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HIGHER-ORDER LINEARIZED MULTISTEP FINITE DIFFERENCE METHODS FOR NON-FICKIAN DELAY REACTION-DIFFUSION EQUATIONS

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Abstract. In this paper, two types of higher-order linearized multistep finite difference schemes are proposed to solve non-Fickian delay reaction-diffusion equations. For the first scheme, the equations are discretized based on the backward differentiation formulas in time and compact finite difference approximations in space. The global convergence of the scheme is proved rigorously with convergence order $\mathcal{O}(\tau^2 + h^4)$ in the maximum norm. Next, a linearized noncompact multistep finite difference scheme is presented and the corresponding error estimate is established. Finally, extensive numerical examples are carried out to demonstrate the accuracy and efficiency of the schemes, and some comparisons with the implicit Euler scheme in the literature are presented to show the effectiveness of our schemes.

Key words. Non-Fickian delay reaction-diffusion equation, linearized compact/noncompact, multistep finite difference scheme, solvability, convergence.

1. Introduction

Nonlinear delay partial differential equations (NDPDEs) are widely used in description of natural phenomena and social behaviors in biology, medicine, control theory, epidemiology, climate models, and many others [6, 16, 20, 39, 45]. These equations have been paid a lot of attention because they provide a powerful tool to reflect the essential characteristics of processes with delayed effects. However, the analytical solutions of most of the delay partial differential equations (DPDEs) can not be explicitly expressed and the theoretical analysis of DPDEs is also difficultly carried out because of the delayed terms. Hence, developing efficient and higher-order numerical methods for DPDEs especial NDPDEs has become an important issue and a hot topic [9, 19, 42, 43].

In this paper, we are dedicated to developing the higher-order numerical approximation to the solution of non-Fickian delay reaction-diffusion equation of the form

$$\frac{\partial u}{\partial t} = D_1\frac{\partial^2 u}{\partial x^2} + \frac{D_2}{\delta}\int_0^t e^{-\frac{t-w}{\delta}}\frac{\partial^2 u}{\partial x^2}(x,w)dw + f(u(x,t), u(x,t-s), x, t), \tag{1}$$

where $(x,t) \in [a,b] \times [0,T]$, D_1, D_2 and δ are positive constants, and $s > 0$ is the delay parameter. The initial condition associated with (1) is given by

$$u(x,t) = \varphi(x,t),\ x \in [a,b],\ t \in [-s,0] \tag{2}$$

and the boundary conditions are specified by

$$u(a,t) = u_a(t),\ u(b,t) = u_b(t),\ t > 0. \tag{3}$$

Equation (1) is called non-Fickian delay reaction-diffusion equation due to certain memory effects taken into account [6, 20]. In the case of $D_2 = 0$, it reduces to a

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regular delayed reaction-diffusion equation, which we frequently encounter in a vast array of fields. For example, if we take

$$f(u(x,t),u(x,t-s),x,t) = -au(x,t) + \frac{bu(x,t-s)}{1+u^m(x,t-s)},$$

then, we reduce the equation (1) to the diffusive Mackey-Glass equation [19, 20], and if we take

$$f(u(x,t),u(x,t-s),x,t) = -au(x,t) + bu(x,t-s)e^{-u^m(x,t-s)},$$

then, we obtain the diffusive Nicholson's blowflies equation [10, 21, 22, 25–29, 32, 33].

As a typical partial integro-differential equation, the equation (1) has been paid more attention and extensively studied [1–4, 6–8, 11–15, 18, 20, 23, 24, 30, 35, 38, 40, 41]. In 1986, Sloan *et al* [34] numerically studied it by the backward Euler and Crank-Nicolson methods. In [13], Fedotov presented an asymptotic method for the analysis of traveling waves in a one-dimensional reaction-diffusion system where the diffusion has a finite velocity with Kolmogorov-Petrovskii-Piskunov kinetics. Araújo [5] investigated the qualitative properties of numerical traveling wave solutions for integro-differential equations. Recently, Khuri *et al* [18] concentrated on the finite difference method and the spline collocation method for the numerical solution of a generalized Fisher integro-differential equation, and Branco *et al* [6] studied the structure of the solution to the non-Fickian delay reaction-diffusion equations from both the theoretical and numerical points of view. In [44], Zhang *et al* constructed a second-order linearized finite difference scheme for the generalized Fisher-Kolmogorov-Petrovskii-Piskunov equation by introducing a new variable which transforms the integro-differential equation into an equivalent coupled system of first-order differential equations. Late then, Kazem [17] considered a meshless method on non-Fickian flows with mixing length growth in porous media based on radial basis functions. Very recently, Li *et al* [20] discussed the long time behavior of non-Fickian delay reaction-diffusion equations and Wang [38] analyzed the finite element method for fully discrete semilinear evolution equations with positive memory based on two-grid discretizations.

However, most of the numerical methods are no more than second-order accuracy, while there are a large number of scenarios where higher-order accurate schemes are a necessity due to the desired accuracy of the simulations. On the other hand, the higher-order schemes allow one to approximate a solution with fewer grid points, while maintaining the same accuracy as a low-order scheme. In certain circumstances, the desired grid point size is based on the ability to resolve the structure of the solution, and not on the accuracy of computation. But, the higher-order finite-difference schemes, typically achieved by computing derivatives with a wider matrix stencil, cause some difficulties near the boundary, just as one must be able to calculate the inner point near the boundary with the same accuracy as the internal scheme, which should be complicated to implement. Based on such a reason, there is a great interest in the higher-order finite difference schemes. Since the 1950s, the compact finite difference schemes have been applied to solve partial differential equations more and more frequently, and more recently, the compact finite difference schemes have been extended to DPDEs, for instance, see [43] by proposing a compact multisplitting scheme for the nonlinear delay convection-reaction-diffusion equations, and [36] by applying the compact difference scheme to delay reaction-diffusion equations based on Crank-Nicolson scheme in temporal direction, and [42] by employing the compact difference scheme combined with extrapolation techniques to solve a class of neutral delay parabolic differential equations. The main advantage is that the compact finite difference schemes exhibit higher-order accuracy but still only rely on the closest neighboring points for computations. The resulting algebraic systems are tridiagonal which can be inverted easily by Thomas algorithm.

To the best of our knowledge, there are few works using the compact finite difference schemes to solve the non-Fickian delay reaction-diffusion equations. Here, the main purpose in our paper is to construct two kinds of efficient numerical schemes for the mentioned equations, where, both schemes are discretized using backward differentiation formulas in the temporal direction, but in the spatial direction, one is based on the fourth-order compact finite difference scheme and the other is based on standard second-order central difference approximation. For simplicity, let us call the former as the compact multistep scheme and the latter as the noncompact multistep scheme, respectively.

Throughout the paper, we assume that the solution $u(x,t)$ to (1)-(3) is sufficiently smooth in the following sense

$$u(x,t) \in \mathcal{C}^{6,4}([a,b]\times[0,T]); \tag{4}$$

and $f(\mu,\nu,x,t)$ has the first-order continuous derivative with respect to the first and second components in the ϵ_0-neighborhood of the solution, where ϵ_0 is a positive constant, and we denote

$$c_1 = \max_{\substack{x\in(a,b),0<t<T,\\ |\epsilon_1|\le\epsilon_0,|\epsilon_2|\le\epsilon_0}} |f_\mu(u(x,t)+\epsilon_1, u(x,t-s)+\epsilon_2, x, t)|, \tag{5}$$

$$c_2 = \max_{\substack{x\in(a,b),0<t<T,\\ |\epsilon_1|\le\epsilon_0,|\epsilon_2|\le\epsilon_0}} |f_\nu(u(x,t)+\epsilon_1, u(x,t-s)+\epsilon_2, x, t)|. \tag{6}$$

The paper is organized as follows. In Section 2, we derive the compact multistep scheme and show the local truncation error. Then, in Section 3 we present the matrix form of the compact multistep scheme and analyze its solvability. Next, we further show the convergence of the compact multistep scheme in Section 4, and propose the noncompact multistep scheme and show the corresponding local truncation error, solvability and convergence in Section 5. Finally, we carry out some numerical experiments to verify the proposed results in Sections 4 and Section 5.

