_{n=1}^{N} e^{\alpha t_n} E^n(\mathbf{u}_h)(\hat{\mathbf{e}}^n) + 2k e^{-\alpha k} \sum_{n=1}^{N} e^{\alpha t_n} \Lambda_h(\hat{\mathbf{e}}^n) \\ = I_1^N + I_2^N, \text{ say.} \end{aligned} \tag{120}$$

The first term on the right hand side of (120) is bounded by (88) and (89). Hence, we need to estimate only the second term. We now rewrite it as

$$\begin{aligned} & e^{\alpha t_n} |\Lambda_h(\boldsymbol{\phi}_h)| \\ =& e^{\alpha t_n} |b(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^{n-1} - \mathbf{u}_h^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{u}_h^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h)| \\ =& e^{\alpha t_n} |b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n - \mathbf{U}^n, \boldsymbol{\phi}_h)| \\ =& e^{\alpha t_n} |-b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{e}^n, \boldsymbol{\phi}_h) + b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) \\ & - b(\mathbf{u}_h^{n-1}, \mathbf{e}^n, \boldsymbol{\phi}_h)|. \end{aligned} \tag{121}$$

A use of (18)along with (6)and (17) in (121) with $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$ yields

$$\begin{aligned} e^{\alpha t_n} |\Lambda_h(\hat{\mathbf{e}}^n)| \leq& e^{\alpha t_n} |b(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \hat{\mathbf{u}}_h^n, \hat{\mathbf{e}}^n) - b(\mathbf{e}^{n-1}, \hat{\mathbf{u}}_h^n, \hat{\mathbf{e}}^n)| \\ \leq& C(\lambda_1) e^{\alpha t_n} \big(\|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\| \|\nabla \mathbf{u}_h^n\| \|\nabla \hat{\mathbf{e}}^n\| \\ & + \|\nabla \mathbf{e}^{n-1}\| \|\nabla \mathbf{u}_h^n\| \|\nabla \hat{\mathbf{e}}^n\|\big). \end{aligned} \tag{122}$$

Hence, we observe that

$$\begin{aligned} |I_2^N| \leq \quad & 2k e^{-\alpha k} \sum_{n=1}^{N} e^{\alpha t_n} |\Lambda_h(\hat{\mathbf{e}}^n)| \leq C(\lambda_1) k e^{-\alpha k} \sum_{n=1}^{N} \Big(e^{\alpha t_n} \|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\| \times \\ & \|\nabla \mathbf{u}_h^n\| \|\nabla \hat{\mathbf{e}}^n\| + e^{\alpha t_n} \|\nabla \mathbf{u}_h^n\| \|\nabla \mathbf{e}^{n-1}\| \Big) \|\nabla \hat{\mathbf{e}}^n\| = |I_4^N| + |I_5^N|, \text{ say.} \end{aligned} \tag{123}$$

Note that from the Taylor's expansion of $u_h(t)$ at t_n in the interval (t_{n-1}, t_n), we obtain

$$\|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\| = \|\int_{t_{n-1}}^{t_n} \nabla \mathbf{u}_{ht}(s) ds\|, \tag{124}$$

and an aplication of the Cauchy-Schwarz inequality yields

$$\|\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|^2 \le k\int_{t_{n-1}}^{t_n} \|\nabla \mathbf{u}_{ht}(s)\|^2\, ds. \tag{125}$$

Using Young's inequality, we bound $|I_4^N|$ as

(126)

$$|I_4^N| \le C(\lambda_1) k e^{-\alpha k}\sum_{n=1}^{N} e^{2\alpha t_n}\|\nabla \mathbf{u}_h^n\|^2 k\int_{t_{n-1}}^{t_n}\|\nabla \mathbf{u}_{ht}(s)\|^2\, ds + \frac{\nu}{6} k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2.$$

With the help of Lemma 11 and Lemma 3.2 in (126), we observe that

$$\begin{aligned}|I_4^N| &\le C(\nu,\alpha,\lambda_1,M_0)k^2 e^{-\alpha k}\sum_{n=1}^{N} e^{2\alpha t_n}\int_{t_{n-1}}^{t_n}\|\nabla \mathbf{u}_{ht}(s)\|^2\, ds + \frac{\nu}{6} k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2 \\ &\le C(\nu,\alpha,\lambda_1,M_0)k^2 e^{\alpha k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} e^{2\alpha t_{n-1}}\|\nabla \mathbf{u}_{ht}(s)\|^2\, ds + \frac{\nu}{6} k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2 \\ &\le\le C(\nu,\alpha,\lambda_1,M_0)k^2 e^{\alpha k}\int_0^{t_N} e^{2\alpha s}\|\nabla \mathbf{u}_{ht}(s)\|^2\, ds + \frac{\nu}{6} k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2 \end{aligned}$$

