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ON THE CONVERGENCE OF β-SCHEMES

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Abstract. Yang's wavewise entropy inequality [19] is verified for β-schemes which, when $m = 2$ and under a mild technique condition, guarantees the convergence of the schemes to the entropy solutions of convex conservation laws in one-dimensional scalar case. These schemes, constructed by S. Osher and S. Chakravarthy [13], are based on unwinding principle and use E-schemes as building blocks with simple flux limiters, without which all of them are even linearly unstable. The total variation diminishing property of these methods was established in the original work of S. Osher and S. Chakravarthy.

Key words. Conservation laws, fully-discrete β-schemes, entropy convergence.

1. Introduction

We consider numerical approximations to the scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $f \in C^1(\mathbb{R})$ is convex, and $u_0 \in BV(\mathbb{R})$. Here BV stands for the subspace of L^1_{loc} consisting of functions with bounded total variation. For the numerical methods concerned, let $\lambda = \frac{\tau}{h}$, where h and τ are spatial and temporal steps respectively, and $u_k^n = u(x_k, t_n)$ be the nodal values of the piecewise constant mesh function $u_h(x,t)$ approximating the solution of (1). The numerical schemes admit conservative form

$$u_k^{n+1} = H(u_{k-p}^n, \cdots, u_{k+p}^n; \lambda) = u_k^n - \lambda(g_{k+\frac{1}{2}}^n - g_{k-\frac{1}{2}}^n), \tag{2}$$

where the numerical flux g is given by

$$g_{k+\frac{1}{2}}^n = g_{k+\frac{1}{2}}[u_k^n], \tag{3}$$

and

$$g_{k+\frac{1}{2}}[v] = g(v_{k-p+1}, v_{k-p+2}, \cdots, v_k, \cdots, v_{k+p}), \tag{4}$$

for any data $\{v_j\}$. The function g is Lipschitz continuous with respect to its $2p$ arguments and is *consistent* with the conservation law in the sense that

$$g(u, u, \cdots, u) \equiv f(u). \tag{5}$$

The schemes that we are interested in are special cases of the general β-schemes when $m = 2$, which were introduced by S. Osher and S. Chakravarthy [1, 13] in the 80s. The entire families of β-schemes are defined for $0 < \beta \le (m\binom{2m}{m})^{-1}$, where m is an integer between 2 and 8. These schemes are $2m-1$ order accurate (except at isolated critical points), variation diminishing, $2m+1$ point band width, conservative approximations to scalar conservation laws. Although the numerical results have been shown [1, 13] to be extremely good, the entropy convergence of these schemes have been open. Our goal of this paper is to show that, when $m = 2$,

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β-schemes indeed persist entropy consistency for homogeneous scalar convex conservation laws. The proof of the convergence is an application of Yang's wavewise entropy inequality (WEI) framework [19], of which he has used to establish the entropy convergence of generalized MUSCL schemes and a class of schemes using flux limiters discussed by Sweby [15]. Recently, by using Yang's convergence criteria that derived from his WEI framework, the author [9, 6] has shown the entropy convergence of van Leer's flux limiter schemes, as well as Osher-Chakravarthy's α schemes for $m = 2$ [1, 13]. The corresponding convergence results of Yang and the author, for semi-discrete schemes, can be found in [7, 8, 10, 18, 17].

The paper is organized as follows. In section 2, we review the notions of the extremum paths, and then we establish the extremum traceableness of general TVD schemes, which is necessary for analyzing the entropy convergence of the schemes that will be given in the next section. In section 3, we present one of Yang's convergence criteria with weaker condition, an important entropy estimate, and finally the main result.

Now we introduce the β-schemes for the case of $m = 2$. Throughout the paper, to improve the readability, we use the shorthand notations of $f_k^n := f(u_k^n)$, $\Delta u_{k\pm\frac{1}{2}}^n = \pm(u_{k\pm 1}^n - u_k^n)$, and $f_{k\pm\frac{1}{2}}^n := \Delta f_{k\pm\frac{1}{2}}^n = \pm(f_{k\pm 1}^n - f_k^n)$. Also, whenever there is no ambiguity in the context, we employ the simplified notations: $u^k := u_k^{n+1}$, $u_k := u_k^n$, $f_k := f_k^n$, and $f_{k\pm\frac{1}{2}}^{\pm} := (f_{k\pm\frac{1}{2}}^n)^{\pm}$, where k and n are the spatial and temporal indexes respectively.

Let $g_{k+\frac{1}{2}}^E := g^E(u_k^n, u_{k+1}^n)$ be the flux of an E-scheme [14] that is characterized by

$$\operatorname{sgn}(\, u_{k+1}^n - u_k^n\,)[g_{k+\frac{1}{2}}^E - f(u)\,] \leq 0, \tag{6}$$

for all u in between u_k^n and u_{k+1}^n. Then the flux differences are defined by

$$f_{k+\frac{1}{2}}^+ = f_{k+1} - g_{k+\frac{1}{2}}^E, \tag{7}$$

and

$$f_{k+\frac{1}{2}}^- = g_{k+\frac{1}{2}}^E - f_k. \tag{8}$$

At the time level $t = t^n$, for all k, we define a series of local CFL numbers

$$\nu_{k+\frac{1}{2}}^+ = \frac{\lambda f_{k+\frac{1}{2}}^+}{u_{k+\frac{1}{2}}}, \qquad \nu_{k+\frac{1}{2}}^- = \frac{\lambda f_{k+\frac{1}{2}}^-}{u_{k+\frac{1}{2}}}. \tag{9}$$

Clearly, we have $\nu_{k+\frac{1}{2}}^+ \geq 0$ and $\nu_{k+\frac{1}{2}}^- \leq 0$. For convenience, we also set the ratios

$$r_k^+ = \frac{f_{k-\frac{1}{2}}^+}{f_{k+\frac{1}{2}}^+}, \qquad r_k^- = \frac{f_{k+\frac{1}{2}}^-}{f_{k-\frac{1}{2}}^-}. \tag{10}$$

The "minmod" operator is given by

$$\operatorname{minmod}(x,y) = \begin{cases} x, \text{ if } |x| \leq |y| \text{ and } xy > 0, \\ y, \text{ if } |x| > |y| \text{ and } xy > 0, \\ 0, \text{ if } xy \leq 0, \end{cases} \tag{11}$$

which can be converted to, divided by x, a monotone increasing function

$$\phi(r) = \max(0, \min(1, r)) = \begin{cases} 1, & \text{if } r \geq 1, \\ r, & \text{if } 0 \leq r \leq 1, \\ 0, & \text{if } r \leq 0, \end{cases} \tag{12}$$

with $r = \frac{y}{x}$. Clearly, $\phi(r)$ has a symmetry property

$$\frac{\phi(r)}{r} = \phi(\frac{1}{r}). \tag{13}$$

This property is very helpful to rewrite a β scheme into an increment form. The operator of "minmod" of three quantities is defined by

$$\text{minmod}[\, x, y, z\,] = \text{minmod}[\,\text{minmod}[\, x, y\,], z\,],$$

which is independent of the order of x, y, and z. Very often we write "mm" for "minmod", when there is no confusion. For $0 < \beta \leq \frac{1}{12}$ and $m = 2$, a β scheme [13], is given by

$$u^k = u_k - \lambda\,(\, g_{k+\frac{1}{2}} - g_{k-\frac{1}{2}}\,), \tag{14}$$

where

(15)

$$\begin{aligned} g_{k+\frac{1}{2}} \quad = \quad & g^E_{k+\frac{1}{2}} - (\frac{1}{12} + \beta)(f^-_{k+\frac{3}{2}})^{(1)} - (\frac{1}{2} - 2\beta)(f^-_{k+\frac{1}{2}})^{(0)} \\ & +(\frac{1}{12} - \beta)(f^-_{k-\frac{1}{2}})^{(-1)} - (\frac{1}{12} - \beta)(f^+_{k+\frac{3}{2}})^{(1)} \\ & +(\frac{1}{2} - 2\beta)(f^+_{k+\frac{1}{2}})^{(0)} + (\frac{1}{12} + \beta)(f^+_{k-\frac{1}{2}})^{(-1)}. \end{aligned}$$

The superscripts shown over the $f^{\pm}$ denote flux limited values of $f^{\pm}$, and are computed as follows:

$$\begin{aligned} (f^-_{k+\frac{3}{2}})^{(1)} \quad & = \quad \text{mm}\,[f^-_{k+\frac{3}{2}}, b\, f^-_{k+\frac{1}{2}}] \\ & = \quad \text{mm}\,[\frac{1}{b}\frac{f^-_{k+\frac{3}{2}}}{f^-_{k+\frac{1}{2}}}, 1]\, b\, f^-_{k+\frac{1}{2}} \\ & = \quad \phi(\frac{r^-_{k+1}}{b})\, b\, f^-_{k+\frac{1}{2}} \end{aligned} \tag{16}$$

$$\begin{aligned} (f^-_{k+\frac{1}{2}})^{(0)} \quad & = \quad \text{mm}\,[f^-_{k+\frac{1}{2}}, b\, f^-_{k+\frac{3}{2}}] \\ & = \quad \text{mm}\,[1, b\,\frac{f^-_{k+\frac{3}{2}}}{f^-_{k+\frac{1}{2}}}]\, f^-_{k+\frac{1}{2}} \\ & = \quad \phi(b\, r^-_{k+1})\, f^-_{k+\frac{1}{2}} \end{aligned} \tag{17}$$

$$(18)\qquad (f^-_{k-\frac12})^{(-1)} = \text{mm}\,[f^-_{k-\frac12}, b\,f^-_{k+\frac12}, b\,f^-_{k+\frac32}]$$
$$= \text{mm}\,[\frac{1}{b}\frac{f^-_{k-\frac12}}{f^-_{k+\frac12}}, 1, \frac{f^-_{k+\frac32}}{f^-_{k+\frac12}}]\,b\,f^-_{k+\frac12}$$
$$= \text{mm}\,[\frac{1}{br^-_k}, \phi(r^-_{k+1})]\,b\,f^-_{k+\frac12}$$
$$= \text{mm}\,[\phi(\frac{1}{b\,r^-_k}), r^-_{k+1}]\,b\,f^-_{k+\frac12}$$

$$(19)\qquad (f^+_{k+\frac32})^{(1)} = \text{mm}\,[f^+_{k+\frac32}, b\,f^+_{k+\frac12}, b\,f^+_{k-\frac12}]$$
$$= \text{mm}\,[\frac{1}{b}\frac{f^+_{k+\frac32}}{f^+_{k+\frac12}}, 1, \frac{f^+_{k-\frac12}}{f^+_{k+\frac12}}]\,b\,f^+_{k+\frac12}$$
$$= \text{mm}\,[\phi(\frac{1}{b\,r^+_{k+1}}), r^+_k]\,b\,f^+_{k+\frac12}$$
$$= \text{mm}\,[\frac{1}{b\,r^+_{k+1}}, \phi(r^+_k)]\,b\,f^+_{k+\frac12}$$

$$(20)\qquad (f^+_{k+\frac12})^{(0)} = \text{mm}\,[f^+_{k+\frac12}, b\,f^+_{k-\frac12}]$$
$$= \text{mm}\,[1, b\,\frac{f^+_{k-\frac12}}{f^+_{k+\frac12}}]\,f^+_{k+\frac12}$$
$$= \phi(b\,r^+_k)\,f^+_{k+\frac12}$$

$$(21)\qquad (f^+_{k-\frac12})^{(-1)} = \text{mm}\,[f^+_{k-\frac12}, b\,f^+_{k+\frac12}]$$
$$= \text{mm}\,[\frac{1}{b}\frac{f^+_{k-\frac12}}{f^+_{k+\frac12}}, 1]\,b\,f^+_{k+\frac12}$$
$$= \phi(\frac{r^+_k}{b})\,b\,f^+_{k+\frac12}$$

In the above, b is a "compression" parameter chosen in the range

$$1 < b \leq 3 + 12\beta.$$

We shall assume for the remainder of the paper that the local CFL numbers satisfy $|\nu^{\pm}_{k+\frac12}| \leq 1$ for all $k \in \mathbb{Z}$.

