INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 1, Pages 48–62

ANALYSIS OF A MIXED-SHEAR-PROJECTED QUADRILATERAL ELEMENT METHOD FOR REISSNER-MINDLIN PLATES

GUOZHU YU, XIAOPING XIE*, AND YUANHUI GUO

Abstract. This paper analyzes an existing 4-node hybrid mixed-shear-projected quadrilateral element MiSP4, presented by Ayad, Dhatt and Batoz (Int. J. Numer. Meth. Engng 1998, 42: 1149-1179) for Reissner-Mindlin plates, which behaves robustly in numerical benchmark tests. This method is based on Hellinger-Reissner variational principle, where continuous piecewise isoparametric bilinear interpolations, as well as the mixed shear interpolation/projection technique of MITC family, are used for the approximations of displacements, and piecewise-independent equilibrium modes are used for the approximations of bending moments/shear stresses. Due to local elimination of the parameters of moments/stresses, the computational cost of MiSP4 element is almost the same as that of the conforming bilinear quadrilateral displacement element. We show that the element is free from shear locking in the sense that the error bound in the derived a priori estimate is independent of the plate thickness.

Key words. Reissner-Mindlin plate, mixed-shear-projected quadrilateral element, shear-locking free.

1. Introduction

Due to avoidance of C^1-continuity difficulty, the Reissner-Mindlin (R-M) plate model is today the dominating two-dimensional model used to calculate the bending of a thick/thin three-dimensional plate of thickness t. It's well-known that for values of t close to zero, the standard low-order finite element discretization of this model suffers from shear locking ([1, 17]).

To overcome the shear locking difficulty and derive 'locking-free' or robust plate bending elements that are valid for the analysis of thick and thin plates, significant efforts are devoted to the development of simple and efficient triangular and quadrilateral finite elements in the past few decades. The most common approach is to modify the variational formulation with some reduction operator so as to weaken the Kirchhoff constraint (see, e.g. [2]-[8], [10], [12], [14]-[16], [18]-[20] and the references therein).

Among the existing elements, the family of finite elements named mixed interpolated tensorial components (MITC) by Bathe et. al [4, 5] is one of the most attractive representative. By virtue of an independent shear approximation and a discrete Mindlin technique along edges, MITC elements define the shear strains in terms of the edge tangential strains that are projected on the element degrees of freedom. As the lowest order quadrilateral MITC element, the 4-node plate element MITC4 is very likely the most used in practice.

Using the same technique of shear interpolation as in the element MITC family, Ayad, Dhatt and Batoz [3] presented an improved formulation for obtaining locking-free quadrilateral element, which is called MiSP4 element. It is based on Hellinger-Reissner variational principle, including variables of displacements, shear

Received by the editors on October 25, 2015.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.

*Corresponding author.

stresses and bending moments. For the approximations of displacements, MiSP4 element uses continuous piecewise isoparametric bilinear interpolations. For the approximations of bending moments/shear stresses, it uses piecewise-independent equilibrium modes. The numerical experiments are presented to show that the MiSP4 element can avoid locking phenomenon, and it also passes the patch test for a general quadrilateral. However, so far there is no uniform error analysis with respect to plate thickness. It should be pointed out that in a very recent paper [8], the shear interpolation treatment was replaced by enhancing a shear-stress-enhanced condition, and the resultant 4-node hybrid finite element scheme was shown to be locking-free.

The main goal of this work is to establish uniform convergence for quadrilateral MiSP4 element. The main tools of our analysis are the self-equilibrium relation, i.e. (21), which contributes to the uniform coercivity of the corresponding bilinear forms, and the properties of shear interpolation proved in [11] for MITC4 element (see Lemma 4.11).

We arrange the rest of this paper as follows. In Section 2 we give weak formulation of the model. Section 3 introduces the finite element spaces for MiSP4 element. We derive in Section 4 uniform error estimates for MiSP4 element. Finally in Section 5 we provide some numerical results to verify the theoretical results.

For convenience, throughout the paper we use the notation $a \lesssim b$ to represent that there exists a generic positive constant C, independent of the mesh parameter h and the plate thickness t, such that $a \leq Cb$. We also abbreviate $a \lesssim b \lesssim a$ as $a \approx b$.

2. Weak problem

The Reissner-Mindlin model for the bending of a clamped isotropic elastic plate in equilibrium reads as: Find $(w, \boldsymbol{\beta}) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$ such that

$$-\mathbf{div}\mathcal{D}\boldsymbol{\epsilon}(\boldsymbol{\beta}) - \lambda t^{-2}(\text{grad } w - \boldsymbol{\beta}) = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$-\lambda t^{-2}\text{div}(\text{grad } w - \boldsymbol{\beta}) = g \quad \text{in} \quad \Omega. \tag{2}$$

Here $\Omega \subset \mathbb{R}^2$, assumed to be a convex polygon for simplicity, is the region occupied by the midsection of the plate with plate thickness t, w and $\boldsymbol{\beta}$ denote respectively the transverse displacement of the midplane and the rotation of the fibers normal to it, $\boldsymbol{\epsilon}(\boldsymbol{\beta})$ is the symmetric part of the gradient of $\boldsymbol{\beta}$, g is the transverse loading, $\mathcal{D}$ is the elastic module tensor defined by

$$\mathcal{D}\mathbf{Q} = \frac{E}{12(1-\nu^2)}[(1-\nu)\mathbf{Q} + \nu \text{tr}(\mathbf{Q})\mathbf{I}]$$

with $\mathbf{Q}$ a 2×2 symmetric matrix, $\lambda = \frac{\kappa E}{2(1+\nu)}$ with E the Young's modulus, ν the Poisson's ratio, and $\kappa = \frac{5}{6}$ the shear correction factor.

Set

$$\mathbb{M} := L^2(\Omega)^{2\times 2}_{sym}, \quad \Gamma := L^2(\Omega)^2, \quad W := H_0^1(\Omega), \quad \Theta := H_0^1(\Omega)^2.$$

When introducing the shear stress vector $\boldsymbol{\gamma} = \lambda t^{-2}(\text{grad } w - \boldsymbol{\beta})$ and the bending moment tensor $\mathbf{M} = -\mathcal{D}\boldsymbol{\epsilon}(\boldsymbol{\beta})$, the model problem (1)-(2) changes into the following system: Find $(\mathbf{M}, \boldsymbol{\gamma}, w, \boldsymbol{\beta}) \in \mathbb{M} \times \Gamma \times W \times \Theta$ such that

$$\mathbf{div}\mathbf{M} - \boldsymbol{\gamma} = 0 \quad \text{in} \quad \Omega, \tag{3}$$

$$\text{div}\boldsymbol{\gamma} + g = 0 \quad \text{in} \quad \Omega, \tag{4}$$

$$\mathbf{M} + \mathcal{D}\boldsymbol{\epsilon}(\boldsymbol{\beta}) = 0 \quad \text{in} \quad \Omega, \tag{5}$$

$$\boldsymbol{\gamma} - \lambda t^{-2}(\text{grad } w - \boldsymbol{\beta}) = 0 \quad \text{in} \quad \Omega. \tag{6}$$

The variational formulation of this system reads: Find $(\mathbf{M}, \boldsymbol{\gamma}, w, \boldsymbol{\beta}) \in \mathbb{M}\times\Gamma\times W\times\Theta$ such that

$$a(\mathbf{M}, \boldsymbol{\gamma}; \mathbf{Q}, \boldsymbol{\tau}) + b(\mathbf{Q}, \boldsymbol{\tau}; w, \boldsymbol{\beta}) = 0 \quad \forall(\mathbf{Q}, \boldsymbol{\tau}) \in \mathbb{M} \times \Gamma, \tag{7}$$

$$b(\mathbf{M}, \boldsymbol{\gamma}; v, \boldsymbol{\zeta}) = -\int_\Omega gv d\mathbf{x} \quad \forall(v, \boldsymbol{\zeta}) \in W \times \Theta, \tag{8}$$

where the bilinear forms

$$\begin{aligned} a(\cdot,\cdot;\cdot,\cdot): &\quad (L^2(\Omega)^{2\times 2}_{sym} \times L^2(\Omega)^2) \times (L^2(\Omega)^{2\times 2}_{sym} \times L^2(\Omega)^2) &\to \mathbb{R}, \\ b(\cdot,\cdot;\cdot,\cdot): &\quad (L^2(\Omega)^{2\times 2}_{sym} \times L^2(\Omega)^2) \times (H^1_0(\Omega) \times H^1_0(\Omega)^2) &\to \mathbb{R} \end{aligned}$$

are defined by

$$a(\mathbf{M}, \boldsymbol{\gamma}; \mathbf{Q}, \boldsymbol{\tau}) := \int_\Omega \mathbf{M} : \mathcal{D}^{-1}\mathbf{Q} d\mathbf{x} + \frac{t^2}{\lambda}\int_\Omega \boldsymbol{\gamma}\cdot\boldsymbol{\tau} d\mathbf{x}, \tag{9}$$

$$b(\mathbf{Q}, \boldsymbol{\tau}; v, \boldsymbol{\zeta}) := \int_\Omega \mathbf{Q} : \boldsymbol{\epsilon}(\boldsymbol{\zeta}) d\mathbf{x} - \int_\Omega \boldsymbol{\tau}\cdot(\text{grad } v - \boldsymbol{\zeta}) d\mathbf{x}. \tag{10}$$

To get further regularity of the solution $(\mathbf{M}, \boldsymbol{\gamma}, w, \boldsymbol{\beta})$, we introduce a weak problem: Find $(w, \boldsymbol{\beta}, \boldsymbol{\gamma}) \in W \times \Theta \times \Gamma$ such that

$$\int_\Omega \boldsymbol{\epsilon}(\boldsymbol{\beta}) : \mathcal{D}\boldsymbol{\epsilon}(\boldsymbol{\zeta}) d\mathbf{x} + \int_\Omega \boldsymbol{\gamma}\cdot(\text{grad } v - \boldsymbol{\zeta}) d\mathbf{x} = \int_\Omega gv d\mathbf{x} \quad \forall(v, \boldsymbol{\zeta}) \in W \times \Theta, \tag{11}$$

$$\int_\Omega \boldsymbol{\tau}\cdot(\text{grad } w - \boldsymbol{\beta}) d\mathbf{x} - \frac{t^2}{\lambda}\int_\Omega \boldsymbol{\gamma}\cdot\boldsymbol{\tau} d\mathbf{x} = 0 \quad \forall\boldsymbol{\tau} \in \Gamma. \tag{12}$$

We recall the following result (see [2, 6]).

