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THE OBSTACLE PROBLEM FOR SHALLOW SHELLS: CURVILINEAR APPROACH

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Abstract. We start with a three-dimensional equilibrium problem involving a linearly elastic solid at small strains subjected to unilateral contact conditions. The reference configuration of the solid is assumed to be a thin shallow shell with a uniform thickness. We focus on the limit when the thickness tends to zero, i.e. when the three-dimensional domain tends to a two-dimensional one. In the generic case, this means that the initial Signorini problem, where the contact conditons hold on the boundary, tends to an obstacle problem, where the contact conditions hold in the domain. When the problem is stated in terms of curvilinear coordinates, the unilateral contact conditions involve a non penetrability inequality which couples the three covariant components of the displacement. We show that nevertheless we can uncouple these components and the contact conditions involve only the transverse covariant component of the displacement at the limit.

Key words. asymptotic analysis, differential geometry, obstacle problem, shallow shells, Signorini conditions.

1. Introduction

The aim of this study was to develop an asymptotic model for a shallow elastic shell which can come into contact with an obstacle. Let us first comment on the so-called obstacle problem. When dealing with the mechanics of a single particle which moves in the presence of some wall, one simply has to ensure that the position of the particle stays on the same side of the wall. But in the case of the mechanics of continuous media, there are two main kinds of problems. First there is the case of a three-dimensional body resting on some support. In this case the contact between the body and the support obviously occurs on a part of the boundary of the body. The corresponding contact conditions have been formulated mathematically by A. Signorini [19] and the equilibrium problem for the body is now classically referred to as a *Signorini problem.* But there exists another case, which seems to be specific to the mechanics of structures, and which can be illustrated by the following example: assuming that a flat membrane clamped at the boundary is pushed up to a wall, then the contact between the membrane and the wall will occur in the membrane, i.e. no longer at the boundary but strictly inside the domain. The corresponding equilibrium problem is the so-called *obstacle problem.*

The present study deals with the justification of the obstacle problem in the case of a shallow shell. The contact conditions will be closely described throughout this study, but we first observe that, due to the existence of a constraint imposed on the position, since the shell cannot cross the obstacle, the contact conditions induce a strong nonlinearity. From the mathematical point of view, this nonlinearity results in a set of constraints which involve inequalities in the displacements and the stresses, and the functional framework is therefore no longer a vector space.

The equilibrium problem has been studied in the case of a plate in [17], where a friction model was added to the description of contact. In the case of a shallow shell involving contact without friction, an asymptotic model has been given and

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justified in the Cartesian framework in [11].
Here we continue to work on these lines by studying this mechanical problem in a system of curvilinear coordinates. Giving and justifying an asymptotic model for a shallow shell in a system of curvilinear coordinates is an interesting result in-itself. In addition the differential geometry framework seems to be more suitable for dealing with the case of general shells. As a matter of fact the Cartesian framework would involve in general several changes of chart, which would make the asymptotic procedure rather complex. It is worth noting that both in the Cartesian and the curvilinear framework, the contact occurs on the boundary in the case of any three-dimensional domain, but the contact will generically occur in the domain in the case of a two-dimensional structural model. In other words the aim here was to prove that a three-dimensional Signorini problem in a domain having a small thickness, namely 2ε in the following, tends to a two-dimensional obstacle problem as ε tends to zero.
This paper is organized as follows.
In section 2 we study a Signorini problem arising in the case of any three-dimensional linearly elastic solid. This is done first in a Cartesian framework and then in a system of curvilinear coordinates. The basic concepts involved are then introduced, in particular the contact conditions, and an existence and uniqueness result for the Signorini problem is recalled, of which the main steps in the proof are outlined.
Section 3 still deals with a three-dimensional domain, which is now the reference configuration of a shallow shell. The contact conditions can be formulated more precisely than in section 2, since the three-dimensional body is now a thin shell. Special attention is paid here to the maps used to build the middle surface and the reference configuration of the shell. The main qualitative situation which results from using curvilinear coordinates is that the contact conditions involve a coupling between all the covariant components of the displacement, whereas only the component normal to the obstacle was involved in the Cartesian case.
In section 4 the three-dimensional domain with a thickness 2ε is changed into a domain with a thickness 2 using a rescaling procedure, and all the data, the unknowns, and the functional framework are rescaled. This makes it possible to perform an asymptotic analysis, which is done in section 5, and to give the limit problem in section 6, where we also give the strong formulation and return to the physical domain in order to have a proper interpretation of the result. The main steps in these sections are classical steps used in most asymptotic analyses; the main difference with previous studies is that the nonlinearity due to the unilateral contact is taken into account here.
The main technical points of interest are given in the Appendix.

2. Formulation of the contact problem in a system of curvilinear coordinates

2.1. The three-dimensional problem in Cartesian coordinates. We first recall the classical contact problem of continuum mechanics. Let $\widehat{\Omega}$ be a domain in $\mathbb{R}^3$, with a system of Cartesian coordinates $\widehat{x} = (\widehat{x}_i)$[1] the closure of which gives the reference configuration of a three-dimensional solid made of an elastic material. When submitted to body forces $\widehat{\boldsymbol{f}} = (\widehat{f}^i) : \widehat{\Omega} \longrightarrow \mathbb{R}^3$, this solid undergoes an elastic displacement field $\widehat{\boldsymbol{u}} = (\widehat{u}_i) : \overline{\widehat{\Omega}} \longrightarrow \mathbb{R}^3$ which solves the set of equilibrium

[1]Latin exponents and indices take their values in the set $\{1;\ 2;\ 3\}$, Greek exponents and indices (except ε) take their values in the set $\{1;\ 2\}$, Einstein's convention for repeated exponents and indices is used and bold letters stand for vectors or vector spaces.

