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A SIMPLE PROOF OF THE COMPLETE CONSENSUS OF DISCRETE-TIME DYNAMICAL NETWORKS WITH TIME-VARYING COUPLINGS

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Abstract. We discuss the complete consensus problem of the discrete-time dynamical networks with time-varying couplings, and provide a simple *analytic* proof for the emergence of asymptotic complete consensus. Our approach is based on the "energy estimate argument" and connectivity of the communication topology. As direct application of our main results, we obtain asymptotic complete consensus for the discrete-time Kuramoto model with local communication topology.

Key words. Complete consensus, network, time-varying coupling

1. Introduction

Consensus problem as a dynamic feature of complex networks is an active recent subject in many different disciplines such as computer sciences, statistical physics, mathematics, biology, communications and control theory, etc. due to its engineering applications in the formation controls of robots, unmanned aerial vehicles and sensor networks [4], [7]. Complete consensus means a status reaching an agreement regarding certain information of interest that depends on the state of all agents. Consensus algorithm is a dynamic interaction rule regulating the mutual information exchange between agents. In reality, information can be exchanged through direct communication or sensing. Communication link between agents is changeable due to the failure of sensing, and range limitations. Hence we need to consider the dynamically changing communication topology for real applications, e.g. nonlinear interactions between consensus dynamics and dynamically changing network structures in biological networks. In this paper, we consider the following consensus algorithm:

$$\omega_i(t+h) = \omega_i(t) + \frac{\lambda h}{N}\sum_{j=1}^{N} c_{ji}(t)\mathcal{F}(\omega_j(t)-\omega_i(t)), \qquad 1 \le i \le N, \tag{1.1}$$

where ω_i is the information of i-th agent, t is a discrete time $h, 2h, \cdots$ and $\mathcal{F}$ is the state coupling function denoting the interaction rule between agents. The time-varying network structure is monitored by the communication matrix $C(t) := (c_{ij}(t))$.

We next briefly review the related theoretical works on the consensus problem for networks with time-varying topologies; Tsitsiklis-Bertsekas-Athans [12] developed a pioneering work on the distributed computation over networks in computer science, and Jadbabaie et al [3] provide a theoretical explanation for the convergence to the Viscek type heading alignment model [13] with time-varying topologies in the realm of flocking context. This Jadbabaie's seminal work has been further generalized in several following literature, for instance, for the undirected information flow, Fax and Murray [2], Olfati-Saber and Murray [10], whereas for a directed information

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flow, Moreau [6], Ren and Beard [8] and Tanner-Jadbabaie-Pappas [11], Fang and Antsaklis [1], etc. In these previous literature, the consensus analysis is mostly based on "*algebraic method*" such as the matrix theory and spectral graph theory to estimate the second eigenvalue(algebraic connectivity) (see [9] for a detailed review).

The purpose of this paper is to present a simple elementary approach for the asymptotic complete consensus based on the energy type estimates. Our proposed approach do not use any explicit spectral information on the eigenvalues. Instead, it is mainly dependent on the elementary inequalities and energy production rates resulting from the basic energy estimates (see Theorem 3.1 and 3.2 in Section 3).

This paper is divided into four sections after this introduction. In Section 2, we present a framework for the asymptotic consensus and several a priori estimates. In Section 3, we present a rigorous complete consensus estimate for the proposed consensus model. In Section 4, we apply main results in Section 3 to the discrete-time Kuramoto model. Finally Section 5 is devoted to the summary of main results, comparison with previous literature and future directions.

2. Preliminaries

In this section, we provide a framework on the complete consensus, and present several a priori estimates for the discrete-time system (1.1).

Let ω_i be the information state of i-th agent whose continuous-time dynamics is governed by the system with time-dependent communications:

$$\frac{d\omega_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} c_{ji}(t)\mathcal{F}(\omega_j - \omega_i), \quad 1 \le i \le N, \tag{2.1}$$

where λ is a positive coupling constant and $c_{ji} = c_{ji}(t)$ is a nonnegative function denoting the communication weight carried from j-th agent to i-th agent, moreover $\mathcal{F}$ is an odd coupling function. The standard discretization procedure for (2.1) reduces to the discrete-time dynamical model (1.1). To relate the communication topology with energy estimates, we associate the system (1.1) with the dynamic graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ at $t = nh \ge 0$:

$$\begin{aligned} &\mathcal{V} \ : \ \text{the set of all nodes,} \\ &\mathcal{E}(t) \ : \ \text{the set of all pairs } (i,j) \in \mathcal{V}\times\mathcal{V} \text{ with } c_{ij} > 0. \end{aligned}$$

