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TWO-LEVEL METHODS BASED ON THREE CORRECTIONS FOR THE 2D/3D STEADY NAVIER-STOKES EQUATIONS

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Abstract. Two-level finite element methods are applied to solve numerically the 2D/3D steady Navier-Stokes equations if a strong uniqueness condition $(\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}} \leq \delta = 1 - \frac{N\|f\|_{-1}}{\nu^2}$ holds, where N is defined in (2.4)-(2.6). Moreover, one-level finite element method is applied to solve numerically the 2D/3D steady Navier-Stokes equations if a weak uniqueness condition $0 < \delta < (\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$ holds. The two-level algorithms are motivated by solving a nonlinear problem on a coarse grid with mesh size H and computing the Stokes, Oseen and Newton correction on a fine grid with mesh size $h << H$. The uniform stability and convergence of these methods with respect to δ and grid sizes h and H are provided. Finally, some numerical tests are made to demonstrate the effectiveness of one-level method and the three two-level methods.

Key words. Navier-Stokes equations, finite element method, Stokes correction, Oseen correction, Newton correction, two-level method.

1. Introduction

In this report we consider the steady incompressible Navier-Stokes equations:

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \text{ in } \Omega, \tag{1.1}$$

$$\text{div}u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,\ \int_\Omega p dx = 0, \tag{1.2}$$

which describes a steady flow of the incompressible viscous Newtonian fluid in a bounded domain. Here Ω is a bounded domain in $R^d(d = 2, 3)$ assumed to have a Lipschitz-continuous boundary $\partial\Omega$, $u : \Omega \to R^d$ and $p : \Omega \to R$ are the velocity and pressure, $\nu > 0$ is the viscosity and f represents the given body forces.

Recently, two-level strategy has been studied for steady semi-linear elliptic equations and nonlinear PDEs by Xu [36, 37], and two-level strategy or multi-level strategy has been studied for the steady Navier-Stokes equations by Layton [23], Layton & Tobiska [28], Layton & Lenferink [25, 26] and Layton, Lee & Peterson [27] and Girault and Lions [7] and He et al [14, 17, 18] and Liu and Hou [29], and two level discretizations of flows of electrically conducting, incompressible fluids has been provided by Ervin, Layton and Maubach in [6] . Moreover, a combination of two-level methods and iterative methods for solving the 2D/3D steady Navier-Stokes equations is provided by He et al [20, 21]. As for the nonstationary Navier-Stokes equations, the two-level finite element semi-discretization scheme has been studied by Girault and Lions [9], and the full discretization of the two-level finite element method in space variable x and the one-level backward Euler scheme

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in time variable t have been discussed by Olshanskii [34] and the full discretization of the two-level finite element method in the space-time variables x and t has been studied by He [10, 11] and He et al. [12] Liu and Hou [30, 31] and Hou and Mei[32]. Recently, some multi-level strategy has been studied for the nonstationary Navier-Stokes equations by He et al. [13, 15, 16].

In this paper, for a larger δ satisfying the strong uniqueness condition $\delta \geq (\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$, we consider three two-level finite element methods by solving a nonlinear Navier-Stokes problem on a coarse grid with mesh size H and computing the Stokes, Oseen and Newton correction on a fine grid with mesh size $h << H$. Moreover, one-level finite element algorithm is applied in the case of the weak uniqueness condition $0 < \delta < (\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$, where $\frac{\|f\|_{-1}}{\|f\|_0}$ is small for a given f. From some stability and convergence analysis with respect to δ of the one-level finite element method, h and H should be of order $O(\delta)$. And from some stability and convergence analysis with respect to δ of the two-level finite element methods, H should be of order $O(\delta^2)$ and h should be of order $O(H^{\frac{3}{2}})$ or $O(\delta^3)$ in the case of the Stokes and Oseen correction and H should be of order $O(\delta^{\frac{3}{2}})$ and h should be of $O(H^{\frac{3}{2}})$ or $O(\delta^{\frac{9}{4}})$. These facts show that h and H should be very small for small δ. Hence, for the finite element approximation of the 2D/3D steady Navier-Stokes equations, it is better to use one-level finite element method in the case of the weak uniqueness condition and the two-level finite element methods in the case of the strong uniqueness condition.

Remark. It follows from the definition that $\nu = \sqrt{(1-\delta)^{-1}N^{-1}\|f\|_{-1}^{-1}}$. Hence, small δ means small ν. For one-level finite element approximation of the 2D/3D steady Navier-Stokes equations, the Stokes, Oseen and Newton iterative methods can be used, the reader can refer to papers [5, 19, 20, 21].

This paper is organized as follows. In §2 an abstract functional setting of the Navier-Stokes problem is given together with some basic assumption **A0** on Ω for the steady Navier-Stokes problem. In §3 some assumptions **A1-A3** concerning the finite element spaces X_μ and M_μ with $\mu = h, H$ are given, and some uniform stability and convergence with respect to δ of the finite element solution (u_μ, p_μ) are recalled. In §4 the uniform stability and convergence with respect to δ of the two-level finite element method based on the Stokes correction on fine grid is given. In §5 the uniform stability and convergence with respect to δ of the two-level finite element method based on the Oseen correction on fine grid is provided. In §6 the uniform stability and convergence of the two-level finite element method based on the Newton correction are proved. In §7, some numerical tests are made to demonstrate the effectiveness of one-level method and the three two-level methods. In §8 some conclusions are made.