equations[2]

$$\widehat{\partial}_i \widehat{\sigma}^{ij} + \widehat{f}^j = 0 \quad \text{in } \widehat{\Omega}, \tag{1}$$

where $\widehat{\boldsymbol{\sigma}} = (\widehat{\sigma}^{ij})$ is the Cauchy stress tensor. In the context of linear elasticity, the constitutive law, i.e. the relation between the stress tensor $\widehat{\boldsymbol{\sigma}}$ and the strain tensor $\widehat{\boldsymbol{e}} = (\widehat{e}_{ij})$, is of the form

$$\widehat{\sigma}^{kl} = \widehat{g}^{ijkl}\ \widehat{e}_{ij}. \tag{2}$$

Due to basic foundations of continuum mechanics, the elasticity tensor $\widehat{g}^{ijkl}$ must satisfy the following symmetry and ellipticity conditions:

$$\widehat{g}^{ijkl} = \widehat{g}^{jikl} = \widehat{g}^{klij} = \widehat{g}^{ijlk}$$

$$\exists\ c > 0 \text{ such that } \widehat{\mathrm{g}}^{\mathrm{ijkl}}\mathrm{X_{ij}X_{kl}} \ \geq\ \mathrm{cX_{ij}X_{ij}}, \quad \forall\ \mathrm{X_{ij}} = \mathrm{X_{ji}} \in \mathbb{R}.$$

We will focus here on the case where the strains satisfy the assumption that we are dealing with small perturbations, which means that the strain tensor reduces to its linear part:

$$\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) = \frac{1}{2}(\widehat{\partial}_i \widehat{v}_j + \widehat{\partial}_j \widehat{v}_i). \tag{3}$$

In addition, for present purposes, it is not restrictive to also assume that the solid is made of a homogeneous isotropic material, so that the fourth order tensor $\widehat{g}^{ijkl}$ is completely given by two Lamé coefficients λ and μ and reads $\widehat{g}^{ijkl} = \lambda\delta^i_j\delta^k_l + \mu(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k)$, where the δ^i_j are the Kronecker symbols.

The equilibrium equations have to be completed by adding boundary conditions. Let us assume that the boundary $\widehat{\mathcal{B}} \equiv \partial\widehat{\Omega}$ is partitioned into $\widehat{\mathcal{B}} = \widehat{\mathcal{B}}_0 \cup \widehat{\mathcal{B}}_+ \cup \widehat{\mathcal{B}}_-$, where $\widehat{\mathcal{B}}_0 \cap \widehat{\mathcal{B}}_+ = \emptyset$, $\widehat{\mathcal{B}}_0 \cap \widehat{\mathcal{B}}_- = \emptyset$, $\widehat{\mathcal{B}}_+ \cap \widehat{\mathcal{B}}_- = \emptyset$ and $\text{area}(\widehat{\mathcal{B}}_0) \neq \emptyset$, corresponding to three different boundary conditions, namely:

- A condition on the displacements: $\widehat{\boldsymbol{u}} = \boldsymbol{0}$ on $\widehat{\mathcal{B}}_0$,
- A condition on the stresses: $\widehat{\boldsymbol{\sigma}}\widehat{\boldsymbol{n}} = \widehat{\boldsymbol{l}}$ on $\widehat{\mathcal{B}}_+$ where $\widehat{\boldsymbol{l}} = (\widehat{l}^i)$ are surface forces and $\widehat{\boldsymbol{n}}$ denotes the outer unit normal to any part of the boundary.
- Unilateral contact conditions on $\widehat{\mathcal{B}}_-$, which means that in addition to the Dirichlet condition on $\widehat{\mathcal{B}}_0$, the solid remains above a given support. It is not restrictive to describe this support as the horizontal plane at the level $\widehat{x}_3 = -h$.

Dirichlet and Neumann boundary conditions are classical, let us describe more closely the unilateral boundary conditions.

i) The first idea is the most intuitive one: if the solid rests on some support, then it cannot enter this support, which means that the displacements of all the points of the corresponding part of the boundary must be such that the solid remains on the same side of the support, which can be regarded as an obstacle.

ii) The second idea is that the solid is simply assumed to rest on the support. This means that it is not glued to the support. The meaning of this physical requirement is that no tensile forces are exerted on this part of the boundary and only compressive forces (e.g. those due to the weight of the solid) are admissible.

iii) The last idea is again based on physical considerations, which means that there are no distance interactions: let us consider a point of $\widehat{\mathcal{B}}_-$; the reaction of the support at this point is equal to zero as long as the point is not in contact with the support, and a non zero reaction is possible only when the point is in contact with the support.

In order to give explicit expressions for points *i), ii), iii)* it is necessary to make a

[2] $\widehat{\partial}_i = \dfrac{\widehat{\partial}}{\partial\widehat{x}_i}$ denotes the derivative with respect to the Cartesian coordinates $(\widehat{x}_i)$.

fundamental assumption, which simply gives an exact meaning to the above mentionned assumption of small perturbations: that the strains and the displacements are sufficiently small for no difference to exist between the normal to part $\widehat{\mathcal{B}}_-$ of the boundary and the normal to the support. In other words, a point of the boundary cannot come into contact after small strains and small displacements if the normal to the boundary at this point is strictly different from the normal to the support. This is closely related to the fact that small strains are usually defined as corresponding to the case where the normal at a point of the reference configuration is identical to the normal at the same material point in the deformed conguration. This assumption will be slightly weakened in the case of a shell, because the domain will be modelled along a given surface. We also assume for simplicity and without significative restrictions that $\widehat{\mathcal{B}}_-$ is the horizontal plane at the level 0. Then, based on these assumptions, the set of conditions *i), ii), iii)* reads:"

$$\left\{\begin{array}{l} \widehat{v}_3 + h \geq 0, \\ \\ \widehat{\sigma}^3 \equiv -\widehat{\sigma}^{3i} n_i \ \geq \ 0, \qquad \text{on} \ \ \widehat{\mathcal{B}}_- \ . \\ \\ \widehat{\sigma}^3\,(\widehat{v}_3 + h) = \ 0, \end{array}\right. \tag{4}$$

This set of conditions consisting of two inequalities and one equality, which must all be satisfied on part of the boundary, is referred to as the Signorini conditions [19].