Lemma 2.1. *The problem (11)-(12) admits a unique solution with*

$$\|w\|_2 + \|\boldsymbol{\beta}\|_2 + \|\boldsymbol{\gamma}\|_0 + t\|\boldsymbol{\gamma}\|_1 \lesssim \|g\|_0. \tag{13}$$

In addition, if Ω is a smoothly bounded domain and $g \in H^1(\Omega)$, then it holds

$$\|w\|_3 \lesssim \|g\|_1. \tag{14}$$

With the above lemma, we obtain some further results [8]:

Theorem 2.2. *Let $(w, \boldsymbol{\beta}, \boldsymbol{\gamma}) \in W \times \Theta \times \Gamma$ be the solution of the problem (11)-(12). Then the following three conclusions (i)-(iii) hold.*
(i) The quadruple $(\mathbf{M} = -\mathcal{D}\boldsymbol{\epsilon}(\boldsymbol{\beta}), \boldsymbol{\gamma}, w, \boldsymbol{\beta}) \in \mathbb{M}\times\Gamma\times W\times\Theta$ is the unique solution of the problem (7)-(8);
(ii) If $\mathbf{M} \in \mathbf{H}(\mathbf{div}; \Omega) := \{\mathbf{Q} \in L^2(\Omega)^{2\times 2}_{sym} : \ \mathbf{div}\mathbf{Q} \in L^2(\Omega)^2\}$, then the equilibrium relation (3) holds;
(iii) Provided that $g \in L^2(\Omega)$, it holds

$$\|w\|_2 + \|\boldsymbol{\beta}\|_2 + \|\mathbf{M}\|_1 + \|\boldsymbol{\gamma}\|_0 + t\|\boldsymbol{\gamma}\|_1 \lesssim \|g\|_0. \tag{15}$$

3. Finite element formulation for MiSP4 method

3.1. Finite element formulation. This subsection is devoted to the finite element formulation of the MiSP4 method on quadrilateral meshes. Let $\mathcal{T}_h$ be a regular family of finite element subdivisions of the polygonal domain Ω. Let $\mathbb{M}_h \subset \mathbb{M}$, $\Gamma_h \subset \Gamma$, $W_h \subset W$, $\Theta_h \subset \Theta$ be finite dimensional spaces for the bending moment, shear stress, transverse displacement, and rotation approximations. Then the corresponding finite element scheme for the problem (7)-(8) reads as: Find $(\mathbf{M}_h, \boldsymbol{\gamma}_h, w_h, \boldsymbol{\beta}_h) \in \mathbb{M}_h \times \Gamma_h \times W_h \times \Theta_h$ such that

$$a(\mathbf{M}_h, \boldsymbol{\gamma}_h; \mathbf{Q}_h, \boldsymbol{\tau}_h) + \tilde{b}(\mathbf{Q}_h, \boldsymbol{\tau}_h; w_h, \boldsymbol{\beta}_h) = 0 \quad \forall (\mathbf{Q}_h, \boldsymbol{\tau}_h) \in \mathbb{M}_h \times \Gamma_h, \tag{16}$$

$$\tilde{b}(\mathbf{M}_h, \boldsymbol{\gamma}_h; v_h, \boldsymbol{\zeta}_h) = -\int_\Omega g v_h d\mathbf{x} \quad \forall (v_h, \boldsymbol{\zeta}_h) \in W_h \times \Theta_h, \tag{17}$$

where

$$\tilde{b}(\mathbf{Q}_h, \boldsymbol{\tau}_h; v_h, \boldsymbol{\zeta}_h) := \int_\Omega \mathbf{Q}_h : \boldsymbol{\epsilon}(\boldsymbol{\zeta}_h) d\mathbf{x} - \int_\Omega \boldsymbol{\tau}_h \cdot (\text{grad } v_h - \boldsymbol{R}_h \boldsymbol{\zeta}_h) d\mathbf{x}, \tag{18}$$

and the reduction operator $\boldsymbol{R}_h : H^1(\Omega)^2 \cap H_0(\text{rot}, \Omega) \to Z_h$ is defined by [11]

$$\int_e \boldsymbol{R}_h \boldsymbol{\psi} \cdot \boldsymbol{t}_e = \int_e \boldsymbol{\psi} \cdot \boldsymbol{t}_e \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h, \tag{19}$$

where

$$H_0(\text{rot}, \Omega) := \{\boldsymbol{\psi} \in L^2(\Omega)^2 : \text{rot}\boldsymbol{\psi} \in L^2(\Omega), \boldsymbol{\psi} \cdot \boldsymbol{t}|_{\partial\Omega} = 0\}, \tag{20}$$

Z_h is to be defined in (30) for MiSP4, and $\boldsymbol{t}_e$ denotes a unit vector tangent to e.

For MiSP4 element, we define

$$\Gamma_h = \mathbf{div}_h \mathbb{M}_h, \quad \text{with } (\mathbf{Q}_h, \boldsymbol{\tau}_h) = (\mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h) \tag{21}$$

for $\mathbf{Q}_h \in \mathbb{M}_h$. Here $\mathbf{div}_h$ denotes the divergence operator piecewise defined with respect to $\mathcal{T}_h$.

From the definition of the space Γ_h, we have an equivalent form of the discrete scheme (16)-(17): Find $(\mathbf{M}_h, w_h, \boldsymbol{\beta}_h) \in \mathbb{M}_h \times W_h \times \Theta_h$ such that

$$a(\mathbf{M}_h, \mathbf{div}_h \mathbf{M}_h; \mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h) + \tilde{b}(\mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h; w_h, \boldsymbol{\beta}_h) = 0 \quad \forall \mathbf{Q}_h \in \mathbb{M}_h, \tag{22}$$

$$\tilde{b}(\mathbf{M}_h, \mathbf{div}_h \mathbf{M}_h; v_h, \boldsymbol{\zeta}_h) = -\int_\Omega g v_h d\mathbf{x} \quad \forall (v_h, \boldsymbol{\zeta}_h) \in W_h \times \Theta_h. \tag{23}$$

3.2. Finite dimensional subspaces. Let $\mathcal{T}_h$ be a conventional quadrilateral mesh of Ω. We denote by h_K the diameter of a quadrilateral $K \in \mathcal{T}_h$, and denote $h := \max_{K \in \mathcal{T}_h} h_K$. Let $Z_i(x_i, y_i)$, $1 \le i \le 4$ be the four vertices of K, and T_i be the sub-triangle of K with vertices Z_{i-1}, Z_i and Z_{i+1} (the index on Z_i is modulo 4). Define

$$\rho_K = \min_{1 \le i \le 4} \{\text{diameter of circle inscribed in } T_i\}.$$

Throughout the paper, we assume that the partition $\mathcal{T}_h$ satisfies the following 'shape-regularity' hypothesis: There exists a constant $\varrho > 2$ independent of h such that for all $K \in \mathcal{T}_h$,

$$h_K \le \varrho \rho_K. \tag{24}$$

Let $\hat{K} = [-1, 1] \times [-1, 1]$ be the reference square with vertices $\hat{Z}_i$, $1 \le i \le 4$. For a quadrilateral $K \in \mathcal{T}_h$, there exists a unique invertible mapping F_K that maps $\hat{K}$ onto K with $F_K(\xi, \eta) \in Q_1^2(\xi, \eta)$ and $F_K(\hat{Z}_i) = Z_i$, $1 \le i \le 4$ (Figure 3.1). Here $\xi, \eta \in [-1, 1]$ are the local isoparametric coordinates.