2. Functional Setting of the Navier-Stokes Equations

Let Ω be a convex polygonal/polyhedral domain in R^d. As in [8, 24], we introduce the following Sobolev spaces,

$$X = H_0^1(\Omega)^d,\ Y = L^2(\Omega)^d,\ M = L_0^2(\Omega) = \{q \in L^2(\Omega)\,;\int_\Omega q(x)dx = 0\}\,.$$

We denote by $(\cdot,\cdot)$, $\|\cdot\|_0$ the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^d$. The space X is equipped with the usual scalar product $(\nabla u, \nabla v)$ and norm $\|\nabla u\|_0$. The subspaces of X and Y are well suited to the incompressible Navier-Stokes equations:

$$V = \{v \in X; \mathrm{div} v = 0 \text{ in } \Omega\},\ V_0 = \{v \in Y; \mathrm{div} v = 0 \text{ and } v\cdot n|_{\partial\Omega} = 0\}.$$

Norms in the Sobolev spaces $H^k(\Omega)$ or $H^k(\Omega)^d$ are denoted by $\|\cdot\|_k$, and seminorms by $|\cdot|_k$ for $k = 1,\ 2$. Also, we denote by $\|\cdot\|_{L^q}$ the norm on space $L^q(\Omega)$ or $L^q(\Omega)^3$ with $1 < q \le \infty$. We define the continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$ respectively by

$$a(u,v) = \nu(\nabla u, \nabla v), \quad \forall u, v \in X\,,$$

and

$$d(v,q) = (q, \mathrm{div} v), \quad \forall (v,q) \in (X, M).$$

Moreover, we define the trilinear form

$$\begin{aligned} b(u,v,w) &= ((u\cdot\nabla)v, w) + \frac{1}{2}((\mathrm{div} u)v, w) \\ &= \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v),\ \forall u, v, w \in X. \end{aligned}$$

For a given $f \in L^2(\Omega)^d$, the variational formulation of problem (1.1)-(1.2) reads as: find a pair $(u,p) \in (X, M)$ such that

$$a(u,v) + d(u,q) - d(v,p) + b(u,u,v) = (f,v),\ \forall (v,q) \in (X,M). \tag{2.1}$$

We make a regularity assumption on the Stokes problem as in [22].

Assumption A0: For a given $g \in L^2(\Omega)^d$ and the Stokes problem

$$-\Delta v + \nabla q = g, \quad \mathrm{div} v = 0 \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0,$$

we assume that (v,q) satisfies the following regularity result:

$$\|Av\|_0 + \|q\|_1 \le c\|g\|_0, \tag{2.2}$$

where $A = -P\Delta$ denotes the Stokes operator and $P : Y \to V_0$ denotes the L^2-orthogonal projection, and c is a positive constant depending only on Ω, which may stand for different value at its different occurrences.

With the above notations, the following estimates hold (see [1, 8, 22, 24, 35])

$$b(u,v,w) = -b(u,w,v),\ \forall u \in X,\ v,\ w \in X, \tag{2.3}$$

$$|b(u,v,w)| \le N\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0,\ \forall u,\ v,\ w \in X, \tag{2.4}$$

$$|b(u,v,w)| \le \frac{N}{2}\|u\|_0(\|\nabla v\|_0\|w\|_{L^\infty} + \|v\|_{L^6}\|\nabla w\|_{L^3}),\ \forall u \in Y,\ v \in X,\ w \in L^\infty(\Omega)^d \cap X, \tag{2.5}$$

$$|b(u,v,w)| \le \frac{N}{2}(\|u\|_{L^\infty}\|\nabla v\|_0 + \|\nabla u\|_{L^3}\|v\|_{L^6})\|w\|_0,\ \forall u \in L^\infty(\Omega)^d \cap X,\ v \in X,\ w \in Y, \tag{2.6}$$

$$\|v\|_0 \le \gamma_0\|\nabla v\|_0,\ \|v\|_{L^3} \le c\|v\|_0^{1/2}\|\nabla v\|_0^{1/2},\ \|v\|_{L^6} \le c\|\nabla v\|_0\ \forall v \in X, \tag{2.7}$$

$$\|\nabla v\|_{L^3} + \|v\|_{L^\infty} \le c\|\nabla v\|_0^{1/2}\|Av\|_0^{1/2},\ \|\nabla v\|_{L^6} + \|v\|_2 \le c\|Av\|_0\ \forall v \in D(A), \tag{2.8}$$

where $D(A) = H^2(\Omega)^d \cap V$ and N is a fixed positive constant depending only on Ω.

The following existence and uniqueness result for problem (2.1) is classical [8, 35]:

Theorem 2.1. Let $f \in X'$ and ν satisfy the following uniqueness condition:

$$0 < \sigma = \frac{N}{\nu^2}\|f\|_{-1} < 1, \tag{2.9}$$

where

$$\|f\|_{-1} = \sup_{v \in X} \frac{(f,v)}{\|\nabla v\|_0}.$$

Then problem (2.1) admits a unique solution $u \in X$ and $p \in M$ such that

$$\nu\|\nabla u\|_0 \le \|f\|_{-1},\ \|p\|_0 \le 3\beta^{-1}\|f\|_{-1}. \tag{2.10}$$

Here the second inequality was deduced by (2.1), (2.4), (2.9)-(2.10) and the inf-sup condition([8, 35]):

$$\beta\|q\|_0 \le \inf_{v \in X} \frac{d(v,q)}{\|\nabla v\|_0},\ \forall q \in M. \tag{2.11}$$

We conclude this section by deriving regularity results depending on ν of the solution u and p .