Using the equilibrium equation (1), the constitutive law (2) with (3) and the boundary conditions on each part of the boundary, the variational form of the equilibrium problem of the solid can be classically written in the system of Cartesian coordinates. Due to the Signorini conditions (4) on $\widehat{\mathcal{B}}_-$, the latter is a variational inequality which reads

$$\left\{\begin{array}{l} \text{Find } \widehat{\boldsymbol{u}} \in \ \widehat{\boldsymbol{K}}(\widehat{\Omega}) \text{ such that for all } \widehat{\boldsymbol{v}} \in \widehat{\boldsymbol{K}}(\widehat{\Omega}) \\ \\ \displaystyle\int_{\widehat{\Omega}} \widehat{g}^{ijkl} \widehat{e}_{kl}(\widehat{\boldsymbol{u}}) \widehat{e}_{ij}(\widehat{\boldsymbol{v}} - \widehat{\boldsymbol{u}}) d\widehat{x} \ \geq \ \int_{\widehat{\Omega}} \widehat{\boldsymbol{f}} \cdot (\widehat{\boldsymbol{v}} - \widehat{\boldsymbol{u}}) d\widehat{x} + \int_{\widehat{\mathcal{B}}_+} \widehat{\boldsymbol{l}} \cdot (\widehat{\boldsymbol{v}} - \widehat{\boldsymbol{u}}) d\widehat{\mathcal{B}}. \end{array}\right. \tag{5}$$

where $d\widehat{\mathcal{B}}$ is the surface element. The set $\widehat{\boldsymbol{K}}(\widehat{\Omega})$ of admissible displacements is a convex cone given by:

$$\widehat{\boldsymbol{K}}(\widehat{\Omega}) = \left\{ \widehat{\boldsymbol{v}} = (\widehat{v}_i) \in \boldsymbol{H}^1(\widehat{\Omega}),\ \widehat{\boldsymbol{v}} = \boldsymbol{0} \text{ on } \widehat{\mathcal{B}}_0,\ \widehat{v}_3 \geq -h \text{ on } \widehat{\mathcal{B}}_- \right\}. \tag{6}$$

Problem (5) with (6) is known to have a single solution. The proof of this existence and uniqueness, which is given in many classical textbooks (e.g. [8]), was first presented in [9] and [20]. The main tool used for this purpose is Korn's inequalities which can be stated as follows:

Korn's inequalities *Let* $\widehat{\Omega} \subset \mathbb{R}^3$ *be a bounded, connected, open set with a Lipschitz boundary. The space*

$$\{\widehat{\boldsymbol{v}} \in \boldsymbol{L}^2(\widehat{\Omega}), \widehat{e}_{ij}(\widehat{\boldsymbol{v}}) \in \boldsymbol{L}^2(\widehat{\Omega})\}$$

coincides with the space $\boldsymbol{H}^1(\widehat{\Omega})$ *and the norm*

$$\left\{ \|\widehat{\boldsymbol{v}}\|^2_{\boldsymbol{L}^2(\widehat{\Omega})} + \sum_{i,j=1}^{3} \|\widehat{e}_{ij}(\widehat{\boldsymbol{v}})\|^2_{L^2(\widehat{\Omega})} \right\}^{1/2}$$

is equivalent to the norm of $\boldsymbol{H}^1(\widehat{\Omega})$. *In addition, if* $\widehat{\boldsymbol{v}} \in \boldsymbol{H}^1(\widehat{\Omega})$ *satisfies the Dirichlet boundary condition* $\widehat{\boldsymbol{v}} = \boldsymbol{0}$ *on part of the boundary of* $\widehat{\Omega}$ *having a positive measure,* *then the semi-norm*

$$\left\{ \sum_{i,j=1}^{3} \|\widehat{e}_{ij}(\widehat{\boldsymbol{v}})\|^2_{L^2(\widehat{\Omega})} \right\}^{1/2}$$

is equivalent to the norm of $\boldsymbol{H}^1(\widehat{\Omega})$.

Korn's inequalities are classical results used in elasticity theory. Their proof can be found in many functional analysis and mechanics textbooks, and the existence and uniqueness of the solution to problem (5) follows from Lax-Milgram lemma (see e.g. [8].

2.2. The three-dimensional curvilinear coordinates setting. Now we consider a domain $\Omega \subset \mathbb{R}^3$ and a system of curvilinear coordinates $x = (x_i)$. Let $\boldsymbol{\Phi} : \overline{\Omega} \longrightarrow \mathbb{R}^3$ be an injective mapping. Let us assume $\boldsymbol{\Phi}$ to be a $\boldsymbol{C}^1$-diffeomorphism. Then the same diffeomorphism maps the domain $\widehat{\Omega} = \boldsymbol{\Phi}(\Omega)$, i.e. $\widehat{x} = \boldsymbol{\Phi}(x)\ \forall x \in \Omega$, and the different parts of the boundary $\mathcal{B} = \partial\Omega$. As in subsection 2.1, it is assumed that $\mathcal{B}$ is partitioned into $\mathcal{B}_0$, $\mathcal{B}_+$ and $\mathcal{B}_-$ where $\mathcal{B}_-$ is the contact zone of $\overline{\Omega}$, so that $\widehat{\mathcal{B}}_0 = \boldsymbol{\Phi}(\mathcal{B}_0), \widehat{\mathcal{B}}_+ = \boldsymbol{\Phi}(\mathcal{B}_+)$ and $\widehat{\mathcal{B}}_- = \boldsymbol{\Phi}(\mathcal{B}_-)$. Let us assume in addition that $\boldsymbol{\Phi}$ keeps the orientation, i.e. $\det\{\nabla\boldsymbol{\Phi}(x)\} > 0$.