This isoparametric bilinear mapping $(x, y) = F_K(\xi, \eta)$ is given by

$$x = \sum_{i=1}^4 x_i N_i(\xi, \eta), \quad y = \sum_{i=1}^4 y_i N_i(\xi, \eta), \tag{25}$$

where

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta),\ N_2 = \frac{1}{4}(1+\xi)(1-\eta),\ N_3 = \frac{1}{4}(1+\xi)(1+\eta),\ N_4 = \frac{1}{4}(1-\xi)(1+\eta).$$

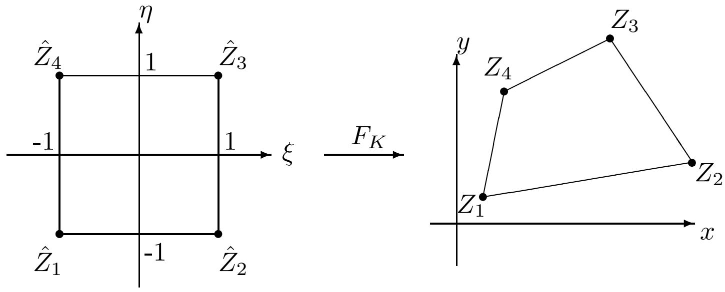

FIGURE 3.1. The mapping F_K

We can rewrite (25) as

$$x = a_0 + a_1\xi + a_2\eta + a_{12}\xi\eta, \quad y = b_0 + b_1\xi + b_2\eta + b_{12}\xi\eta, \tag{26}$$

with

$$\begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_{12} & b_{12} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}.$$

The Jacobi matrix and the Jacobian of the transformation F_K are respectively given by

$$DF_K(\xi,\eta) = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} a_1 + a_{12}\eta & a_2 + a_{12}\xi \\ b_1 + b_{12}\eta & b_2 + b_{12}\xi \end{pmatrix},$$

$$J_K = det(DF_K) = J_0 + J_1\xi + J_2\eta,$$

where

$$J_0 = a_1b_2 - a_2b_1,\ J_1 = a_1b_{12} - a_{12}b_1,\ J_2 = a_{12}b_2 - a_2b_{12}.$$

Remark 3.1. *Notice that when K is a parallelogram, we have $a_{12} = b_{12} = 0$, and F_K is reduced to an affine mapping. Especially, when K is a rectangle, we further have $a_2 = b_1 = 0$.*

For element MiSP4, the continuous isoparametric bilinear interpolations are used for the transverse displacement and rotation approximations, i.e. the transverse displacement space W_h and rotation space Θ_h are chosen as

$$W_h := \{v_h \in H_0^1(\Omega) \cap C(\bar{\Omega}) : v_h|_K \circ F_K \in Q_1(\hat{K}) \ \ \forall K \in \mathcal{T}_h\}, \tag{27}$$

$$\Theta_h := \{\boldsymbol{\zeta}_h \in (H_0^1(\Omega) \cap C(\bar{\Omega}))^2 : \boldsymbol{\zeta}_h|_K \circ F_K \in Q_1(\hat{K})^2 \ \ \forall K \in \mathcal{T}_h\}. \tag{28}$$

Here $Q_1(\hat{K})$ denotes the set of bilinear polynomials on $\hat{K}$. For the approximation of bending moment tensor, we define

$$\mathbb{M}_h := \{\mathbf{Q}_h \in L^2(\Omega)^{2\times 2}_{sym} : (\mathbf{Q}_h|_K \circ F_K)_{i,j} \in Q_1(\hat{K}) \ \ \forall K \in \mathcal{T}_h, i,j = 1,2\}. \tag{29}$$

We take the space Z_h as

$$Z_h := \left\{\boldsymbol{\psi}_h \in H_0(\mathrm{rot},\Omega) : \boldsymbol{\psi}_h|_K \circ F_K = span\left\{DF_K^{-t}\begin{pmatrix} 1 & \eta & 0 & 0 \\ 0 & 0 & 1 & \xi \end{pmatrix}\right\} \ \ \forall K \in \mathcal{T}_h\right\}. \tag{30}$$

4. Error analysis for MiSP4

This section is denoted to the error estimates for the MiSP4 element. The corresponding subspaces in this section are defined as in subsection 3.2. We first give the following properties for the operator $\boldsymbol{R}_h$.

Lemma 4.1. [13, Theorem III.4.4] *The operator $\boldsymbol{R}_h$ satisfies*

$$\|\boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}\|_0 \lesssim h\|\boldsymbol{\eta}\|_1, \forall \boldsymbol{\eta} \in H^1(\Omega)^2 \cap H_0(rot, \Omega).$$

Lemma 4.2. [11, Lemma 2.1] *The following property holds*

$$grad\ W_h = \{\boldsymbol{\psi}_h \in Z_h, rot\ \boldsymbol{\psi}_h = 0\}.$$

With the property in Lemma 4.2 and the definition of $\boldsymbol{R}_h$ in (19), we obvious have $\boldsymbol{R}_h(\text{grad } v_h) = \text{grad } v_h, \forall v_h \in W_h$, and so the expression (18) can be rewritten as

$$\tilde{b}(\mathbf{Q}_h, \boldsymbol{\tau}_h; v_h, \boldsymbol{\zeta}_h) = \int_\Omega \mathbf{Q}_h : \boldsymbol{\epsilon}(\boldsymbol{\zeta}_h) d\mathbf{x} - \int_\Omega \boldsymbol{\tau}_h \cdot \boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h) d\mathbf{x}. \tag{31}$$

In general, for any $\mathbf{Q} \in (H^1(\Omega))^{2\times2}_{sym} + \mathbb{M}_h$, $v \in (H^2(\Omega) \cap H^1_0(\Omega)) + W_h$, $\boldsymbol{\zeta} \in H^1_0(\Omega)^2 + \Theta_h$, we define

$$\tilde{b}(\mathbf{Q}, \mathbf{div}_h\mathbf{Q}; v, \boldsymbol{\zeta}) := \int_\Omega \mathbf{Q} : \boldsymbol{\epsilon}(\boldsymbol{\zeta}) d\mathbf{x} - \int_\Omega \mathbf{div}_h\mathbf{Q} \cdot \boldsymbol{R}_h(\text{grad } v - \boldsymbol{\zeta}) d\mathbf{x}, \tag{32}$$

and define two mesh-dependent semi-norms for the finite dimensional spaces:

$$|||\mathbf{Q}|||_{h,1} := \|\mathbf{Q}\|_0 + (t+h)\|\mathbf{div}_h\mathbf{Q}\|_0, \tag{33}$$

$$|||(v, \boldsymbol{\zeta})|||_{h,2} := \|\boldsymbol{\epsilon}(\boldsymbol{\zeta})\|_0 + (t+h)^{-1}\|\boldsymbol{R}_h(\text{grad } v - \boldsymbol{\zeta})\|_0. \tag{34}$$

It is easy to see that $|||\cdot|||_{h,1}$ is a norm on $(H^1(\Omega))^{2\times2}_{sym} + \mathbb{M}_h$, and $|||(\cdot,\cdot)|||_{h,2}$ is a norm on $W_h \times \Theta_h$. With this definition of mesh-dependent semi-norms, we can easily check the continuity results in Lemma 4.3. While the corresponding coercivity results are deduced in Lemma 4.4-4.6. Lemma 4.5 is a preparation for Lemma 4.6.

Lemma 4.3. *For any $\mathbf{M}, \mathbf{Q} \in (H^1(\Omega))^{2\times2}_{sym} + \mathbb{M}_h$, $v \in (H^2(\Omega) \cap H^1_0(\Omega)) + W_h$, $\boldsymbol{\zeta} \in H^1_0(\Omega)^2 + \Theta_h$, it holds uniformly the continuity condition*

$$a(\mathbf{M}, \mathbf{div}_h\mathbf{M}; \mathbf{Q}, \mathbf{div}_h\mathbf{Q}) \lesssim |||\mathbf{M}|||_{h,1}|||\mathbf{Q}|||_{h,1}, \tag{35}$$

$$\tilde{b}(\mathbf{Q}, \mathbf{div}_h\mathbf{Q}; v, \boldsymbol{\zeta}) \lesssim |||\mathbf{Q}|||_{h,1}|||(v, \boldsymbol{\zeta})|||_{h,2}. \tag{36}$$

Lemma 4.4. *It holds uniformly the discrete coercivity condition*

$$a(\mathbf{Q}_h, \mathbf{div}_h\mathbf{Q}_h; \mathbf{Q}_h, \mathbf{div}_h\mathbf{Q}_h) \gtrsim |||\mathbf{Q}_h|||_{h,1}, \quad \forall \mathbf{Q}_h \in \mathbb{M}_h. \tag{37}$$

Proof. The proof immediately follows from the inverse inequality $\|\mathbf{div}_h\mathbf{Q}_h\|_0 \lesssim h^{-1}\|\mathbf{Q}_h\|_0$. □

Lemma 4.5. *The following two conclusions hold:*
(1) For any given $\boldsymbol{\zeta}_h \in \Theta_h$, there exists $\mathbf{Q}^1_h \in \mathbb{M}_h$, such that

$$(\mathbf{Q}^1_h, \boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)) = \|\mathbf{Q}^1_h\|^2_0 \approx \|\boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)\|^2_0, \ \textit{and}\ \mathbf{div}_h\mathbf{Q}^1_h = 0; \tag{38}$$

(2)For any given $v_h \in W_h$, $\boldsymbol{\zeta}_h \in \Theta_h$, there exists $\mathbf{Q}^2_h \in \mathbb{M}_h$, such that

$$(\mathbf{div}_h\mathbf{Q}^2_h, \boldsymbol{R}_h(grad\ v_h - \boldsymbol{\zeta}_h)) = -(t^2+h^2)\|\mathbf{div}_h\mathbf{Q}^2_h\|^2_0 \approx -\frac{1}{t^2+h^2}\|\boldsymbol{R}_h(grad\ v_h - \boldsymbol{\zeta}_h)\|^2_0, \tag{39}$$

and

$$\|\mathbf{div}_h \mathbf{Q}_h^2\|_0 \approx h^{-1}\|\mathbf{Q}_h^2\|_0. \tag{40}$$

Proof. The proof is similar to that in [8].

(1) Given $\boldsymbol{\zeta}_h \in \Theta_h$, choose $\mathbf{Q}_h^1$ as the 5-parameter PS element in [8]. The proof for (38) can be found in [8, Lemma 4.4].