2.1. A framework for complete consensus. In this part, we list main assumptions on the communication topology and the coupling function:

- ($\mathcal{H}1$) The switching communication topology $C(t) = (c_{ij}(t)), t = nh$ is symmetric and bounded:

$$c_{ij}(t) = c_{ji}(t) \le C_u < \infty, \quad \forall\, i,j,t, \qquad C_l := \inf_{i,j,t}\{c_{ij}(t) \ : \ c_{ij}(t) > 0\} > 0.$$

- ($\mathcal{H}2$) The accumulative switching communication topologies contains infinitely often completely connected paths in the sense that for some divergent sequence $\{T_i\}_{i=1}^{\infty}$,

$$1 \le T_1 < T_2 < \cdots < T_n \to \infty,$$

$$\cup_{j=T_i}^{T_{i+1}} \mathcal{E}(j) \quad \text{contains a connected path,}$$

  i.e., every two nodes in $\mathcal{V}$ can be reachable from each other in the time interval $[T_i, T_{i+1})$.

- ($\mathcal{H}3$) The state coupling function $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function satisfying the following properties: For a given $\omega \in \mathbb{R}^d$,
  - (i) $\mathcal{F}(\omega) = -\mathcal{F}(-\omega), \quad \mathcal{F}(\omega)\cdot\omega \geq 0.$
  - (ii) There exist positive constants K_l and K_u such that
$$||\mathcal{F}(\omega)|| \leq K_u||\omega||, \qquad \mathcal{F}(\omega)\cdot\omega \geq K_l||\omega||^2.$$

We next briefly discuss the above assumptions. The condition ($\mathcal{H}1$) has been employed in previous literature [3]. This symmetric property of the communication topology C is rather restricted, but is essential in the energy estimate in Section 3. For the assumption ($\mathcal{H}2$), it is well-known that the existence of connected paths in C is a sufficient condition to ensure the asymptotic consensus (see [3]). On the other hand, the condition ($\mathcal{H}3$) means the coupling function $\mathcal{F}$ has at most sublinear growth. For the simplicity of notation, we set
$$\omega(n) := \omega(nh), \qquad n \geq 0.$$

2.2. Basic a priori estimates. We set the first three moments m_i as follows: For $n \geq 0$,
$$m_0(n) := \sum_{i=1}^N 1 = N, \qquad m_1(n) := \sum_{i=1}^N \omega_i(n), \qquad m_2(n) := \sum_{i=1}^N ||\omega_i(n)||^2.$$

Lemma 2.1. *Suppose the main assumptions* ($\mathcal{H}1$)$-$($\mathcal{H}3$) *hold, and we also assume the time-step size h satisfies*
$$\lambda h K_u^2 C_u < K_l.$$
Let ω_i be the solution to the system (1.1). *Then the first and second moments $m_i, i = 1, 2$ satisfy*

(i) $m_1(n+1) = m_1(n)$.

(ii) $m_2(n+1) + \dfrac{\lambda h}{N}\Big|K_l - \lambda h K_u^2 C_u\Big| \displaystyle\sum_{i,j=1}^N c_{ji}(n)||\omega_j(n) - \omega_i(n)||^2 \leq m_2(n).$

Proof. (i) For the conservation of m_1, we first establish
$$m_1(n+1) = m_1(n) + \frac{\lambda h}{2N}\sum_{1\leq i,j\leq N}(c_{ji} - c_{ij})\mathcal{F}(\omega_j(n) - \omega_i(n)).$$

We add the equation (1.1) with respect to n to find
$$\begin{aligned} m_1(n+1) &= \sum_{i=1}^N \omega_i(n+1) \\ &= \sum_{i=1}^N \omega_i(n) + \frac{\lambda h}{N}\sum_{1\leq i,j\leq N} c_{ji}\mathcal{F}(\omega_j(n) - \omega_i(n)) \\ &= m_1(n) + \frac{\lambda h}{2N}\sum_{1\leq i,j\leq N}(c_{ji} - c_{ij})\mathcal{F}(\omega_j(n) - \omega_i(n)). \end{aligned}$$

We now use the symmetric property $c_{ij} = c_{ji}$ to find
$$m_1(n+1) = m_1(n).$$

(ii) It follows from (1.1) that
$$m_2(n+1) \quad = \quad m_2(n) + \frac{2\lambda h}{N}\sum_{i,j=1}^N c_{ji}(n)\omega_i(n)\cdot\mathcal{F}(\omega_j(n) - \omega_i(n))$$

$$
\begin{aligned}
&+ \quad \frac{(\lambda h)^2}{N^2} \sum_{i,j,k=1}^{N} c_{ji}(n) c_{ki}(n) \mathcal{F}(\omega_j(n) - \omega_i(n)) \cdot \mathcal{F}(\omega_k(n) - \omega_i(n)) \\
&:= \quad m_2(n) + \mathcal{I}_1(n) + \mathcal{I}_2(n).
\end{aligned}
$$