Let us introduce the three-dimensional covariant basis $\boldsymbol{g}_i = \partial_i\boldsymbol{\Phi}$, the contravariant basis $\boldsymbol{g}^i$ and the three-dimensional Christoffel's symbols $\mathcal{G}^k_{ij}$,

$$\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta^j_i, \quad \mathcal{G}^k_{ij} = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^k.$$

The metric tensor is given by either its covariant components $g_{ij} = (\boldsymbol{g}_i \cdot \boldsymbol{g}_j)$ or its contravariant components $g^{ij} = (\boldsymbol{g}^i \cdot \boldsymbol{g}^j)$, and let $g = \det\{g_{ij}\}$ be its determinant. Any displacement field is given in the contravariant basis by its covariant components $\widehat{\boldsymbol{v}}(\widehat{x}) = v_i(x)\boldsymbol{g}^i(x)$ and any volume or surface force is given by its contravariant components $\widehat{\boldsymbol{f}}(\widehat{x}) = f^i(x)\boldsymbol{g}_i(x)$, $\widehat{\boldsymbol{l}}(\widehat{x}) = l^i(x)\boldsymbol{g}_i(x)$.

The contact between the solid and the support is again formulated in terms of Signorini conditions on $\mathcal{B}_-$, which now read

$$\text{(7)} \qquad \left\{ \begin{array}{l} v_i\boldsymbol{g}^i|_3 + \boldsymbol{\Phi}|_3 + h \geq 0, \\[2ex] \sigma_3 \ \equiv -\left[(\boldsymbol{\sigma n})^i \boldsymbol{g}_i\right]|_3 \ \geq \ 0, \qquad \text{on } \mathcal{B}_- \ . \\[2ex] \sigma_3 \left(v_i\boldsymbol{g}^i|_3 + \boldsymbol{\Phi}|_3 + h\right) = \ 0, \end{array} \right.$$

where $\boldsymbol{v}|_3$ denotes the third Cartesian component of any vector $\boldsymbol{v}$ given in the contravariant basis and $\boldsymbol{n}$ is the unit normal to the boundary. Let g^{ijkl} denote the contravariant coefficients of the fourth order three-dimensional elasticity tensor. We are still dealing with a linear, homogeneous isotropic material with two Lamé coefficients (λ, μ); g^{ijkl} can therefore be written as follows:

$$g^{ijkl} = \lambda g^{ij} g^{kl} + \mu\Big(g^{ik}g^{jl} + g^{il}g^{jk}\Big).$$

Let $e_{i||j}(\boldsymbol{v})$ be the covariant components of the linearized strain tensor given by:

$$\text{(8)} \qquad e_{i||j}(\boldsymbol{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i) - \mathcal{G}^p_{ij} v_p.$$

Lastly, let us assume that the solid is subjected to body forces $\widehat{\boldsymbol{f}}(\widehat{x})$, to surface forces $\widehat{\boldsymbol{l}}(\widehat{x})$ on $\boldsymbol{\Phi}(\mathcal{B}_+)$, and that it is clamped on $\boldsymbol{\Phi}(\mathcal{B}_0)$. Let $d\mathcal{B}$ be the surface element. Then, using (7) and (8), the variational equilibrium problem (5) set in $\widehat{\Omega}$ can be transformed into a new variational inequality in Ω, which reads

$$\left\{\begin{array}{l}\text{Find } \boldsymbol{u} \in \boldsymbol{K}(\Omega) \text{ such that for all } \boldsymbol{v} \in \boldsymbol{K}(\Omega) \\ \displaystyle\int_\Omega g^{ijkl} e_{k||l}(\boldsymbol{u}) e_{i||j}(\boldsymbol{v}-\boldsymbol{u})\sqrt{g}dx \\ \displaystyle\geq \int_\Omega f^i(v_i-u_i)\sqrt{g}dx + \int_{\mathcal{B}_+} l^i(v_i-u_i)\sqrt{g}|\nabla\boldsymbol{\Phi}^{-T}\boldsymbol{n}|d\mathcal{B}.\end{array}\right. \tag{9}$$

The functional framework $\boldsymbol{K}(\Omega)$ is still a convex cone, since the same functions as those of $\widehat{\boldsymbol{K}}(\widehat{\Omega})$ are now written in terms of curvilinear coordinates, which means that they are changed by the diffeomorphism $\boldsymbol{\Phi}$ and must satisfy the non-penetrability condition pertaining in (7). This gives:

$$\boldsymbol{K}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \mathcal{B}_0,\ v_i\boldsymbol{g}^i|_3 \geq -\boldsymbol{\Phi}|_3 - h \text{ on } \mathcal{B}_-\right\}. \tag{10}$$

Based on a new version of Korn's inequality which is similar to that of subsection 2.1 but in which the components of the strain tensor $(\widehat{e}_{ij})$ are replaced by those of $(e_{i||j})$ (see [5]), this problem has a single solution.

3. The shallow shell model

The shell is first defined as a particular three-dimensional domain, so that everything that has been written in section 2 will apply, but the special shape of the domain will lead to some specific statements, either from the geometrical point of view (modelling the reference configuration) or from the mechanical point of view (writing the contact conditions).