Using (9)-(13), we can rewrite the expressions

$$[-(\frac{1}{12}-\beta)b\ \text{mm}\,[\phi(\frac{1}{br^+_{k+1}}), r^+_k] + (\frac12 - 2\beta)\phi(b\,r^+_k) + (\frac{1}{12}+\beta)b\,\phi(\frac{r^+_k}{b})]\nu^+_{k+\frac12}u_{k+\frac12}$$

as

$$[-(\frac{1}{12}-\beta)b\ \text{mm}\,[\frac{1}{r^+_k}\phi(\frac{1}{br^+_{k+1}}), 1] + (\frac12 - 2\beta)b\,\phi(\frac{1}{b\,r^+_k}) + (\frac{1}{12}+\beta)\phi(\frac{b}{r^+_k})]\nu^+_{k-\frac12}u_{k-\frac12};$$

and

$$[(\frac{1}{12}+\beta)b\,\phi(\frac{r^-_k}{b}) + (\frac12 - 2\beta)\phi(b\,r^-_k) - (\frac{1}{12}-\beta)b\ \ \text{mm}\,[\phi(\frac{1}{br^-_{k-1}}), r^-_k]\,]\nu^-_{k-\frac12}u_{k-\frac12}$$

as

$$[(\frac{1}{12}+\beta)\phi(\frac{b}{r_k^-})+(\frac{1}{2}-2\beta)b\,\phi(\frac{1}{b\,r_k^-})-(\frac{1}{12}-\beta)b\ \text{mm}\ [\frac{1}{r_k^-}\phi(\frac{1}{b\,r_{k-1}^-}),1]\,]\nu_{k+\frac{1}{2}}^-u_{k+\frac{1}{2}}.$$

With these alternative representations, it is easy to see that the schemes (14)-(15) can be written in an increment form

$$u^k = u_k - C_{k-\frac{1}{2}}u_{k-\frac{1}{2}} + D_{k+\frac{1}{2}}u_{k+\frac{1}{2}}, \tag{22}$$

with

(23)

$$\begin{aligned} C_{k-\frac{1}{2}} &= \nu_{k-\frac{1}{2}}^+[\,-(\frac{1}{12}-\beta)b\ \text{mm}\ [\frac{1}{r_k^+}\phi(\frac{1}{r_{k+1}^+}),1]+(\frac{1}{2}-2\beta)b\,\phi(\frac{1}{b\,r_k^+}) \\ &+ (\frac{1}{12}+\beta)\phi(\frac{b}{r_k^+})+1+(\frac{1}{12}-\beta)b\ \text{mm}\ [\frac{1}{b\,r_k^+},\phi(r_{k-1}^+)] \\ &- (\frac{1}{2}-2\beta)\phi(b\,r_{k-1}^+)-(\frac{1}{12}+\beta)b\,\phi(\frac{r_{k-1}^+}{b})\,], \end{aligned}$$

and

(24)

$$\begin{aligned} D_{k+\frac{1}{2}} &= -\nu_{k+\frac{1}{2}}^-[\,1-(\frac{1}{12}+\beta)b\,\phi(\frac{r_{k+1}^-}{b})-(\frac{1}{2}-2\beta)\phi(b\,r_{k+1}^-) \\ &+ (\frac{1}{12}-\beta)b\ \text{mm}\ [\frac{1}{b\,r_k^-},\phi(r_{k+1}^-)]+(\frac{1}{12}+\beta)\phi(\frac{b}{r_k^-}) \\ &+ (\frac{1}{2}-2\beta)b\,\phi(\frac{1}{b\,r_k^-})-(\frac{1}{12}-\beta)b\ \text{mm}\ [\frac{1}{r_k^-}\phi(\frac{1}{b\,r_{k-1}^-}),1]\,]. \end{aligned}$$

This form of the schemes provides a convenient way of checking extremum traceability and TVD property of the schemes [9, 5].

2. A key property of general TVD schemes

In this section, we introduce the notions of Yang's extremum paths [19]. These concepts ware introduced by Yang (see Definition 2.13 [19]) in order to track the extrema in the computational domain. We then give sufficient conditions that guarantee a TVD scheme to be extremum traceable. The following two definitions are relevant, we restate them so that the paper is reasonably self-contained. Throughout the paper, we refer to [19] for the definitions, lemmas and theorems that we have quoted in the context.

Consider a numerical solution u defined on the set of grid points $X := \{(x_j, t_n) : j \in \mathbb{Z}, n \in \mathbb{Z}^+\}$. A finite set of successive grid points $\{x_q, \cdots, x_r\}$ with $r \geq q$ is said to be the *stencil of a spatial maximum*, or simply *an $\overline{E}$-stencil* of u at the time t_n, provided $u_q^n = \cdots = u_r^n, u_{q-1}^n < u_q^n$ and $u_{r+1}^n < u_r^n$. Notions of *$\underline{E}$-stencils* for minima and *E-stencils* for general extrema are defined similarly.

Definition 2.1 (see Definition 2.13 [19])**.** A nonempty subset of X denoted by $\overline{E}_{t_n,t_m}, n \leq m$, is called *a ridge of the numerical solution u from t_n to t_m* if

(i) for all ν, $n \leq \nu \leq m$, the set

$$P_{\overline{E}}(\nu) := \{x_j : (x_j, t_\nu) \in \overline{E}_{t_n,t_m}\} = \{x_{q^\nu}, \cdots, x_{r^\nu}\}$$

is not empty and is an $\overline{E}$-stencil of u at t_ν;

(ii) for all ν, $n \le \nu \le m-1$,

$$P_{\overline{E}}(\nu) \cup P_{\overline{E}}(\nu+1) = \{x_j : \min(q^\nu, q^{\nu+1}) \le j \le \max(r^\nu, r^{\nu+1})\}.$$

The set $P_{\overline{E}}(\nu)$ is called *the x-projection of* $\overline{E}_{t_n,t_m}$ at t_ν. The value of u along the ridge is denoted by $V_{\overline{E}}(\nu) : V_{\overline{E}}(\nu) = u_j^\nu$ for $q^\nu \le j \le r^\nu$.

If, for all ν, $n \le \nu \le m$, the $\overline{E}$-stencil in the item (i) of the definition is replaced by an $\underline{E}$-stencil, then the set is called a *trough* of u from t_n to t_m and is denoted by $\underline{E}_{t_n,t_m}$. The related notions $P_{\underline{E}}(\nu)$ and $V_{\underline{E}}(\nu)$ are defined similarly. Ridges and troughs are also called *extremum paths*. When we do not distinguish between ridges and troughs, we use E_{t_n,t_m}, $P_E(\nu)$, and $V_E(\nu)$ for either type. We write

$$E^1_{t_n,t_m} < (\le) E^2_{t_n,t_m}, \text{ if } \max P_{E^1}(\nu) < (\le) \max P_{E^2}(\nu) \text{ for } n \le \nu \le m.$$

Definition 2.2 (see Definition 2.14 [19])**.** A scheme is said to be *extremum traceable* if there exists a constant $c \ge 1$ such that for each numerical solution u of the scheme and each integer $N > 0$, there exists a finite or infinite collection of extremum paths $\{E^l_{t_0,t_N}\}_{l=l_1}^{l_2}$ with the following properties:

(i) $\{P_{E^l}(N)\}_{l=l_1}^{l_2}$ is precisely the set of E-stencils of u_j^n at the time t_N arranged in ascending spatial coordinates.

(ii) If $E^l_{t_0,t_N}$ is a ridge (trough), then $V_{E_l}(n)$ is a non increasing (non decreasing) function of n.

(iii) Let $P_{E^l}(n) = \{x_{q^l(n)}, \cdots, x_{r^l(n)}\}$ for $1 \le n \le N$. If $P_{E^l}(n) \cap P_{E^l}(n+1) = \emptyset$, then

$$|\, u^n_{q^l(n+1)} - u^n_{r^l(n)} \,| \le c\,|\, V_{E^l}(n+1) - V_{E^l}(n) \,| \quad \text{when} \quad q^l(n+1) > r^l(n),$$

$$|\, u^n_{r^l(n+1)} - u^n_{q^l(n)} \,| \le c\,|\, V_{E^l}(n+1) - V_{E^l}(n) \,| \quad \text{when} \quad q^l(n) > r^l(n+1).$$

(iv) If $l_2 > l_1$, then $E^l_{t_0,t_N} < E^{l+1}_{t_0,t_N}$ for $l_1 \le l \le l_2 - 1$.

Because of the true solution of (1) has TVD property, it is very important to maintain such property in the developing of a numerical scheme that approximate (1). Recall that an extremum traceable scheme is TVD [19] and a scheme equipped with extremum traceability prevents new extrema values that generate spurious oscillations of the solutions other than those which propagate from the previous time-level. Also, the numerical solutions computed by a TVD scheme will converge to a weak solution [11, 12] of (1). For general TVD schemes, we are able to show the following result, which is a condition to use Yang's WEI convergence criterion (Theorem 3.5). With this condition, we will carry out the entropy convergence analysis of the schemes of (14)-(15) in the next section.

Theorem 2.3 (see Theorem 2.3 [9])**.** *The sufficient conditions for the schemes* (14)-(15) *to be extremum traceable are the following inequalities:*

$$0 \le C_{k+\frac{1}{2}}, \quad 0 \le D_{k+\frac{1}{2}}, \quad 0 \le C_{k+\frac{1}{2}} + D_{k+\frac{1}{2}} \le 1, \text{ for all } k; \tag{1}$$

there is a positive constant μ such that, if u_k is a space extremum, then

$$\max\{C_{k\pm\frac{1}{2}}, C_{k\pm\frac{3}{2}}, D_{k\pm\frac{1}{2}}, D_{k+\frac{3}{2}}\} \le \frac{\mu}{4} < \frac{1}{4}, \tag{2}$$

where $C_{k+\frac{1}{2}}$ and $D_{k+\frac{1}{2}}$ are given by (23)-(24).

The inequalities of (1) are well known sufficient conditions, introduced by Harten [5], for the schemes (14)-(15) to be TVD. In terms of the local CFL numbers, we can state this theorem as follows.