We next estimate the terms $\mathcal{I}_i, i = 1, 2$ separately.
Case 1 ($\mathcal{I}_1$): By direct calculation, we have

$$
\begin{aligned}
\mathcal{I}_1 &= \frac{2\lambda h}{N} \sum_{i,j=1}^{N} c_{ji}(n) \omega_i(n) \cdot \mathcal{F}(\omega_j(n) - \omega_i(n)) \\
&= -\frac{2\lambda h}{N} \sum_{i,j=1}^{N} c_{ji}(n) \omega_j(n) \cdot \mathcal{F}(\omega_j(n) - \omega_i(n)) \\
&= -\frac{\lambda h}{N} \sum_{i,j=1}^{N} c_{ji}(n) (\omega_j(n) - \omega_i(n)) \cdot \mathcal{F}(\omega_j(n) - \omega_i(n)) \\
&\le -\frac{\lambda h K_l}{N} \sum_{i,j=1}^{N} c_{ji}(n) ||\omega_j(n) - \omega_i(n)||^2.
\end{aligned}
$$

Case 2 ($\mathcal{I}_2$): By direct calculation, we have

$$
\begin{aligned}
|\mathcal{I}_2| &\le \frac{(\lambda h)^2 K_u^2}{N^2} \sum_{i,j,k=1}^{N} c_{ji}(n) c_{ki}(n) ||\omega_j(n) - \omega_i(n)|| ||\omega_k(n) - \omega_i(n)|| \\
&\le \frac{(\lambda h)^2 K_u^2}{2N^2} \sum_{i,j,k=1}^{N} \Big(c_{ji}^2(n) ||\omega_j(n) - \omega_i(n)||^2 + c_{ki}^2(n) ||\omega_k(n) - \omega_i(n)||^2 \Big) \\
&\le \frac{(\lambda h)^2 K_u^2 C_u}{N} \sum_{i,j=1}^{N} c_{ji}(n) ||\omega_j(n) - \omega_i(n)||^2.
\end{aligned}
$$

We finally combine the estimates for $\mathcal{I}_1$ and $\mathcal{I}_2$ to find the desired results. $\square$

3. Asymptotic consensus estimates

In this section, we present asymptotic complete consensus estimates under the "*spatial-temporal connectivity*" assumption ($\mathcal{H}2$). For the case of the space-time connectivity assumption ($\mathcal{H}2$) and the time interval $T_i - T_{i-1} > 1$, we show that the states ω_i and ω_j become asymptotically equal as time goes on.

3.1. Asymptotic consensus estimate I. Before we present a asymptotic consensus estimate without decay rate, we first study a vertical estimate which controls the time-variation in terms of spatial estimates.

Lemma 3.1. (Vertical estimate) *In each time zone $[T_{n-1}, T_n)$, time-variation of fluctuations is uniformly bounded by the spatial variations, more precisely, we have*

$$
\sum_{k=T_{n-1}+1}^{T_n-1} \sum_{i=1}^{N} ||\omega_i(k) - \omega_i(k-1)||^2 \le \frac{(\lambda h C_u K_u)^2}{N} \sum_{k=T_{n-1}+1}^{T_n-1} \sum_{(i,j)\in\mathcal{E}(k)} ||\omega_j(k) - \omega_i(k)||^2.
$$

Proof. For $i \in \{1, \cdots, N\}$ and $1 \le k \le T_n - T_{n-1} - 1$, we use the system (1.1) to find

$$
\begin{aligned}
&||\omega_i(T_{n-1}+k) - \omega_i(T_{n-1}+k-1)|| \\
&\quad \le \frac{\lambda h C_u K_u}{N} \sum_{j\in\mathcal{E}_i(T_{n-1}+k-1)} ||\omega_j(T_{n-1}+k-1) - \omega_i(T_{n-1}+k-1)||.
\end{aligned}
$$

where $\mathcal{E}_i(t) := \{j \ : \ (j,i) \in \mathcal{E}(t)\}$.