Theorem 2.2. If $f \in Y$, Assumption **A0** and (2.9) hold, then the solution (u,p) of problem (2.1) satisfies the following regularity:

$$\nu\|\nabla u\|_0 + \|p\|_0 \le c\|f\|_{-1},\ \nu\|Au\|_0 + \|p\|_1 \le c\|f\|_0. \tag{2.12}$$

Proof. We deduce from problems (2.1), Assumption **A0** and (2.6)-(2.10) that

$$\begin{aligned}\nu\|Au\|_0 + \|p\|_1 &\le c\|f\|_0 + c\frac{N}{2}(\|u\|_{L^\infty}\|\nabla u\|_0 + \|\nabla u\|_{L^3}\|u\|_{L^6})\\ &\le c\|f\|_0 + cN\|\nabla u\|^{\frac{3}{2}}\|Au\|_0^{\frac{1}{2}} \le \frac{\nu}{2}\|Au\|_0 + c\|f\|_0 + c\nu^{-1}N^2\|\nabla u\|^3\\ &\le \frac{\nu}{2}\|Au\|_0 + c\|f\|_0 + c\frac{N^2\|f\|_{-1}^2}{\nu^4}\|f\|_{-1}\\ &\le \frac{\nu}{2}\|Au\|_0 + c\|f\|_0. \end{aligned} \tag{2.13}$$

Combining (2.13) with (2.10) yields (2.12). The proof ends.

In the final part of this section, we will give some estimates in the 2D case:

$$\|v\|_{L^4} \le c\|v\|_0^{1/2}\|\nabla v\|_0^{1/2}\ \forall v \in X,\ \|v\|_{L^\infty} \le c\|v\|_0^{1/2}\|Av\|_0^{1/2}\ \forall v \in D(A), \tag{2.14}$$

which is useful in the error estimates of the finite element solutions (u_μ, p_μ) for the 2D case.

3. Finite Element Galerkin Approximation

From now on, H is a real positive parameter tending to 0. Also, τ_H is a uniformly regular partition of Ω into triangles or tetrahedra with diameters bounded by H. Conforming velocity-pressure finite element space pair (X_H, M_H) is constructed based upon the partition τ_H. Next, the fine mesh partition τ_h can be thought of as generated from τ_H by a mesh refinement process, see e.g. [33], and therefore nested. Similarly, we can establish the conforming velocity-pressure finite element space pair (X_h, M_h) based on τ_h. It is not necessary for the algorithm, nor needed for the results of our convergence theorems to hold. However, we shall assume them nested since it will simplify our analysis substantially, i.e. $(X_H, M_H) \subset (X_h, M_h) \subset (X, M)$. Further, we assume (X_μ, M_μ), $\mu = h$, and H satisfy the usual approximation properties(see [8]):

Assumption A1: There exists a mapping $r_\mu \in \mathcal{L}(D(A); X_\mu)$ such that

$$\|\nabla(r_\mu v - v)\|_0 \le c\mu\|Av\|_0, \forall v \in D(A);$$

The orthogonal projection operator $\rho_\mu : M \to M_\mu$ satisfies:

$$\|\rho_\mu p\|_0 \le \|p\|_0,\ \|q-\rho_\mu q\|_0 \le c\mu\|q\|_1,\ \forall q \in H^1(\Omega)\cap M;$$

Assumption A2: There exists a constant $\beta_1 > 0$ such that

$$\sup_{v_\mu \in X_\mu} \frac{d(v_\mu, q_\mu)}{\|\nabla v_\mu\|_0} \ge \beta_1\|q_\mu\|_0;$$

Assumption A3: The following inverse inequality holds

$$\|\nabla v_\mu\|_0 \le c\mu^{-1}\|v_\mu\|_0,\ \forall v_\mu \in X_\mu.$$

We give an example of the spaces X_μ and M_μ such that Assumptions **A1-A3** are satisfied. For more examples, refer to [3, 4, 8, 35] and the "mini-element" of Arnold, Brezzi and Fortin [2].

We define the discrete analogue of the space V as

$$V_\mu = \{v_\mu \in X_\mu; d(v_\mu, q_\mu) = 0,\ \forall q_\mu \in M_\mu\}.$$

Next, we define the L^2-orthogonal projector $P_\mu : L^2(\Omega)^3 \to V_\mu$ by

$$(P_\mu u, v_\mu) = (u, v_\mu),\ \forall v_\mu \in V_\mu.$$

Also, we can define the discrete Stokes operator $A_\mu = -P_\mu\Delta_\mu$ through the condition

$$(-\Delta_\mu u_\mu, v_\mu) = (\nabla u_\mu, \nabla v_\mu)\ \forall u_\mu,\ v_\mu \in X_\mu.$$

The finite element Galerkin approximation of (2.1) based on (X_μ, M_μ) reads :

Find $(u_\mu, p_\mu) \in (X_\mu, M_\mu)$ such that for all $(v, q) \in (X_\mu, M_\mu)$

$$a(u_\mu, v) + d(u_\mu, q) - d(v, p_\mu) + b(u_\mu, u_\mu, v) = (f, v). \tag{3.1}$$

A similar argument to that used in [8] yields the following existence and uniqueness results.

Theorem 3.1. Suppose that Assumptions **A0-A3** and the uniqueness condition (2.9) are valid. Then, the finite element Galerkin approximation problem (3.1) possesses a unique solution $(u_\mu, p_\mu) \in (X_\mu, M_\mu)$ which satisfies

$$\nu\|\nabla u_\mu\|_0 \le \|f\|_{-1}. \tag{3.2}$$

In order to derive error estimates of the finite element solution (u_μ, p_μ), we also define the Galerkin projection $(R_\mu, Q_\mu) = (R_\mu(u,p), Q_\mu(u,p)) : (X, M) \to (X_\mu, M_\mu)$ by requiring

$$a(R_\mu - u, v_\mu) - d(v_\mu, Q_\mu - p) + d(R_\mu - u, q_\mu) = 0,\ \forall (u,p) \in (X, M),\ (v_\mu, q_\mu) \in (X_\mu, M_\mu). \tag{3.3}$$

Note that, due to Assumption **A3**, (R_μ, Q_μ) is well defined. Now, we will recall the following approximate properties in [20].