(2) Given $v_h \in W_h$, $\boldsymbol{\zeta}_h \in \Theta_h$, for any $K \in \mathcal{T}_h$, $\boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h)|_K$ can be expressed as

$$\begin{aligned}
&\boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h)|_K \\
= &\frac{1}{J_K}\begin{pmatrix} b_2 + b_{12}\xi & -(b_1 + b_{12}\eta) \\ -(a_2 + a_{12}\xi) & a_1 + a_{12}\eta \end{pmatrix}\begin{pmatrix} 1 & \eta & 0 & 0 \\ 0 & 0 & 1 & \xi \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix},
\end{aligned}$$

here $(c_1, c_2, c_3, c_4)^T$ depends on $v_h, \boldsymbol{\zeta}_h$. Some calculations show

$$\begin{aligned}
&\|\boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h)\|_{0,K}^2 \\
=&\frac{4}{J_K(\xi_1,\eta_1)}\Big[(b_2c_1 - b_1c_3)^2 + \frac{1}{3}(b_2c_2 - b_{12}c_3)^2 + \frac{1}{3}(b_{12}c_1 - b_1c_4)^2 \\
&\quad + \frac{1}{9}(b_{12}c_2 - b_{12}c_4)^2 + (a_2c_1 - a_1c_3)^2 + \frac{1}{3}(a_2c_2 - a_{12}c_3)^2 \\
&\quad + \frac{1}{3}(a_{12}c_1 - a_1c_4)^2 + \frac{1}{9}(a_{12}c_2 - a_{12}c_4)^2\Big] \\
=&\frac{C_1}{J_K(\xi_1,\eta_1)}\left[(b_2c_1 - b_1c_3)^2 + (b_2c_2 - b_{12}c_3)^2 + (a_2c_1 - a_1c_3)^2 + (a_{12}c_1 - a_1c_4)^2\right].
\end{aligned}$$

Take $\mathbf{Q}_h|_K = \begin{pmatrix} c_1\xi + c_3\eta + c_2\xi\eta \\ c_1\xi + c_3\eta + c_4\xi\eta \\ 0 \end{pmatrix}$, then we have

$$\mathbf{div}_h \mathbf{Q}_h|_K = \frac{1}{J_K}\begin{pmatrix} (b_2c_1 - b_1c_3) + (b_{12}c_1 - b_1c_2)\xi + (b_2c_2 - b_{12}c_3)\eta \\ -(a_2c_1 - a_1c_3) - (a_{12}c_1 - a_1c_4)\xi + (a_2c_4 - a_{12}c_3)\eta \end{pmatrix}$$

and

$$\begin{aligned}
\|\mathbf{div}_h \mathbf{Q}_h\|_{0,K}^2 =&\frac{4}{J_K(\xi_2,\eta_2)}\Big[(b_2c_1 - b_1c_3)^2 + \frac{1}{3}(b_2c_2 - b_{12}c_3)^2 + \frac{1}{3}(b_{12}c_1 - b_1c_4)^2 \\
&\quad + (a_2c_1 - a_1c_3)^2 + \frac{1}{3}(a_2c_4 - a_{12}c_3)^2 + \frac{1}{3}(a_{12}c_1 - a_1c_4)^2\Big] \\
=&\frac{C_2}{J_K(\xi_2,\eta_2)}\left[(b_2c_1 - b_1c_3)^2 + (b_2c_2 - b_{12}c_3)^2\right. \\
&\quad \left. + (a_2c_1 - a_1c_3)^2 + (a_{12}c_1 - a_1c_4)^2\right].
\end{aligned}$$

On the other hand, it holds

$$
\begin{aligned}
&\int_K \mathbf{div}_h \mathbf{Q}_h \cdot \boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h) d\mathbf{x} \\
=&\frac{4}{J_K(\xi_3,\eta_3)} \Big[(b_2c_1 - b_1c_3)^2 + \frac{1}{3}(b_2c_2 - b_{12}c_3)^2 + \frac{1}{3}(b_{12}c_1 - b_1c_4)^2 \\
&\qquad + (a_2c_1 - a_1c_3)^2 + \frac{1}{3}(a_2c_4 - a_{12}c_3)^2 + \frac{1}{3}(a_{12}c_1 - a_1c_4)^2 \Big] \\
=&\frac{C_3}{J_K(\xi_3,\eta_3)} \left[(b_2c_1 - b_1c_3)^2 + (b_2c_2 - b_{12}c_3)^2 + (a_2c_1 - a_1c_3)^2 + (a_{12}c_1 - a_1c_4)^2 \right].
\end{aligned}
$$

Let $C_0 = -\frac{C_3}{C_2}\frac{J_K(\xi_2,\eta_2)}{J_K(\xi_3,\eta_3)}\frac{1}{t^2+h^2}$, and choose $\mathbf{Q}_h^2|_K = C_0\mathbf{Q}_h|_K$, i.e. $\mathbf{div}_h\mathbf{Q}_h^2|_K = C_0\mathbf{div}_h\mathbf{Q}_h|_K$, then a summation over all elements in $\mathcal{T}_h$ completes the proof for (39). The result (40) follows from the construction of $\mathbf{Q}_h^2$. ☐

Lemma 4.6. *It holds the inf-sup condition*

$$
\sup_{\mathbf{Q}_h\in\mathbb{M}_h} \frac{\tilde{b}(\mathbf{Q}_h, \mathbf{div}_h\mathbf{Q}_h; v_h, \boldsymbol{\zeta}_h)}{|||\mathbf{Q}_h|||_{h,1}} \gtrsim |||(v_h,\boldsymbol{\zeta}_h)|||_{h,2}, \ \forall (v_h,\boldsymbol{\zeta}_h) \in W_h \times \Theta_h. \tag{41}
$$

Proof. For $\boldsymbol{\zeta}_h \in \Theta_h$, from (38) there exists a positive constant C_4 and $\mathbf{Q}_h^1 \in \mathbb{M}_h$, such that

$$
(\mathbf{Q}_h^1, \boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)) = \|\mathbf{Q}_h^1\|_0^2 = C_4\|\boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)\|_0^2, \text{ and } \mathbf{div}_h\mathbf{Q}_h^1 = 0. \tag{42}
$$

For $v_h \in W_h$, $\boldsymbol{\zeta}_h \in \Theta_h$, from (39) for any positive constant C_5 there exists $\mathbf{Q}_h^2 \in \mathbb{M}_h$, such that

$$
\begin{aligned}
(\mathbf{div}_h\mathbf{Q}_h^2, \boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h)) =& - C_5(t^2+h^2)\|\mathbf{div}_h\mathbf{Q}_h^2\|_0^2 \\
=& - C_5^{-1}(t^2+h^2)^{-1}\|\boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h)\|_0^2,
\end{aligned} \tag{43}
$$

and there exists a positive constant C_6 independent of h and t, such that

$$
\|\mathbf{div}_h\mathbf{Q}_h^2\|_0^2 = C_6h^{-2}\|\mathbf{Q}_h^2\|_0^2. \tag{44}
$$

Let $\mathbf{Q}_h = \mathbf{Q}_h^1 + \mathbf{Q}_h^2$, then we have

$$
\begin{aligned}
&\tilde{b}(\mathbf{Q}_h, \mathbf{div}_h\mathbf{Q}_h; v_h, \boldsymbol{\zeta}_h) \\
=&(\mathbf{Q}_h^1 + \mathbf{Q}_h^2, \boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)) - (\mathbf{div}_h\mathbf{Q}_h^1 + \mathbf{div}_h\mathbf{Q}_h^2, \boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h)) \\
=&(\mathbf{Q}_h^1, \boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)) + (\mathbf{Q}_h^2, \boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)) - (\mathbf{div}_h\mathbf{Q}_h^2, \boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h)) \\
\geq&\|\mathbf{Q}_h^1\|_0^2 - \|\mathbf{Q}_h^2\|_0\|\boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)\|_0 + C_5(t^2+h^2)\|\mathbf{div}_h\mathbf{Q}_h^2\|_0^2 \\
\geq&\|\mathbf{Q}_h^1\|_0^2 - \frac{C_4}{2}\|\boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)\|_0^2 - \frac{1}{2C_4}\|\mathbf{Q}_h^2\|_0^2 + C_5(t^2+h^2)\|\mathbf{div}_h\mathbf{Q}_h^2\|_0^2 \\
\geq&\|\mathbf{Q}_h^1\|_0^2 - \frac{C_4}{2}\|\boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)\|_0^2 - \frac{h^2}{2C_4C_6}\|\mathbf{div}_h\mathbf{Q}_h^2\|_0^2 + C_5(t^2+h^2)\|\mathbf{div}_h\mathbf{Q}_h^2\|_0^2 \\
\geq&\frac{C_4}{2}\|\mathbf{Q}_h^1\|_0^2 + \frac{C_5}{2}(t^2+h^2)\|\mathbf{div}_h\mathbf{Q}_h^2\|_0^2 \text{ (by taking } C_5 = \tfrac{1}{C_4C_6}) \\
\approx&\|\boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)\|_0^2 + (t^2+h^2)^{-1}\|\boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h)\|_0^2 \\
\approx&\|\mathbf{Q}_h^1 + \mathbf{Q}_h^2\|_0^2 + (t^2+h^2)\|\mathbf{div}_h\mathbf{Q}_h^1 + \mathbf{div}_h\mathbf{Q}_h^2\|_0^2 = \|\mathbf{Q}_h\|_0^2 + (t^2+h^2)\|\mathbf{div}_h\mathbf{Q}_h\|_0^2.
\end{aligned}
$$

This immediately indicates the desired result. ☐

With the above continuity and coercivity results, we can obtain the following error estimates for MiSP4 element by following the standard error analysis.