We next square the above inequality to find

$$\begin{aligned}&||\omega_i(T_{n-1}+k) - \omega_i(T_{n-1}+k-1)||^2 \\ &\quad \le \frac{(\lambda h C_u K_u)^2}{N} \sum_{j \in \mathcal{E}_i(T_{n-1}+k-1)} ||\omega_j(T_{n-1}+k-1) - \omega_i(T_{n-1}+k-1)||^2.\end{aligned}$$

Here we used the fact that

$$\Big(\sum_{k=1}^M |a_k|\Big)^2 \le M \sum_{k=1}^M |a_k|^2, \qquad |\mathcal{E}_i(T_{n-1}+k-1)| \le N.$$

Finally we take a sum with respect to $i \in \{1, \cdots, N\}$ and $1 \le k \le T_n - T_{n-1} - 1$ to get

$$\sum_{k=T_{n-1}+1}^{T_n-1} \sum_{i=1}^N ||\omega_i(k)-\omega_i(k-1)||^2 \le \frac{(\lambda h C_u K_u)^2}{N} \sum_{k=T_{n-1}+1}^{T_n-1} \sum_{(i,j)\in\mathcal{E}(k)} ||\omega_j(k)-\omega_i(k)||^2.$$

□

Theorem 3.1. *Suppose the main assumptions* $(\mathcal{H}1) - (\mathcal{H}3)$ *hold, and we assume that the parameters satisfy*

$$\lambda h K_u^2 C_u < K_l. \tag{3.1}$$

Let ω_i *be the global solution to the system* (1.1). *Then for each* i, j*, we have*

$$\lim_{n\to\infty} ||\omega_i(n) - \omega_j(n)|| = 0.$$

Proof. By the assumption (3.1) and Lemma 3.1, we have

$$m_2(k+1) + \frac{\lambda h}{N}\Big|K_l - \lambda h K_u^2 C_u\Big| \sum_{i,j=1}^N c_{ji}(k)||\omega_j(k) - \omega_i(k)||^2 \le m_2(k). \tag{3.2}$$

We now iterate the above relation (3.2) with respect to $k = 0, \cdots, n$ to find

$$m_2(n+1) + \frac{\lambda h C_l}{N}\Big|K_l - \lambda h K_i^2 C_u\Big| \sum_{k=0}^n \sum_{(i,j)\in \ \mathcal{E}(k)} ||\omega_j(k) - \omega_i(k)||^2 \le m_2(0).$$

By letting $n \to \infty$, we have

$$\frac{\lambda h C_l}{N}\Big|K_l - \lambda h K_u^2 C_u\Big| \sum_{k=0}^\infty \sum_{(i,j)\in \ \mathcal{E}(k)} ||\omega_j(k) - \omega_i(k)||^2 < \infty.$$

In particular, this yields

$$\lim_{n\to\infty} \sum_{k=T_n}^{T_{n+1}-1} \sum_{(i,j)\in \ \mathcal{E}(k)} ||\omega_j(k) - \omega_i(k)||^2 = 0. \tag{3.3}$$

Let (i,j) be any pair of nodes, and $n \gg 1$. Then since $\cup_{k=T_q}^{T_{q+1}-1}\mathcal{E}(k),\ q \ge 0$ contains a completely connected path, for each i, j, there exist m such that

$$\begin{aligned}&n \in [T_m, T_{m+1}), \\ &i = i_0, i_1, \cdots i_M = j,\ n := t_0, t_1, \cdots t_M = n \quad \text{satisfying} \\ &\omega_i(n) = \omega_{i_0}(t_0) \to \omega_{i_1}(t_1) \to \cdots \to \omega_{i_M}(t_M) = \omega_j(n),\end{aligned}$$

where i_k and t_k may not be different each other. Then by the standard triangle inequality, elementary inequality $\left(\sum_{k=1}^{M}|a_k|\right)^2 \le M\sum_{k=1}^{M}|a_k|^2$, and Lemma 3.1, we have

$$\begin{aligned} ||\omega_i(n)-\omega_j(n)||^2 &\le M\sum_{l=1}^{M}||\omega_{i_l}(t_l)-\omega_{i_{l-1}}(t_{l-1})||^2 \\ &\le \mathcal{O}(1)\sum_{k=T_m}^{T_{m+1}-1}\sum_{(i,j)\in\mathcal{E}(k)}||\omega_j(k)-\omega_i(k)||^2, \end{aligned} \tag{3.4}$$

where $\mathcal{O}(1)$ is a bounded constant depending on M, λ, h, C_u, N but independent of n. Let ε be given, then there exists a positive integer $L = L(\varepsilon) \gg 1$ such that

$$n \ge L(\varepsilon) \quad \Longrightarrow \quad \sum_{k=T_m}^{T_{m+1}-1}\sum_{(i,j)\in\ \mathcal{E}(k)}||\omega_j(k)-\omega_i(k)||^2 < \varepsilon.$$