2. Derivation of the compact multistep scheme

We firstly divide the region $\Omega\times(0,T]$, where $\Omega=(a,b)$. Take a positive number M, and let $h=\frac{b-a}{M}$. Suppose the lag s is integral multiple of time step τ, $\tau=\frac{s}{n}$. Denote $x_i=a+ih$, $t_k=k\tau$ and define $\Omega_{h\tau}=\Omega_h\times\Omega_\tau$, where $\Omega_h=\{x_i|0\le i\le M\}$, $\Omega_\tau=\{t_k|-n\le k\le N\}$, $N=[\frac{T}{\tau}]$, $U_i^k=u(x_i,t_k), 0\le i\le M, -n\le k\le N$. Suppose $\mathcal{W}=\{v_i^k|0\le i\le M,-n\le k\le N\}$ is the grid function space defined on $\Omega_{h\tau}$. We denote by

$$\begin{aligned}
v_i^{k+\frac12} &= \frac12(v_i^k+v_i^{k+1}),\ \delta_t v_i^{k+\frac12}=\frac1\tau(v_i^{k+1}-v_i^k),\\
\delta_t^+ v_i^k &= \frac12(3\delta_t v_i^{k+\frac12}-\delta_t v_i^{k-\frac12}),\\
\delta_x v_{i+\frac12}^k &= \frac1h(v_{i+1}^k-v_i^k),\ \delta_x^2 v_i^k=\frac1h(\delta_x v_{i+\frac12}^k-\delta_x v_{i-\frac12}^k),\\
\mathcal{A}v_i^k &= \frac1{12}(v_{i+1}^k+10v_i^k+v_{i-1}^k).
\end{aligned}$$

Let $z(x,t)=\int_0^t g(x,t,w)dw$, where $g(x,t,w)=\sigma(t,w)\frac{\partial^2 u}{\partial x^2}(x,w)$ with $\sigma(t,w)=e^{-\frac{t-w}{\delta}}$. It easily turns out that

$$\frac{\partial u}{\partial t}(x,t)=D_1\frac{\partial^2 u}{\partial x^2}(x,t)+\frac{D_2}{\delta}z(x,t)+f(u(x,t),u(x,t-s),x,t). \tag{7}$$

Now let us introduce an interesting lemma.

Lemma 2.1 ([43]). *Suppose $p(x)\in C^6[x_{i-1},x_{i+1}]$, then*

$$\frac{1}{12}[p''(x_{i-1})+10p''(x_i)+p''(x_{i+1})]-\frac{1}{h^2}[p(x_{i-1})-2p(x_i)+p(x_{i+1})]=\frac{h^4}{240}p^{(6)}(\omega_i),$$

where $\omega\in(x_{i-1},x_{i+1})$.

Considering (7) at the point (x_i,t_{k+1}), we have

$$\begin{aligned}&\frac{\partial u}{\partial t}(x_i,t_{k+1})\\ &=D_1\frac{\partial^2 u}{\partial x^2}(x_i,t_{k+1})+\frac{D_2}{\delta}z(x_i,t_{k+1})+f(u(x_i,t_{k+1}),u(x_i,t_{k+1}-s),x_i,t_{k+1}),\end{aligned} \tag{8}$$

which can be, by Taylor's expansion, reduced to

$$\frac{\partial u}{\partial t}(x_i,t_{k+1})=\delta_t^+U_i^{k+1}+\frac{1}{3}\tau^2\frac{\partial^3 u}{\partial t^3}(x_i,\xi_i^{k+1}),\ \xi_i^{k+1}\in(t_{k-1},t_{k+1}). \tag{9}$$

Again, by Taylor's expansion at point $(2U_i^k-U_i^{k-1},U_i^{k+1-n},x_i,t_{k+1})$, we obtain

$$\begin{aligned}&f(u(x_i,t_{k+1}),u(x_i,t_{k+1}-s),x_i,t_{k+1})\\ &=f(2U_i^k-U_i^{k-1},U_i^{k+1-n},x_i,t_{k+1})\\ &\quad+(u(x_i,t_{k+1})-2U_i^k+U_i^{k-1})f_\mu(\eta_i^k,\zeta_i^k,x_i,t_{k+1})\\ &\quad+(u(x_i,t_{k+1}-s)-U_i^{k+1-n})f_\nu(\eta_i^k,\zeta_i^k,x_i,t_{k+1})\\ &=f(2U_i^k-U_i^{k-1},U_i^{k+1-n},x_i,t_{k+1})\\ &\quad+\tau^2\tfrac{\partial^2 u}{\partial t^2}(x_i,\rho_k)f_\mu(\eta_i^k,\zeta_i^k,x_i,t_{k+1}),\end{aligned} \tag{10}$$

where ρ_k, η_i^k and ξ_i^k are some numbers such that $\rho_k\in(t_{k-1},t_{k+1})$, η_i^k is between $u(x_i,t_{k+1})$ and $2U_i^k-U_i^{k-1}$, ζ_i^k is between $u(x_i,t_{k+1}-s)$ and U_i^{k+1-n}. According to the composite trapezoidal rule [31], we obtain

$$\begin{aligned}z(x_i,t_{k+1})&=\textstyle\int_0^{t_{k+1}}g(x_i,t_{k+1},w)dw=\sum_{j=0}^k\int_{t_j}^{t_{j+1}}g(x_i,t_{k+1},w)dw\\ &=\tfrac{\tau}{2}\textstyle\sum_{j=0}^k\left(g(x_i,t_{k+1},t_j)+g(x_i,t_{k+1},t_{j+1})\right)+r_1^k(g),\\ &=\tfrac{\tau}{2}\textstyle\sum_{j=0}^k\Big(\sigma(t_{k+1},t_j)\frac{\partial^2 u}{\partial x^2}(x_i,t_j)+\sigma(t_{k+1},t_{j+1})\frac{\partial^2 u}{\partial x^2}(x_i,t_{j+1})\Big)\\ &\quad+r_1^k(g),\end{aligned} \tag{11}$$

where $r_1^k(g)=-\frac{\tau^3}{12}\sum\limits_{j=0}^k\frac{\partial g}{\partial w}|_{(x_i,t_{k+1},\eta_j)}$, $\eta_j\in(t_j,t_{j+1})$. Performing operator $\mathcal{A}$ on both sides of the equation (8), we obtain

$$\begin{aligned}\mathcal{A}\tfrac{\partial u}{\partial t}(x_i,t_{k+1})\ &=\ D_1\mathcal{A}\tfrac{\partial^2 u}{\partial x^2}(x_i,t_{k+1})+\tfrac{D_2}{\delta}\mathcal{A}z(x_i,t_{k+1})\\ &\qquad+\mathcal{A}f(u(x_i,t_{k+1}),u(x_i,t_{k+1}-s),x_i,t_{k+1}).\end{aligned} \tag{12}$$

Using Lemma 2.1, the spatial discretization can be carried out as follows

$$\mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i,t_k)=\delta_x^2U_i^k+\frac{h^4}{240}\frac{\partial^6 u}{\partial x^6}(\theta_i^k,t_k),\ \theta_i^k\in(x_{i-1},x_{i+1}). \tag{13}$$

Combining (11) and (13) and noting the linearity of the operator $\mathcal{A}$, we have

(14)

$$\begin{aligned}&\mathcal{A}z(x_i,t_{k+1})=\mathcal{A}\int_0^{t_{k+1}}g(x_i,t_{k+1},w)dw\\&=\tfrac{\tau}{2}\textstyle\sum_{j=0}^{k}\left(\mathcal{A}g(x_i,t_{k+1},t_j)+\mathcal{A}g(x_i,t_{k+1},t_{j+1})\right)+\mathcal{A}r_1^k(g)\\&=\tfrac{\tau}{2}\textstyle\sum_{j=0}^{k}\Big(\sigma(t_{k+1},t_j)\mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i,t_j)+\sigma(t_{k+1},t_{j+1})\mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i,t_{j+1})\Big)+\mathcal{A}r_1^k(g)\\&=\tfrac{\tau}{2}\textstyle\sum_{j=0}^{k}\Big(\sigma(t_{k+1},t_j)\delta_x^2U_i^j+\sigma(t_{k+1},t_{j+1})\delta_x^2U_i^{j+1}\Big)+r_2^k(g)\end{aligned}$$

where

(15)

$$r_2^k(g)=\mathcal{A}r_1^k(g)+\frac{\tau h^4}{480}\sum_{j=0}^{k}\left(\sigma(t_{k+1},t_j)\frac{\partial^6 u}{\partial x^6}(\theta_i^j,t_{k+1}).+\sigma(t_{k+1},t_{j+1})\frac{\partial^6 u}{\partial x^6}(\theta_i^{j+1},t_{k+1})\right).$$