3.1. The middle surface S^ε and the reference configuration $\overline{\boldsymbol{\Phi}^\varepsilon(\Omega^\varepsilon)}$ of the shell. Let ω be a bounded, connected subset of $\mathbb{R}^2$ with curvilinear coordinates (x_1, x_2) and Lipschitz boundary $\partial\omega$. Let ε be a small positive parameter, which is the half-thickness of the shell, and let $\boldsymbol{\varphi}^\varepsilon$ be a smooth enough injective mapping depending on ε and defined on $\overline{\omega}$ with values in $\mathbb{R}^3$. More specifically we require the following smoothness: $\boldsymbol{\varphi}^\varepsilon : \overline{\omega} \longrightarrow \mathbb{R}^3$, $\boldsymbol{\varphi}^\varepsilon \in \boldsymbol{C}^3(\overline{\omega})$. The image $\boldsymbol{\varphi}^\varepsilon(\overline{\omega})$ is a surface embedded in $\mathbb{R}^3$ which defines the middle surface of the shell:

$$S^\varepsilon = \boldsymbol{\varphi}^\varepsilon(\overline{\omega}).$$

First we build the local basis of S^ε. The two vectors $\boldsymbol{a}_1^\varepsilon(x_1,x_2) = \dfrac{\partial\boldsymbol{\varphi}^\varepsilon(x_1,x_2)}{\partial x_1}$ and $\boldsymbol{a}_2^\varepsilon(x_1,x_2) = \dfrac{\partial\boldsymbol{\varphi}^\varepsilon(x_1,x_2)}{\partial x_2}$ are assumed to be linearly independent, so that they form a basis of the tangent plane to S^ε. The unit normal to S^ε is given by $\boldsymbol{a}_3^\varepsilon = \boldsymbol{a}^{3,\varepsilon} = \dfrac{\boldsymbol{a}_1^\varepsilon \times \boldsymbol{a}_2^\varepsilon}{|\boldsymbol{a}_1^\varepsilon \times \boldsymbol{a}_2^\varepsilon|}$, and the three vectors $(\boldsymbol{a}_i^\varepsilon)$ define the two dimensional covariant basis of the middle surface S^ε of the shell.
The two dimensional contravariant basis $(\boldsymbol{a}^{i,\varepsilon})$ is obtained using the relations $\boldsymbol{a}_i^\varepsilon \cdot \boldsymbol{a}^{j,\varepsilon} = \delta_i^j$.
The two fundamental forms $a_{\alpha\beta}^\varepsilon$ and $b_{\alpha\beta}^\varepsilon$, which are the metric tensor and the curvature tensor respectively, and the two dimensional Christoffel's symbols $\Gamma_{\alpha\beta}^{\sigma,\varepsilon}$ associated with S^ε, are given by:

$$a_{\alpha\beta}^\varepsilon = \boldsymbol{a}_\alpha^\varepsilon \cdot \boldsymbol{a}_\beta^\varepsilon, \quad b_{\alpha\beta}^\varepsilon = \partial_\alpha \boldsymbol{a}_\beta^\varepsilon \cdot \boldsymbol{a}_3^\varepsilon, \quad \Gamma_{\alpha\beta}^{\sigma,\varepsilon} = \partial_\alpha \boldsymbol{a}_\beta^\varepsilon \cdot \boldsymbol{a}^{\sigma,\varepsilon}.$$

We can now model a three-dimensional domain with a mapping $\mathbf{\Phi}^\varepsilon : \overline{\Omega^\varepsilon} \longrightarrow \mathbb{R}^3$ defined from $\boldsymbol{\varphi}^\varepsilon$ as follows: Let $x^\varepsilon = (x_1, x_2, x_3^\varepsilon)$ be a point of the cylinder $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$, then

$$\mathbf{\Phi}^\varepsilon(x^\varepsilon) = \boldsymbol{\varphi}^\varepsilon(x_1, x_2) + x_3^\varepsilon \boldsymbol{a}_3^\varepsilon(x_1, x_2),\ (x_1, x_2) \in \omega,\ x_3^\varepsilon \in (-\varepsilon, \varepsilon). \tag{11}$$

Hence the physical domain in the Cartesian framework is $\widehat{\Omega}^\varepsilon = \mathbf{\Phi}^\varepsilon(\Omega)^\varepsilon$ and the reference configuration of the shallow shell is defined as the set $\overline{\widehat{\Omega}^\varepsilon}$. It has been proved that the mapping $\mathbf{\Phi}^\varepsilon$ preserves the orientation, i.e. $\det\{\nabla\mathbf{\Phi}^\varepsilon(x^\varepsilon)\} > 0$, for ε small enough. It is now straightforward to introduce the three-dimensional covariant basis $\boldsymbol{g}_i^\varepsilon = \partial_i \mathbf{\Phi}^\varepsilon$, the three-dimensional contravariant basis $\boldsymbol{g}^{i,\varepsilon}$ with $\boldsymbol{g}_i^\varepsilon \cdot \boldsymbol{g}^{j,\varepsilon} = \delta_i^j$, and the three-dimensional Christoffel's symbols $\mathcal{G}_{ij}^{k,\varepsilon} = \partial_i \boldsymbol{g}_j^\varepsilon \cdot \boldsymbol{g}^{k,e}$. Based on the definition of the mapping (11), we have

$$\begin{cases} \boldsymbol{g}_\alpha^\varepsilon = \boldsymbol{a}_\alpha^\varepsilon + x_3^\varepsilon \partial_\alpha \boldsymbol{a}_3^\varepsilon, \quad \boldsymbol{g}_3^\varepsilon = \boldsymbol{a}_3^\varepsilon, \\ \\ \mathcal{G}_{33}^{k,\varepsilon} = \mathcal{G}_{i3}^{3,\varepsilon} = 0, \end{cases}$$

and the metric tensor takes the simple polynomial form:

$$g_{\alpha\beta}^\varepsilon = a_{\alpha\beta}^\varepsilon + x_3^\varepsilon b_{\alpha\beta}^\varepsilon + (x_3^\varepsilon)^2 c_{\alpha\beta}^\varepsilon, \quad c_{\alpha\beta}^\varepsilon = a^{\tau\sigma,e} b_{\alpha\tau}^\varepsilon b_{\sigma\beta}^\varepsilon.$$

Let $\widehat{\boldsymbol{v}}^\varepsilon$ be any vector-valued function defined in $\widehat{\Omega}^\varepsilon$. The following correspondence between $\boldsymbol{v}^\varepsilon$ and a function $\widehat{\boldsymbol{v}}^\varepsilon$ defined in $\widehat{\Omega}^\varepsilon$ by $\boldsymbol{v}^\varepsilon = \widehat{\boldsymbol{v}}^\varepsilon \circ \mathbf{\Phi}^\varepsilon$ will now be used.