(127)

$$\le C(\nu,\alpha,\lambda_1,M_0)k^2 e^{2\alpha t_n} + \frac{\nu}{6} k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2.$$

A use of the Young's inequality yields

$$\begin{aligned}|I_5^N| &\le C(\lambda_1) k e^{-\alpha k}\sum_{n=1}^{N} e^{\alpha t_n}\|\nabla \mathbf{u}_h^n\|\|\nabla \mathbf{e}^{n-1}\|\|\nabla \hat{\mathbf{e}}^n\| \\ &\le C(\nu,\lambda_1) k e^{-\alpha k}\sum_{n=1}^{N} e^{-2\alpha t_{n-1}}\|\nabla \mathbf{u}_h^n\|^2\|\nabla \hat{\mathbf{e}}^{n-1}\|^2 + \frac{\nu}{6} k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2 \\ &\le C(\nu,\lambda_1) k e^{-\alpha k}\sum_{n=0}^{N-1} e^{-2\alpha t_n}\|\nabla \mathbf{u}_h^n\|^2\|\nabla \hat{\mathbf{e}}^n\|^2 + \frac{\nu}{6} k e^{-\alpha k}\sum_{n=1}^{N}\|\nabla \hat{\mathbf{e}}^n\|^2. \qquad (128)\end{aligned}$$

Substitute (88)-(89) and (127)-(128) in (120). As in the estimate of (93), we now apply Gronwall's lemma to complete the rest of the proof. □

Now a use of Theorems 3.1, 5.1, 5.2, 5.4 and (110) completes the proof of the following Theorem.

Theorem 6.1. *Under the assumptions of Theorems 3.1 and 5.1, the following holds true:*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \le C e^{C t_N}\left(\frac{h^2}{\sqrt{\kappa}} + k\right)$$

and

$$\|\nabla(\mathbf{u}(t_n) - \mathbf{U}^n)\| + \|(p(t_n) - P^n)\| \le \frac{C}{\sqrt{\kappa}} e^{C t_N}(h + k).$$

As in the Theorem 5.4, under uniqueness condition, the estimates in the Theorem 6.1 are valid uniformly for all $n > 0$.

7. Numerical Experiments

Earlier in [27], three examples are considered depending on the forcing function $\mathbf{f}$, that is, when $\mathbf{f}$ is bounded in $L^\infty(\mathbf{L}^2)$, $\mathbf{f}=0$ and $\mathbf{f}=O(e^{-\alpha_1 t})$ and computational experiments which are based on backward Euler method are conducted. In this section, we focus on several numerical experiments with varying κ, using (P_2-P_0) mixed finite element space (see, [4]) for spatial discretization and backward Euler scheme for time discretization, which confirm our theoretical findings.

Now, consider the following finite dimensional subspaces $\mathbf{H}_h$ and L_h of $\mathbf{H}_0^1$ and L^2 respectively, as:

$$\mathbf{H}_h = \{\mathbf{v} \in (H_0^1(\Omega))^2 \cap \big(C(\bar{\Omega})\big)^2 : \mathbf{v}|_K \in (P_2(K))^2, K \in \mathcal{T}_h\},$$
$$L_h = \{q \in L^2(\Omega) : q|_K \in P_0(K), K \in \mathcal{T}_h\},$$

where $\mathcal{T}_h$ denotes the triangulation of the domain $\bar{\Omega}$. Then, apply the completely discrete finite element formulation for the problem (1)-(3) using backward Euler method (11) as: given $\mathbf{U}^{n-1}$, find the pair $(\mathbf{U}^n, P^n)$ satisfying:

$$(129)\qquad (\mathbf{U}^n,\mathbf{v}_h) + (\kappa+\nu\Delta t)\ a(\mathbf{U}^n,\mathbf{v}_h) + \Delta t\ b(\mathbf{U}^n,\mathbf{U}^n,\mathbf{v}_h) - \Delta t\ (P^n,\nabla\cdot\mathbf{v}_h)$$
$$= (\mathbf{U}^{n-1},\mathbf{v}_h) + \kappa a(\mathbf{U}^{n-1},\mathbf{v}_h) + \Delta t\ (\mathbf{f}(t_n),\mathbf{v}_h)\ \ \forall \mathbf{v}_h \in \mathbf{H}_h,$$
$$(\nabla\cdot\mathbf{U}^n, w_h) = 0 \qquad \forall w_h \in L_h.$$

Table 1. Errors and Convergence rates for backward Euler method with $k = \mathbb{O}(h^2)$.

h	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{L^2}}$	Rate	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{H}^1}$	Rate	$\Vert p(t_n)-P^n\Vert$	Rate
1/2	0.0237483		0.2014946		1.1766897	
1/4	0.0079066	1.5866855	0.1211721	0.7084543	0.6506894	0.8546928
1/8	0.0022462	1.8155237	0.0657439	0.8821283	0.3437042	0.9208014
1/16	0.0005736	1.9693621	0.0341358	0.9455711	0.1769821	0.9575635

Table 2. Numerical convergence rates for velocity in $\mathbf{L^2}$-norm with variation in κ for Example 1.