Substituting (9), (10) and (13)–(15) into (12), we obtain

$$\begin{aligned}&\mathcal{A}\delta_t^+U_i^{k+1}\\=\ &D_1\delta_x^2U_i^{k+1}+\tfrac{D_2\tau}{2\delta}\textstyle\sum_{j=0}^{k}\Big(\sigma(t_{k+1},t_j)\delta_x^2U_i^j+\sigma(t_{k+1},t_{j+1})\delta_x^2U_i^{j+1}\Big)\\&+\mathcal{A}f(2U_i^k-U_i^{k-1},U_i^{k+1-n},x_i,t_{k+1})+R_i^k\end{aligned} \tag{16}$$

where

(17)

$$\begin{aligned}R_i^k\ =\ &-\tfrac{1}{3}\tau^2\mathcal{A}\tfrac{\partial^3 u}{\partial t^3}(x_i,\xi_i^{k+1})+D_1\tfrac{h^4}{240}\tfrac{\partial^6 u}{\partial x^6}(\theta_i^{k+1},t_{k+1})\\&+\mathcal{A}\tau^2\tfrac{\partial^2 u}{\partial t^2}(x_i,\rho_k)f_\mu(\eta_i^k,\zeta_i^k,x_i,t_{k+1})+\tfrac{D_2}{\delta}r_2^k(g)\\=\ &\tau^2\left(-\tfrac{1}{3}\mathcal{A}\tfrac{\partial^3 u}{\partial t^3}(x_i,\xi_i^{k+1})+\mathcal{A}\tfrac{\partial^2 u}{\partial t^2}(x_i,\rho_k)f_\mu-D_2\tfrac{\tau}{12\delta}\sum_{j=0}^{k}\tfrac{\partial g}{\partial w}|_{(x_i,t_{k+1},\eta_j)}\right)\\&+\tfrac{h^4}{240}\Bigg(D_1\tfrac{\partial^6 u}{\partial x^6}(\theta_i^{k+1},t_{k+1})+D_2\tfrac{\tau}{2\delta}\sum_{j=0}^{k}(\sigma(t_{k+1},t_j)\tfrac{\partial^6 u}{\partial x^6}(\theta_i^j,t_{k+1})\\&+\sigma(t_{k+1},t_{j+1})\tfrac{\partial^6 u}{\partial x^6}(\theta_i^{j+1},t_{k+1}))\Bigg).\end{aligned}$$

Noticing the initial and boundary conditions (2) and (3), we have

$$U_0^k=u_a(t_k),\ U_N^k=u_b(t_k),\ 1\le k\le N, \tag{18}$$

$$U_i^k=\varphi(x_i,t_k),\ 0\le i\le M,\ -n\le k\le 0. \tag{19}$$

Combining the boundary and initial value conditions (18) and (19), omitting the small term R_i^k, and replacing U_i^k by u_i^k in (16), we can construct the compact difference scheme as follows

$$\begin{aligned}&\mathcal{A}\delta_t^+u_i^{k+1}\\=\ &D_1\delta_x^2u_i^{k+1}+\tfrac{D_2\tau}{2\delta}\textstyle\sum_{j=0}^{k}\Big(\sigma(t_{k+1},t_j)\delta_x^2u_i^j+\sigma(t_{k+1},t_{j+1})\delta_x^2u_i^{j+1}\Big)\\&+\mathcal{A}f(2u_i^k-u_i^{k-1},u_i^{k+1-n},x_i,t_{k+1}),\ 1\le i\le M-1,0\le k\le N-1,\end{aligned} \tag{20}$$

$$u_0^k=u_a(t_k),\ u_N^k=u_b(t_k),\ 1\le k\le N, \tag{21}$$

$$u_i^k = \varphi(x_i, t_k), \ -n \le k \le 0. \tag{22}$$

From the estimate of R_i^k, we can easily estimate the local truncation error.

Lemma 2.2. *Under the assumption* (4)-(6)*, the local truncation error of the scheme* (20)-(22) *satisfies*

$$|R_i^k| \le \widehat{c}(\tau^2 + h^4), \ 1 \le i \le M, \ 0 \le k \le N,$$

where $\widehat{c}$ is a positive constant independent of τ and h.

3. Matrix form of the compact multistep scheme

Multiplying (20) by 2τ and noting $\mathcal{A} = 1 + \frac{h^2}{12}\delta_x^2$, we have

$$\begin{aligned}
&\left(3(1 + \frac{h^2}{12}\delta_x^2) - 2D_1\tau\delta_x^2 - \frac{D_2}{\delta}\tau^2\delta_x^2\right) u_i^{k+1} \\
&= \left(1 + \frac{h^2}{12}\delta_x^2\right)(4u_i^k - u_i^{k-1}) + \sum_{j=0}^{k} \sigma_j (u_{i+1}^j - 2u_i^j + u_{i-1}^j) \\
&\quad +2\tau\left(1 + \tfrac{h^2}{12}\delta_x^2\right) f(2u_i^k - u_i^{k-1}, u_i^{k+1-n}, x_i, t_{k+1})
\end{aligned}$$

for $1 \le i \le M-1, 0 \le k \le N-1$, where the parameters

$$\sigma_0 = \frac{D_2\tau^2}{\delta h^2} e^{-\frac{(k+1)\tau}{\delta}}, \ \sigma_j = \frac{2D_2\tau^2}{\delta h^2} e^{-\frac{(k+1-j)\tau}{\delta}}, \ j \ge 1.$$

Let

$$\lambda = \frac{1}{4\delta h^2}(\delta h^2 - 8D_1\delta\tau - 4D_2\tau^2)$$

and

$$f_i^{k+1} = f(2u_i^k - u_i^{k-1}, u_i^{k+1-n}, x_i, t_{k+1}).$$

The matrix form of the above scheme reads

$$A\mathbf{U}^{k+1} = \frac{1}{3}B\mathbf{U}^k + \frac{1}{12}B\mathbf{U}^{k-1} + \sum_{j=0}^{k} \sigma_j C\mathbf{U}^j + \mathbf{F}^{k+1}, \tag{23}$$

where the tridiagonal matrices in $\mathbb{R}^{(M-1)\times(M-1)}$ are given by

$$A = \begin{pmatrix} 3-2\lambda & \lambda & & & \\ \lambda & 3-2\lambda & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda & 3-2\lambda & \lambda \\ & & & \lambda & 3-2\lambda \end{pmatrix}, \tag{24}$$

$$B = \begin{pmatrix} 10 & 1 & & & \\ 1 & 10 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 10 & 1 \\ & & & 1 & 10 \end{pmatrix}, C = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} \tag{25}$$

and the column vectors in $\mathbb{R}^{M-1}$ are given by $\mathbf{U} = (u_1, u_2, \cdots, u_{M-1})^T$ and

(26)
$$\mathbf{F}^{k+1} = \begin{pmatrix} \frac{1}{3}u_0^k + \frac{1}{12}u_0^{k-1} + \sum\limits_{j=0}^{k}\sigma_j u_0^j - \lambda u_0^{k+1} + \frac{\tau}{6}(f_0^{k+1} + 10f_1^{k+1} + f_2^{k+1}) \\ \frac{\tau}{6}(f_1^{k+1} + 10f_2^{k+1} + f_3^{k+1}) \\ \vdots \\ \frac{\tau}{6}(f_{M-3}^{k+1} + 10f_{M-2}^{k+1} + f_{M-1}^{k+1}) \\ \frac{1}{3}u_M^k + \frac{1}{12}u_M^{k-1} + \sum\limits_{j=0}^{k}\sigma_j u_M^j - \lambda u_M^{k+1} + \frac{\tau}{6}(f_{M-2}^{k+1} + 10f_{M-1}^{k+1} + f_M^{k+1}) \end{pmatrix},$$

where $k \geq 1$.

It easily knows that the coefficient matrix A of the multistep finite difference scheme (23) is diagonally dominant. Thus, the matrix A is nonsingular and the solution of the scheme (20)-(22) is determined uniquely. It can be written as follows.