Corollary 2.4. *The sufficient conditions for the schemes* (14)-(15) *to be extremum traceable are the following inequalities:*

$$\nu^{+}_{k+\frac{1}{2}} - \nu^{-}_{k+\frac{1}{2}} \le \frac{1}{3} \tag{3}$$

for all k, and when u_k is an extremum,

$$\max\{\nu^{+}_{k\pm\frac{1}{2}},\, \nu^{+}_{k\pm\frac{3}{2}},\, -\nu^{-}_{k+\frac{3}{2}},\, -\nu^{-}_{k\pm\frac{1}{2}}\} \le \frac{1}{10}. \tag{4}$$

Proof. Indeed, for all k, in reference of (23)-(24), we have $C_{k+\frac{1}{2}} \ge 0$, $D_{k+\frac{1}{2}} \ge 0$, and

$$C_{k+\frac{1}{2}} + D_{k+\frac{1}{2}} \le (\nu^{+}_{k+\frac{1}{2}} - \nu^{-}_{k+\frac{1}{2}})\frac{27}{12} \le 1.$$

When u_k is an extremum, $C_{k\pm\frac{1}{2}}$, and $D_{k\pm\frac{1}{2}}$ are reduced to

$$C_{k-\frac{1}{2}} = \nu^{+}_{k-\frac{1}{2}}[\,1 - (\frac{1}{2} - 2\beta)\phi(r^{+}_{k-1}) - (\frac{1}{12} + \beta)b\,\phi(\frac{r^{+}_{k-1}}{b})\,] \le \nu^{+}_{k-\frac{1}{2}},$$

$$D_{k+\frac{1}{2}} = -\nu^{-}_{k+\frac{1}{2}}[\,1 - (\frac{1}{12} + \beta)b\,\phi(\frac{r^{-}_{k+1}}{b}) - (\frac{1}{2} - 2\beta)\phi(b\,r^{-}_{k+1})\,] \le -\nu^{-}_{k+\frac{1}{2}},$$

$$\begin{aligned} C_{k+\frac{1}{2}} &= \nu^{+}_{k+\frac{1}{2}}[\,1 - (\frac{1}{12} - \beta)b\,\mathrm{mm}[\,\frac{1}{r^{+}_{k+1}}\phi(\frac{1}{r^{+}_{k+2}}), 1\,] + (\frac{1}{2} - 2\beta)b\,\phi(\frac{1}{b\,r^{+}_{k+1}}) \\ &\quad + (\frac{1}{12} + \beta)\phi(\frac{b}{r^{+}_{k+1}})\,] \\ &\le \nu^{+}_{k+\frac{1}{2}}[\,1 + (\frac{1}{12} + \beta) + (\frac{1}{2} - 2\beta)b\,] \le \frac{27}{12}\nu^{+}_{k+\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} D_{k-\frac{1}{2}} &= -\nu^{-}_{k-\frac{1}{2}}[\,1 + (\frac{1}{12} + \beta)\phi(\frac{b}{r^{-}_{k-1}}) + (\frac{1}{2} - 2\beta)b\,\phi(\frac{1}{b\,r^{-}_{k-1}}) \\ &\quad - (\frac{1}{12} - \beta)b\,\mathrm{mm}[\,\frac{1}{r^{-}_{k-1}}\phi(\frac{1}{b\,r^{-}_{k-2}}), 1\,]\,] \le -\frac{36}{12}\nu^{-}_{k-\frac{1}{2}}. \end{aligned}$$

Following the four inequalities of the above , we have arrived the desired estimations

$$C_{k\pm\frac{1}{2}} + D_{k\pm\frac{1}{2}} \le 1, \quad 2C_{k-\frac{1}{2}} + D_{k-\frac{1}{2}} \le 1, \quad C_{k+\frac{1}{2}} + 2D_{k+\frac{1}{2}} \le 1,$$

as well as,

$$C_{k+\frac{1}{2}} + D_{k+\frac{1}{2}} + D_{k+\frac{3}{2}} + C_{k-\frac{1}{2}} \le 1,\; C_{k-\frac{1}{2}} + D_{k-\frac{1}{2}} + D_{k+\frac{1}{2}} + C_{k-\frac{3}{2}} \le 1,$$

and so on. □

Consider a subfamily of E-fluxes given by

$$g^{E}(x,y) = \begin{cases} f(x) & \text{if } s \le x \le y, \\ f(y) & \text{if } x \le y \le s, \end{cases} \tag{5}$$

where s is a sonic point of $f(\cdot)$: $f'(s) = 0$. It is clear that both Godunov [4] and Engquist-Osher [3] fluxes:

$$g^{God}(u_j, u_{j+1}) = \begin{cases} \min_{u_j \le w \le u_{j+1}} f(w) & \text{when} \quad u_j \le u_{j+1}, \\ \max_{u_j \ge w \ge u_{j+1}} f(w) & \text{when} \quad u_j \ge u_{j+1}, \end{cases} \tag{6}$$

and

$$g^{EO}(u_j,u_{j+1}) = \int_0^{u_j} \max(f'(w),0)dw + \int_0^{u_{j+1}} \min(f'(w),0)dw + f(0), \tag{7}$$

are members of the fluxes given by (5). For β-schemes with $m = 2$, we conclude their extremum traceability in the following lemma.

Lemma 2.5. *The schemes* (14)-(15)*, with the building blocks given by the members of* (5)*, are extremum traceable, provided that*

$$\nu^{+}_{k+\frac{1}{2}} - \nu^{-}_{k+\frac{1}{2}} \le \frac{1}{3} \tag{8}$$

for all k, and when u_k is an extremum, $\lambda \max_{u_{k-2}\le w\le u_{k+2}} |\, f'(w)\,| \le \frac{1}{10}$.

3. the convergence of β-schemes

The following separation property characterizes that, at spatial extrema, the values of maximum (minimum) values of the numerical solutions are not increasing (decreasing). The similar conditions have been used to check TVD property of a scheme by E. Tadmor [16]. Lemma 3.2 verifies that β-schemes satisfy this separation property.

Assumption 3.1. *The numerical fluxes* $g^n_{k+\frac{1}{2}}$, $-\infty < k < \infty$, *satisfy*

$$g^n_{k+\frac{1}{2}} \ge f(u^n_k) \ge g^n_{k-\frac{1}{2}} \quad if \quad u^n_k \ge u^n_{k\pm1}$$

and

$$g^n_{k+\frac{1}{2}} \le f(u^n_k) \le g^n_{k-\frac{1}{2}} \quad if \quad u^n_k \le u^n_{k\pm1}$$

Lemma 3.2. *The scheme* (14)-(15) *satisfies the Assumption* 3.1.

Proof. If $u_k \ge u_{k\pm1}$, then

$$\begin{aligned}
g_{k+\frac{1}{2}} &= g^E_{k+\frac{1}{2}} - (\frac{1}{12}+\beta)(f^-_{k+\frac{3}{2}})^{(1)} - (\frac{1}{2}-2\beta)(f^-_{k+\frac{1}{2}})^{(0)} \\
&\ge g^E_{k+\frac{1}{2}} - (\frac{1}{12}+\beta)b\, f^-_{k+\frac{1}{2}} - (\frac{1}{2}-2\beta)f^-_{k+\frac{1}{2}} \\
&= g^E_{k+\frac{1}{2}} + [-(\frac{1}{12}+\beta)b - (\frac{1}{2}-2\beta)]f^-_{k+\frac{1}{2}} \\
&\ge g^E_{k+\frac{1}{2}} - f^-_{k+\frac{1}{2}} \\
&= f_k,
\end{aligned}$$

and

$$\begin{aligned}
g_{k-\frac{1}{2}} &= g^E_{k-\frac{1}{2}} + (\frac{1}{2}-2\beta)(f^+_{k-\frac{1}{2}})^{(0)} + (\frac{1}{12}+\beta)(f^+_{k-\frac{3}{2}})^{(-1)} \\
&\le g^E_{k-\frac{1}{2}} + (\frac{1}{2}-2\beta)f^+_{k-\frac{1}{2}} + (\frac{1}{12}+\beta)b\, f^+_{k-\frac{1}{2}} \\
&= g^E_{k-\frac{1}{2}} + [(\frac{1}{2}-2\beta) + (\frac{1}{12}+\beta)b\,]f^+_{k-\frac{1}{2}} \\
&\le g^E_{k-\frac{1}{2}} + f^+_{k-\frac{1}{2}} \\
&= f_k.
\end{aligned}$$

Similarly, we can show that if $u^n_k \le u^n_{k\pm1}$, then $g^n_{k+\frac{1}{2}} \le f(u^n_k) \le g^n_{k-\frac{1}{2}}$. □

Let $f[w;L,R]$ be the linear function interpolating $f(w)$ at $w = L$ and $w = R$. In this section, we assume that $f''(w) \ge 0$. In reference of (2), we denote $\tilde{v}_j = H(v_{j-p},\cdots,v_{j+p};\lambda)$ and $\bar{v}_j = \frac{v_j+\tilde{v}_j}{2}$ for any collection of data $\{v_j\}$.

Definition 3.3 (see Definition 2.20 [19])**.** We call an ordered pair of numbers $\{L, R\}$ a rarefying pair if $L < R$ and $f[w; L, R] > f(w)$ when $L < w < R$. We call a collection of data $\Gamma = \{v_j\}_{j=I-p}^{J+p}$ an ε-rarefying collection of the scheme to the rarefying pair $\{L, R\}$ if, for $\varepsilon > 0$,

(i) $L = v_I \le v_{I+1} \le \cdots \le v_J = R$;
(ii) $\tilde{v}_I \le \tilde{v}_{I+1} \le \cdots \le \tilde{v}_J$, $|L - \tilde{v}_I| < \varepsilon$, $|R - \tilde{v}_J| < \varepsilon$;
(iii) either $v_{I-1} \ge v_I$ or $v_I = v_{I+1}$; and either $v_{J+1} \le v_J$ or $v_{J-1} = v_J$.

Clearly, the conditions of (i) and (ii) imply that

$$\bar{v}_I \le \bar{v}_{I+1} \le \cdots \le \bar{v}_J,\ |L - \bar{v}_I| < \frac{\varepsilon}{2},\ \text{and } |R - \bar{v}_J| < \frac{\varepsilon}{2}.$$

We define the piecewise constant function g_Γ associated with the ε-rarefying collection Γ as follows:

$$g_\Gamma(w) = g_{j+\frac{1}{2}}[v] \quad \text{for } w \in (\bar{v}_j, \bar{v}_{j+1}), \quad I \le j \le J-1. \tag{9}$$

Definition 3.4. For the given rarefying pair $\{L, R\}$, a 0-rarefying collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ of the scheme that satisfies

$$L = v_{I-2} = v_{I-1} = v_I = v_{I+1} \le \cdots \le v_{J-1} = v_J = v_{J+1} = v_{J+2} = R \tag{10}$$

is called a normal collection.

Theorem 3.5 (see Theorem 2.21[19])**.** *An extremum traceable scheme that satisfies Assumption* 3.1 *converges for convex conservation laws if, for every rarefying pair* $\{L, R\}$ *and* ε*-rarefying collection to the pair,*

$$\int_L^R f[w; L, R]\, dw - \int_{\bar{v}_I}^{\bar{v}_J} g_\Gamma(w)\, dw > \delta \tag{11}$$

for some constant $\delta > 0$ *depending only on the exact flux* f*, the numerical flux function* g*, and the two numbers* L *and* R*, provided that* ε *is sufficiently small.*

For the class of β-schemes concerned, the condition on ε-rarefying collections in theorem 3.5 can be weakened by normal collections.