Theorem 4.7. *Given $g \in L^2(\Omega)$, let $(\mathbf{M}, \boldsymbol{\gamma} = \mathbf{div}\mathbf{M}, w, \boldsymbol{\beta}) \in \mathbb{M} \times \Gamma \times W \times \Theta$ be the solution of the problem* (7)-(8)*. Then the discretization problem* (22)-(23) *admits a unique solution $(\mathbf{M}_h, w_h, \boldsymbol{\beta}_h) \in \mathbb{M}_h \times W_h \times \Theta_h$ such that*

$$\begin{aligned} & |||\mathbf{M} - \mathbf{M}_h|||_{h,1} + |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_{h,2} \\ \lesssim & \inf_{\mathbf{Q}_h \in \mathbb{M}_h} |||\mathbf{M} - \mathbf{Q}_h|||_{h,1} \\ & + \inf_{(v_h, \boldsymbol{\zeta}_h) \in W_h \times \Theta_h} |||(w - v_h, \boldsymbol{\beta} - \boldsymbol{\zeta}_h)|||_{h,2} + ht\|\boldsymbol{\gamma}\|_1 + h\|\boldsymbol{\gamma}\|_0. \end{aligned}$$

Proof. Since

$$\begin{aligned} & a(\mathbf{M}, \boldsymbol{\gamma}; \mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h) + \tilde{b}(\mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h; w, \boldsymbol{\beta}) \\ & - (\mathbf{div}_h \mathbf{Q}_h, \text{grad } w - \boldsymbol{\beta} - \boldsymbol{R}_h(\text{grad } w - \boldsymbol{\beta})) = 0, \forall \mathbf{Q}_h \in \mathbb{M}_h, \end{aligned}$$

$$a(\mathbf{M}_h, \mathbf{div}_h \mathbf{M}_h; \mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h) + \tilde{b}(\mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h; w_h, \boldsymbol{\beta}_h) = 0, \forall \mathbf{Q}_h \in \mathbb{M}_h,$$

then for all $\mathbf{Q}_h \in \mathbb{M}_h$, it holds

$$\begin{aligned} a(\mathbf{M} - \mathbf{M}_h, \boldsymbol{\gamma} - \mathbf{div}_h \mathbf{M}_h; \mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h) + \tilde{b}(\mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h; w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) \\ -(\mathbf{div}_h \mathbf{Q}_h, \text{grad } w - \boldsymbol{\beta} - \boldsymbol{R}_h(\text{grad } w - \boldsymbol{\beta})) = 0. \end{aligned}$$

Denote

$$Z_h(g) = \{\mathbf{Q}_h \in \mathbb{M}_h : \tilde{b}(\mathbf{Q}_h, \mathbf{div}_h \mathbf{Q}_h; v_h, \boldsymbol{\zeta}_h) = -(g, v_h),\ \forall (v_h, \boldsymbol{\zeta}_h) \in W_h \times \Theta_h\}.$$

Let $\tilde{\mathbf{Q}}_h$ be any element of $Z_h(g)$. Since $\tilde{\mathbf{Q}}_h - \mathbf{M}_h \in Z_h(0)$, then

$$\begin{aligned} & |||\tilde{\mathbf{Q}}_h - \mathbf{M}_h|||_{h,1}^2 \\ \lesssim \ & a(\tilde{\mathbf{Q}}_h - \mathbf{M}_h, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h); \tilde{\mathbf{Q}}_h - \mathbf{M}_h, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h)) \\ = \ & a(\tilde{\mathbf{Q}}_h - \mathbf{M}, \mathbf{div}_h \tilde{\mathbf{Q}}_h - \boldsymbol{\gamma}; \tilde{\mathbf{Q}}_h - \mathbf{M}_h, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h)) \\ & + a(\mathbf{M} - \mathbf{M}_h, \boldsymbol{\gamma} - \mathbf{div}_h \mathbf{M}_h; \tilde{\mathbf{Q}}_h - \mathbf{M}_h, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h)) \\ = \ & a(\tilde{\mathbf{Q}}_h - \mathbf{M}, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}); \tilde{\mathbf{Q}}_h - \mathbf{M}_h, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h)) \\ & - \tilde{b}(\tilde{\mathbf{Q}}_h - \mathbf{M}_h, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h); w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) \\ & + (\mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h), \text{grad } w - \boldsymbol{\beta} - \boldsymbol{R}_h(\text{grad } w - \boldsymbol{\beta})) \\ = \ & a(\tilde{\mathbf{Q}}_h - \mathbf{M}, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}); \tilde{\mathbf{Q}}_h - \mathbf{M}_h, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h)) \\ & - \tilde{b}(\tilde{\mathbf{Q}}_h - \mathbf{M}_h, \mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h); w - v_h, \boldsymbol{\beta} - \boldsymbol{\zeta}_h) \\ & + \tfrac{t^2}{\lambda}(\mathbf{div}_h(\tilde{\mathbf{Q}}_h - \mathbf{M}_h), \boldsymbol{\gamma} - \boldsymbol{R}_h \boldsymbol{\gamma}) \\ \lesssim \ & |||\tilde{\mathbf{Q}}_h - \mathbf{M}_h|||_{h,1}(|||\tilde{\mathbf{Q}}_h - \mathbf{M}|||_{h,1} + |||(w - v_h, \boldsymbol{\beta} - \boldsymbol{\zeta}_h)|||_{h,2} + ht\|\boldsymbol{\gamma}\|_1). \end{aligned}$$

So we have

$$|||\tilde{\mathbf{Q}}_h - \mathbf{M}_h|||_{h,1} \lesssim |||\tilde{\mathbf{Q}}_h - \mathbf{M}|||_{h,1} + |||(w - v_h, \boldsymbol{\beta} - \boldsymbol{\zeta}_h)|||_{h,2} + ht\|\boldsymbol{\gamma}\|_1.$$

Then, by using the triangle inequality, we get

(45)
$$|||\mathbf{M} - \mathbf{M}_h|||_{h,1} \lesssim \inf_{\tilde{\mathbf{Q}}_h \in Z_h(g)} |||\tilde{\mathbf{Q}}_h - \mathbf{M}|||_{h,1} + \inf_{(v_h, \boldsymbol{\zeta}_h) \in W_h \times \Theta_h} |||(w - v_h, \boldsymbol{\beta} - \boldsymbol{\zeta}_h)|||_{h,2} + ht\|\boldsymbol{\gamma}\|_1.$$

For any $\mathbf{Q}_h \in \mathbb{M}_h$, there exists $\bar{\mathbf{Q}}_h \in \mathbb{M}_h$, such that, for all $(v_h, \boldsymbol{\zeta}_h) \in W_h \times \Theta_h$,

$$\begin{aligned} \tilde{b}(\bar{\mathbf{Q}}_h, \mathbf{div}_h \bar{\mathbf{Q}}_h; v_h, \boldsymbol{\zeta}_h) = & \tilde{b}(\mathbf{M} - \mathbf{Q}_h, \mathbf{div}_h(\mathbf{M} - \mathbf{Q}_h); v_h, \boldsymbol{\zeta}_h) \\ & - (\boldsymbol{\gamma}, \text{grad } v_h - \boldsymbol{\zeta}_h - \boldsymbol{R}_h(\text{grad } v_h - \boldsymbol{\zeta}_h)) \end{aligned}$$

and

$$\begin{aligned}
&|||\bar{\mathbf{Q}}_h|||_{h,1} \\
\lesssim\ & \sup_{(v_h,\boldsymbol{\zeta}_h)\in W_h\times\Theta_h} \frac{\tilde{b}(\bar{\mathbf{Q}}_h, \mathbf{div}_h\bar{\mathbf{Q}}_h; v_h, \boldsymbol{\zeta}_h)}{|||(v_h,\boldsymbol{\zeta}_h)|||_{h,2}} \\
\lesssim\ & \sup_{(v_h,\boldsymbol{\zeta}_h)\in W_h\times\Theta_h} \left\{ \frac{\tilde{b}(\mathbf{M}-\mathbf{Q}_h, \mathbf{div}_h(\mathbf{M}-\mathbf{Q}_h); v_h, \boldsymbol{\zeta}_h)}{|||(v_h,\boldsymbol{\zeta}_h)|||_{h,2}} \right. \\
& \qquad \left. - \frac{(\boldsymbol{\gamma}, \mathrm{grad}\ v_h - \boldsymbol{\zeta}_h - \boldsymbol{R}_h(\mathrm{grad}\ v_h - \boldsymbol{\zeta}_h))}{|||(v_h,\boldsymbol{\zeta}_h)|||_{h,2}} \right\} \\
=\ & \sup_{(v_h,\boldsymbol{\zeta}_h)\in W_h\times\Theta_h} \frac{\tilde{b}(\mathbf{M}-\mathbf{Q}_h, \mathbf{div}_h(\mathbf{M}-\mathbf{Q}_h); v_h, \boldsymbol{\zeta}_h) + (\boldsymbol{\gamma}, \boldsymbol{\zeta}_h - \boldsymbol{R}_h\boldsymbol{\zeta}_h)}{|||(v_h,\boldsymbol{\zeta}_h)|||_{h,2}} \\
\lesssim\ & |||\mathbf{M}-\mathbf{Q}_h|||_{h,1} + h\|\boldsymbol{\gamma}\|_0.
\end{aligned}$$