Theorem 3.1. *The difference scheme* (20)-(22) *has a unique solution.*

4. Convergence analysis of the compact multistep scheme

First of all, let us introduce some useful notations and lemmas. Let $\mathcal{V} = \{v|v = (v_0, v_1, ..., v_M), v_0 = v_M = 0\}$ be the grid function space defined on Ω_h. For any $v, w \in \mathcal{V}$, we define the inner products and corresponding norms as follows.

$$(v, w) = h\sum_{i=1}^{M-1} v_i w_i,\ \|v\| = \sqrt{(v, v)},\ |v|_1 = \sqrt{(\delta_x v, \delta_x v)},\ \|v\|_\infty = \max_{0\leq i\leq M}|v_i|.$$

Lemma 4.1 (Gronwall inequality). *Let $\{\mathcal{F}^k | k \geq 0\}$ be a non-negative sequence, and satisfy*

$$\mathcal{F}^{k+1} \leq \mathcal{A} + \mathcal{B}\tau\sum_{i=1}^{k}\mathcal{F}^l,\ k = 0, 1, \cdots,$$

where $\mathcal{A}$ and $\mathcal{B}$ are non-negative constants, then

$$\mathcal{F}^{k+1} \leq \mathcal{A}\exp(\mathcal{B}k\tau),\ k = 0, 1, 2, \cdots.$$

Lemma 4.2 ([37]). *For any $v \in \mathcal{V}$, we have*

$$\|v\|_\infty \leq \frac{\sqrt{b-a}}{2}|v|_1, \tag{27}$$

$$\|v\| \leq \frac{b-a}{\sqrt{6}}|v|_1, \tag{28}$$

$$\frac{2}{3}\|v\|^2 \leq (\mathcal{A}v, v) \leq \|v\|^2. \tag{29}$$

Now, we give the convergence theorem. Let $e_i^k = U_i^k - u_i^k,\ 0 \leq i \leq M,\ 0 \leq k \leq N$.

Theorem 4.3 (Convergence). *Under the assumption* (4)-(6)*, there exists a positive number $\overline{C}$ such that*

$$\|e^k\|_\infty \leq \overline{C}(\tau^2 + h^4),\ 0 \leq i \leq M,\ 0 \leq k \leq N, \tag{30}$$

where

$$\overline{C} = \frac{\widehat{c}}{2}\sqrt{\frac{3T}{2c_3}}(b-a)\exp\left(\frac{5\sqrt{6}c_1^2(b-a)T}{2c_3} + \frac{3T}{2\sqrt{6}c_3}c_2^2(b-a) + \frac{kD_2T}{\delta c_3}\right),$$

with $c_3 = \max\left\{D_1, \frac{D_2\tau}{2\delta}\right\}$.

Proof. Subtracting (20)–(22) from (16), (18), (19), respectively, and operating $h\delta_t e_i^{k+\frac{1}{2}}$ on both sides of these equations, and adding them for i from 1 to $M-1$, we obtain

(31)

$$\begin{aligned}
&(\mathcal{A}\delta_t^+ e^k, \delta_t e^{k+\frac{1}{2}}) \\
&= D_1(\delta_x^2 e^{k+1}, \delta_t e^{k+\frac{1}{2}}) + (R^k, \delta_t e^{k+\frac{1}{2}}) \\
&\quad +(\mathcal{A}[f(2U^k - U^{k-1}, U^{k+1-n}, x, t_{k+1}) - f(2u^k - u^{k-1}, u^{k+1-n}, x, t_{k+1})], \delta_t e^{k+\frac{1}{2}}) \\
&\quad + \tfrac{D_2\tau}{2\delta}\sum_{j=0}^{k}(\sigma(t_{k+1}, t_j)(\delta_x^2 e^j, \delta_t e^{k+\frac{1}{2}}) + \sigma(t_{k+1}, t_{j+1})(\delta_x^2 e^{j+1}, \delta_t e^{k+\frac{1}{2}})).
\end{aligned}$$

Namely,

(32)

$$\begin{aligned}
&(\mathcal{A}\delta_t^+ e^k, \delta_t e^{k+\frac{1}{2}}) - D_1(\delta_x^2 e^{k+1}, \delta_t e^{k+\frac{1}{2}}) - \tfrac{D_2\tau}{2\delta}(\delta_x^2 e^{k+1}, \delta_t e^{k+\frac{1}{2}}) \\
&= \tfrac{D_2\tau}{\delta}\sum_{j=1}^{k}\sigma(t_{k+1}, t_j)(\delta_x^2 e^j, \delta_t e^{k+\frac{1}{2}}) + (R^k, \delta_t e^{k+\frac{1}{2}}) \\
&\quad +(\mathcal{A}[f(2U^k - U^{k-1}, U^{k+1-n}, x, t_{k+1}) - f(2u^k - u^{k-1}, u^{k+1-n}, x, t_{k+1})], \delta_t e^{k+\frac{1}{2}}).
\end{aligned}$$

The errors in the initial time-interval and at the space-boundary are

$$e_i^k = 0,\ 0 \le i \le M,\ -n \le k \le 0, \tag{33}$$

$$e_i^k = 0,\ i = 0, M,\ 1 \le k \le N. \tag{34}$$

In what follows, we adopt the mathematical induction method to prove the convergence theorem. Obviously, $\|e^k\|_\infty = 0$ in the case of $-n \le k \le 0$. Suppose that (52) is valid for $0 < k \le l$, we will prove that (52) is also true for $k = l + 1$.

The first term on the left hand side of (32) can be estimated as follows, by using Hölder inequality and the definition of $\delta_t^+ e_i^k$,

$$\begin{aligned}
&(\mathcal{A}\delta_t^+ e^k, \delta_t e^{k+\frac{1}{2}}) \\
=\ &(\mathcal{A}(\tfrac{3}{2}\delta_t e^{k+\frac{1}{2}} - \tfrac{1}{2}\delta_t e^{k-\frac{1}{2}}), \delta_t e^{k+\frac{1}{2}}) \\
=\ &\tfrac{3}{2}(\mathcal{A}\delta_t e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) - \tfrac{1}{2}(\mathcal{A}\delta_t e^{k-\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) \\
\ge\ &(\mathcal{A}\delta_t e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) + \tfrac{1}{4}\left[(\mathcal{A}\delta_t e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) - (\mathcal{A}\delta_t e^{k-\frac{1}{2}}, \delta_t e^{k-\frac{1}{2}})\right].
\end{aligned}\tag{35}$$

Thanks to the discrete Green formula, we can estimate $(\delta_x^2 e^{k+1}, \delta_t e^{k+\frac{1}{2}})$ in the second and third terms on the left hand side of (32) as follows

$$\begin{aligned}
&-(\delta_x^2 e^{k+1}, \delta_t e^{k+\frac{1}{2}}) \\
=\ &\tfrac{1}{\tau}(\delta_x e^{k+1}, \delta_x(e^{k+1} - e^k)) \\
\ge\ &\tfrac{1}{\tau}\left[(\delta_x e^{k+1}, \delta_x e^{k+1}) - \tfrac{1}{2}(\delta_x e^{k+1}, \delta_x e^{k+1}) - \tfrac{1}{2}(\delta_x e^k, \delta_x e^k)\right] \\
=\ &\tfrac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2).
\end{aligned}\tag{36}$$

Again, by the discrete Green formula again, the first term in right-hand side of (32) can be controlled by

$$
\begin{aligned}
& \tfrac{D_2\tau}{\delta}\sum_{j=1}^{k}\sigma(t_{k+1},t_j)(\delta_x^2 e^j,\delta_t e^{k+\frac12}) \\
= \;& \tfrac{D_2}{\delta}\sum_{j=1}^{k}\sigma(t_{k+1},t_j)(\delta_x e^j,\delta_x(e^k-e^{k+1})) \\
& \le \tfrac{D_2}{2\delta}\sum_{j=1}^{k}\sigma(t_{k+1},t_j)(|e^j|_1^2+2|e^k|_1^2+2|e^{k+1}|_1^2) \\
& \le \tfrac{D_2}{\delta}(|e^{j_0}|_1^2+|e^k|_1^2+|e^{k+1}|_1^2)\sum_{j=1}^{k}\sigma(t_{k+1},t_j) \\
& \le \tfrac{kD_2}{\delta}(|e^{j_0}|_1^2+|e^k|_1^2+|e^{k+1}|_1^2),
\end{aligned}
\tag{37}
$$

where $|e^{j_0}|_1=\max\{|e^1|_1,|e^2|_1,\cdots,|e^k|_1\}$ and

$$
\sum_{j=1}^{k}\sigma(t_{k+1},t_j)=\sum_{j=1}^{k}e^{-\frac{t_{k+1}-t_j}{\delta}}=e^{-\frac{(k+1)\tau}{\delta}}(e^{\frac{\tau}{\delta}}+e^{\frac{2\tau}{\delta}}+\cdots+e^{\frac{k\tau}{\delta}})=\frac{e^{-\frac{k\tau}{\delta}}-1}{1-e^{\frac{\tau}{\delta}}}<k
$$

is employed. The second term can be estimated by using Hölder inequality as follows

$$
(R^k,\delta_t e^{k+\frac12})\le\frac{1}{2\varepsilon}\|R^k\|^2+\frac{\varepsilon}{2}\|\delta_t e^{k+\frac12}\|^2. \tag{38}
$$