Lemma 3.6. *An extremum traceable scheme of the form* (14)-(15) *converges for convex conservation laws, provided that for each rarefying pair* $\{L, R\}$ *there is a constant* $\delta > 0$ *such that the inequality* (11) *holds for all normal collections of the scheme to the pair* $\{L, R\}$*.*

Proof. Let $\Lambda = \{\kappa_{P-2}, \cdots, \kappa_{Q+2}\}$ be an arbitrary ε-rarefying collection of the scheme to the pair $\{L, R\}$. Let

$$S' = \int_{\bar{\kappa}_P}^{\bar{\kappa}_Q} g_\Lambda(w)\, dw = \sum_{j=P}^{Q-1} (\bar{\kappa}_{j+1} - \bar{\kappa}_j)\, g_{j+\frac{1}{2}}[\kappa]. \tag{12}$$

by (i) and (iii) of Definition 3.3, either κ_P or κ_{P+1} is a minimum. In either case, Assumption 3.1 and the condition (ii) of Definition 3.3 imply that

$$\varepsilon > |L - \tilde{\kappa}_P| = |\tilde{\kappa}_P - \kappa_P| = \lambda |g_{P+\frac{1}{2}}[\kappa] - g_{P-\frac{1}{2}}[\kappa]| \ge \lambda |g_{P\pm\frac{1}{2}}[\kappa] - f(L)|. \tag{13}$$

Similarly, we have

$$\varepsilon > |R - \tilde{\kappa}_Q| \ge \lambda |g_{Q\pm\frac{1}{2}}[\kappa] - f(R)|. \tag{14}$$

Next, we construct a normal collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ as follows. First, let $I = P-1$ and $J = Q+1$ and we also set $v_{I-2} = v_{I-1} = v_I = L$, $v_J = v_{J+1} = v_{J+2} = R$, and $v_j = \kappa_j$ for $I+1 \le j \le J-1$. Then, we have

$$g_{I\pm\frac{1}{2}}[v] = f(L) \quad \text{and} \quad g_{J\pm\frac{1}{2}}[v] = f(R), \tag{15}$$

which imply that,

$$\bar{v}_I = \tilde{v}_I = v_I = L, \qquad \bar{v}_J = \tilde{v}_J = v_J = R. \tag{16}$$

Thus, the normality of $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ is justified by the non-decreasing relation of

$$\tilde{v}_I \leq \tilde{v}_{I+1} \leq \cdots \leq \tilde{v}_J.$$

Indeed, we notice that the following relationship

$$\tilde{v}_{I+3} \leq \tilde{v}_{I+4} \leq \cdots \leq \tilde{v}_{J-4} \leq \tilde{v}_{J-3},$$

is directly inherited from the condition (ii) of the given ε-rarefying collection of Λ

$$\tilde{\kappa}_{P+2} \leq \tilde{\kappa}_{P+3} \leq \cdots \leq \tilde{\kappa}_{Q-3} \leq \tilde{\kappa}_{Q-2}.$$

Also, using the definition of the numerical flux, we can verify that $(f^-_{P-\frac{1}{2}})^{(-1)} = 0$, $(f^+_{P+\frac{3}{2}})^{(1)} = 0$, $(f^+_{P+\frac{1}{2}})^{(0)} = 0$, and $(f^+_{P-\frac{1}{2}})^{(-1)} = 0$, which imply that

$$\begin{aligned} g_{P+\frac{1}{2}} &= g^E_{P+\frac{1}{2}} - (\frac{1}{12} + \beta)(f^-_{P+\frac{3}{2}})^{(1)} - (\frac{1}{2} - 2\beta)(f^-_{P+\frac{1}{2}})^{(0)} \\ &= g^E_{I+\frac{3}{2}} - (\frac{1}{12} + \beta)(f^-_{I+\frac{5}{2}})^{(1)} - (\frac{1}{2} - 2\beta)(f^-_{I+\frac{3}{2}})^{(0)} \\ &= g_{I+\frac{3}{2}}. \end{aligned}$$

Likewise, $(f^-_{Q+\frac{1}{2}})^{(1)} = 0$, $(f^-_{Q-\frac{1}{2}})^{(0)} = 0$, $(f^-_{Q-\frac{3}{2}})^{(-1)} = 0$, and $(f^+_{Q+\frac{1}{2}})^{(1)} = 0$, imply that

$$\begin{aligned} g_{Q-\frac{1}{2}} &= g^E_{Q-\frac{1}{2}} + (\frac{1}{2} - 2\beta)(f^+_{Q-\frac{1}{2}})^{(0)} + (\frac{1}{12} + \beta)(f^+_{Q-\frac{3}{2}})^{(-1)} \\ &= g^E_{J-\frac{3}{2}} + (\frac{1}{2} - 2\beta)(f^+_{J-\frac{3}{2}})^{(0)} + (\frac{1}{12} + \beta)(f^+_{J-\frac{5}{2}})^{(-1)} \\ &= g_{J-\frac{3}{2}}. \end{aligned}$$

Thus, we have $\tilde{v}_{I+2} = \tilde{\kappa}_{P+1}$, and $\tilde{v}_{J-2} = \tilde{\kappa}_{Q-1}$ as well. Therefore, we only need to verify that

$$\tilde{v}_I \leq \tilde{v}_{I+1} \leq \tilde{v}_{I+2} \quad \text{and} \quad \tilde{v}_{J-2} \leq \tilde{v}_{J-1} \leq \tilde{v}_J.$$

We will show that $\tilde{v}_I \leq \tilde{v}_{I+1}$ and $\tilde{v}_{I+1} \leq \tilde{v}_{I+2}$. The proof of $\tilde{v}_{J-2} \leq \tilde{v}_{J-1} \leq \tilde{v}_J$ is similar and we omit the details. Notice that the following is the consequence of the definition of the scheme and the Assumption

$$\begin{aligned} \tilde{v}_{I+1} &= v_{I+1} - \lambda(g_{I+\frac{3}{2}} - g_{I+\frac{1}{2}}) \\ &= v_{I+1} - \lambda(g_{I+\frac{3}{2}} - g^E_{I+\frac{1}{2}}) \\ &\geq v_{I+1} \geq v_I = \tilde{v}_I, \end{aligned}$$

and $\tilde{v}_{I+1} \leq \tilde{v}_{I+2}$ follows from the fact that $g_{P-\frac{1}{2}} \geq f_P = f_{I+1}$, and $g_{P+\frac{1}{2}} = g_{I+\frac{3}{2}}$. Indeed,

$$\begin{aligned} \tilde{v}_{I+2} - \tilde{v}_{I+1} &= \tilde{\kappa}_{P+1} - \tilde{v}_{I+1} = \tilde{\kappa}_{P+1} - \tilde{\kappa}_P + \tilde{\kappa}_P - \tilde{v}_{I+1} \\ &\geq \tilde{\kappa}_P - \tilde{v}_{I+1} \\ &= \kappa_P - \lambda(g_{P+\frac{1}{2}} - g_{P-\frac{1}{2}}) - v_{I+1} + \lambda(g_{I+\frac{3}{2}} - g_{I+\frac{1}{2}}) \\ &= \lambda(g_{P-\frac{1}{2}} - g_{I+\frac{1}{2}}) = \lambda(g_{P-\frac{1}{2}} - f_{I+1}) \geq 0. \end{aligned}$$

Secondly, let G be the Lipschitz constant of the numerical flux g, and $K = \max\{|f(L)|, |f(R)|\} + G(R-L)$. Denote

$$S = \int_L^R g_\Gamma(w)\, dw = \sum_{j=I}^{J-1} (\bar{v}_{j+1} - \bar{v}_j) g_{j+\frac{1}{2}}[v], \tag{17}$$

then a-priori estimate $|S - S'| \le 3K\varepsilon$ holds. Let δ' be a constant such that for all normal collections of the scheme to the pair $\{L, R\}$ the inequality (11) holds for $\delta = \delta'$. Thus, for $\delta = \delta'$, the inequality (11) also holds for the normal collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$. Therefore, for $\delta = \frac{\delta'}{2}$, the inequality (11) holds for all ε-collection of the scheme to the pair $\{L, R\}$ provided that $\varepsilon \le \frac{\delta}{3K}$.

It remains to show the a-priori estimate. First, we notice that $\bar{\kappa}_j = \bar{v}_j$ for $P + 1 \le j \le Q - 2$, Therefore the terms of the difference

$$S - S' = \sum_{j=I}^{J-1} (\bar{v}_{j+1} - \bar{v}_j) g_{j+\frac{1}{2}}[v] - \sum_{j=P}^{Q-1} (\bar{\kappa}_{j+1} - \bar{\kappa}_j)\, g_{j+\frac{1}{2}}[\kappa]$$

from $j = P + 1$ to $j = Q - 2$ are all diminished. For the remaining terms, we use the relationship of Λ and Γ and (13)-(16) to yield the following estimates.

$$|\bar{v}_{I+1} - \bar{\kappa}_{I+1}| < \frac{\varepsilon}{2}, \quad |\bar{v}_{J-1} - \bar{\kappa}_{J-1}| < \frac{\varepsilon}{2}, \tag{18}$$

$$|\bar{v}_{I+1} - \bar{v}_I| = |\bar{v}_{I+1} - L| \le |\bar{v}_{I+1} - \bar{\kappa}_{I+1}| + |\bar{\kappa}_{I+1} - L| < \varepsilon, \tag{19}$$

and

$$|\bar{v}_J - \bar{v}_{J-1}| = |\bar{v}_{J-1} - R| \le |\bar{v}_{J-1} - \bar{\kappa}_{J-1}| + |\bar{\kappa}_Q - R| < \varepsilon. \tag{20}$$

Finally, using the fact that $\bar{v}_{I+2} = \bar{\kappa}_{P+1}$, $\bar{v}_{J-2} = \bar{\kappa}_{Q-1}$, $g_{I+\frac{3}{2}}[v] = g_{P+\frac{1}{2}}[\kappa]$, $g_{J-\frac{3}{2}}[v] = g_{Q-\frac{1}{2}}[\kappa]$, and (18)-(20), we have derived the desired estimate as follows.

$$\begin{aligned}
|S - S'| &= |(\bar{v}_{I+1} - \bar{v}_I) g_{I+\frac{1}{2}}[v] + (\bar{v}_J - \bar{v}_{J-1}) g_{J-\frac{1}{2}}[v] \\
&+ (\bar{v}_{I+2} - \bar{v}_{I+1}) g_{I+\frac{3}{2}}[v] - (\bar{\kappa}_{P+1} - \bar{\kappa}_P) g_{P+\frac{1}{2}}[\kappa] \\
&+ (\bar{v}_{J-1} - \bar{v}_{J-2}) g_{J-\frac{3}{2}}[v] - (\bar{\kappa}_Q - \bar{\kappa}_{Q-1}) g_{Q-\frac{1}{2}}[\kappa]| \\
&\le |\bar{v}_{I+1} - \bar{v}_I||g_{I+\frac{1}{2}}[v]| + |\bar{v}_J - \bar{v}_{J-1}||g_{J-\frac{1}{2}}[v]| \\
&+ |\bar{v}_{I+1} - \bar{\kappa}_P||g_{I+\frac{3}{2}}[v]| + |\bar{v}_{J-1} - \bar{\kappa}_Q||g_{Q-\frac{1}{2}}[\kappa]| \\
&< (\varepsilon + \varepsilon + \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon) K = 3K\varepsilon,
\end{aligned} \tag{21}$$

and the proof is completed. □

For a normal collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$, we denote the vertex $(v_j, f(v_j))$ by V_j and the area of convex polygon $V_{j_1} V_{j_2} \cdots V_{j_r}$ by $S_{j_1,\ldots,j_r}$. Let $\sigma_\Gamma = \max_{I-2\le j\le J+2} |\nu^{\pm}_{j\pm\frac{1}{2}}|$, and let

$$\alpha_j = \begin{cases} 0.5 & \text{if } \Delta v_{j-2} = \Delta v_{j+1} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

When the building block of the schemes (14)-(15) is the subclass of E-schemes with the fluxes defined by (5), we have the following very important inequality (22), which will enable us to prove the mail result of Theorem 3.9. The proof of lemma 3.7 will be given shortly.

Lemma 3.7. *Let $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ be a normal collection to a rarefying pair $\{L, R\}$. Then the numerical solutions of the schemes* (14)-(15) *for convex conservation laws* (1) *satisfy, for a sufficiently small σ_Γ, the following inequality*

$$\int_L^R (f[w; L, R] - g_\Gamma) dw \ge S_{I,I+1,\ldots,J} - \sum_{j=I+1}^{J-1} \alpha_j S_{j-1,j,j+1}. \tag{22}$$

Lemma 3.8 (see Lemma 3.7 [19])**.** *We have*

$$S_{I,I+1,\dots,J} - \sum_{j=I+1}^{J-1} S_{j-1,j,j+1} \geq S_{I,i,i+1,J} - (S_{I,i,i+1} + S_{i,i+1,J})$$

for $I < i < J-1$.

Let $\sigma = \lambda \max_w |f'(w)|$. For the class of fully-discrete β-schemes when $m = 2$, equipped with above lemmas, we have obtained the following entropy convergence result. The proof is similar to the one given by Jiang [9] for van Leer's flux limiter schemes and we omit the proof.