S No	h	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{L^2}}$ $\kappa=1$	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{L^2}}$ $\kappa=0.01$	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{L^2}}$ $\kappa=0.0001$	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{L^2}}$ $\kappa=0.00000001$
1	1/4	1.3640461	1.5866855	1.5852818	1.5852666
2	1/8	1.7542327	1.8155237	1.8154086	1.8154075
3	1/16	1.9111161	1.9693621	1.9694188	1.9694193

Using basis functions, we approximate the velocity and pressure as

$$(130)\qquad \mathbf{U}^n = \sum_{j=1}^{ng} \begin{pmatrix} \mathbf{u}_j^{nx} \\ \mathbf{u}_j^{ny} \end{pmatrix} \boldsymbol{\phi}_j^{\mathbf{u}}(\mathbf{x}), \quad P^n = \sum_{j=1}^{ne} p_j^n \phi_j^p(\mathbf{x}),$$

where $\boldsymbol{\phi}_j^{\mathbf{u}}(\mathbf{x})$ and $\phi_j^p(\mathbf{x})$ form bases for $\mathbf{H}_h$ and L_h with cardinality ng and ne, respectively. Here, $\mathbf{u}_j^{nx}$ and $\mathbf{u}_j^{ny}$ represent the x and y component of the approximate velocity field, respectively, at time $t=t_n$. Using (130), the basis functions for $\mathbf{H}_h$ and L_h in (129), we obtain a system of nonlinear algebraic equations, which is solved using Newton's method.

Example 1: Choose the right hand side function f in such a way that the exact solution $(\mathbf{u},p) = ((u_1,u_2),p)$ is

$u_1 = 0.1e^{-t}x^2(x-1)^2(y^3-2y^2+y)$, $u_2 = -0.1e^{-t}y^2(y-1)^2(x^3-2x^2+x)$, $p = 4.8e^{-t}(y^3-2y)$.

We choose $\nu = 1$, $\kappa = 0.01$ with $\Omega = (0,1)\times(0,1)$ and time $t = [0,1]$. Here, $\bar{\Omega}$ is subdivided into triangles with mesh size h.

TABLE 3. Numerical convergence rates for velocity in $\mathbf{H}^1$-norm with variation in κ for Example 1.

S No	h	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{H}^1}$ $\kappa=1$	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{H}^1}$ $\kappa=0.01$	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{H}^1}$ $\kappa=.0001$	$\Vert\mathbf{u}(t_n)-\mathbf{U}^n\Vert_{\mathbf{H}^1}$ $\kappa=0.00000001$
1	1/4	0.5114209	0.7336837	0.7321703	0.7321540
2	1/8	0.8243502	0.8821283	0.881730	0.8817265
3	1/16	0.9302921	0.94557101	0.9454654	0.9454644

TABLE 4. Numerical convergence rates for pressure in L^2-norm with variation in κ for Example 1.

S No	h	$\Vert p(t_n)-P^n\Vert$ $\kappa=1$	$\Vert p(t_n)-P^n\Vert$ $\kappa=0.01$	$\Vert p(t_n)-P^n\Vert$ $\kappa=0.0001$	$\Vert p(t_n)-P^n\Vert$ $\kappa=0.00000001$
1	1/4	0.8514019	0.8546928	0.8546353	0.8546348
2	1/8	0.9224011	0.9208014	0.9208148	0.9208149
3	1/16	0.9591777	0.9575635	0.9575832	0.9575834

The theoretical analysis provides a convergence rate of $\mathcal{O}(h^2)$ in $\mathbf{L^2}$-norm, of $\mathcal{O}(h)$ in $\mathbf{H}^1$-norm for the velocity and of $\mathcal{O}(h)$ in L^2-norm for the pressure term. Table 1 presents numerical errors and computed convergence rates obtained on successively refined meshes for the first order backward Euler method. These computational results agree with optimal convergence rates obtained in Theorem 5.4. Further, when $\kappa \to 0$ the order of convergence for velocity and pressure terms are given through Tables 2, 3 and 4 which again confirm our theoretical results given in Theorem 5.4.

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