Choose $\tilde{\mathbf{Q}}_h = \bar{\mathbf{Q}}_h + \mathbf{Q}_h$, then $\tilde{\mathbf{Q}}_h \in Z_h(g)$. Thus we get

$$\begin{aligned}
|||\mathbf{M}-\tilde{\mathbf{Q}}_h|||_{h,1} =& |||\mathbf{M}-\mathbf{Q}_h-\bar{\mathbf{Q}}_h|||_{h,1} \le |||\mathbf{M}-\mathbf{Q}_h|||_{h,1} + |||\bar{\mathbf{Q}}_h|||_{h,1} \\
\lesssim& |||\mathbf{M}-\mathbf{Q}_h|||_{h,1} + h\|\boldsymbol{\gamma}\|_0.
\end{aligned}$$

This estimate and (45) imply

$$\begin{aligned}
|||\mathbf{M}-\mathbf{M}_h|||_{h,1} \lesssim & \inf_{\mathbf{Q}_h\in\mathbb{M}_h} |||\mathbf{M}-\mathbf{Q}_h|||_{h,1} \\
& + \inf_{(v_h,\boldsymbol{\zeta}_h)\in W_h\times\Theta_h} |||(w-v_h, \boldsymbol{\beta}-\boldsymbol{\zeta}_h)|||_{h,2} + ht\|\boldsymbol{\gamma}\|_1 + h\|\boldsymbol{\gamma}\|_0.
\end{aligned} \tag{46}$$

On the other hand, from the coercivity and continuity properties we get

$$\begin{aligned}
&|||(v_h - w_h, \boldsymbol{\zeta}_h - \boldsymbol{\beta}_h)|||_{h,2} \\
\lesssim\ & \sup_{\mathbf{Q}_h\in\mathbb{M}_h} \frac{\tilde{b}(\mathbf{Q}_h, \mathbf{div}_h\mathbf{Q}_h; v_h - w_h, \boldsymbol{\zeta}_h - \boldsymbol{\beta}_h)}{|||\mathbf{Q}_h|||_{h,1}} \\
=\ & \sup_{\mathbf{Q}_h\in\mathbb{M}_h} \left\{ \frac{-a(\mathbf{M}-\mathbf{M}_h, \mathbf{div}_h(\mathbf{M}-\mathbf{M}_h); \mathbf{Q}_h, \mathbf{div}_h\mathbf{Q}_h)}{|||\mathbf{Q}_h|||_{h,1}} \right. \\
& \frac{-\tilde{b}(\mathbf{Q}_h, \mathbf{div}_h\mathbf{Q}_h; w - v_h, \boldsymbol{\beta}-\boldsymbol{\zeta}_h)}{|||\mathbf{Q}_h|||_{h,1}} \\
& \left. + \frac{(\mathbf{div}_h\mathbf{Q}_h, \mathrm{grad}\ w - \boldsymbol{\beta} - \boldsymbol{R}_h(\mathrm{grad}\ w - \boldsymbol{\beta}))}{|||\mathbf{Q}_h|||_{h,1}} \right\} \\
\lesssim\ & |||\mathbf{M}-\mathbf{M}_h|||_{h,1} + |||(w - v_h, \boldsymbol{\beta}-\boldsymbol{\zeta}_h)|||_{h,2} + ht\|\boldsymbol{\gamma}\|_1.
\end{aligned} \tag{47}$$

This inequality and (46) imply

$$\begin{aligned}
&|||(w - w_h, \boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,2} \\
\lesssim\ & \inf_{\mathbf{Q}_h\in\mathbb{M}_h} |||\mathbf{M}-\mathbf{Q}_h|||_{h,1} \\
& + \inf_{(v_h,\boldsymbol{\zeta}_h)\in W_h\times\Theta_h} |||(w-v_h, \boldsymbol{\beta}-\boldsymbol{\zeta}_h)|||_{h,2} + ht\|\boldsymbol{\gamma}\|_1 + h\|\boldsymbol{\gamma}\|_0.
\end{aligned} \tag{48}$$

A combination of (46) and (48) completes the proof. □

Next we consider the approximation properties of finite element spaces. Lemma 4.8 gives the error estimates for space $\mathbf{M}_h$, and Lemma 4.12 is for space $W_h \times \Theta_h$. We need to notice here the key for Lemma 4.12 is the property of the operator $\boldsymbol{R}_h$ described in Lemma 4.11. Finally the convergence theorem, i.e. Theorem 4.13, follows from these lemmas.

Lemma 4.8. *Given $g \in L^2(\Omega)$, let $(\mathbf{M}, \boldsymbol{\gamma} = \mathbf{div}\mathbf{M}, w, \boldsymbol{\beta}) \in \mathbb{M} \times \Gamma \times W \times \Theta$ be the solution of the problem* (7)-(8). *It holds*

$$\inf_{\mathbf{Q}_h \in \mathbb{M}_h} |||\mathbf{M} - \mathbf{Q}_h|||_{h,1} \lesssim h\left(\|\mathbf{M}\|_1 + \|\boldsymbol{\gamma}\|_0 + t\|\boldsymbol{\gamma}\|_1\right).$$

Proof. For the exact solution $\mathbf{M}$, first let $\mathbf{Q}_h^1$ be its piecewise constant L^2 projection, then

$$\|\mathbf{M} - \mathbf{Q}_h^1\|_0 \lesssim h\|\mathbf{M}\|_1.$$

For the exact solution $\boldsymbol{\gamma}$, secondly choose $\mathbf{Q}_h^2$ satisfying:
(1) $\mathbf{div}_h\mathbf{Q}_h^2$ is the piecewise constant L^2 projection of $\boldsymbol{\gamma}$, then

$$\|\boldsymbol{\gamma} - \mathbf{div}_h\mathbf{Q}_h^2\|_0 \approx h\|\boldsymbol{\gamma}\|_1, \quad \|\mathbf{div}_h\mathbf{Q}_h^2\|_0 \lesssim \|\boldsymbol{\gamma}\|_0;$$

(2) $\|\mathbf{Q}_h^2\|_0 \approx h\|\mathbf{div}_h\mathbf{Q}_h^2\|_0$, then $\|\mathbf{Q}_h^2\|_0 \lesssim h\|\boldsymbol{\gamma}\|_0$.
Take $\mathbf{Q}_h = \mathbf{Q}_h^1 + \mathbf{Q}_h^2$, then we get the desired result

$$\begin{aligned}
|||\mathbf{M} - \mathbf{Q}_h|||_{h,1} &\leq \|\mathbf{M} - \mathbf{Q}_h^1\|_0 + \|\mathbf{Q}_h^2\|_0 + (h+t)\|\boldsymbol{\gamma} - \mathbf{div}_h\mathbf{Q}_h^2\|_0 \\
&\lesssim h\|\mathbf{M}\|_1 + h\|\boldsymbol{\gamma}\|_0 + h\|\boldsymbol{\gamma} - \mathbf{div}_h\mathbf{Q}_h^2\|_0 + t\|\boldsymbol{\gamma} - \mathbf{div}_h\mathbf{Q}_h^2\|_0 \\
&\lesssim h\|\mathbf{M}\|_1 + h\|\boldsymbol{\gamma}\|_0 + h\|\boldsymbol{\gamma}\|_0 + th\|\boldsymbol{\gamma}\|_1 \\
&\lesssim h\left(\|\mathbf{M}\|_1 + \|\boldsymbol{\gamma}\|_0 + t\|\boldsymbol{\gamma}\|_1\right).
\end{aligned}$$

□

Remark 4.9. *We note that with the same technique as in Lemma 4.8, the condition $t \lesssim h$ in* [8, Lemma 3.2] *and in* [8, Theorem 4.3] *can be removed.*

Assumption 4.10. [11] *The mesh $\mathcal{T}_h$ is a refinement of a coarser partition $\mathcal{T}_{2h}$, obtained by jointing the midpoints of each opposite edge in each $K_{2h} \in \mathcal{T}_{2h}$ (called macroelement). In addition, $\mathcal{T}_{2h}$ is a similar refinement of a still coarser regular partition $\mathcal{T}_{4h}$.*

Lemma 4.11. [11, Lemma 3.2, 3.4] *Given $g \in L^2(\Omega)$, let $(\mathbf{M}, \boldsymbol{\gamma} = \mathbf{div}\mathbf{M}, w, \boldsymbol{\beta}) \in \mathbb{M} \times \Gamma \times W \times \Theta$ be the solution of the problem* (7)-(8). *Then under Assumption 4.10, there exist $(\hat{w}, \hat{\boldsymbol{\beta}}) \in W_h \times \Theta_h$ and operator $\mathbf{\Pi} : H^1(\Omega)^2 \cap H_0(rot, \Omega) \to Z_h$ satisfying*

$$\|\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\|_1 \lesssim h\|\boldsymbol{\beta}\|_2, \tag{49}$$

$$\boldsymbol{R}_h(grad\ \hat{w} - \hat{\boldsymbol{\beta}}) = \mathbf{\Pi}(grad\ w - \boldsymbol{\beta}), \tag{50}$$

and

$$\|\boldsymbol{\eta} - \mathbf{\Pi}\boldsymbol{\eta}\|_0 \lesssim h\|\boldsymbol{\eta}\|_1, \forall \boldsymbol{\eta} \in H^1(\Omega)^2 \cap H_0(rot, \Omega). \tag{51}$$