3.2. The boundary conditions.

3.2.1. Bilateral boundary conditions imposed on $\mathcal{B}_+^\varepsilon$ and $\mathcal{B}_0^\varepsilon$. In the same way as with the general three-dimensional solid, the boundary $\partial\Omega^\varepsilon$ of the domain Ω^ε is assumed to be partitioned into three parts. In line with the particular shape of the domain, these parts are now the "lateral" one $\mathcal{B}_0^\varepsilon$, the " upper" one $\mathcal{B}_+^\varepsilon$ and the "lower" one $\mathcal{B}_-^\varepsilon$, i.e.:

$$\partial\Omega^\varepsilon = \mathcal{B}_0^\varepsilon \cup \mathcal{B}_+^\varepsilon \cup \mathcal{B}_-^\varepsilon, \quad \mathcal{B}_0^\varepsilon = \partial\omega \times (-\varepsilon, \varepsilon), \quad \mathcal{B}_+^\varepsilon = \omega \times \{\varepsilon\}, \quad \mathcal{B}_-^\varepsilon = \omega \times \{-\varepsilon\}.$$

The following assumptions are adopted:
1) the upper part, $\mathcal{B}_+^\varepsilon$, is loaded by a surface force $\boldsymbol{l}^\varepsilon \in \boldsymbol{L}^2(\mathcal{B}_+^\varepsilon)$,
2) the lateral part, $\mathcal{B}_0^\varepsilon$, is clamped, that is $\boldsymbol{u}^\varepsilon = \mathbf{0}$ on $\mathcal{B}_0^\varepsilon$.

Remark 1. *This choice of boundary conditions is not restrictive: only very slight changes would result from applying clamping conditions on a non zero measure part of $\mathcal{B}_0^\varepsilon$ and stress free conditions on the complementary (of course the case where non zero surface forces are applied on the part of $\mathcal{B}_0^\varepsilon$ which is not clamped would generally give rise to singularities which are beside the point in the present study, and can therefore be ruled out).*

3.2.2. Unilateral contact conditions on $\mathcal{B}_-^\varepsilon$. We still restrict our attention to the case of a plane horizontal obstacle. It was established in [11] that the most appropriate setting consists in taking this plane to be at the level $-\varepsilon$.

For the sake of clarity, we first write the non penetrability condition in the Cartesian framework. Let O be the origin of the three-dimensional coordinate system. Let $\widehat{\mathbf{M}}^\varepsilon$ be a generic point of S^ε and $\hat{x}_-^\varepsilon$ be the corresponding point of $\mathcal{B}_-^\varepsilon$. The position of $\hat{x}_-^\varepsilon$ in the Cartesian system is then given by (see [11]):

$$O\hat{x}_-^\varepsilon = O\widehat{\mathbf{M}}^\varepsilon - \varepsilon \boldsymbol{a}_3^\varepsilon(\hat{x}_-^\varepsilon).$$

The non penetrability condition of a point of $\mathcal{B}_-^\varepsilon$ into the horizontal plane at the level $-\varepsilon$ therefore reads:

$$(O\hat{x}_-^\varepsilon - \varepsilon \boldsymbol{a}_3^\varepsilon(\hat{x}_-^\varepsilon) + \widehat{\boldsymbol{v}}^\varepsilon(\hat{x}_-^\varepsilon))|_3 \geq -\varepsilon,$$

where $\widehat{\boldsymbol{v}}^\varepsilon(\hat{x}_-^\varepsilon)$ stands for the displacement at the point $\hat{x}_-^\varepsilon$. Using the map $\boldsymbol{\Phi}^\varepsilon$ defined above this can also be written

$$(\boldsymbol{\Phi}^\varepsilon(x_1, x_2, -\varepsilon) + \boldsymbol{v}^\varepsilon(\hat{x}_-^\varepsilon))|_3 \geq -\varepsilon,$$

From the mechanical point of view, the unilateral contact conditions are obtained by adding two conditions, taking the stresses at the boundary into account as explained in subsection 2.1 in the case of any three-dimensional solid.

In terms of curvilinear coordinates, the unilateral contact conditions on $\mathcal{B}_-$ can be written as in subsection 2.2, using the particular position of the obstacle and the specific mapping $\boldsymbol{\Phi}^\varepsilon$.

$$\text{(12)} \qquad \begin{cases} i)\ v_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}|_3 + \boldsymbol{\Phi}^\varepsilon|_3 + \varepsilon \geq 0, \\ ii)\ \sigma_3^\varepsilon \ \equiv -\Big((\boldsymbol{\sigma}^\varepsilon \boldsymbol{a}_3^\varepsilon)^i \cdot \boldsymbol{g}_i^\varepsilon\Big)|_3 \ \geq \ 0, & \text{on } \ \mathcal{B}_- \ . \\ iii)\ \sigma_3^\varepsilon \Big(v_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}|_3 + \boldsymbol{\Phi}^\varepsilon|_3 + \varepsilon\Big) = \ 0, \end{cases}$$