Hence for sufficiently large n, the estimate (3.4) implies

$$||\omega_i(n)-\omega_j(n)||^2 \le \mathcal{O}(1)\varepsilon, \quad \text{i.e.,} \quad \lim_{n\to\infty}||\omega_i(n)-\omega_j(n)|| = 0.$$

□

Remark 3.1. *Note that the assumption* (3.1) *is not restrictive since the time-step size h can be chosen to be sufficiently small so that*

$$\lambda h \ll 1.$$

3.2. Asymptotic consensus estimate II. In this part, we consider the special situation of $(\mathcal{H}2)$ which is

$$T_n = nh,$$

so that the communication topology is connected at each instant $t = nh$ (the symmetry and existence of a spanning tree imply the connectivity of $C(n) = (c_{ij}(n))$: For each i, j and n, there is a shortest directed path from i to j satisfying

$$(\mathcal{H}2)': \ i = k_0 \to k_1 \to \cdots \to k_{d_{ij}} = j, \qquad (k_l, k_{l+1}) \in \mathcal{E}(n). \tag{3.5}$$

We set $d_{ij}(n)$ to be the smallest length among all directed path from i to j at time $t = nh$. We set the diameter of the communication matrix $C = (c_{ij})$ by

$$\mathrm{diam}(C(n)) := \max_{1\le i,j\le N} d_{ij}(n).$$

Then it is easy to see that

$$\mathrm{diam}(C(n)) \le N.$$

Lemma 3.2. (Horizontal estimate) *Suppose the spatial connectivity assumption $(\mathcal{H}2)'$ holds. Then there exists a positive constant K_c such that for each n,*

$$K_c(n)\sum_{1\le l,k\le N}||\omega_l(n)-\omega_k(n)||^2 \le \sum_{(l,k)\in\ \mathcal{E}(n)}||\omega_l(n)-\omega_k(n)||^2 \le \sum_{1\le l,k\le N}||\omega_l(n)-\omega_k(n)||^2.$$

where the positive constant $K_c(n)$ is given by the following relation:

$$K_c(n) := \frac{1}{1+\mathrm{diam}(C(n))(N-|\mathcal{E}(n)|)} \ge \frac{1}{1+N^2}. \tag{3.6}$$

Proof. We first show that for any $(l,k) \in \mathcal{E}^c(n)$,

$$||\omega_l - \omega_k||^2 \text{ can be controlled by the quantity } \sum_{(l,k)\in\ \mathcal{E}(n)} ||\omega_l - \omega_k||^2.$$

For such a pair $(l,k) \in \mathcal{E}^c(n)$, we can find the shortest path from l to k, for example,

$$l = i_0 \to \cdots \to i_m = k, \quad c_{i_j i_{j+1}} \neq 0, \quad j = 0, \cdots, m-1.$$

We now use the standard triangle inequality to see

$$||\omega_l - \omega_k|| \le ||\omega_{i_0} - \omega_{i_1}|| + ||\omega_{i_1} - \omega_{i_2}|| + \cdots + ||\omega_{i_{m-1}} - \omega_{i_m}||.$$

This yields

$$\begin{aligned} ||\omega_l - \omega_k||^2 &\le m(||\omega_{i_0} - \omega_{i_1}||^2 + ||\omega_{i_1} - \omega_{i_2}||^2 + \cdots + ||\omega_{i_{m-1}} - \omega_{i_m}||^2) \\ &\le \mathrm{diam}(C(n))(||\omega_l - \omega_{i_1}||^2 + ||\omega_{i_1} - \omega_{i_2}||^2 + \cdots + ||\omega_{i_{m-1}} - \omega_k||^2) \\ &\le \mathrm{diam}(C(n)) \sum_{(l,k)\in\mathcal{E}(n)} ||\omega_l - \omega_k||^2. \end{aligned}$$

We now sum over all pairs $(l,k) \in \mathcal{E}^c(n)$ to find

$$\sum_{(l,k)\in\mathcal{E}^c(n)} ||\omega_l - \omega_k||^2 \le \mathrm{diam}(C(n))|\mathcal{E}^c(n)| \sum_{(l,k)\in\mathcal{E}(n)} ||\omega_l - \omega_k||^2. \tag{3.7}$$

We use the above estimate (3.7) to obtain

$$\begin{aligned} \sum_{1\le k,l\le N} ||\omega_l - \omega_k||^2 &= \sum_{(l,k)\in\mathcal{E}(n)} ||\omega_l - \omega_k||^2 + \sum_{(l,k)\in\mathcal{E}^c(n)} ||\omega_l - \omega_k||^2 \\ &\le (1 + \mathrm{diam}(C(n))|\mathcal{E}^c(n)|) \sum_{(l,k)\in\mathcal{E}(n)} ||\omega_k - \omega_l||^2. \end{aligned}$$