Theorem 3.9. *The numerical solutions of the schemes* (14)-(15)*, for the convex problems* (1)*, converge to the entropy solution provided that* $g^E(\cdot,\cdot)$ *is a numerical flux given by* (5)*, and* σ *is sufficiently small.*

Remark. The proof the lemma 3.7 is under the condition that the values of $(f^-_{j-\frac{1}{2}})^{(-1)}$ and $(f^+_{j+\frac{3}{2}})^{(1)}$ are simultaneously taken as either first, second, or the last argument of their min mod operators. We believe that the lemma is also true for the general case at least for large values of b, since the values of $(f^-_{j-\frac{1}{2}})^{(-1)}$ and $(f^+_{j+\frac{3}{2}})^{(1)}$ are mostly return to their unlimited ones respectively [13] for larger values of b.

Proof of Lemma 3.7. In the following, we keep the same notations $f^\pm_{j+\frac{1}{2}}$ and $r^\pm_j$ for $\{v_j\}$ instead of $\{u_j\}$. We also use

$$f'_{j+\frac{1}{2}} := \frac{f(v_{j+1}) - f(v_j)}{v_{j+1} - v_j} \tag{23}$$

to denote the divided difference.

To justify the inequality (22), it suffices to show the following inequality:

$$\int_L^R g_\Gamma(w)dw - \sum_{j=I}^{J-1} \int_{v_j}^{v_{j+1}} f[w; v_j, v_{j+1}]dw \leq \sum_{j=I+1}^{J-1} \alpha_j S_{j-1,j,j+1}. \tag{24}$$

Let $f_s := f(v_s)$, $\Delta v_{s+\frac{1}{2}} := v_{k+1} - v_s$, and $f'_{s+\frac{1}{2}} := (f_{k+1} - f_s)/\Delta v_{s+\frac{1}{2}}$, where v_s is a sonic point ($f'(v_s) = 0$). Without loss of generality, let $v_k \leq v_s \leq v_{k+1}$ for some integer k with $I \leq k \leq J-1$. We also let $\frac{1}{12} f'_{s+\frac{1}{2}} \leq \beta f'_{k+\frac{3}{2}}$, otherwise, for the given $\{v_i\}$ and β, we add v'_{k+1}, such that $v_k \leq v_s \leq v'_{k+1} \leq v_{k+1}$ and the inequality $\frac{1}{2} f'_{s+\frac{1}{2}} \leq \beta f'_{k+\frac{3}{2}}$ holds for v_s, v'_{k+1} and v_{k+1}. Then $\Gamma' = \Gamma \cup \{v'_{k+1}\}$ is also a normal collection and if (24) holds for Γ', it will holds for Γ as well (we subtract the triangle area $S_{v_k, v'_{k+1}, v_{k+1}}$ from both sides of (24) that holds for Γ'). Now for any $g^E(\cdot,\cdot)$ given by (5), we have

$$f^+_{j+\frac{1}{2}} = 0, \quad \text{for } I \leq j \leq k-1;$$

$$f^+_{j+\frac{1}{2}} = f'_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}}, \quad \text{for } J-1 \geq j \geq k+1;$$

$$f^-_{j+\frac{1}{2}} = 0, \quad \text{for } J-1 \geq j \geq k+1;$$

and

$$f^-_{j+\frac{1}{2}} = f'_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}}, \quad \text{for } I \leq j \leq k-1.$$

Denote $f_j := f(v_j)$, $\Delta v_{j\pm\frac{1}{2}} = \pm(v_{j\pm1} - v_j)$, $\Delta v_{s-\frac{1}{2}} := v_s - v_k$, $f'_{s-\frac{1}{2}} := (f_s - f_k)/\Delta v_{s-\frac{1}{2}}$. Then, by (9), we have

(25)

$$\begin{aligned}\text{LHS of (24)} &= \sum_{j=I}^{J-1} g_{j+\frac{1}{2}}\Delta\bar{v}_{j+\frac{1}{2}} - \sum_{j=I}^{J-1}\int_{v_j}^{v_{j+1}} f[w;v_j,v_{j+1}]dw \\ &= \sum_{j=I}^{J-1} g_{j+\frac{1}{2}}\frac{\Delta v_{j+\frac{1}{2}}+\Delta\tilde{v}_{j+\frac{1}{2}}}{2} - \sum_{j=I}^{J-1}\frac{f_j+f_{j+1}}{2}\Delta v_{j+\frac{1}{2}} \\ &= \frac{1}{2}(P_{(j\le k-2)}+P_{k-1}+P_k+P_{k+1}+P_{(j\ge k+2)}),\end{aligned}$$

and the definitions of $P_{(j\le k-2)}, P_{k-1}, P_k, P_{k+1}$ and $P_{(j\ge k+2)}$ will be given in order.

Recall the numerical flux is defined by

$$\begin{aligned} g_{j+\frac{1}{2}} &= g^E_{j+\frac{1}{2}} - (\frac{1}{12}+\beta)(f^-_{j+\frac{3}{2}})^{(1)} - (\frac{1}{2}-2\beta)(f^-_{j+\frac{1}{2}})^{(0)} \\ &\quad +(\frac{1}{12}-\beta)(f^-_{j-\frac{1}{2}})^{(-1)} - (\frac{1}{12}-\beta)(f^+_{j+\frac{3}{2}})^{(1)} \\ &\quad +(\frac{1}{2}-2\beta)(f^+_{j+\frac{1}{2}})^{(0)} + (\frac{1}{12}+\beta)(f^+_{j-\frac{1}{2}})^{(-1)},\end{aligned}$$

and using the increment form (22)-(24), we have

$$\Delta\tilde{v}_{j+\frac{1}{2}} = \Delta v_{j+\frac{1}{2}} - (C_{j+\frac{1}{2}}+D_{j+\frac{1}{2}})\Delta v_{j+\frac{1}{2}} + D_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}} + C_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}} \ge 0.$$

For $j\le k-2$, we consider the value of $(f^-_{j-\frac{1}{2}})^{(-1)} = \mathrm{mm}[\,f^-_{j-\frac{1}{2}}, b\,f^-_{j+\frac{1}{2}}, b\,f^-_{j+\frac{3}{2}}\,]$ in three cases separately. In other words, case 1. $(f^-_{j-\frac{1}{2}})^{(-1)} = f^-_{j-\frac{1}{2}}$, case 2. $(f^-_{j-\frac{1}{2}})^{(-1)} = b\,f^-_{j+\frac{1}{2}}$, and case 3. $(f^-_{j-\frac{1}{2}})^{(-1)} = b\,f^-_{j+\frac{3}{2}}$. Using the relations

$$\begin{aligned}&\sum_{j=I}^{k-2}(\frac{1}{6}-2\beta)f'_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}\Delta v_{j+\frac{1}{2}} \\ = &\sum_{j=I}^{k-2}(\frac{1}{6}-2\beta)f'_{j+\frac{1}{2}}\Delta v_{j+\frac{1}{2}}\Delta v_{j+\frac{3}{2}} - (\frac{1}{6}-2\beta)f'_{k-\frac{3}{2}}\Delta v_{k-\frac{3}{2}}\Delta v_{k-\frac{1}{2}},\end{aligned}$$

and

$$\sum_{j=I}^{k-2}2\beta f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}} = \sum_{j=I}^{k-2}2\beta f'_{j+\frac{3}{2}}\Delta v^2_{j+\frac{3}{2}} - 2\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}},$$

for case 1., we have

$$\begin{aligned}&\sum_{j=I}^{k-2}[\,-(\frac{1}{6}-2\beta)f'_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}} - 4\beta f'_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}} + 4\beta f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}} \\ &+(\frac{1}{6}-2\beta)(f^-_{j-\frac{1}{2}})^{(-1)}\Delta v_{j+\frac{1}{2}}\,] \\ \le\ &\sum_{j=I}^{k-2}[\,(\frac{1}{6}-2\beta)(f'_{j+\frac{1}{2}}-f'_{j+\frac{3}{2}})\Delta v_{j+\frac{1}{2}}\Delta v_{j+\frac{3}{2}} + 2\beta f'_{j+\frac{3}{2}}(\Delta v_{j+\frac{1}{2}}-\Delta v_{j+\frac{3}{2}})^2\,] \\ &-2\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}} - (\frac{1}{6}-2\beta)f'_{k-\frac{3}{2}}\Delta v_{k-\frac{3}{2}}\Delta v_{k-\frac{1}{2}} \\ \le\ &-2\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}} - (\frac{1}{6}-2\beta)f'_{k-\frac{3}{2}}\Delta v_{k-\frac{3}{2}}\Delta v_{k-\frac{1}{2}};\end{aligned}$$

for case 2., we have

$$
\begin{aligned}
&\sum_{j=I}^{k-2}[\,-(\frac{1}{6}+2\beta)f'_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}}+4\beta f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}}+(\frac{1}{6}-2\beta)(f^-_{j-\frac{1}{2}})^{(-1)}\Delta v_{j+\frac{1}{2}}\,]\\
\le&\sum_{j=I}^{k-2}[\,-(\frac{1}{6}+2\beta)f'_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}}+4\beta f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}}+(\frac{1}{6}-2\beta)f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}}\,]\\
=&\sum_{j=I}^{k-2}(\frac{1}{6}+2\beta)[\,-f'_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}}\Delta v_{j+\frac{3}{2}}+f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}}\,]\\
\le&\sum_{j=I}^{k-2}(\frac{1}{12}+\beta)f'_{j+\frac{3}{2}}(\Delta v_{j+\frac{1}{2}}-\Delta v_{j+\frac{3}{2}})^2-(\frac{1}{12}+\beta)f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}}\\
\le&-(\frac{1}{12}+\beta)f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}};
\end{aligned}
$$

for case 3., we have

$$
\begin{aligned}
&\sum_{j=I}^{k-2}[\,-(\frac{1}{6}+2\beta)f'_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}}+4\beta f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}}\\
&+(\frac{1}{6}-2\beta)(f^-_{j-\frac{1}{2}})^{(-1)}\Delta v_{j+\frac{1}{2}}\,]\\
\le\ &\sum_{j=I}^{k-2}[\,-(\frac{1}{6}+2\beta)f'_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}}+4\beta f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}}\\
&+(\frac{1}{6}-2\beta)f'_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}}\Delta v_{j+\frac{3}{2}}\,]\\
=\ &\sum_{j=I}^{k-2}4\beta[\,-f'_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}}\Delta v_{j+\frac{3}{2}}+f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}}\,]\\
\le\ &\sum_{j=I}^{k-2}2\beta f'_{j+\frac{3}{2}}(\Delta v_{j+\frac{1}{2}}-\Delta v_{j+\frac{3}{2}})^2-2\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}}\\
\le\ &-2\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}}.
\end{aligned}
$$

Let

$$
T:=[\,f_{k-1}-(\frac{1}{12}+\beta)f'_{k-\frac{1}{2}}\Delta v_{k-\frac{1}{2}}-(\frac{1}{2}-2\beta)f'_{k-\frac{3}{2}}\Delta v_{k-\frac{3}{2}}\,]D_{k-\frac{1}{2}}\Delta v_{k-\frac{1}{2}},
$$

then

$$
\begin{aligned}
&\sum_{j=I}^{k-2}[\,f_{j+1}\Delta v_{j+\frac{3}{2}}-(\frac{1}{12}+\beta)f'_{j+\frac{3}{2}}\Delta v^2_{j+\frac{3}{2}}-(\frac{1}{2}-2\beta)f'_{j+\frac{1}{2}}\Delta v_{j+\frac{1}{2}}\Delta v_{j+\frac{3}{2}}\,]D_{j+\frac{3}{2}}\\
=&\sum_{j=I}^{k-2}[\,f_j\Delta v_{j+\frac{1}{2}}-(\frac{1}{12}+\beta)f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}}-(\frac{1}{2}-2\beta)f'_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}\Delta v_{j+\frac{1}{2}}\,]D_{j+\frac{1}{2}}+T.
\end{aligned}
$$

We also denote

$$\text{case 1.}\quad T_{(j\le k-2)} := -2\beta f'_{k-\frac12}\Delta v^2_{k-\frac12} - (\frac{1}{6}-2\beta)f'_{k-\frac32}\Delta v_{k-\frac32}\Delta v_{k-\frac12} + T,$$

$$\text{case 2.}\quad T_{(j\le k-2)} := -(\frac{1}{12}+\beta)f'_{k-\frac12}\Delta v^2_{k-\frac12} + T,$$

$$\text{case 3.}\quad T_{(j\le k-2)} := -2\beta f'_{k-\frac12}\Delta v^2_{k-\frac12} + T;$$

and

$$\begin{aligned} d_j \quad := \quad & -(\frac{7}{12}+3\beta)f'_{j+\frac12}\Delta v^2_{j+\frac12} + (\frac{1}{12}+\beta)f'_{j+\frac32}\Delta v_{j+\frac12}\Delta v_{j+\frac32} \\ - \quad & (\frac{1}{2}-2\beta)f'_{j-\frac12}\Delta v_{j-\frac12}\Delta v_{j+\frac12}. \end{aligned}$$

With these definitions and the convexity of the flux in mind, we have derived the following estimates.