The right hand side of (31) can be estimated by

(39)

$$
\begin{aligned}
&(\mathcal{A}[f(2U^k-U^{k-1},U^{k+1-n},x,t_{k+1})-f(2u^k-u^{k-1},u^{k+1-n},x,t_{k+1})],\delta_t e^{k+\frac12}) \\
&\le(\mathcal{A}(c_1|2e^k-e^{k-1}|)+c_2|e^{k+1-n}|,\delta_t e^{k+\frac12}) \\
&\le\tfrac{1}{2\varepsilon}h\sum_{i=1}^{M-1}(c_1|2e_i^k-e_i^{k-1}|+c_2|e_i^{k+1-n})^2+\tfrac{\varepsilon}{2}\|\delta_t e^{k+\frac12}\|^2 \\
&\le\tfrac{1}{2\varepsilon}h\sum_{i=1}^{M-1}(2c_1^2|2e_i^k-e_i^{k-1}|^2+2c_2^2|e_i^{k+1-n}|^2)+\tfrac{\varepsilon}{2}\|\delta_t e^{k+\frac12}\|^2 \\
&\le\tfrac{1}{\varepsilon}h\sum_{i=1}^{M-1}[c_1^2(8(e_i^k)^2+2(e_i^{k-1})^2)+c_2^2(e_i^{k+1-n})^2]+\tfrac{\varepsilon}{2}\|\delta_t e^{k+\frac12}\|^2 \\
&\le\tfrac{2c_1^2}{\varepsilon}(4\|e^k\|^2+\|e^{k-1}\|^2)+\tfrac{c_2^2}{\varepsilon}\|e^{k+1-n}\|^2+\tfrac{\varepsilon}{2}\|\delta_t e^{k+\frac12}\|^2.
\end{aligned}
$$

Let $c_3=\max\{D_1,\frac{D_2\tau}{2\delta}\}$. Plugging (35)–(39) into (32), we have

$$
\begin{aligned}
&\tfrac{c_3}{2\tau}(|e^{k+1}|_1^2-|e^k|_1^2)+(\mathcal{A}\delta_t e^{k+\frac12},\delta_t e^{k+\frac12}) \\
&\quad+\tfrac14\left[(\mathcal{A}\delta_t e^{k+\frac12},\delta_t e^{k+\frac12})-(\mathcal{A}\delta_t e^{k-\frac12},\delta_t e^{k-\frac12})\right] \\
&\le\tfrac{1}{2\varepsilon}\|R^k\|^2+\tfrac{\varepsilon}{2}\|\delta_t e^{k+\frac12}\|^2 \\
&\quad+\tfrac{2c_1^2}{\varepsilon}(4\|e^k\|^2+\|e^{k-1}\|^2)+\tfrac{c_2^2}{\varepsilon}\|e^{k+1-n}\|^2+\tfrac{\varepsilon}{2}\|\delta_t e^{k+\frac12}\|^2 \\
&\quad+\tfrac{kD_2}{\delta}(|e^{j_0}|_1^2+|e^k|_1^2+|e^{k+1}|_1^2),\ 0\le k\le l.
\end{aligned}
\tag{40}
$$

By Lemma 2.1, it is known that

$$
\|R^k\|^2=h\sum_{i=1}^{M-1}(R_i^k)^2\le(b-a)\widehat{c}^2(\tau^2+h^4)^2. \tag{41}
$$

Employing Lemma 4.2, taking $\varepsilon = \frac{2}{3}$ and using (41), we obtain

$$\begin{aligned}
&\tfrac{c_3}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2) + \tfrac{1}{4}\left[(\mathcal{A}\delta_t e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) - (\mathcal{A}\delta_t e^{k-\frac{1}{2}}, \delta_t e^{k-\frac{1}{2}})\right] \\
&\le \tfrac{3\hat{c}^2}{4}(b-a)(\tau^2+h^4)^2 + \tfrac{3}{\sqrt{6}}c_1^2(b-a)(4|e^k|_1^2 + |e^{k-1}|_1^2) \\
&\quad + \tfrac{3}{2\sqrt{6}}c_2^2(b-a)|e^{k+1-n}|_1^2 + \tfrac{kD_2}{\delta}(|e^{j_0}|_1^2 + |e^k|_1^2 + |e^{k+1}|_1^2).
\end{aligned} \tag{42}$$

Multiplying the inequality above by $\frac{2\tau}{c_3}$ on both sides of them and summing up for k, we have

$$\begin{aligned}
|e^{k+1}|_1^2 \le\ & \left(\tfrac{5\sqrt{6}c_1^2(b-a)}{c_3} + \tfrac{3}{\sqrt{6}c_3}c_2^2(b-a) + \tfrac{2kD_2}{\delta c_3}\right)\tau \sum_{m=1}^{k+1}|e^m|_1^2 \\
& + \tfrac{3\hat{c}^2}{2c_3}(b-a)T(\tau^2+h^4)^2.
\end{aligned} \tag{43}$$

Utilizing Gronwall's inequality in Lemma 4.1, then we get from (43) that

$$|e^{k+1}|_1^2 \le \frac{3\hat{c}^2}{2c_3}(b-a)T\exp\left(\frac{5\sqrt{6}c_1^2(b-a)T}{c_3} + \frac{3T}{\sqrt{6}c_3}c_2^2(b-a) + \frac{2kD_2T}{\delta c_3}\right)(\tau^2+h^4)^2. \tag{44}$$

Using Lemma 4.2, we have

$$\begin{aligned}
\|e^{k+1}\|_\infty &\le \tfrac{\sqrt{b-a}}{2}|e^{k+1}|_1 \\
&\le \tfrac{\hat{c}}{2}\sqrt{\tfrac{3T}{2c_3}}(b-a)(\tau^2+h^4) \\
&\quad \times \exp\left(\tfrac{5\sqrt{6}c_1^2(b-a)T}{2c_3} + \tfrac{3T}{2\sqrt{6}c_3}c_2^2(b-a) + \tfrac{kD_2T}{\delta c_3}\right).
\end{aligned} \tag{45}$$

By the inductive principle, the desired result is obtained. □

5. Non-compact multistep scheme

The compact multistep scheme in (20)-(22) requires the solution $u(x,t) \in C_{x,t}^{6,3}(\overline{\Omega} \times [-s,T])$. In this section, another scheme, where the spatial derivative is approximated by standard central difference quotient, is presented. The scheme has second-order accuracy in spatial direction when we assume $u(x,t) \in C_{x,t}^{4,3}(\overline{\Omega}\times[-s,T])$. The discretization in temporal direction is the same to the derivation of the compact multistep scheme (20)-(22). We call the scheme as non-compact multistep scheme. Throughout this paper, C always denotes a generic constant but independent of the time step τ and the space step h.

5.1. Derivation of non-compact multistep scheme. Applying Taylor's expansion, we have

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_k) = \delta_x^2 U_i^k - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\varsigma_i^k, t_k), \tag{46}$$

where $\varsigma_i^k \in (x_{i-1}, x_{i+1})$. Plugging (46) into (8) (11), and combining it with (9) and (10), we obtain

$$\begin{aligned}
\delta_t^+ U_i^{k+1} =\ & D_1\delta_x^2 U_i^{k+1} + \tfrac{D_2\tau}{2\delta}\textstyle\sum_{j=0}^k\left(\sigma(t_{k+1},t_j)\delta_x^2 U_i^j + \sigma(t_{k+1},t_{j+1})\delta_x^2 U_i^{j+1}\right) \\
& + f(2U_i^k - U_i^{k-1}, U_i^{k+1-n}, x_i, t_{k+1}) + \widetilde{R}_i^k,
\end{aligned} \tag{47}$$

where

$$
\begin{aligned}
\widetilde{R}_i^k \;=\;& \tau^2\Big(-\tfrac{1}{3}\tau^2\tfrac{\partial^3 u}{\partial t^3}(x_i,\xi_i^{k+1})+\tfrac{\partial^2 u}{\partial t^2}(x_i,\rho_k)f_\mu(\eta_i^k,\zeta_i^k,x_i,t_{k+1})\Big)\\
& -\tfrac{D_2\tau}{12\delta}\sum_{j=0}^{k}\tfrac{\partial g}{\partial w}|_{(x_i,t_{k+1},\eta_j)})\\
& -\tfrac{h^2}{12}(D_1\tfrac{\partial^4 u}{\partial x^4}(\varsigma_i^{k+1},t_{k+1})+\tfrac{D_2\tau}{2\delta}\sum_{j=0}^{k}(\sigma(t_{k+1},t_j)\tfrac{\partial^4 u}{\partial x^4}(\varsigma_i^{j},t_{k+1})\\
& +\sigma(t_{k+1},t_{j+1})\tfrac{\partial^4 u}{\partial x^4}(\varsigma_i^{j+1},t_{k+1}))).
\end{aligned}\tag{48}
$$