Lemma 4.12. *Given $g \in L^2(\Omega)$, let $(\mathbf{M}, \boldsymbol{\gamma} = \mathbf{div}\mathbf{M}, w, \boldsymbol{\beta}) \in \mathbb{M} \times \Gamma \times W \times \Theta$ be the solution of the problem* (7)-(8). *Then under Assumption 4.10, it holds*

$$\inf_{(v_h, \boldsymbol{\zeta}_h) \in W_h \times \Theta_h} |||(w - v_h, \boldsymbol{\beta} - \boldsymbol{\zeta}_h)|||_{h,2} \lesssim h\|\boldsymbol{\beta}\|_2 + \frac{ht^2}{t+h}\|\boldsymbol{\gamma}\|_1. \tag{52}$$

Proof. Choose $(v_h, \boldsymbol{\zeta}_h) = (\hat{w}, \hat{\boldsymbol{\beta}})$, with $(\hat{w}, \hat{\boldsymbol{\beta}}) \in W_h \times \Theta_h$ as in Lemma 4.11, then we can get

$$\begin{aligned}
& \inf_{(v_h,\boldsymbol{\zeta}_h)\in W_h\times\Theta_h} |||(w-v_h,\boldsymbol{\beta}-\boldsymbol{\zeta}_h)|||_{h,2} \\
= & \inf_{(v_h,\boldsymbol{\zeta}_h)\in W_h\times\Theta_h} \|\boldsymbol{\epsilon}(\boldsymbol{\beta})-\boldsymbol{\epsilon}(\boldsymbol{\zeta}_h)\|_0 + \tfrac{1}{t+h}\|\boldsymbol{R}_h(\text{grad } w-\boldsymbol{\beta})-\boldsymbol{R}_h(\text{grad } v_h-\boldsymbol{\zeta}_h)\|_0 \\
\leq & \|\boldsymbol{\epsilon}(\boldsymbol{\beta})-\boldsymbol{\epsilon}(\hat{\boldsymbol{\beta}})\|_0 + \tfrac{1}{t+h}\|\boldsymbol{R}_h(\text{grad } w-\boldsymbol{\beta})-\boldsymbol{R}_h(\text{grad } \hat{w}-\hat{\boldsymbol{\beta}})\|_0 \\
= & \|\boldsymbol{\epsilon}(\boldsymbol{\beta})-\boldsymbol{\epsilon}(\hat{\boldsymbol{\beta}})\|_0 + \tfrac{1}{t+h}\|\boldsymbol{R}_h(\text{grad } w-\boldsymbol{\beta})-\boldsymbol{\Pi}(\text{grad } w-\boldsymbol{\beta})\|_0 \\
\leq & \|\boldsymbol{\epsilon}(\boldsymbol{\beta})-\boldsymbol{\epsilon}(\hat{\boldsymbol{\beta}})\|_0 + \tfrac{1}{t+h}\|\boldsymbol{R}_h(\text{grad } w-\boldsymbol{\beta})-(\text{grad } w-\boldsymbol{\beta})\|_0 \\
& +\frac{1}{t+h}\|(\text{grad } w-\boldsymbol{\beta})-\boldsymbol{\Pi}(\text{grad } w-\boldsymbol{\beta})\|_0 \\
\lesssim & h\|\boldsymbol{\beta}\|_2 + \tfrac{ht^2}{t+h}\|\lambda t^{-2}(\text{grad } w-\boldsymbol{\beta})\|_1 \\
\lesssim & h\|\boldsymbol{\beta}\|_2 + \tfrac{ht^2}{t+h}\|\boldsymbol{\gamma}\|_1.
\end{aligned}$$

□

Theorem 4.13. *Given* $g \in L^2(\Omega)$, *let* $(\mathbf{M}, \boldsymbol{\gamma} = \mathbf{div}\mathbf{M}, w, \boldsymbol{\beta}) \in \mathbb{M} \times \Gamma \times W \times \Theta$ *and* $(\mathbf{M}_h, \boldsymbol{\gamma}_h, w_h, \boldsymbol{\beta}_h) \in \mathbb{M}_h \times \Gamma_h \times W_h \times \Theta_h$ *be the solutions of the problems* (7)-(8) *and* (16)-(17) *respectively. Then under Assumption 4.10, it holds the error estimate*
(53)

$$|||\mathbf{M}-\mathbf{M}_h|||_{h,1} + |||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,2} \lesssim h(\|\mathbf{M}\|_1+\|\boldsymbol{\beta}\|_2+\|\boldsymbol{\gamma}\|_0+t\|\boldsymbol{\gamma}\|_1) \lesssim h\|g\|_0.$$

Furthermore, it holds

$$\begin{aligned}
& \|\mathbf{M}-\mathbf{M}_h\|_0 + (t+h)\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 + \|w-w_h\|_1 + \|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_1 \\
\lesssim & h(\|\mathbf{M}\|_1 + \|w\|_2 + \|\boldsymbol{\beta}\|_2 + t\|\boldsymbol{\gamma}\|_1 + \|\boldsymbol{\gamma}\|_0) \lesssim h\|g\|_0.
\end{aligned} \tag{54}$$

Proof. The estimate (53) follows from the Theorem 4.7, Lemma 4.8 and Lemma 4.12.

For the second estimate, we only need to estimate $\|w - w_h\|_1$. In fact,

$$\begin{aligned}
& \|\text{grad } w - \text{grad } w_h\|_0 \\
= & \|\text{grad } w - \boldsymbol{R}_h\text{grad } w + \boldsymbol{R}_h(\text{grad } w - \text{grad } w_h - \boldsymbol{\beta} + \boldsymbol{\beta}_h) + \boldsymbol{R}_h(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\|_0 \\
\leq & \|\text{grad } w - \boldsymbol{R}_h\text{grad } w\|_0 + \|\boldsymbol{R}_h(\text{grad } w - \text{grad } w_h - \boldsymbol{\beta} + \boldsymbol{\beta}_h)\|_0 + \|\boldsymbol{R}_h(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\|_0 \\
\lesssim & h(\|w\|_2 + \|\mathbf{M}\|_1 + \|\boldsymbol{\beta}\|_2 + t\|\boldsymbol{\gamma}\|_1 + \|\boldsymbol{\gamma}\|_0).
\end{aligned}$$

□

5. Numerical results

We compute a square plate with analytical solution to show the convergence. This example is taken from [9, 15]. The domain is the unit square $(0,1)^2$, the material parameters are taken as $E = 1.0$, $\nu = 0.3$ and $\kappa = \frac{5}{6}$. The exact solution is: the first component of the rotation $\beta_1 = 100y^3(y-1)^3x^2(x-1)^2(2x-1)$, the second component of the rotation $\beta_2 = 100x^3(x-1)^3y^2(y-1)^2(2y-1)$, and the displacement $w = 100(\frac{1}{3}x^3(x-1)^3y^3(y-1)^3 - \frac{2t^2}{5(1-\nu)}[y^3(y-1)^3x(x-1)(5x^2-5x+1)+x^3(x-1)^3y(y-1)(5y^2-5y+1)])$. Therefore, the transverse load $g = \frac{200E}{1-\nu^2}(x^3(x-1)^3(5y^2-5y+1)+y^3(y-1)^3(5x^2-5x+1)+x(x-1)y(y-1)(5x^2-5x+1)(5y^2-5y+1))$. For the plate thickness t, we consider four cases: $t = 1.0, 0.1, 0.001, 1e-8$.

The results for MiSP4 method under the uniform meshes (Figure 5.1) are reported in Table 5.1. These results are conformable to the error estimates in Theorem 4.13.

We note that the error analysis for MiSP4 element requires the partitions of domain to satisfy Assumption 4.10. However, numerical results in Table 5.2 show that