Lemma 3.2. The Galerkin projection $(R_\mu, Q_\mu) = (R_\mu(u,p), Q_\mu(u,p))$ satisfies

$$\nu\|R_\mu(u,p) - u\|_0 + \mu(\nu\|\nabla(R_\mu(u,p) - u)\|_0 + \|Q_\mu(u,p) - p\|_0) \le c\mu(\nu\|\nabla u\|_0 + \|p\|_0), \tag{3.4}$$

for all $(u, p) \in (X, M)$ and

$$\nu\|R_\mu(u,p) - u\|_0 + \mu(\nu\|\nabla(R_\mu(u,p) - u)\|_0 + \|Q_\mu(u,p) - p\|_0) \le c\mu^2(\nu\|Au\|_0 + \|p\|_1), \tag{3.5}$$

for all $(u, p) \in (D(A),\ H^1(\Omega)\cap M)$.

From Theorem 3.1 and Lemma 3.2, the stability and convergence of the finite element solution (u_μ, p_μ) can be obtained, see [20].

Theorem 3.3. Suppose that Assumptions **A0-A3** and the uniqueness condition (2.9) are valid. Then, the finite element solution $(u_\mu, p_\mu) \in (X_\mu, M_\mu)$ satisfies the following stability and error estimates:

$$\nu\|\nabla u_\mu\|_0 \le \|f\|_{-1},\ \nu\|A_\mu u_\mu\|_0 \le c\|f\|_0. \tag{3.14}$$

Moreover, we assume that $\mu \le (\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}\delta$ holds. Then, the error $(u - u_\mu, p - p_\mu)$ satisfies the following uniform bound:

$$\delta\nu\|u - u_\mu\|_0 + \mu(\nu\|\nabla(u - u_\mu)\|_0 + \|p - p_\mu\|_0) \le c_1\mu^2\|f\|_0, \tag{3.15}$$

for some positive constant c_1.

4. Two-level method based on the Stokes correction

For slightly large δ, we shall recall the two-level method(Method I) based on the Stokes correction [18, 20, 25] and study the uniform stability and convergence of the finite element solution (u^h, p^h) based on Method I. Method I can be divided into the following two steps:

Step 1. Find a global coarse grid solution $(u_H, p_H) \in (X_H, M_H)$ defined by

$$a(u_H, v) + b(u_H, u_H, v) - d(v, p_H) + d(u_H, q) = (f, v), \tag{4.1}$$

for all $(v, q) \in (X_H, M_H)$.

Step 2a. Find a fine grid solution $(u^h, p^h) \in (X_h, M_h)$ defined by the following Stokes problem:

$$a(u^h, v) - d(v, p^h) + d(u^h, q) + b(u_H, u_H, v) = (f, v)\ \forall (v, q) \in (X_h, M_h). \tag{4.2}$$

Lemma 4.1. Under the assumptions of Theorem 3.3, then

$$\nu\|\nabla u^h\|_0 \le 2\|f\|_{-1},\ \nu\|A_h u^h\|_0 \le c\|f\|_0, \tag{4.3}$$

$$\nu\|\nabla(u^h - u_h)\|_0 + \|p^h - p_h\|_0 \le c\delta^{-1}H^2(\frac{\|f\|_0}{\|f\|_{-1}})^{\frac{1}{2}}\|f\|_0. \tag{4.4}$$

Proof. From (2.4), (2.6)-(2.9) and Lemma 3.2, we obtain

$$\begin{aligned}
\nu\|\nabla u^h\|_0 &\le \|f\|_{-1} + N\|\nabla u_H\|_0^2 \\
&\le \|f\|_{-1} + \frac{N}{\nu^2}\|f\|_{-1}^2 \le 2\|f\|_{-1}, \\
\nu\|A_h u^h\|_0 &\le \|f\|_0 + cN(\|\nabla u_H\|_0\|u_H\|_{L^\infty} + \|\nabla u_H\|_{L^3}\|u_H\|_{L^6}) \\
&\le \|f\|_0 + cN\|\nabla u_H\|_0\|A_H u_H\|_0 \le \|f\|_0 + c\frac{N}{\nu^2}\|f\|_{-1}\|f\|_0 \\
&\le c\|f\|_0,
\end{aligned}$$

which is (4.3).

Next, setting $(e, \eta) = (u_h - u^h, p_h - p^h)$, we derive from (3.1) and (5.2) that

(4.5)
$$a(e, v) - d(v, \eta) + d(e, q) + b(u_h - u_H, u_h, v) + b(u_H, u_h - u_H, v) = 0\ \forall (v, q) \in (X_h, M_h),$$

Setting $(v,q)=(e,\eta)$ in (4.5), using (2.3)-(2.7) and (3.6), we obtain

$$\begin{aligned}\nu\|\nabla e\|_0 &\le cN\|u_h-u_H\|_0(\|\nabla u_h\|_0^{\frac12}\|A_hu_h\|_0^{\frac12}+\|\nabla u_H\|_0^{\frac12}\|A_Hu_H\|_0^{\frac12})\\ &\le c\frac{N}{\nu^2}\nu\|u_h-u_H\|_0\|f\|_{-1}^{\frac12}\|f\|_0^{\frac12}\\ &\le c\delta^{-1}H^2(\frac{\|f\|_0}{\|f\|_{-1}})^{\frac12}\|f\|_0. \end{aligned}\tag{4.6}$$

Moreover, it follows from Assumption **A3** and (4.5)-(4.6) that

$$\begin{aligned}\|\eta\|_0 &\le c\nu\|\nabla e\|_0+cN\|u_h-u_H\|_0(\|\nabla u_h\|_0^{\frac12}\|A_hu_h\|_0^{\frac12}+\|\nabla u_H\|_0^{\frac12}\|A_Hu_H\|_0^{\frac12})\\ &\le c\delta^{-1}H^2(\frac{\|f\|_0}{\|f\|_{-1}})^{\frac12}\|f\|_0.\end{aligned}\tag{4.7}$$

Hence, by combining (4.6) with (4.7), we have completed the proof of (4.4). The proof ends.