Omitting the small term $\widetilde{R}_i^k$, replacing the exact solution U_i^k with the numerical approximation u_i^k in (47), and combining with the boundary condition (18) and initial value condition (19), we get the non-compact difference scheme as follows

$$
\begin{aligned}
&\delta_t^+ u_i^{k+1}\\
&= D_1\delta_x^2 u_i^{k+1}+\tfrac{D_2\tau}{2\delta}\textstyle\sum_{j=0}^{k}\Big(\sigma(t_{k+1},t_j)\delta_x^2u_i^j+\sigma(t_{k+1},t_{j+1})\delta_x^2u_i^{j+1}\Big)\\
&\quad +f(2u_i^k-u_i^{k-1},u_i^{k+1-n},x_i,t_{k+1}),\ 1\le i\le M-1, 0\le k\le N-1,
\end{aligned}\tag{49}
$$

$$u_0^k=u_a(t_k),\ u_N^k=u_b(t_k),\ 1\le k\le N,\tag{50}$$

$$u_i^k=\varphi(x_i,t_k),\ -n\le k\le 0.\tag{51}$$

5.2. Local truncation error and convergence analysis. Now we can easily estimate $\widehat{R}_i^k$ from (48) as follows.

Lemma 5.1. *Under the assumption* (4)-(6)*, the local truncation error of the scheme* (49)-(51) *satisfies*

$$|\widetilde{R}_i^k|\le \widetilde{c}(\tau^2+h^2),\ 1\le i\le M,\ 0\le k\le N,$$

where $\widetilde{c}$ is a certain positive constant.

Since the proof of the convergence is similar to Theorem 5.2, we omit it in detail.

Theorem 5.2 (Convergence)**.** *Under the assumption* (4)–(6)*, there exists a positive number C such that*

$$\|e^k\|_\infty\le C(\tau^2+h^2),\ 0\le i\le M,\ 0\le k\le N.\tag{52}$$

6. Numerical examples

In this section, we carry out several numerical examples to check the performance of the algorithms in our paper. For ease of comparison, the difference scheme presented in [6] for the equations (1)-(3) is listed out as follows,

(53)

$$\delta_t u_i^{k+\frac{1}{2}}=D_1\delta_x^2u_i^{k+1}+\tau\tfrac{D_2}{\delta}\textstyle\sum_{j=1}^{k+1}e^{-\frac{t_{k+1}-t_j}{\delta}}\delta_x^2u_i^j+f(u_i^{k+1},u_i^{k+1-n},x_i,t_{k+1}),$$

$1\le i\le M-1, 0\le k\le N-1,$

$$u_i^k=\varphi(x_i,t_k),\ 0\le i\le M,\ -n\le k\le 0,\tag{54}$$

$$u_0^k=u_a(t_k),\ u_M^k=u_b(t_k),\ 1\le k\le N.\tag{55}$$

For the sake of brevity, we redescribe the algorithms as follows:

Scheme I : *the scheme* (53) – (55);
Scheme II : *the scheme* (49) – (51);
Scheme III : *the scheme* (20) – (22).

TABLE 1. Errors in L_∞-norm, the convergence rates and CPU time (s) of Schemes I–III with $(h/2, \tau/2)$ for Example 6.1, where $D_1 = 1$, $D_2 = 10$, $T = 10$, $\delta = 5$, $s = 1$.

		Scheme III			Scheme II			Scheme I		
h	τ	$E_\infty(h,\tau)$	Ord_1	CPU	$E_\infty(h,\tau)$	Ord_1	CPU	$E_\infty(h,\tau)$	Ord_1	CPU
$\frac{1}{20}$	$\frac{1}{20}$	3.43e-4	*	6.2e-2	1.93e-2	*	5.7e-2	5.96e-2	*	6.1e-2
$\frac{1}{40}$	$\frac{1}{40}$	9.04e-5	1.92	2.7e-1	4.84e-3	2.00	2.3e-1	3.48e-2	0.78	2.3e-1
$\frac{1}{80}$	$\frac{1}{80}$	2.29e-5	1.98	1.1	1.21e-3	2.00	1.0	1.87e-2	0.90	1.0
$\frac{1}{160}$	$\frac{1}{160}$	5.75e-6	2.00	5.6	3.02e-4	2.00	5.1	9.64e-3	0.95	5.2

* *Could not obtain datum from the second column, and likewise for others.*

The maximum norm errors of the numerical solution are computed by

$$E_\infty(h,\tau) = \max_{0\le n\le N} \|U^n - u^n\|_\infty,$$

and numerical convergence rates of Schemes I-III are denoted by

$$Ord_1 = \log_2\left(\frac{E_\infty(h,\tau)}{E_\infty(h/2,\tau/2)}\right), \quad Ord_2 = \log_2\left(\frac{E_\infty(h,\tau)}{E_\infty(h/2,\tau/4)}\right).$$

Example 6.1. *To demonstrate the efficiency of the schemes in the article, firstly, we consider equations* (1)-(3) *as follows*

$$\begin{cases} \frac{\partial u}{\partial t} = D_1\frac{\partial^2 u}{\partial x^2} + \frac{D_2}{\delta}\int_0^t e^{-\frac{t-w}{\delta}}\frac{\partial^2 u}{\partial x^2}(x,w)dw + f(u(x,t), u(x,t-s), x, t), \\ (x,t) \in [0,1]\times(0,T], \\ u(0,t) = u(1,t) = 0, t > 0, \\ u(x,t) = \exp(\frac{t}{\delta})\sin(\pi x), (x,t)\in[0,1]\times[-s,0], \end{cases} \tag{56}$$

where

$$\begin{aligned} &f(u(x,t), u(x,t-s), x, t) \\ &= -u(x,t)[1-u(x,t-s)] - \frac{\pi^2 D_2}{2}e^{-\frac{t}{\delta}}\sin\pi x \\ &\quad + \left(\pi^2 D_1 + \frac{\pi^2 D_2}{2} + \frac{1}{\delta} + 1\right)e^{\frac{t}{\delta}}\sin\pi x - e^{\frac{2t-s}{\delta}}(\sin\pi x)^2. \end{aligned}$$

The exact solution of the above problem is $u(x,t) = e^{\frac{t}{\delta}}\sin(\pi x)$. In this example, we investigate the global errors, CPU time and the convergence orders of Schemes I, II and III, respectively.

Firstly, we set the parameters as $D_1 = 1$, $D_2 = 10$, $T = 10$, $\delta = 5$, $s = 1$. Table 1 demonstrates the errors in L_∞-norm, convergence rates and CPU time of the three kinds of schemes with $\tau = h$. We observe that the global convergence rates of Scheme II and Scheme III are second-order, while Scheme I is just first-order. Obviously, Scheme III is better than the other two algorithms in terms of CPU time and the accuracy. From Table 2, we see that the global convergence rates of Scheme III is fourth-order while Scheme I and Scheme II are second-order when the spatial step-size and the temporal step-size are reduced by a factor of 1/2 and 1/4, respectively. In Fig. 1, (a) and (b), we see that Scheme III is the most efficient method for the simulation of the non-Fickian diffusion equation compared with the other two schemes.

In order to demonstrate the efficiency of Scheme III further, we take another set of the parameters, $D_1 = 1$, $D_2 = 1$, $T = 30$, $\delta = 15$, $s = 2$ to compute the example again, and the similar results are observed in Tables 3, 4 and in Fig. 1, (c), (d).

Table 2. Errors in L_∞-norm, the convergence rates and CPU time (s) of Schemes I–III with $(h/2, \tau/4)$ for Example 6.1, where $D_1 = 1$, $D_2 = 10$, $T = 10$, $\delta = 5$, $s = 1$.