Therefore we have

$$\sum_{(l,k)\in\mathcal{E}(n)} ||\omega_l - \omega_k||^2 \ge \frac{1}{1 + \mathrm{diam}(C(n))|\mathcal{E}^c(n)|} \sum_{1\le l,k\le N} ||\omega_l - \omega_k||^2.$$

□

We introduce an average quantity and fluctuations around it as follows.

$$\omega_c(n) := \frac{1}{N}\sum_{i=1}^N \omega_i(n), \qquad \hat{\omega}_i(n) := \omega_i(n) - \omega_c(n), \qquad \hat{m}_2(n) := \sum_{i=1}^N ||\hat{\omega}_i(n)||^2.$$

In the following theorem, we show that the asymptotic consensus occurs for any initial configurations as long as the communication topology is symmetric and connected at each instant.

Theorem 3.2. *Suppose the main assumptions* $(\mathcal{H}1), (\mathcal{H}2)'$ *and* $(\mathcal{H}3)$ *hold, and the parameters satisfy*

$$\lambda h K_u^2 C_u < K_l.$$

Let ω_i *be the global solution to the system* (1.1). *Then we have an exponential asymptotic consensus:*

$$\hat{m}_2(n) \le \hat{m}_2(0) e^{-K_d n}, \qquad n \ge 0,$$

where K_d *is a positive constant explicitly defined by*

$$K_d := 2\lambda h K_c C_l \Big| K_l - \lambda h K_u^2 C_u \Big|.$$

Proof. It follows from Lemma 2.1 that

$$\hat{m}_2(n+1) + \frac{\lambda h C_l}{N}\Big|K_l - \lambda h K_u^2 C_u\Big| \sum_{(i,j)\in\mathcal{E}(n)} ||\hat{\omega}_j(n) - \hat{\omega}_i(n)||^2 \le \hat{m}_2(n).$$

We use the above inequality and Lemma 3.2 to find

$$\hat{m}_2(n+1) + \frac{\lambda h K_c(n) C_l}{N}\Big|K_l - \lambda h K_u^2 C_u\Big| \sum_{1\le i,j\le N} ||\hat{\omega}_j(n) - \hat{\omega}_i(n)||^2 \le \hat{m}_2(n).$$

On the other hand, note that the first moment of $\hat{\omega}_i$ is zero, hence we have

$$\sum_{1\le i,j\le N} ||\hat{\omega}_j(n) - \hat{\omega}_i(n)||^2 = 2N\hat{m}_2(n).$$

This yields

$$\hat{m}_2(n+1) \le \hat{m}_2(n)\{1 - 2K_c(n)\lambda h C_l(K_l - \lambda h K_u^2 C_u)\}.$$

We iterate the above inequality to obtain the following estimate:

$$\begin{aligned} \hat{m}_2(n) &\le \hat{m}_2(0)(1-K_d)^n \\ &\le \hat{m}_2(0)e^{-K_d n}, \end{aligned}$$

where we used an inequality $1 - x \le e^{-x}, x \ge 0$ and K_d is a positive constant defined by

$$K_d := 2\lambda h K_c C_l\Big|K_l - \lambda h K_u^2 C_u\Big| \ge \frac{2\lambda h C_l}{1+N^2}\Big|K_l - \lambda h K_u^2 C_u\Big|.$$

□

4. A discrete-time Kuramoto model with local communication topology

In this section, we show that how the consensus estimates in previous section can be applied to a physical model arising from synchronization problem in statistical physics.

Consider the discrete-time Kuramoto model with local communication topology [5]: For $i = 1, \cdots, N,\ n = 0, 1, \cdots,$

$$\theta_i(n+1) = \theta_i(n) + \frac{\lambda h}{N}\sum_{j=1}^{N} c_{ji}(n)\sin(\theta_j(n) - \theta_i(n)). \tag{4.1}$$

We define maximal and minimal phase indices $M(n)$ and $m(n)$ at time $t = nh$:

$$\begin{aligned} &\theta_{M(n)}(n) := \max_{1\le i\le N}\theta_i(n), \quad \theta_{m(n)}(n) := \min_{1\le i\le N}\theta_i(n), \\ &D_\theta(n) := \theta_{M(n)}(n) - \theta_{m(n)}(n). \end{aligned} \tag{4.2}$$

Before we present a complete phase synchronization estimate, we consider the monotonicity of extremal fluctuations.