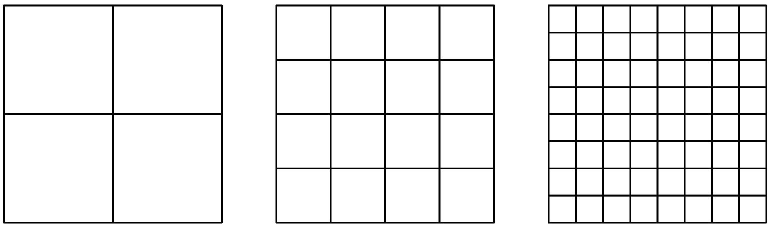

FIGURE 5.1. Uniform mesh

TABLE 5.1. Results of error on uniform mesh with MiSP4

t		4×4	8×8	16×16	32×32	64×64	rate
1	$\lvert w-w_h\rvert_1$	0.2806	0.1460	0.0736	0.0369	0.0184	0.9819
	$\lvert\boldsymbol{\beta}-\boldsymbol{\beta}_h\rvert_1$	0.0771	0.0383	0.0191	0.0095	0.0048	1.0039
	$\lVert\mathbf{M}-\mathbf{M}_h\rVert_0$	0.0062	0.0020	0.0008	0.0003	0.0002	1.2977
	$\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0877	0.0458	0.0231	0.0116	0.0058	0.9799
	$(t+h)\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.1187	0.0539	0.0252	0.0121	0.0059	1.0812
0.1	$\lvert w-w_h\rvert_1$	0.0117	0.0052	0.0025	0.0012	0.0006	1.0610
	$\lvert\boldsymbol{\beta}-\boldsymbol{\beta}_h\rvert_1$	0.0775	0.0384	0.0191	0.0095	0.0048	1.0057
	$\lVert\mathbf{M}-\mathbf{M}_h\rVert_0$	0.0061	0.0020	0.0008	0.0003	0.0002	1.2957
	$\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0870	0.0458	0.0231	0.0116	0.0058	0.9771
	$(t+h)\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0395	0.0127	0.0044	0.0017	0.0007	1.4504
0.001	$\lvert w-w_h\rvert_1$	0.0095	0.0041	0.0019	0.0009	0.0005	1.0896
	$\lvert\boldsymbol{\beta}-\boldsymbol{\beta}_h\rvert_1$	0.0777	0.0384	0.0191	0.0095	0.0048	1.0065
	$\lVert\mathbf{M}-\mathbf{M}_h\rVert_0$	0.0061	0.0020	0.0008	0.0003	0.0002	1.2944
	$\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0866	0.0460	0.0234	0.0117	0.0059	0.9704
	$(t+h)\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0307	0.0082	0.0021	0.0005	0.0001	1.9555
1e-8	$\lvert w-w_h\rvert_1$	0.0095	0.0041	0.0019	0.0009	0.0005	1.0896
	$\lvert\boldsymbol{\beta}-\boldsymbol{\beta}_h\rvert_1$	0.0777	0.0384	0.0191	0.0095	0.0048	1.0065
	$\lVert\mathbf{M}-\mathbf{M}_h\rVert_0$	0.0061	0.0020	0.0008	0.0003	0.0002	1.2944
	$\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0866	0.0460	0.0234	0.0117	0.0059	0.9703
	$(t+h)\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0306	0.0081	0.0021	0.0005	0.0001	1.9703

this assumption seems not to be absolutely necessary for the uniform convergence, as is similar to the MITC4 element [11]. Here the used partitions (Figure 5.2) do not satisfy Assumption 4.10.

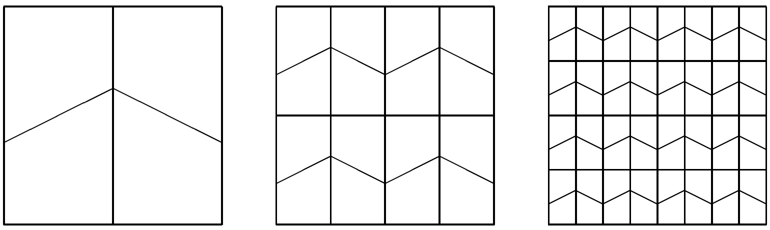

FIGURE 5.2. Quadrilateral mesh

TABLE 5.2. Results of error on quadrilateral mesh with MiSP4

t		4×4	8×8	16×16	32×32	64×64	rate
1	$\lvert w-w_h\rvert_1$	0.2873	0.1693	0.0881	0.0445	0.0223	0.9217
	$\lvert\boldsymbol{\beta}-\boldsymbol{\beta}_h\rvert_1$	0.0924	0.0528	0.0255	0.0122	0.0060	0.9872
	$\lVert\mathbf{M}-\mathbf{M}_h\rVert_0$	0.0066	0.0032	0.0012	0.0005	0.0002	1.1968
	$\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0899	0.0531	0.0277	0.0140	0.0070	0.9203
	$(t+h)\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.1285	0.0645	0.0306	0.0147	0.0072	1.0398
0.1	$\lvert w-w_h\rvert_1$	0.0118	0.0064	0.0031	0.0015	0.0008	0.9898
	$\lvert\boldsymbol{\beta}-\boldsymbol{\beta}_h\rvert_1$	0.0834	0.0496	0.0253	0.0122	0.0060	0.9506
	$\lVert\mathbf{M}-\mathbf{M}_h\rVert_0$	0.0065	0.0031	0.0012	0.0005	0.0002	1.1925
	$\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0930	0.0574	0.0285	0.0141	0.0070	0.9318
	$(t+h)\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0493	0.0181	0.0059	0.0022	0.0009	1.4475
0.001	$\lvert w-w_h\rvert_1$	0.0096	0.0051	0.0024	0.0012	0.0006	1.0151
	$\lvert\boldsymbol{\beta}-\boldsymbol{\beta}_h\rvert_1$	0.0835	0.0475	0.0245	0.0120	0.0060	0.9525
	$\lVert\mathbf{M}-\mathbf{M}_h\rVert_0$	0.0066	0.0032	0.0013	0.0006	0.0003	1.1477
	$\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0947	0.0702	0.0470	0.0355	0.0310	0.4031
	$(t+h)\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0408	0.0152	0.0051	0.0019	0.0009	1.3908
1e-8	$\lvert w-w_h\rvert_1$	0.0096	0.0051	0.0024	0.0012	0.0006	1.0151
	$\lvert\boldsymbol{\beta}-\boldsymbol{\beta}_h\rvert_1$	0.0835	0.0475	0.0245	0.0120	0.0060	0.9525
	$\lVert\mathbf{M}-\mathbf{M}_h\rVert_0$	0.0066	0.0032	0.0013	0.0006	0.0003	1.1466
	$\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0947	0.0703	0.0470	0.0356	0.0315	0.3976
	$(t+h)\lVert\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\rVert_0$	0.0407	0.0151	0.0051	0.0019	0.0008	1.3976

Acknowledgments

The work of the first author was partly supported by National Natural Science Foundation of China (11401492 and 11226333) and the Fundamental Research Funds for the Central Universities (2682015CX053). The work of the second author was partly supported by National Natural Science Foundation of China (11171239) and Major Research Plan of National Natural Science Foundation of China (91430105).

References

[1] D.N. Arnold. Discretization by finite elements of a model parameter dependent problem. Numerische Mathematik, 37(3) (1981) 405–421.

[2] D.N. Arnold and R.S. Falk. A uniformly accurate finite element method for the Reissner-Mindlin plate. SIAM Journal on Numerical Analysis, 26(6) (1989) 1276–1290.

[3] R. Ayad, G. Dhatt, and J.L. Batoz. A new hybrid-mixed variational approach for Reissner–Mindlin plates. The MiSP model. International Journal for Numerical Methods in Engineering, 42(7) (1998) 1149–1179.

[4] K.J. Bathe, F. Brezzi, and S.W. Cho. The mitc7 and mitc9 plate bending elements. Computers & Structures, 32(3) (1989) 797–814.

[5] K.J. Bathe and E.N. Dvorkin. A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation. International Journal for Numerical Methods in Engineering, 21(2) (1985) 367–383.

[6] F. Brezzi and M. Fortin. Numerical approximation of Mindlin-Reissner plates. Mathematics of Computation, 47(175) (186) 151–158.

[7] F. Brezzi, M. Fortin, and R. Stenberg. Error analysis of mixed-interpolated elements for Reissner-Mindlin plates. Mathematical Models and Methods in Applied Sciences, 1(2) (1991) 125–151.

[8] C. Carstensen, X. Xie, G. Yu, and T. Zhou. A priori and a posteriori analysis for a locking-free low order quadrilateral hybrid finite element for Reissner-Mindlin plates. Computer Methods in Applied Mechanics and Engineering, 200(9-12) (2011) 1161–1175.
[9] C. Chinosi and C. Lovadina. Numerical analysis of some mixed finite element methods for Reissner-Mindlin plates. Computational Mechanics, 16(1) (1995) 36–44.
[10] R. Durán and E. Liberman. On mixed finite element methods for the Reissner-Mindlin plate model. Mathematics of Computation, 58(198) (1992) 561–573.
[11] R.G. Durán, E. Hernández, L. Hervella-Nieto, E. Liberman, and R. Rodríguez. Error estimates for low-order isoparametric quadrilateral finite elements for plates. SIAM Journal on Numerical Analysis, 41 (2003) 1751–1772.
[12] R.S. Falk and T. Tu. Locking-free finite elements for the Reissner-Mindlin plate. Mathematics of Computation, 69(231) (2000) 911–928.
[13] V. Girault and P.A. Raviart. Finite element methods for Navier-Stokes equations, Theory and algorithms, volume 5 of Springer Series in Computational Mathematics, 1986.
[14] Y. Guo, G. Yu, and X. Xie. Uniform analysis of a stabilized hybrid finite element method for Reissner-Mindlin plates. Science China Mathematics, 56(8):1727–1742, 2013.
[15] J. Hu and Z.C. Shi. Error analysis of quadrilateral wilson element for Reissner–Mindlin plate. Computer Methods in Applied Mechanics and Engineering, 197(6) (2008) 464–475.
[16] J. Hu and Z.C. Shi. Analysis for quadrilateral MITC elements for the Reissner-Mindlin plate problem. Mathematics of Computation, 78(266) (2009) 673–711.
[17] T.J.R. Hughes. The finite element method: linear static and dynamic finite element analysis. Prentice-hall, 1987.
[18] T.J.R. Hughes, M. Cohen, and M. Haroun. Reduced and selective integration techniques in the finite element analysis of plates. Nuclear Engineering and Design, 46(1) (1978) 203–222.
[19] P.B. Ming and Z.C. Shi. Analysis of some low order quadrilateral Reissner-Mindlin plate elements. Mathematics of Computation, 75(255)(2006) 1043–1065.
[20] O.C. Zienkiewicz, R.L. Taylor, and J.M. Too. Reduced integration technique in general analysis of plates and shells. International Journal for Numerical Methods in Engineering, 3(2) (1971) 275–290.

School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan, China, 610031
E-mail: yuguozhumail@gmail.com

School of Mathematics, Sichuan University, Chengdu, Sichuan, China, 610064
E-mail: xpxie@scu.edu.cn

Experiment Center, China West Normal University, Nanchong, Sichuan, China, 637009
E-mail: gyh6209@sina.com