		Scheme III			Scheme II			Scheme I		
h	τ	$E_\infty(h,\tau)$	Ord_2	CPU	$E_\infty(h,\tau)$	Ord_2	CPU	$E_\infty(h,\tau)$	Ord_2	CPU
$\frac{1}{20}$	$\frac{1}{40}$	6.76e-5	*	2.1e-1	1.97e-2	*	2.0e-1	2.01e-2	*	2.0e-1
$\frac{1}{40}$	$\frac{1}{160}$	4.23e-6	3.99	3.2	4.92e-3	2.00	3.1	5.03e-3	2.00	3.1
$\frac{1}{80}$	$\frac{1}{640}$	2.65e-7	4.00	59	1.23e-3	2.00	55	1.26e-3	2.00	55
$\frac{1}{160}$	$\frac{1}{2560}$	1.65e-8	4.00	1510	3.08e-4	2.00	1040	3.15e-4	2.00	1038

Table 3. Errors in L_∞-norm, the convergence rates and CPU time (s) of Schemes I–III with $(h/2, \tau/2)$ for Example 6.1, where $D_1 = D_2 = 1$, $T = 30$, $\delta = 15$, $s = 2$.

		Scheme III			Scheme II			Scheme I		
h	τ	$E_\infty(h,\tau)$	Ord_1	CPU	$E_\infty(h,\tau)$	Ord_1	CPU	$E_\infty(h,\tau)$	Ord_1	CPU
$\frac{1}{20}$	$\frac{1}{20}$	5.91e-5	*	4.5e-1	5.38e-2	2.01	4.5e-1	2.55e-3	*	4.3e-2
$\frac{1}{40}$	$\frac{1}{40}$	2.72e-5	1.12	1.9	1.33e-2	2.00	1.9	9.40e-5	4.76	3.0e-1
$\frac{1}{80}$	$\frac{1}{80}$	7.56e-6	1.84	8.3	3.32e-3	2.00	8.1	2.48e-4	-1.40	2.0
$\frac{1}{160}$	$\frac{1}{160}$	1.94e-6	1.96	43	8.30e-4	2.00	40	1.98e-4	0.33	13
$\frac{1}{320}$	$\frac{1}{320}$	4.88e-7	1.99	277	2.07e-4	2.00	239	1.17e-4	0.75	89
$\frac{1}{640}$	$\frac{1}{640}$	1.22e-7	2.00	1832	5.19e-5	2.00	1562	6.33e-5	0.89	715

Table 4. Errors in L_∞-norm, the convergence rates and CPU time (s) of Schemes I–III with $(h/2, \tau/4)$ for Example 6.1, where $D_1 = D_2 = 1$, $T = 30$, $\delta = 15$, $s = 2$.

		Scheme III			Scheme II			Scheme I		
h	τ	$E_\infty(h,\tau)$	Ord_2	CPU	$E_\infty(h,\tau)$	Ord_2	CPU	$E_\infty(h,\tau)$	Ord_2	CPU
$\frac{1}{20}$	$\frac{1}{40}$	3.45e-5	*	1.7	5.39e-2	*	1.7	3.64e-3	*	1.9e-1
$\frac{1}{40}$	$\frac{1}{160}$	2.15e-6	4.00	28	1.33e-2	2.01	28	9.09e-4	2.00	3.1
$\frac{1}{80}$	$\frac{1}{640}$	1.34e-7	4.00	499	3.33e-3	2.00	505	2.27e-4	2.00	54
$\frac{1}{160}$	$\frac{1}{2560}$	8.39e-9	4.00	11464	8.32e-4	2.00	9380	5.68e-5	2.00	1002

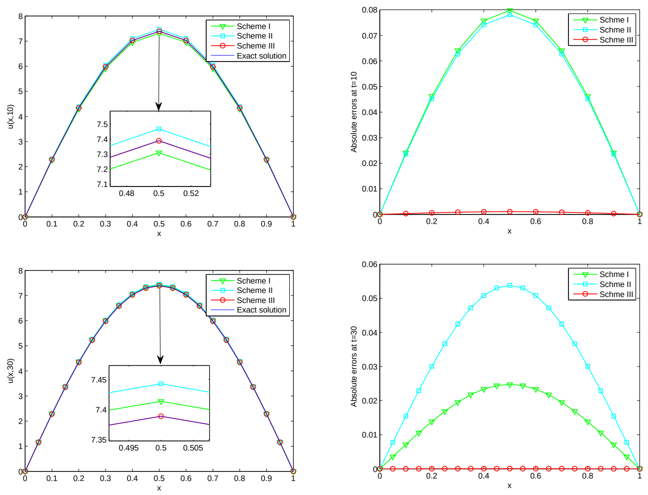

Figure 1. Numerical solutions and corresponding error curves at time $t = 10$ with $h = \tau = 1/10$, $D_1 = 1$, $D_2 = 10$, $\delta = 5$ (a)-(b) and at time $t = 30$ with $h = \tau = 1/20$, $D_1 = D_2 = 1$, $\delta = 15$ (c)-(d), respectively in Example 6.1.

TABLE 5. Errors in L_∞-norm, the convergence rates and CPU time (s) of Schemes I–III with $(h/2, \tau/2)$ for Example 6.2.

		Scheme III			Scheme II			Scheme I		
h	τ	$E_\infty(h,\tau)$	Ord_1	CPU	$E_\infty(h,\tau)$	Ord_1	CPU	$E_\infty(h,\tau)$	Ord_1	CPU
$\frac{1}{20}$	$\frac{1}{20}$	6.04e-2	$*$	2.0e-2	9.43e-2	$*$	1.8e-2	8.14e-1	$*$	1.6e-2
$\frac{1}{40}$	$\frac{1}{40}$	1.74e-2	1.80	8.4e-2	2.60e-2	1.86	7.4e-2	5.18e-1	6.50e-1	6.6e-2
$\frac{1}{80}$	$\frac{1}{80}$	4.57e-3	1.92	3.6e-1	6.73e-3	1.95	3.2e-1	2.93e-1	8.24e-1	3.0e-1
$\frac{1}{160}$	$\frac{1}{160}$	1.16e-3	1.98	1.9	1.70e-3	1.98	1.6	1.55e-1	9.18e-1	1.5

TABLE 6. Errors in L_∞-norm, the convergence rates and CPU time (s) of Schemes I–III with $(h/2, \tau/4)$ for Example 6.2.

		Scheme III			Scheme II			Scheme I		
h	τ	$E_\infty(h,\tau)$	Ord_2	CPU	$E_\infty(h,\tau)$	Ord_2	CPU	$E_\infty(h,\tau)$	Ord_2	CPU
$\frac{1}{20}$	$\frac{1}{40}$	1.71e-2	$*$	6.1e-2	1.66e-1	$*$	5.8e-2	4.24e-1	$*$	5.5e-2
$\frac{1}{40}$	$\frac{1}{160}$	1.15e-3	3.90	8.7e-1	4.56e-2	1.87	8.4e-1	1.16e-1	1.87	8.1e-1
$\frac{1}{80}$	$\frac{1}{640}$	7.20e-5	3.99	14	1.15e-2	1.99	14	2.92e-2	1.99	14
$\frac{1}{160}$	$\frac{1}{2560}$	4.50e-6	4.00	289	2.87e-3	2.00	267	7.31e-3	2.00	258

Example 6.2. *In this example, we consider the initial boundary problem with nonzero boundary condition as follows*

$$
(57)\qquad \begin{cases} \frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + \frac{D_2}{\delta}\int_0^t +u^2(x,t)e^{-\frac{t-w}{\delta}}\frac{\partial^2 u}{\partial x^2}(x,w)dw \\ \qquad +u(x,t-0.1) + f(x,t), \quad (x,t) \in [-1,1]\times[-0.1,5], \\ u(-1,t) = -1, u(1,t) = -1, t > 0, \\ u(x,t) = t^2\cos\pi x, (x,t) \in [-1,1]\times[-0.1,0], \end{cases}
$$

with $D_1 = D_2 = \delta = 1$ *and*

$$
\begin{aligned} f(x,t) &= [\pi^2 D_1 t^2 + 2t - (t-0.1)^2]\cos(\pi x) - t^4\cos^2(\pi x) \\ &\quad + D_2\pi^2(t^2 - 2t\delta + 2\delta^2 - 2\delta^2 e^{-\frac{t}{\delta}})\cos\pi x, \end{aligned}
$$

such that the above problem has an exact solution $u(x,t) = t^2\cos\pi x$.