Lemma 4.1. *Suppose that the parameters λ, h and the communication topology (c_{ji}), initial configurations satisfy*

$$\begin{aligned} &(i)\ 0 < \lambda h < 1, \qquad c_{ji}(n) \in \{0, 1\}. \\ &(ii)\ D_\theta(0) < \pi. \end{aligned}$$

Then we have

$$\sup_{n\ge 0} D_\theta(n) \le D_\theta(0).$$

Proof. (Proof by induction): Suppose that the size of $D_\theta(n)$ satisfies

$$D_\theta(n) < \pi.$$

Then we will show that

$$D_\theta(n+1) \le D_\theta(n).$$

For this, we consider the evolution of $\theta_{M(n)}$ and $\theta_{m(n)}$ separately.
Case 1: We will show

$$\theta_{M(n+1)}(n+1) \le \theta_{M(n)}(n).$$

Without loss of generality, we may assume

$$\theta_{M(n+1)}(n+1) \ne \theta_{M(n)}(n).$$

We next consider two cases:

$$\text{Either } \theta_{M(n+1)}(n) = \theta_{M(n)}(n) \quad \text{or} \quad \theta_{M(n+1)}(n) \ne \theta_{M(n)}(n).$$

Subcase 1.1: Suppose $\theta_{M(n+1)}(n) = \theta_{M(n)}(n)$. It follows from (4.1) that

$$\theta_{M(n+1)}(n+1) = \theta_{M(n+1)}(n) + \frac{\lambda h}{N}\sum_{j=1}^{N} c_{jM(n+1)}(n)\sin(\theta_j(n) - \theta_{M(n+1)}(n)).$$

We now replace $\theta_{M(n+1)}(n)$ by $\theta_{M(n)}(n)$ to find

$$\theta_{M(n+1)}(n+1) - \theta_{M(n)}(n) = \frac{\lambda h}{N}\sum_{j=1}^{N} c_{jM(n+1)}(n)\sin(\theta_j(n) - \theta_{M(n)}(n)) \le 0.$$

Since $-\pi \le \theta_j(n) - \theta_{M(n)}(n) \le 0$, the above relation yields

$$\theta_{M(n+1)}(n+1) \le \theta_{M(n)}(n).$$

Subcase 1.2: Suppose $\theta_{M(n+1)}(n) \ne \theta_{M(n)}(n)$.
If $\theta_{M(n+1)}(n+1) > \theta_{M(n)}(n)$, then

$$\begin{aligned}\theta_{M(n)}(n) &< \theta_{M(n+1)}(n+1)\\ &= \theta_{M(n+1)}(n) + \frac{\lambda h}{N}\sum_{j=1}^{N} c_{jM(n+1)}(n)\sin(\theta_j(n) - \theta_{M(n+1)}(n)).\end{aligned}$$

This yields

$$\begin{aligned}0 &< \theta_{M(n)}(n) - \theta_{M(n+1)}(n)\\ &< \frac{\lambda h}{N}\sum_{j=1}^{N} c_{jM(n+1)}(n)\sin(\theta_j(n) - \theta_{M(n+1)}(n))\\ &= \frac{\lambda h}{N}\sum_{j\in\mathcal{N}_{M(n+1)}(n)} \sin(\theta_j(n) - \theta_{M(n+1)}(n)),\end{aligned} \tag{4.3}$$

where the set $\mathcal{N}_{M(n+1)}(n)$ is the communication neighbor of $M(n+1)$ at time $t = nh$:

$$\mathcal{N}_{M(n+1)}(n) = \{j \in \{1, \cdots, N\} \mid c_{jM(n+1)}(n) = 1\}.$$

On the other hand, we set

$$\mathcal{D}(n) = \{j \in \{1, \cdots, N\} \mid \theta_j(n) - \theta_{M(n+1)}(n) \ge 0\}.$$

Then it is easy to see that

$$|\mathcal{N}_{M(n+1)}(n) \cap \mathcal{D}(n)| \le N.$$

We now return to (4.3):

$$\begin{aligned}
0 &< \theta_{M(n)}(n) - \theta_{M(n+1)}(n) \\
&= \frac{\lambda h}{N} \sum_{j \in \mathcal{N}_{M(n+1)}(n) \cap \mathcal{D}(n)} \sin(\theta_j(n) - \theta_{M(n+1)}(n)) \\
&\le \frac{\lambda h}{N} \sum_{j \in \mathcal{N}_{M(n+1)}(n) \cap \mathcal{D}(n)} (\theta_j(n) - \theta_{M(n+1)}(n)) \quad \text{using } \sin x \le x \text{ for } x \ge 0 \\
&\le \frac{\lambda h}{N} \sum_{j \in \mathcal{N}_{M(n+1)}(n) \cap \mathcal{D}(n)} (\theta_{M(n)}(n) - \theta_{M(n+1)}(n)) \\
&= \frac{\lambda h}{N} |\mathcal{N}_{M(n+1)}(n) \cap \mathcal{D}(n)| (\theta_{M(n)}(n) - \theta_{M(n+1)}(n)) \le (\lambda h)(\theta_{M(n)}(n) - \theta_{M(n+1)}(n)).
\end{aligned}$$