From Lemma 4.1 and Theorem 3.3, we obtain the following error estimate result.

Theorem 4.2. Under the condition of Theorem 3.3, if δ is sufficiently large such that $\delta\ge(\frac{\|f\|_{-1}}{\|f\|_0})^{\frac12}$, (u^h,p^h) provided by Method I satisfies the following error estimates:

$$\nu\|\nabla(u-u^h)\|_0+\|p-p^h\|_0\le c(h+H^2\frac{\|f\|_0}{\|f\|_{-1}})\|f\|_0. \tag{4.8}$$

5. Two-level method based on the Oseen correction

For slightly large δ, we shall recall the two-level method(Method II) based on the Oseen correction [18, 20, 25] and study the uniform stability and convergence of the finite element solution (u^h,p^h) based on Method II. Method II can be divided into Step 1 and the following two step:

Step 2b. Find a fine grid solution $(u^h,p^h)\in(X_h,M_h)$ defined by the following Oseen problem:

$$a(u^h,v)-d(v,p^h)+d(u^h,q)+b(u_H,u^h,v)=(f,v)\ \forall(v,q)\in(X_h,M_h). \tag{5.1}$$

Lemma 5.1. Under the assumptions of Theorem 3.3, then

$$\nu\|\nabla u^h\|_0\le\|f\|_{-1},\ \nu\|A_hu^h\|_0\le c\|f\|_0, \tag{5.2}$$

$$\nu\|\nabla(u^h-u_h)\|_0+\|p^h-p_h\|_0\le c\delta^{-1}H^2(\frac{\|f\|_0}{\|f\|_{-1}})^{\frac12}\|f\|_0. \tag{5.3}$$

Proof. From (2.4), (2.6)-(2.9), (5.1) and Lemma 3.2, we obtain

$$\begin{aligned}\nu\|\nabla u^h\|_0 &\le\|f\|_{-1},\\ \nu\|A_hu^h\|_0 &\le\|f\|_0+\frac12N(\|u_H\|_{L^\infty}\|\nabla u^h\|_0+\|\nabla u_H\|_{L^3}\|u^h\|_{L^6})\\ &\le\|f\|_0+cN\|\nabla u_H\|_0^{\frac12}\|A_Hu_H\|_0^{\frac12}\|\nabla u^h\|_0\le\|f\|_0+c\frac{N}{\nu^2}\|f\|_{-1}\|f\|_0\\ &\le c\|f\|_0,\end{aligned}$$

which is (5.2).

Next, setting $(e,\eta)=(u_h-u^h,p_h-p^h)$, we derive from (3.1) and (5.1) that

$$a(e,v)-d(v,\eta)+d(e,q)+b(u_h-u_H,u_h,v)+b(u_H,e,v)=0\ \forall(v,q)\in(X_h,M_h), \tag{5.4}$$

Setting $(v,q)=(e,\eta)$ in (5.4), using (2.3)-(2.7) and (3.6), we obtain

$$
\begin{aligned}
\nu\|\nabla e\|_0 &\le cN\|u_h-u_H\|_0\|\nabla u_h\|_0^{\frac{1}{2}}\|A_hu_h\|_0^{\frac{1}{2}}\\
&\le c\frac{N}{\nu^2}\nu\|u_h-u_H\|_0\|f\|_{-1}^{\frac{1}{2}}\|f\|_0^{\frac{1}{2}}\\
&\le c\delta^{-1}H^2(\frac{\|f\|_0}{\|f\|_{-1}})^{\frac{1}{2}}\|f\|_0. \qquad (5.5)
\end{aligned}
$$

Moreover, it follows from Assumption **A3** and (5.4)-(5.5) that

$$
\begin{aligned}
\|\eta\|_0 &\le c\nu\|\nabla e\|_0+cN\|u_h-u_H\|_0\|\nabla u_h\|_0^{\frac{1}{2}}\|A_hu_h\|_0^{\frac{1}{2}}+cN\|\nabla u_H\|_0\|\nabla e\|_0\\
&\le c\delta^{-1}H^2(\frac{\|f\|_0}{\|f\|_{-1}})^{\frac{1}{2}}\|f\|_0. \qquad (5.6)
\end{aligned}
$$

Hence, by combining (5.5) with (5.6), we have completed the proof of (5.3). The proof ends.

From Lemma 5.1 and Theorem 3.3, we obtain the following error estimate result.

Theorem 5.2. Under the condition of Theorem 4.2, then (u^h,p^h) provided by Method II satisfies the following error estimates:

$$
\nu\|\nabla(u-u^h)\|_0+\|p-p^h\|_0\le c(h+H^2\frac{\|f\|_0}{\|f\|_{-1}})\|f\|_0. \qquad (5.7)
$$

Remark 5.1. We find from Lemmas 4.1 and 5.1 that for small δ with $\delta<(\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$, H should be of order $O(\delta^2)$ and h should be of order $O(H^{\frac{3}{2}})$ or $O(\delta^3)$ for Method I and Method II. These facts show that for small δ, h and H should be very small and Methods I and II are not suitable to the 2D/3D steady Navier-Stokes equations. For large δ satisfying $\delta\ge(\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$, we conclude from Theorems 3.3, 4.2 and 5.2, that Method I and Method II have the almost same uniform stability and convergence as the one-level finite element method if $h=O(H^2\frac{\|f\|_0}{\|f\|_{-1}})$ is chosen. However, Method I and Method II are simpler than one-level finite element method.

6. Two-level method based on the Newton correction

In this section, we consider the uniform stability and convergence of the two-level method(Method III) based on the Newton correction [20, 23]. Method III can be described as Step 1 and the following step:

Step 2c. Find a fine grid solution $(u^h,p^h)\in(X_h,M_h)$ defined by the following problem:

(6.1)

$$
a(u^h,v)-d(v,p^h)+d(u^h,q)+b(u^h,u_H,v)+b(u_H,u^h,v)=(f,v)+b(u_H,u_H,v).
$$

for all $(v,q)\in(X_h,M_h)$.