3.3. The equilibrium equations. When the three-dimensional body in Ω^ε is subjected to body forces $\boldsymbol{f}^\varepsilon \in \boldsymbol{L}^2(\Omega^\varepsilon)$, is submitted to loaded boundary conditions at $\mathcal{B}_+^\varepsilon$, clamped boundary conditions at $\mathcal{B}_0^\varepsilon$, and is in unilateral contact at $\mathcal{B}_-^\varepsilon$, the displacement field $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ is the solution of the following variational problem

$$\text{(13)} \qquad \begin{cases} \text{Find } \boldsymbol{u}^\varepsilon \in \ \boldsymbol{K}^\varepsilon(\Omega^\varepsilon) \text{ such that for all } \boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{K}^\varepsilon(\Omega^\varepsilon) \\ \displaystyle\int_{\Omega^\varepsilon} g^{ijkl,\varepsilon} e_{k||l}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{i||j}^\varepsilon(\boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon)\sqrt{g^\varepsilon} dx^\varepsilon \\ \displaystyle\geq \int_{\Omega^\varepsilon} f^{i,\varepsilon}(v_i^\varepsilon - u_i^\varepsilon)\sqrt{g^\varepsilon} dx^\varepsilon + \int_{\mathcal{B}_+^\varepsilon} l^{i,\varepsilon}(v_i^\varepsilon - u_i^\varepsilon)\sqrt{g^\varepsilon} d\mathcal{B}^\varepsilon \end{cases}$$

where the components of the elasticity tensor $g^{ijkl,\varepsilon}$ and of the linearized strain tensor $e_{i||j}^\varepsilon(\boldsymbol{v}^\varepsilon)$ are defined as previously:

$$e_{i||j}^\varepsilon(\boldsymbol{v}^\varepsilon) = \frac{1}{2}\Big(\frac{\partial v_j^\varepsilon}{\partial x_i^\varepsilon} + \frac{\partial v_i^\varepsilon}{\partial x_j^\varepsilon}\Big) - \mathcal{G}_{ij}^{p,\varepsilon} v_p^\varepsilon,$$

$$g^{ijkl,\varepsilon} = \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu\Big(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}\Big).$$

The set of admissible displacements is still a convex cone defined as

$$\text{(14)} \qquad \boldsymbol{K}^\varepsilon(\Omega^\varepsilon) = \left\{\boldsymbol{v}^\varepsilon \in \boldsymbol{H}^1(\Omega^\varepsilon),\ \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \mathcal{B}_0^\varepsilon,\ v_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}|_3 \geq -\boldsymbol{\Phi}^\varepsilon|_3 - \varepsilon \text{ on } \mathcal{B}_-^\varepsilon\right\}.$$

The Korn's inequality which exists for this problem is simply adapted from that presented in subsection 2.2, which makes it possible to prove the existence and uniqueness of the solution to problem (13) with any fixed $\varepsilon > 0$.

4. Formulation in the fixed domain Ω

The formulation of the problem set in the variable domain $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ with thickness 2ε (and therefore where the volume tends to zero with ε) is tranformed into a problem set in the cylinder $\Omega = \omega \times (-1, 1)$, i.e. in a cylindrical domain which no longer depends on ε. The transformation of the problem requires three steps.

• **Step 1: A simple geometrical transformation.** The domain Ω is obtained by performing the dilatation:

$$\left\{ \begin{array}{l} \Omega^\varepsilon \ni x^\varepsilon = (x_1, x_2, x_3^\varepsilon) \longrightarrow x = (x_1, x_2, x_3) \in \Omega, \\ \\ x_3 = \dfrac{1}{\varepsilon} x_3^\varepsilon. \end{array} \right. \tag{15}$$

With obvious notations, the boundary of domain Ω is $\partial\Omega = \mathcal{B}_0 \cup \mathcal{B}_+ \cup \mathcal{B}_-$, $\mathcal{B}_0 = \partial\omega \times (-1, 1)$, $\mathcal{B}_+ = \omega \times \{1\}$, $\mathcal{B}_- = \omega \times \{-1\}$.

Each function $v^\varepsilon(x^\varepsilon)$ defined in the domain Ω^ε is associated with new functions $v(\varepsilon)(x)$ defined in the domain Ω by

$$v^\varepsilon(x^\varepsilon) = v(\varepsilon)(x), \text{ with the relation (15).}$$

The covariant basis $(\boldsymbol{a}_i^\varepsilon, \boldsymbol{g}_i^\varepsilon)$, the contravariant basis $(\boldsymbol{a}^{i,\varepsilon}, \boldsymbol{g}^{i,\varepsilon})$ and the Christoffel's symbols $\Gamma_{ij}^{k,\varepsilon}$, $\mathcal{G}_{ij}^{k,\varepsilon}$, are therefore associated with the new functions $\boldsymbol{a}_i(\varepsilon)$, $\boldsymbol{g}_i(\varepsilon)$, $\boldsymbol{a}^i(\varepsilon)$, $\boldsymbol{g}^i(\varepsilon)$, $\Gamma_{ij}^k(\varepsilon)$, $\mathcal{G}_{ij}^k(\varepsilon)$ by:

$$\left\{ \begin{array}{ll} \boldsymbol{a}_i^\varepsilon(x^\varepsilon) = \boldsymbol{a}_i(\varepsilon)(x), & \boldsymbol{a}^{i,\varepsilon}(x^\varepsilon) = \boldsymbol{a}^i(\varepsilon)(x), \\ \boldsymbol{g}_i^\varepsilon(x^\varepsilon) = \boldsymbol{g}_i(\varepsilon)(x), & \boldsymbol{g}^{i,\varepsilon}(x^\varepsilon) = \boldsymbol{g}^i(\varepsilon)(x), \\ \Gamma_{ij}^{k,\varepsilon}(x^\varepsilon) = \Gamma_{ij}^k(\varepsilon)(x), & \mathcal{G}_{ij}^{k,\varepsilon}(x^\varepsilon) = \mathcal{G}_{ij}^k(\varepsilon)(x). \end{array} \right. \tag{16}$$

• **Step 2: Assumptions on the data.** In order to obtain a limit model when ε tends to zero in problem (13), we recall (see [3]) that we assume that the mapping $\boldsymbol{\varphi}^\varepsilon : \overline{\omega} \longrightarrow \mathbb{R}^3$ which defines the middle surface S^ε of the shell has to be of the form:

$$\boldsymbol{\varphi}^\varepsilon = (\varphi_1, \varphi_2, \varepsilon\varphi_3) \tag{17}$$

where the three functions φ_1, φ_2, φ_3 are independent of ε.