Tables 5 presents the maximum errors, CPU time and convergence orders of the three kinds of schemes with different step-sizes. We observe that Scheme III and Scheme II are second-order accurate while Scheme I is just first-order when temporal step-size and the temporal step-size are both reduced by a half. From Table 6, we know that Scheme III is fourth-order while Scheme I and Scheme II are just second-order when temporal step-size and the temporal step-size are both reduced by the factor of 1/2 and 1/4, respectively. We can also see that the numerical results are consistent with the theoretical results in Theorems 4.3 and 5.2. From these tables and figures, we conclude that Scheme III is, obviously, the most efficient among the three schemes when the solution of the equations is smooth enough.

Example 6.3. *In this example, we apply our schemes to the non-Fickian Mackey-Glass equation (c.f., [20])*

$$
(58)\qquad \begin{cases} \frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + \frac{D_2}{\delta}\int_0^t e^{-\frac{t-w}{\delta}}\frac{\partial^2 u}{\partial x^2}(x,w)dw - \alpha u(x,t) \\ \qquad +\frac{\beta u(x,t-s)}{1+u^m(x,t-s)}, \quad (x,t) \in [a,b]\times[-s,0], \\ u(a,t) = u(b,t) = 0, t > 0, \\ u(x,t) = (2+\cos t)x(1-x), (x,t) \in [a,b]\times[-s,0], \end{cases}
$$

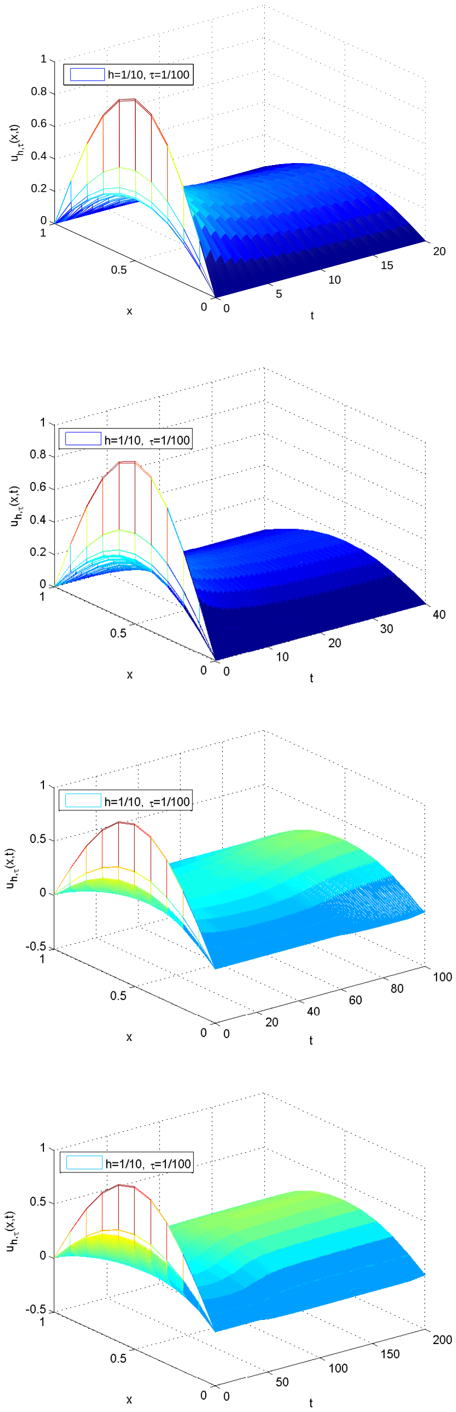

FIGURE 2. Numerical simulations for the equation (58) with $h = 1/10$, $\tau = 1/100$ at time $t = 20$ for (a), $t = 40$ for (b), $t = 100$ for (c) and $t = 200$ for (d) in Example 6.3.

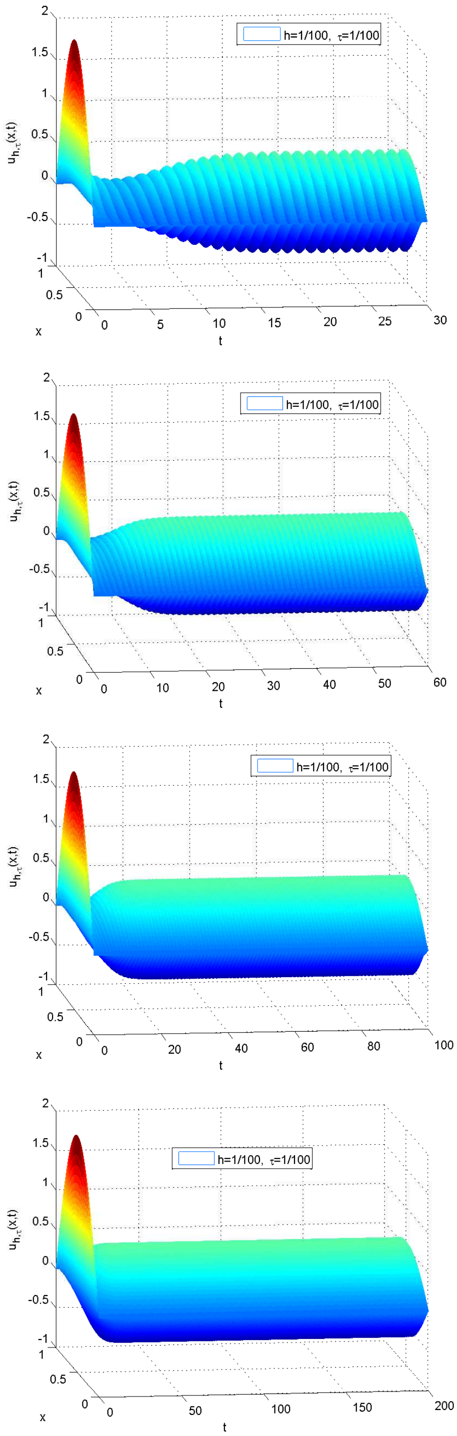

FIGURE 3. Numerical simulations for the equation (59) with $h = \tau = 1/100$, at time $t = 30$ for (a) , $t = 60$ for (b), $t = 100$ for (c) and $t = 200$ for (d) in Example 6.4.

with $D_1 = D_2 = \delta = s = 1$, $m = 2$, $a = 0$, $b = 1$, $\alpha = 100$, $\beta = 120$. The numerical surfaces are displayed in Fig. 2 with $h = 1/10$, $\tau = 1/100$ at time $t = 20$ for (a), 40 for (b), 100 for (c), and 200 for (d). We observe that Fig. 2 (a) is coincided with the paper [20] if the same angle of view is used.

Example 6.4. *In the last example, the non-Fickian Nicholson's type equation is considered (c.f., [20])*

$$\begin{cases} \frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + \frac{D_2}{\delta} \int_0^t e^{-\frac{t-w}{\delta}} \frac{\partial^2 u}{\partial x^2}(x,w)dw - \alpha u(x,t) \\ \qquad +\beta u(x,t-s)e^{-u^m(x,t-s)}, \quad (x,t) \in [a,b] \times [-s,0], \\ u(a,t) = u(b,t) = 0, t > 0, \\ u(x,t) = (2 + \sin t)\sin(\pi x), (x,t) \in [a,b] \times [-s,0], \end{cases} \tag{59}$$

with $D_1 = D_2 = \delta = s = 1$, $m = 2$, $a = 0$, $b = 1$, $\alpha = 2$, and $\beta = 16$. We see that Fig. 3 (a) with $t = 30$ is the same to the numerical surface in paper [20]. To understand the evolving surface of the solution further, we solve the problem at $t = 60, 100, 200$, respectively, which are plotted in Fig. 3 (b)-(d).

7. Concluding remarks

We have formulated two kinds of effective numerical schemes for non-Fickian delay reaction-diffusion equations. Both of the schemes are based on the backward differentiation formulas in temporal direction. The spatial direction is discretized by the fourth-order compact difference scheme and second-order central difference approximation, respectively. The error estimates of the schemes are established rigorously. Numerical simulations have shown that both the schemes display the desired accuracies and the higher order compact multistep scheme is the most efficient.

The schemes in our paper are also available to solve the evolution equations with general positive-type memory term as follows

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + \frac{D_2}{\delta} \int_0^t \beta(t-s)\mathcal{B}u(x,w)dw + f(u(x,t), u(x,t-s), x, t), \tag{60}$$

where β is the positive-definite kernel without singularity and $\mathcal{B}$ is a second-order self-adjoint positive-definite linear elliptic differential operator. In addition, the extension of the results in our work to the higher dimensional case on a rectangular domain is also available. Some alternate direction technique is necessary. We leave them as our future work.

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