Lemma 6.1. Under the condition of Theorem 4.2, (u^h,p^h) provided by Method III satisfies the following stability and error estimates:

(6.2)

$$
\nu\|\nabla u^h\|_0\le\|f\|_{-1}+N\|\nabla(u^h-u_H)\|_0^2,\ \nu\|A_hu^h\|_0\le c\|f\|_0+cN\|\nabla(u^h-u_H)\|_0^2,
$$

$$
\nu\|\nabla(u_h-u^h)\|_0+\|p_h-p^h\|_0\le c\delta^{-\frac{3}{2}}H^{\frac{5}{2}}\frac{\|f\|_0}{\|f\|_{-1}}\|f\|_0, \qquad (6.3)
$$

in the 3D case, and

$$\nu\|\nabla(u_h-u^h)\|_0+\|p_h-p^h\|_0\le c|\log h|(\delta^{-2}H^3\frac{\|f\|_0}{\|f\|_{-1}}\|f\|_0, \tag{6.4}$$

in the 2D case.

Proof. From (2.4), (2.6)-(2.9), (3.17), (6.1) and Theorem 3.3, we obtain

$$\nu\|\nabla u^h\|_0\le\|f\|_{-1}+N\|\nabla(u^h-u_H)\|_0^2,$$

$$\begin{aligned}\nu\|A_hu^h\|_0&\le\|f\|_0+cN\|\nabla u^h\|_0^{\frac12}\|A_hu^h\|_0^{\frac12}\|\nabla u_H\|_0+cN\|\nabla u_H\|_0^{\frac32}\|A_Hu_H\|_0^{\frac12}\\&\le\|f\|_0+\frac{\nu}{2}\|A_hu^h\|_0+\frac{\nu}{2}\|A_Hu_H\|_0+c\nu^{-1}N^2\|\nabla u^h\|_0\|\nabla u_H\|_0^2+c\nu^{-1}N^2\|\nabla u_H\|_0^3\\&\le c\|f\|_0+\frac{\nu}{2}\|A_hu^h\|_0+c\nu\|\nabla u^h\|_0,\end{aligned}$$

which imply (6.2).

Next, setting $(e,\eta)=(u_h-u^h,p_h-p^h)$, we derive from (3.1) and (6.1) that

$$\begin{aligned}&a(e,v)-d(v,\eta)+d(e,q)+b(e,u_h,v)+b(u^h,e,v)\\&\quad+b(u^h-u_H,u^h-u_H,v)=0\ \forall(v,q)\in(X_h,M_h),\end{aligned}\tag{6.5}$$

Setting $(v,q)=(e,\eta)$ in (6.5) and using (2.3)-(2.7) and (3.6), we obtain

$$\begin{aligned}\nu\delta\|\nabla e\|_0^2&\le a(e,e)+b(e,u_H,e)\\&\le|b(u_h-u_H,u_h-u_H,e)|\\&\le cN\|\nabla e\|_0\|\nabla(u_h-u_H)\|_0^{\frac32}\|u_h-u_H\|_0^{\frac12}.\end{aligned}\tag{6.6}$$

Combining (6.6) with Theorem 3.3 and Theorem 4.3 and using (2.9), we obtain

$$\begin{aligned}\nu\delta\|\nabla e^m\|_0&\le cN\|\nabla(u_h-u_H)\|_0^{\frac32}\|u_h-u_H\|_0^{\frac12}\\&\le c\frac{N}{\nu^2}\delta^{-\frac12}H^{\frac52}\|f\|_0^2\\&\le c\delta^{-\frac12}H^{\frac52}\|f\|_0^2\|f\|_{-1}.\end{aligned}\tag{6.7}$$

Moreover, it follows from Assumption **A3**, (2.9), (6.5) and (6.7) that

$$\begin{aligned}\|\eta\|_0&\le c\nu\|\nabla e\|_0+cN\|\nabla e\|_0(\|\nabla u_h\|_0+\|\nabla(u_h-u_H)\|_0)\\&\quad+cN\|\nabla(u_h-u_H)\|_0^{\frac32}\|u_h-u_H\|_0^{\frac12}\\&\le cN\|\nabla(u_h-u_H)\|_0^{\frac32}\|u_h-u_H\|_0^{\frac12}+c\nu\|\nabla e\|_0\\&\le c\delta^{-\frac32}H^{\frac52}\frac{\|f\|_0}{\|f\|_{-1}}\|f\|_0.\end{aligned}\tag{6.8}$$

Hence, by combining (6.7) with (6.8), we have completed the proof of (6.3). Moreover, in the 2D case, we can use the estimate:

$$\|v_h\|_{L^\infty}\le c|\log h|\|\nabla v_h\|_0,\ \|u_h-u_H\|_{L^4}\le c\|u_h-u_H\|_0^{\frac12}\|\nabla(u_h-u_H\|_0^{\frac12},$$

in the estimates of the trilinear terms in (6.7)-(6.8). Thus, we can deduce (6.4). The proof ends.

From Lemma 6.1 and Theorem 3.3, we obtain the following error estimate result.

Theorem 6.2. Under the condition of Theorem 4.2, then (u^h,p^h) provided by Method III satisfies the following error estimate:

$$\nu\|\nabla(u-u^h)\|_0+\|p-p^h\|_0\le c(h+H^{\frac52}(\frac{\|f\|_0}{\|f\|_{-1}})^2)\|f\|_0, \tag{6.9}$$

in the 3D case and

$$\nu\|\nabla(u-u^h)\|_0+\|p-p^h\|_0\le c(h+H^3(\frac{\|f\|_0}{\|f\|_{-1}})^2)\|f\|_0, \tag{6.10}$$

in the 2D case.