In conclusion, we have

$$0 < \theta_{M(n)}(n) - \theta_{M(n+1)}(n) \le (\lambda h)(\theta_{M(n)}(n) - \theta_{M(n+1)}(n)) < \theta_{M(n)}(n) - \theta_{M(n+1)}(n).$$

This gives a contradiction. Therefore we have

$$\theta_{M(n+1)}(n+1) \le \theta_{M(n)}(n).$$

By induction, we also have

$$\theta_{M(n)}(n) \le \theta_{M(0)}(0).$$

Case 2: By the similar arguments as in Case 1, we have

$$\theta_{m(n)}(n) \ge \theta_{m(0)}(0).$$

Hence Case 1 and Case 2 yield

$$\theta_{M(n+1)}(n+1) \le \theta_{M(n)}(n), \quad \theta_{m(n+1)}(n+1) \ge \theta_{m(n)}(n),$$

i.e., we have

$$D_\theta(n+1) = \theta_{M(n+1)}(n+1) - \theta_{m(n+1)}(n+1) \le \theta_{M(n)}(n) - \theta_{m(n)}(n) = D_\theta(n).$$

Hence if $D_\theta(n) < \pi$, then

$$D_\theta(n+1) \le D_\theta(n) < \pi.$$

This again implies

$$D_\theta(n) \le D_\theta(0), \quad \text{for all } n.$$

□

Lemma 4.2. *Suppose that the parameters λ, h and the communication topology (c_{ji}), and initial configurations satisfy*

(i) $c_{ji}(n) \in \{0, 1\}$.
(ii) $0 < \lambda h < 1, \quad D_\theta(0) < \pi$.

Then we have

$$(\theta_j(n) - \theta_i(n)) \sin(\theta_j(n) - \theta_i(n)) \ge \frac{\sin D_\theta(0)}{D_\theta(0)} |\theta_j(n) - \theta_i(n)|^2.$$

Proof. Without loss of generality, we assume

$$\theta_j(n) - \theta_i(n) \ge 0.$$

Then it follows from Lemma 4.1 that

$$0 \le \theta_j(n) - \theta_i(n) \le D_\theta(n) \le D_\theta(0) < \pi.$$

Hence we have

$$\sin(\theta_j(n) - \theta_i(n)) \geq \frac{\sin D_\theta(0)}{D_\theta(0)}(\theta_j(n) - \theta_i(n)) \geq 0.$$

This again yields the desired result. □

Lemma 4.1 and 4.2 yield the following proposition.

Proposition 4.1. *Suppose that the parameters and initial configurations satisfy*

(i) $0 < \lambda h < 1, \quad D_\theta(0) < \pi.$
(ii) $c_{ji}(n) \in \{0, 1\}, \quad$ *and* $(c_{ji}(n))$ *is spatial-temporal connected as in* $(\mathcal{H}2)$.

Then we have asymptotic complete consensus:

$$\lim_{n\to\infty} |\theta_i(n) - \theta_j(n)| = 0.$$

Proof. We apply Theorem 3.1 with

$$C_l = 1, \quad C_u = 1, \quad K_u = 1, \quad K_l = \frac{\sin D_\theta(0)}{D_\theta(0)} < 1,$$

to get the desired result. □

5. Conclusion

In this paper, we presented a simple alternative approach for the asymptotic complete consensus based on the energy type estimates. For the case of spatial-temporal connected communication topology, we show that two states of different agents approach to the consensus along divergent time sequences. In contrast, for the case of spatial connected topology, the asymptotic consensus occurs exponentially fast. We have applied our estimates to the discrete-time Kuramoto model. Since our estimates are simply based on energy estimates, we do not need any conventional spectral information on the communication topology. Under the spatial-temporal connected condition on the communication topology, we recover the well-known Jadbabaie-Lin-Morse's type consensus theorem in [3]. Although our approach in present form do not give new results, but we believe our simple approach can be useful for other consensus problems.

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