(26)

$$\begin{aligned}
&P_{(j\le k-2)} := \sum_{j=I}^{k-2}\{g_{j+\frac12}[\,\Delta v_{j+\frac12} + \Delta\tilde v_{j+\frac12}\,] - (f_j + f_{j+1})\Delta v_{j+\frac12}\} \\
&= \sum_{j=I}^{k-2}\{[\,g^E_{j+\frac12} - (\frac{1}{12}+\beta)(f^-_{j+\frac32})^{(1)} - (\frac{1}{2}-2\beta)(f^-_{j+\frac12})^{(0)} \\
&\quad + (\frac{1}{12}-\beta)(f^-_{j-\frac12})^{(-1)}\,][\,2\Delta v_{j+\frac12} - D_{j+\frac12}\Delta v_{j+\frac12} \\
&\quad + D_{j+\frac32}\Delta v_{j+\frac32}\,] - (f_j + f_{j+1})\Delta v_{j+\frac12}\} \\
&\le \sum_{j=I}^{k-2}\{[\,g^E_{j+\frac12} - (\frac{1}{12}+\beta)f^-_{j+\frac32} - (\frac{1}{2}-2\beta)f^-_{j+\frac12} \\
&\quad + (\frac{1}{12}-\beta)(f^-_{j-\frac12})^{(-1)}\,][\,2\Delta v_{j+\frac12} - D_{j+\frac12}\Delta v_{j+\frac12} \\
&\quad + D_{j+\frac32}\Delta v_{j+\frac32}\,] - (f_j + f_{j+1})\Delta v_{j+\frac12}\} \\
&\le \sum_{j=I}^{k-2}\{[\,2f_{j+1} - (f_j + f_{j+1})\,]\Delta v_{j+\frac12} - (\frac{1}{6}+2\beta)f'_{j+\frac32}\Delta v_{j+\frac32}\Delta v_{j+\frac12} \\
&\quad - (1-4\beta)f'_{j+\frac12}\Delta v^2_{j+\frac12} + (\frac{1}{6}-2\beta)(f^-_{j-\frac12})^{(-1)}\Delta v_{j+\frac12}\} \\
&\quad - \sum_{j=I}^{k-2}[\,f_{j+1}\Delta v_{j+\frac12} - (\frac{1}{12}+\beta)f'_{j+\frac32}\Delta v_{j+\frac32}\Delta v_{j+\frac12} - (\frac{1}{2}-2\beta)f'_{j+\frac12}\Delta v^2_{j+\frac12}\,]D_{j+\frac12} \\
&\quad + \sum_{j=I}^{k-2}[\,f_{j+1}\Delta v_{j+\frac32} - (\frac{1}{12}+\beta)f'_{j+\frac32}\Delta v^2_{j+\frac32} - (\frac{1}{2}-2\beta)f'_{j+\frac12}\Delta v_{j+\frac12}\Delta v_{j+\frac32}\,]D_{j+\frac32} \\
&= \sum_{j=I}^{k-2}[\,-(\frac{1}{6}-2\beta)f'_{j+\frac32}\Delta v_{j+\frac32}\Delta v_{j+\frac12} - 4\beta f'_{j+\frac32}\Delta v_{j+\frac32}\Delta v_{j+\frac12} + 4\beta f'_{j+\frac12}\Delta v^2_{j+\frac12} \\
&\quad + (\frac{1}{6}-2\beta)(f^-_{j-\frac12})^{(-1)}\Delta v_{j+\frac12}\,] + T + \sum_{j=I}^{k-2}[\,-(\frac{7}{12}+3\beta)f'_{j+\frac12}\Delta v^2_{j+\frac12} \\
&\quad + (\frac{1}{12}+\beta)f'_{j+\frac32}\Delta v_{j+\frac12}\Delta v_{j+\frac32} - (\frac{1}{2}-2\beta)f'_{j-\frac12}\Delta v_{j-\frac12}\Delta v_{j+\frac12}\,]D_{j+\frac12}
\end{aligned}$$

$$\leq \left\{\begin{array}{ll}\text{case 1.} & T_{(j\leq k-2)}\\ \text{case 2.} & T_{(j\leq k-2)}\\ \text{case 3.} & T_{(j\leq k-2)}\end{array}\right. + \sum_{j=I}^{k-2} d_j D_{j+\frac{1}{2}}.$$

Similarly, for $j \geq k+2$, we consider the value of $(f^+_{j+\frac{3}{2}})^{(1)} = \mathrm{mm}[\, f^+_{j+\frac{3}{2}}, b\, f^+_{j+\frac{1}{2}}, b\, f^+_{j-\frac{1}{2}}\,]$ in three cases. For case 1., we have $(f^+_{j+\frac{3}{2}})^{(1)} = f^+_{j+\frac{3}{2}}$ and

$$\begin{aligned}
&\sum_{j=k+2}^{J-1} \{[\, f_j - (\frac{1}{12}-\beta)(f^+_{j+\frac{3}{2}})^{(1)} + (\frac{1}{2}-2\beta)f^+_{j+\frac{1}{2}} + (\frac{1}{12}+\beta)f^+_{j-\frac{1}{2}}\,]2\Delta v_{j+\frac{1}{2}}\\
&-(f_j+f_{j+1})\Delta v_{j+\frac{1}{2}}\}\\
\leq\ &\sum_{j=k+2}^{J-1}[\,-4\beta f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}} + 4\beta f'_{j+\frac{1}{2}}\Delta v_{j+\frac{1}{2}}\Delta v_{j+\frac{3}{2}}\,]\\
&+(\frac{1}{6}+2\beta)f'_{k+\frac{3}{2}}\Delta v_{k+\frac{3}{2}}\Delta v_{k+\frac{5}{2}}\\
\leq\ &-2\beta f'_{k+\frac{5}{2}}\Delta v^2_{k+\frac{5}{2}} + (\frac{1}{6}+2\beta)f'_{k+\frac{3}{2}}\Delta v_{k+\frac{3}{2}}\Delta v_{k+\frac{5}{2}};
\end{aligned}$$

for case 2., we have $(f^+_{j+\frac{3}{2}})^{(1)} = bf^+_{j+\frac{1}{2}}$ and

$$\begin{aligned}
&\sum_{j=k+2}^{J-1} \{[\, f_j - (\frac{1}{12}-\beta)(f^+_{j+\frac{3}{2}})^{(1)} + (\frac{1}{2}-2\beta)f^+_{j+\frac{1}{2}} + (\frac{1}{12}+\beta)f^+_{j-\frac{1}{2}}\,]2\Delta v_{j+\frac{1}{2}}\\
&-(f_j+f_{j+1})\Delta v_{j+\frac{1}{2}}\}\\
\leq\ &\sum_{j=k+2}^{J-1}[\,-(\frac{1}{6}+2\beta)f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}} + (\frac{1}{6}+2\beta)f'_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}\Delta v_{j+\frac{1}{2}}\,]\\
\leq\ &(\frac{1}{12}+\beta)f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{3}{2}};
\end{aligned}$$

for case 3., we have $(f^+_{j+\frac{3}{2}})^{(1)} = bf^+_{j-\frac{1}{2}}$ and

$$\begin{aligned}
&\sum_{j=k+2}^{J-1} \{[\, f_j - (\frac{1}{12}-\beta)(f^+_{j+\frac{3}{2}})^{(1)} + (\frac{1}{2}-2\beta)f^+_{j+\frac{1}{2}} + (\frac{1}{12}+\beta)f^+_{j-\frac{1}{2}}\,]2\Delta v_{j+\frac{1}{2}}\\
&-(f_j+f_{j+1})\Delta v_{j+\frac{1}{2}}\}\\
\leq\ &\sum_{j=k+2}^{J-1}[\,-4\beta f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}} + 4\beta f'_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}\Delta v_{j+\frac{1}{2}}\,]\\
\leq\ &2\beta f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{3}{2}}.
\end{aligned}$$

let

$$J := [\, f_{k+2} + (\frac{1}{2}-2\beta)f^+_{k+\frac{5}{2}} + (\frac{1}{12}+\beta)f^+_{k+\frac{3}{2}}\,]C_{k+\frac{3}{2}}\Delta v_{k+\frac{3}{2}},$$

then we have

$$\begin{aligned}
&\sum_{k+2}^{J-1}[\, f_j + (\frac{1}{2}-2\beta)f^+_{j+\frac{1}{2}} + (\frac{1}{12}+\beta)f^+_{j-\frac{1}{2}}\,]C_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}\\
=\ &\sum_{k+2}^{J-1}[\, f_{j+1} + (\frac{1}{2}-2\beta)f^+_{j+\frac{3}{2}} + (\frac{1}{12}+\beta)f^+_{j+\frac{1}{2}}\,]C_{j+\frac{1}{2}}\Delta v_{j+\frac{1}{2}} + J.
\end{aligned}$$

With the following notations,

$$\begin{aligned} c_j &:= (\frac{7}{12}+3\beta)f'_{j+\frac{1}{2}}\Delta v^2_{j+\frac{1}{2}} + (\frac{7}{12}-3\beta)f'_{j+\frac{3}{2}}\Delta v_{j+\frac{1}{2}}\Delta v_{j+\frac{3}{2}} \\ &- (\frac{1}{12}+\beta)f'_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}\Delta v_{j+\frac{1}{2}}; \end{aligned}$$

and

$$\begin{aligned} \text{case 1.}\quad J_{(j\geq k+2)} &:= -2\beta f'_{k+\frac{5}{2}}\Delta v^2_{k+\frac{5}{2}} + (\frac{1}{6}+2\beta)f'_{k+\frac{3}{2}}\Delta v_{k+\frac{3}{2}}\Delta v_{k+\frac{5}{2}} + J, \\ \text{case 2.}\quad J_{(j\geq k+2)} &:= (\frac{1}{12}+\beta)f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{3}{2}} + J, \\ \text{case 3.}\quad J_{(j\geq k+2)} &:= 2\beta f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{3}{2}} + J; \end{aligned}$$

we obtain

$$P_{(j\geq k+2)} := \sum_{j=k+2}^{J-1}\{g_{j+\frac{1}{2}}[\,\Delta v_{j+\frac{1}{2}} + \Delta\tilde{v}_{j+\frac{1}{2}}\,] - (f_j + f_{j+1})\Delta v_{j+\frac{1}{2}}\}$$

$$\begin{aligned} &= \sum_{j=k+2}^{J-1}\{[\,g^E_{j+\frac{1}{2}} - (\frac{1}{12}-\beta)(f^+_{j+\frac{3}{2}})^{(1)} + (\frac{1}{2}-2\beta)(f^+_{j+\frac{1}{2}})^{(0)} \\ &\quad +(\frac{1}{12}+\beta)(f^+_{j-\frac{1}{2}})^{(-1)}\,][\,2\Delta v_{j+\frac{1}{2}} - C_{j+\frac{1}{2}}\Delta v_{j+\frac{1}{2}} + C_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}\,] \\ &\quad -(f_j + f_{j+1})\Delta v_{j+\frac{1}{2}}\} \\ &\leq \sum_{j=k+2}^{J-1}\{[\,f_j - (\frac{1}{12}-\beta)(f^+_{j+\frac{3}{2}})^{(1)} + (\frac{1}{2}-2\beta)f^+_{j+\frac{1}{2}} + (\frac{1}{12}+\beta)f^+_{j-\frac{1}{2}}\,] \\ &\quad \times[\,2\Delta v_{j+\frac{1}{2}} - C_{j+\frac{1}{2}}\Delta v_{j+\frac{1}{2}} + C_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}\,] \\ &\quad -(f_j + f_{j+1})\Delta v_{j+\frac{1}{2}}\} \\ &\leq \left\{\begin{array}{ll} \text{case 1.} & J_{(j\geq k+2)} \\ \text{case 2.} & J_{(j\geq k+2)} \\ \text{case 3.} & J_{(j\leq k+2)} \end{array}\right. + \sum_{j=k+2}^{J-1} c_j C_{j+\frac{1}{2}}. \end{aligned}$$