Remark 6.1. We find from Lemma 6.1 that for small δ with $\delta<(\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$, H should be of order $O(\delta^{\frac{3}{2}})$ and h should be of $O(H^{\frac{3}{2}})$ or $O(\delta^{\frac{9}{4}})$. These facts show that h and H should be very small for small δ. and Method III is not suitable to the 2D/3D steady Navier-Stokes equations. For large δ satisfying $\delta\ge(\frac{\|f\|_{-1}}{\|f\|_0})^{\frac{1}{2}}$, we conclude from Theorems 3.3 and 6.2 that Method III has the almost same uniform stability and convergence as the one-level finite element method if $h=O(H^{\frac{5}{2}}(\frac{\|f\|_0}{\|f\|_{-1}})^2)$ for the 3D case and if $h=O(H^3(\frac{\|f\|_0}{\|f\|_{-1}})^2)$ for the 2D case. Also, we find from Theorems 3.3, 4.2, 5.2 and 6.2 that Methods I, II and III are better than one-level finite element method and Method III is better than Methods I, II and one-level finite element method.

7. Numerical Analysis

In this section, we concentrate on the performance of the one-level finite element method and three two-level finite element methods described in this article.

For the purpose of numerical comparisons, we consider the spatial domain in R^2 as $(0,1)\times(0,1)$. The finite element subspace (X_h, M_h) of (X, M) is characterized by a uniformly triangulation τ_h with the mini-element P_1b-P_1 for the stationary Navier-Stokes equations. We give the following exact solution

$$p(x)=10(2x_1-1)(2x_2-1),$$

$$u_1(x)=10x_1^2(x_1-1)^2x_2(x_2-1)(2x_2-1), u_2(x)=-10x_1(x_1-1)(2x_1-1)x_2^2(x_2-1)^2.$$

Note that the right hand $f(x)=(f_1(x),f_2(x))$ is determined by the stationary Navier-Stokes equations (1.1).

Firstly, we compare the accuracy of the one-level method and the two-level methods. Observed from Theorems 3.3, 4.2, 5.2 and 6.2, optimal error estimates are obtained for these methods. Accuracy is measured by comparing the numerical solution to a discretized version of the exact solution in H^1-norm for the velocity and L^2-norm for the pressure. In presenting these computations, we fix the fine grid and then choose the coarse grid. From the error analysis for the two-level methods, we can choose the fine mesh as fine as $h\sim O(H^2)$, $h\sim O(H^2)$, and $h\sim O(H^3)$, respectively. All definitions require a choice of the fixed tolerance as 1.0e-6. In Tables 1-4, the corresponding results are reported for these methods with different mesh scales. As shown by the tables, the one-level method is about first order accurate in both H^1-norm and L^2-norm for the velocity and pressure. Also, three two-level methods almost have the same convergence rate as the one-level method. The most important thing is to show that three two-level method is more efficient than the one-level method through comparing the computational time. Furthermore, Methods I, II and III are respectively the best, the second, and the third choose from the point of view of fast computation without lost any accuracy.

Table 1: The one-level Method:($\nu = 1$).

$1/h$	$CPU(s)$	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	u_{H^1}	p_{L^2}
16	0.969	0.168279	0.00653095		
25	2.359	0.106371	0.00313554	1.0278	1.6441
36	4.468	0.0734393	0.00176451	1.0160	1.5767
49	8.719	0.053785	0.00111395	1.0103	1.4919
64	15.75	0.0411008	0.00077312	1.0071	1.3676

Table 2: The Method I:($\nu = 1$).

$1/h$	$1/H$	$CPU(s)$	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	u_{H^1}	p_{L^2}
16	4	0.39	0.168282	0.00652517		
25	5	0.829	0.106372	0.00312588	1.0278	1.6491
36	6	1.64	0.0734397	0.00175089	1.0160	1.5895
49	7	2.875	0.0537852	0.00109662	1.0103	1.5176
64	8	4.75	0.0411008	0.00075281	1.0071	1.4086

Table 3: The Method II:($\nu = 1$).

$1/h$	$1/H$	$CPU(s)$	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	u_{H^1}	p_{L^2}
16	4	0.406	0.16828	0.00652713		
25	5	1.187	0.106371	0.00312964	1.0278	1.6470
36	6	1.75	0.0734394	0.00175644	1.0160	1.5841
49	7	3.391	0.053785	0.00110411	1.0103	1.5058
64	8	6.563	0.0411007	0.000761899	1.0071	1.3891

Table 4: The Method III:($\nu = 1$).

$1/h$	$1/H$	$CPU(s)$	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	u_{H^1}	p_{L^2}
16	3	1.016	0.16828	0.00652941		
25	3	1.578	0.106371	0.00313133	1.0278	1.6466
36	3	3.282	0.0734401	0.00175618	1.0160	1.5860
49	4	5.281	0.0537854	0.00110549	1.0103	1.5013
64	4	10.11	0.0411013	0.000760592	1.0071	1.4002

On the other hand, to establish a reference point for the viscosity of the possible impact from these methods, we provide the results of the one-level method with different small viscosity $\nu = 0.0001,\ 0.00005,\ 0.00001$ for the given mesh $1/h = 36$. Observed from Fig.1, there are no obvious negative impact on the results along with the different viscosity. Moreover, we compare three two-level finite element methods by choosing an appropriate choice of mesh widths and viscosity based on some stability and convergence analysis with respect to δ. Note that all nonlinear problems based on two-level methods are solved by computing Oseen iterations on coarse mesh H until the norm of the difference in successive iterates is within a fixed tolerance[19]. Then the linear problems are solved by computing one step correction based on the Stokes, Oseen and Newton schemes. Especially, the black pictures in Fig 3 and 4 mean that the Method I can not work in case $\nu = 0.01$ and $\nu = 0.001$, and Methods II can not work in case $\nu = 0.001$. From Figures 2-4, the results are shown that Method III can solve the Navier-Stokes equations for the relative small viscosity among three two-level methods. Moreover, Method II can only solve the stationary Navier-Stokes equations accurately with large viscosity.