Now, we compute the $(k-1)$th, kth and $(k+1)$th terms of the LHS of (24) defined by P_{k-1}, P_k and P_{k+1} respectively as follows.

$$P_{k-1} := g_{k-\frac{1}{2}}[\,\Delta v_{k-\frac{1}{2}} + \Delta\tilde{v}_{k-\frac{1}{2}}\,] - (f_{k-1} + f_k)\Delta v_{k-\frac{1}{2}}$$

$$\begin{aligned} &= [\,g^E_{k-\frac{1}{2}} - (\frac{1}{12}+\beta)(f^-_{k+\frac{1}{2}})^{(1)} - (\frac{1}{2}-2\beta)(f^-_{k-\frac{1}{2}})^{(0)} \\ &\quad +(\frac{1}{12}-\beta)(f^-_{k-\frac{3}{2}})^{(-1)}\,][2\Delta v_{k-\frac{1}{2}} - D_{k-\frac{1}{2}}\Delta v_{k-\frac{1}{2}} \\ &\quad +D_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}}\,] - (f_{k-1} + f_k)\Delta v_{k-\frac{1}{2}} \\ &\leq [\,f_k - (\frac{1}{12}+\beta)f^-_{k+\frac{1}{2}} - (\frac{1}{2}-2\beta)f^-_{k-\frac{1}{2}} + (\frac{1}{12}-\beta)(f^-_{k-\frac{3}{2}})^{(-1)}\,] \\ &\quad [\,2\Delta v_{k-\frac{1}{2}} - D_{k-\frac{1}{2}}\Delta v_{k-\frac{1}{2}} + D_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}}] - (f_{k-1} + f_k)\Delta v_{k-\frac{1}{2}} \\ &\leq \left\{\begin{array}{ll} \text{case 1.} & P_{k-1} \\ \text{case 2.} & P_{k-1} \\ \text{case 3.} & P_{k-1} \end{array}\right. - [\,f_k - (\frac{1}{12}+\beta)f^-_{k+\frac{1}{2}} - (\frac{1}{2}-2\beta)f^-_{k-\frac{1}{2}}\,]D_{k-\frac{1}{2}}\Delta v_{k-\frac{1}{2}} \\ &\quad +[\,f_k - (\frac{1}{12}+\beta)f^-_{k+\frac{1}{2}} - (\frac{1}{2}-2\beta)f^-_{k-\frac{1}{2}}\,]D_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}}, \end{aligned}$$

where

$$\begin{aligned}
\text{case 1.}\quad P_{k-1} &:= 4\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}} - (\frac{1}{6}+2\beta)f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k-\frac{1}{2}} \\
&\quad +(\frac{1}{6}-2\beta)f'_{k-\frac{3}{2}}\Delta v_{k-\frac{3}{2}}\Delta v_{k-\frac{1}{2}}, \\
\text{case 2.}\quad P_{k-1} &:= (\frac{1}{6}+2\beta)f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}} - (\frac{1}{6}+2\beta)f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k-\frac{1}{2}}, \\
\text{case 3.}\quad P_{k-1} &:= 4\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}} - 4\beta f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k-\frac{1}{2}}.
\end{aligned}$$

$$\begin{aligned}
& P_k := g^E_{k+\frac{1}{2}}[\,2\Delta v_{k+\frac{1}{2}} - D_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}} - C_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}}\,] \\
& -(f_k+f_{k+1})\Delta v_{k+\frac{1}{2}} \\
=\ & f_s[\,2\Delta v_{k+\frac{1}{2}} - D_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}} - C_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}}\,] \\
& -(f_k+f_{k+1})\Delta v_{k+\frac{1}{2}} \\
=\ & f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k+\frac{1}{2}} - f'_{s+\frac{1}{2}}\Delta v_{s+\frac{1}{2}}\Delta v_{k+\frac{1}{2}} \\
& -f_s\Delta v_{k+\frac{1}{2}}D_{k+\frac{1}{2}} - f_s\Delta v_{k+\frac{1}{2}}C_{k+\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
& P_{k+1} := [\,g^E_{k+\frac{3}{2}} - (\frac{1}{12}-\beta)(f^+_{k+\frac{5}{2}})^{(1)} + (\frac{1}{2}-2\beta)(f^+_{k+\frac{3}{2}})^{(0)} \\
& +(\frac{1}{12}+\beta)(f^+_{k+\frac{1}{2}})^{(-1)}\,][\,2\Delta v_{k+\frac{3}{2}} - C_{k+\frac{3}{2}}\Delta v_{k+\frac{3}{2}} + C_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}}\,] \\
& -(f_{k+1}+f_{k+2})\Delta v_{k+\frac{3}{2}} \\
\le\ & [\,f_{k+1} - (\frac{1}{12}-\beta)(f^+_{k+\frac{5}{2}})^{(1)} + (\frac{1}{2}-2\beta)f^+_{k+\frac{3}{2}} + (\frac{1}{12}+\beta)f^+_{k+\frac{1}{2}}\,] \\
& [\,2\Delta v_{k+\frac{3}{2}} - C_{k+\frac{3}{2}}\Delta v_{k+\frac{3}{2}} + C_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}}] - (f_{k+1}+f_{k+2})\Delta v_{k+\frac{3}{2}} \\
\le\ & \begin{cases} \text{case 1.} & P_{k+1} \\ \text{case 2.} & P_{k+1} \\ \text{case 3.} & P_{k+1} \end{cases} \\
& -[\,f_{k+1}\Delta v_{k+\frac{3}{2}} + (\frac{1}{2}-2\beta)f^+_{k+\frac{3}{2}}\Delta v_{k+\frac{3}{2}} \\
& +(\frac{1}{12}+\beta)f^+_{k+\frac{1}{2}}\Delta v_{k+\frac{3}{2}}\,]C_{k+\frac{3}{2}} \\
& +[\,f_{k+1}\Delta v_{k+\frac{1}{2}} + (\frac{1}{2}-2\beta)f^+_{k+\frac{3}{2}}\Delta v_{k+\frac{1}{2}} \\
& +(\frac{1}{12}+\beta)f^+_{k+\frac{1}{2}}\Delta v_{k+\frac{1}{2}}\,]C_{k+\frac{1}{2}},
\end{aligned}$$

where,

$$\begin{aligned}
\text{case 1.}\quad P_{k+1} &:= -4\beta f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{3}{2}} - (\frac{1}{6}-2\beta)f'_{k+\frac{5}{2}}\Delta v_{k+\frac{3}{2}}\Delta v_{k+\frac{5}{2}} \\
&\quad +(\frac{1}{6}+2\beta)f'_{s+\frac{1}{2}}\Delta v_{s+\frac{1}{2}}\Delta v_{k+\frac{3}{2}}, \\
\text{case 2.}\quad P_{k+1} &:= -(\frac{1}{6}+2\beta)f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{3}{2}} + (\frac{1}{6}+2\beta)f'_{s+\frac{1}{2}}\Delta v_{s+\frac{1}{2}}\Delta v_{k+\frac{3}{2}}, \\
\text{case 3.}\quad P_{k+1} &:= -4\beta f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{3}{2}} + 4\beta f'_{s+\frac{1}{2}}\Delta v_{s+\frac{1}{2}}\Delta v_{k+\frac{3}{2}}.
\end{aligned}$$

Next, we combine the estimations of $T_{(j\le k-2)}, T_{(j\ge k+2)}, P_{k-1}, P_k$, and P_{k+1} into one estimation. We consider three cases separately. Let

$$\begin{aligned}
d_{k-1} &:= -(\frac{7}{12}+3\beta)f'_{k-\frac12}\Delta v^2_{k-\frac12}-(\frac12-2\beta)f'_{k-\frac32}\Delta v_{k-\frac32}\Delta v_{k-\frac12}\\
&\quad +(\frac{1}{12}+\beta)f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k-\frac12},\\
d_k &:= -(\frac{13}{12}+\beta)f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k+\frac12}-(\frac12-2\beta)f'_{k-\frac12}\Delta v_{k-\frac12}\Delta v_{k+\frac12},\\
c_k &:= (\frac{13}{12}+\beta)f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac12}+(\frac12-2\beta)f'_{k+\frac32}\Delta v_{k+\frac12}\Delta v_{k+\frac32},\\
c_{k+1} &:= (\frac{7}{12}+3\beta)f'_{k+\frac32}\Delta v^2_{k+\frac32}+(\frac12-2\beta)f'_{k+\frac52}\Delta v_{k+\frac32}\Delta v_{k+\frac52}\\
&\quad -(\frac{1}{12}+\beta)f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac32}.
\end{aligned}$$

First, for the case 1.

(27)

$$\begin{aligned}
&\text{case 1.}\,T(k) := \text{case 1.}\,T_{(j\le k-2)}+P_k\\
&+\text{case 1.}\,P_{k-1}+\text{case 1.}\,P_{k+1}+\text{case 1.}\,T_{(j\ge k+2)}\\
= \;& -2\beta f'_{k-\frac12}\Delta v^2_{k-\frac12}-(\frac16-2\beta)f'_{k-\frac32}\Delta v_{k-\frac32}\Delta v_{k-\frac12}\\
&+f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k+\frac12}-f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac12}\\
&+4\beta f'_{k-\frac12}\Delta v^2_{k-\frac12}-(\frac16+2\beta)f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k-\frac12}+(\frac16-2\beta)f'_{k-\frac32}\Delta v_{k-\frac32}\Delta v_{k-\frac12}\\
&-4\beta f'_{k+\frac32}\Delta v^2_{k+\frac32}-(\frac16-2\beta)f'_{k+\frac52}\Delta v_{k+\frac32}\Delta v_{k+\frac52}+(\frac16+2\beta)f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac32}\\
&-2\beta f'_{k+\frac52}\Delta v^2_{k+\frac52}+(\frac16+2\beta)f'_{k+\frac32}\Delta v_{k+\frac32}\Delta v_{k+\frac52}\\
&+d_{k-1}D_{k-\frac12}+d_kD_{k+\frac12}+c_kC_{k+\frac12}+c_{k+1}C_{k+\frac32}\\
\le\; & \text{case 1.}\,T_1(k)+T_2(k)\\
\le\; & T_2(k),
\end{aligned}$$

where

$$\begin{aligned}
&\text{case 1.}\,T_1(k) := 2\beta f'_{k-\frac12}\Delta v^2_{k-\frac12}+f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k+\frac12}-f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac12}\\
&-(\frac16+2\beta)f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k-\frac12}-4\beta f'_{k+\frac32}\Delta v^2_{k+\frac32}-(\frac16-2\beta)f'_{k+\frac52}\Delta v_{k+\frac32}\Delta v_{k+\frac52}\\
&+(\frac16+2\beta)f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac32}-2\beta f'_{k+\frac52}\Delta v^2_{k+\frac52}+(\frac16+2\beta)f'_{k+\frac32}\Delta v_{k+\frac32}\Delta v_{k+\frac52},
\end{aligned}$$

and

$$T_2(k) \;:=\; d_{k-1}D_{k-\frac12}+d_kD_{k+\frac12}+c_kC_{k+\frac12}+c_{k+1}C_{k+\frac32}.$$

Also, in the estimation of case 1. $T_1(k)$, we use the following inequalities.

$$(\frac16+2\beta)f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac32}\le(\frac{1}{12}+\beta)f'_{s+\frac12}\Delta v^2_{s+\frac12}+(\frac{1}{12}+\beta)f'_{s+\frac12}\Delta v^2_{k+\frac32},$$

and

$$-(\frac16+2\beta)f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k-\frac12}\le-(\frac{1}{12}+\beta)f'_{s-\frac12}\Delta v^2_{s-\frac12}-(\frac{1}{12}+\beta)f'_{s-\frac12}\Delta v^2_{k-\frac12},$$

where we have used inequality of $2ab \le a^2 + b^2$ for any real numbers a and b. Using the fact that $\Delta v_{k+\frac{1}{2}} = \Delta v_{s+\frac{1}{2}} + \Delta v_{s-\frac{1}{2}}$, we have

$$f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k+\frac{1}{2}} = f'_{s-\frac{1}{2}}\Delta v^2_{s-\frac{1}{2}} + f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{s+\frac{1}{2}},$$

$$-f'_{s+\frac{1}{2}}\Delta v_{s+\frac{1}{2}}\Delta v_{k+\frac{1}{2}} = -f'_{s+\frac{1}{2}}\Delta v^2_{s+\frac{1}{2}} - f'_{s+\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{s+\frac{1}{2}},$$

and

$$\begin{aligned}
& \text{case } 1.\, T_1(k) \\
= \;& -(\frac{1}{6} - 2\beta)(f'_{k+\frac{5}{2}} - f'_{k+\frac{3}{2}})\Delta v_{k+\frac{3}{2}}\Delta v_{k+\frac{5}{2}} + 2\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}} \\
& + f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k+\frac{1}{2}} - f'_{s+\frac{1}{2}}\Delta v_{s+\frac{1}{2}}\Delta v_{k+\frac{1}{2}} \\
& -(\frac{1}{6} + 2\beta) f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k-\frac{1}{2}} \\
& +(\frac{1}{6} + 2\beta) f'_{s+\frac{1}{2}}\Delta v_{s+\frac{1}{2}}\Delta v_{k+\frac{3}{2}} - 4\beta f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{3}{2}} \\
& +4\beta f'_{k+\frac{3}{2}}\Delta v_{k+\frac{3}{2}}\Delta v_{k+\frac{5}{2}} - 2\beta f'_{k+\frac{5}{2}}\Delta v^2_{k+\frac{5}{2}} \\
\le \;& 2\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}} + f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k+\frac{1}{2}} - f'_{s+\frac{1}{2}}\Delta v_{s+\frac{1}{2}}\Delta v_{k+\frac{1}{2}} \\
& -(\frac{1}{6} + 2\beta) f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k-\frac{1}{2}} \\
& +(\frac{1}{12} + \beta) f'_{s+\frac{1}{2}}\Delta v^2_{s+\frac{1}{2}} + (\frac{1}{12} + \beta) f'_{s+\frac{1}{2}}\Delta v^2_{k+\frac{3}{2}} - 2\beta f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{3}{2}} \\
\le \;& 2\beta f'_{k-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}} + f'_{s-\frac{1}{2}}\Delta v^2_{s-\frac{1}{2}} + f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{s+\frac{1}{2}} \\
& -(\frac{11}{12} - \beta) f'_{s+\frac{1}{2}}\Delta v^2_{s+\frac{1}{2}} - f'_{s+\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{s+\frac{1}{2}} - (\frac{1}{12} + \beta) f'_{s-\frac{1}{2}}\Delta v^2_{s-\frac{1}{2}} \\
& -(\frac{1}{12} + \beta) f'_{s-\frac{1}{2}}\Delta v^2_{k-\frac{1}{2}} - \beta f'_{k+\frac{3}{2}}\Delta v^2_{k+\frac{1}{2}} + \frac{1}{12} f'_{s+\frac{1}{2}}\Delta v^2_{k+\frac{3}{2}} \\
\le \;& (2\beta f'_{k-\frac{1}{2}} - (\frac{1}{12} + \beta) f'_{s-\frac{1}{2}})\Delta v^2_{k-\frac{1}{2}} + (\frac{11}{12} - \beta) f'_{s-\frac{1}{2}}\Delta v^2_{s-\frac{1}{2}} \\
\le \;& 0.
\end{aligned}$$

Second, for the case 2., using the following relationships

$$\begin{aligned}
f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{k+\frac{1}{2}} &= f'_{s-\frac{1}{2}}\Delta v^2_{s-\frac{1}{2}} + f'_{s-\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{s+\frac{1}{2}} \\
&\le (\frac{1}{12} + \beta) f'_{s-\frac{1}{2}}\Delta v^2_{s-\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
-f'_{s+\frac{1}{2}}\Delta v_{s+\frac{1}{2}}\Delta v_{k+\frac{1}{2}} &= -f'_{s+\frac{1}{2}}\Delta v^2_{s+\frac{1}{2}} - f'_{s+\frac{1}{2}}\Delta v_{s-\frac{1}{2}}\Delta v_{s+\frac{1}{2}} \\
&\le -(\frac{1}{12} + \beta) f'_{s+\frac{1}{2}}\Delta v^2_{s+\frac{1}{2}},
\end{aligned}$$

we have,

$$
\begin{aligned}
& \text{case } 2.\, T_1(k) \\
:= \;& \text{case } 2.\, T_{(j\le k-2)} + \text{case } 2.\, P_{k-1} + P_k + \text{case } 2.\, P_{k+1} + \text{case } 2.\, T_{(j\ge k+2)} \\
= \;& -(\frac{1}{12}+\beta) f'_{k-\frac12}\Delta v^2_{k-\frac12} + (\frac{1}{6}+2\beta) f'_{k-\frac12}\Delta v^2_{k-\frac12} \\
& -(\frac{1}{6}+2\beta) f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k-\frac12} \\
& + f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k+\frac12} - f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac12} \\
& -(\frac{1}{6}+2\beta) f'_{k+\frac32}\Delta v^2_{k+\frac32} + (\frac{1}{6}+2\beta) f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac32} \\
& +(\frac{1}{12}+\beta) f'_{k+\frac32}\Delta v^2_{k+\frac32} \\
= \;& (\frac{1}{12}+\beta) f'_{k-\frac12}\Delta v^2_{k-\frac12} - (\frac{1}{6}+2\beta) f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k-\frac12} \\
& + f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k+\frac12} - f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac12} \\
& -(\frac{1}{12}+\beta) f'_{k+\frac32}\Delta v^2_{k+\frac32} + 2(\frac{1}{12}+\beta) f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac32} \\
\le \;& (\frac{1}{12}+\beta) f'_{k-\frac12}\Delta v^2_{k-\frac12} - 2(\frac{1}{12}+\beta) f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k-\frac12} \\
& +(\frac{1}{12}+\beta) f'_{s-\frac12}\Delta v^2_{s-\frac12} - (\frac{1}{12}+\beta) f'_{s+\frac12}\Delta v^2_{s+\frac12} \\
& +2(\frac{1}{12}+\beta) f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac32} - (\frac{1}{12}+\beta) f'_{k+\frac32}\Delta v^2_{k+\frac32} \\
\le \;& 0.
\end{aligned}
$$

Thus,

$$
\text{case } 2.\, T(k) := \text{case } 2.\, T_1(k) + T_2(k) \le T_2(k). \tag{28}
$$

Third, for the case 3., we have,

$$
\begin{aligned}
& \text{case } 3.\, T_1(k) \\
:= \;& \text{case } 3.\, T_{(j\le k-2)} + \text{case } 3.\, P_{k-1} + P_k + \text{case } 3.\, P_{k+1} + \text{case } 3.\, T_{(j\ge k+2)} \\
= \;& -2\beta f'_{k-\frac12}\Delta v^2_{k-\frac12} + 4\beta f'_{k-\frac12}\Delta v^2_{k-\frac12} - 4\beta f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k-\frac12} \\
& + f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k+\frac12} - f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac12} \\
& -4\beta f'_{k+\frac32}\Delta v^2_{k+\frac32} + 4\beta f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac32} + 2\beta f'_{k+\frac32}\Delta v^2_{k+\frac32} \\
= \;& 2\beta f'_{k-\frac12}\Delta v^2_{k-\frac12} - 4\beta f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{k-\frac12} \\
& +2\beta f'_{s-\frac12}\Delta v^2_{s-\frac12} + (1-2\beta) f'_{s-\frac12}\Delta v^2_{s-\frac12} + f'_{s-\frac12}\Delta v_{s-\frac12}\Delta v_{s+\frac12} \\
& -2\beta f'_{s+\frac12}\Delta v^2_{s+\frac12} - (1-2\beta) f'_{s+\frac12}\Delta v^2_{s+\frac12} - f'_{s+\frac12}\Delta v_{s-\frac12}\Delta v_{s+\frac12} \\
& -2\beta f'_{k+\frac32}\Delta v^2_{k+\frac32} + 4\beta f'_{s+\frac12}\Delta v_{s+\frac12}\Delta v_{k+\frac32} \\
\le \;& 2\beta f'_{s-\frac12}(\Delta v_{k-\frac12} - \Delta v_{s-\frac12})^2 - 2\beta f'_{s+\frac12}(\Delta v_{k+\frac32} - \Delta v_{s+\frac12})^2 \\
\le \;& 0,
\end{aligned}
$$

and

$$
\text{case } 3.\, T(k) := \text{case } 3.\, T_1(k) + T_2(k) \le T_2(k). \tag{29}
$$

Finally, using (25), (26), (27), (27), (27), (27), (27), (28), and (29), we have

$$\begin{aligned}\text{LHS of (24)} &= \sum_{j=I}^{J-1} g_{j+\frac{1}{2}}\Delta\bar{v}_{j+\frac{1}{2}} - \sum_{j=I}^{J-1}\int_{v_j}^{v_{j+1}} f[w;v_j,v_{j+1}]dw \\ &= \frac{1}{2}(P_{(j\le k-2)} + P_{k-1} + P_k + P_{k+1} + P_{(j\ge k+2)}) \\ &\le \frac{1}{2}\{\sum_{j=I}^{k-2} d_j D_{j+\frac{1}{2}} + \sum_{j=k+2}^{J-1} c_j C_{j+\frac{1}{2}} + T_2(k)\} \\ &= \frac{1}{2}\{\sum_{j=I}^{k} d_j D_{j+\frac{1}{2}} + \sum_{j=k}^{J-1} c_j C_{j+\frac{1}{2}}\}.\end{aligned}$$

Clearly, for sufficiently small σ_Γ, it is feasible that

$$\text{LHS of (24)} \le \frac{1}{2}\sum_{j=I+1}^{J-1} S_{j-1,j,j+1} \le \sum_{j=I+1}^{J-1} \alpha_j S_{j-1,j,j+1}.$$

Thus, we have completed the proof of Lemma 3.7. □

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