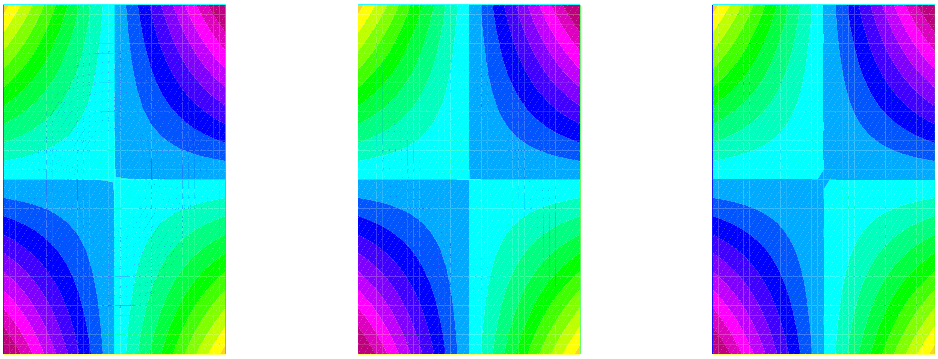

Fig 1. The one-level method with $\nu = 0.0001,\ 0.00005,\ 0.00001$ and $h = 1/36$.

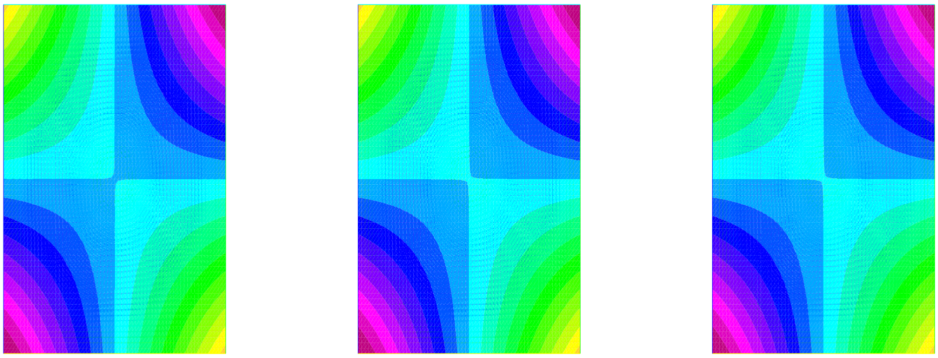

Fig 2. Methods I-III with $\nu = 1$ and $h = 1/64$.

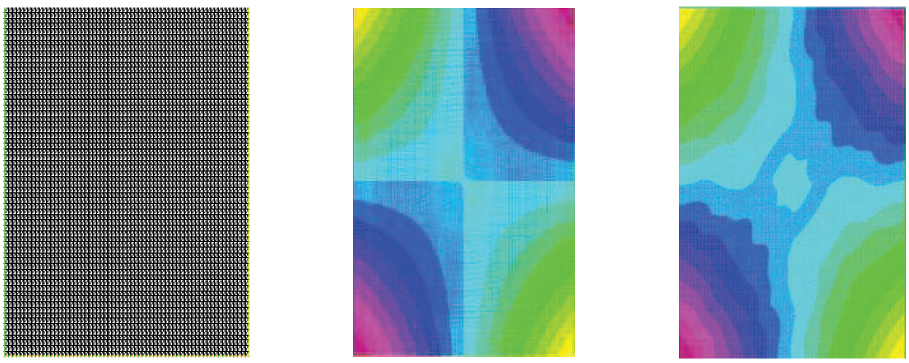

Fig 3. Methods I-III with $\nu = 0.01$ and $h = 1/64$.

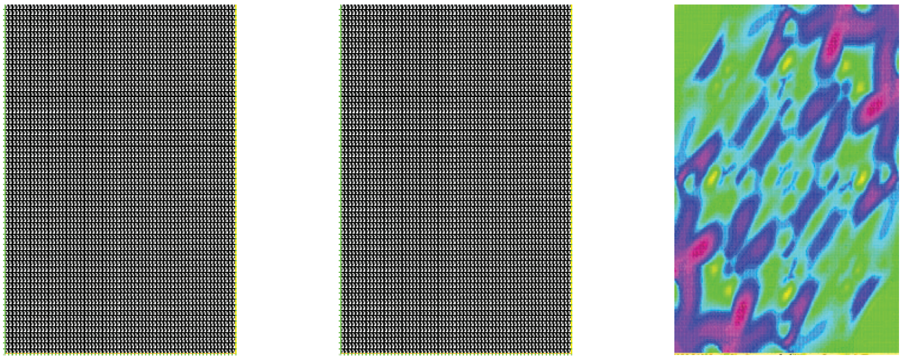

Fig 4. Methods I-III with $\nu = 0.001$ and $h = 1/64$.

8. Conclusion

In this paper, we have analyzed the uniform stability and convergence analysis with respect to δ of the one-level finite element method and the two-level finite element methods. For the finite element approximation of the 2D/3D steady Navier-Stokes equations, it is better to use one-level finite element method with $h = O(\delta)$ in the case of the weak uniqueness condition and the two-level finite element methods in the case of the strong uniqueness condition, where for the Stokes and Oseen correction h should be of order $O(H^2)$ and for the Newton correction h should be of order $O(H^{\frac{5}{2}})$ in the 3D case and order $O(H^3)$ in the 2D case. In particular, the two-level method based on the Newton correction on fine grid is of the better convergence rate with respect to H than the two-level methods based on the Stokes and Oseen corrections on fine grid and more suitable to solve the steady 2D/3D Navier-Stokes equations for larger δ (or large ν).

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