In addition, for the sake of coherence with respect to the contact conditions between the solid and the support, we must also assume that φ_3 is positive.

Remark 2. *The form of mapping $\boldsymbol{\varphi}^\varepsilon$ chosen here is one which can be used to draw up any shallow shell model. Previous studies have dealt with mappings of the form $\boldsymbol{\varphi}^\varepsilon = (x_1,\ x_2,\ \varphi^\varepsilon)$. The most general choice is done here in the case of a unilateral problem, but in that of bilateral problems (problems in which unilateral contact conditions are removed and only classical Dirichlet and Neumann boundary conditions remain) it can be used to extend the justification of the shallow shell model given in* [2, 3] *to more general charts.*

The last assumption about the data concerns the external forces and the material parameters. It has been established in [7] that the appropriate scaling of the forces is as follows:
- the volume forces are of the form $\varepsilon^2 f^\alpha \boldsymbol{g}_\alpha(\varepsilon) + \varepsilon^3 f^3 \boldsymbol{g}_3(\varepsilon)$,
- the surface force are of the form $\varepsilon^3 l^\alpha \boldsymbol{g}_\alpha(\varepsilon) + \varepsilon^4 l^3 \boldsymbol{g}_3(\varepsilon)$,

and it was proved in [14] in the case of plates that this is the only possible way of obtaining a limit shell problem.
- Since the Lamé constants (λ, μ) are assumed to be *independent of* ε, the coefficients of the elasticity tensor with scaled contravariant components $g^{ijkl}(\varepsilon)$ are expressed by

$$g^{ijkl}(\varepsilon) = \lambda g^{ij}(\varepsilon) g^{kl}(\varepsilon) + \mu\Big(g^{ik}(\varepsilon) g^{jl}(\varepsilon) + g^{il}(\varepsilon) g^{jk}(\varepsilon)\Big),$$

where $g(\varepsilon) = \det(g_{ij}(\varepsilon))$, $g_{ij}(\varepsilon) = \boldsymbol{g}_i(\varepsilon) \cdot \boldsymbol{g}_j(\varepsilon)$ are the covariant components of the metric tensor and $g^{ij}(\varepsilon) = \boldsymbol{g}^i(\varepsilon) \cdot \boldsymbol{g}^j(\varepsilon)$ are the contravariant components of the metric tensor.

• **Step 3: Scaling of the unknowns.** Lastly, the displacement field $\boldsymbol{u}(\varepsilon)$ and the test-functions $\boldsymbol{v}(\varepsilon)$ are scaled as follows:

$$\left\{\begin{array}{ll} u^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), & u^\varepsilon_3(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), \\ v^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 v_\alpha(x), & v^\varepsilon_3(x^\varepsilon) = \varepsilon v_3(x). \end{array}\right. \tag{18}$$

Assumption (18) has been communly adopted, as in [2] for example. By performing these scaling procedures, it is intended to make the first term $u^0_3(\varepsilon)$ and the two first terms $u^0_\alpha(\varepsilon), u^1_\alpha(\varepsilon)$ vanish from expansions of the form $\boldsymbol{u}^\varepsilon(x^\varepsilon) = \boldsymbol{u}^0(\varepsilon)(x) + \varepsilon \boldsymbol{u}^1(\varepsilon)(x) + ...$ This has been rigorously established in [14, 15] in the case of elastic plate models, in both the linear and nonlinear cases. The displacement field therefore reads $\varepsilon^2 u_\alpha(\varepsilon)\boldsymbol{g}^\alpha(\varepsilon) + \varepsilon u_3(\varepsilon)\boldsymbol{g}^3(\varepsilon)$. Along with this scaling procedure, $(e_{i||j}(\varepsilon))$ denotes the scaled covariant components of the linearized strain tensor defined by

$$e^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon(x^\varepsilon)) = e_{i||j}(\varepsilon)(\boldsymbol{u}(\varepsilon)(x)). \tag{19}$$

4.1. The contact condition in the fixed domain Ω. Upon performing the geometrical transformation (15), the support becomes the horizontal plane at level -1 in the Cartesian framework. The non-penetrability condition (12-i) imposed on $\mathcal{B}^\varepsilon_-$ which can therefore be written on $\mathcal{B}_-$ reads:

$$\varepsilon^2 v_\alpha \boldsymbol{g}^\alpha(\varepsilon)|_3 + \varepsilon v_3 \boldsymbol{g}^3(\varepsilon)|_3 \geq -\varepsilon\varphi_3 + \varepsilon\Big(\boldsymbol{g}_3(\varepsilon)|_3 - 1\Big) \text{ on } \mathcal{B}_-$$

or equivalently

$$\varepsilon v_\alpha \boldsymbol{g}^\alpha(\varepsilon)|_3 + v_3 \boldsymbol{a}^3(\varepsilon)|_3 \geq -\varphi_3 + \Big(\boldsymbol{a}_3(\varepsilon)|_3 - 1\Big) \text{ on } \mathcal{B}_-.$$

4.2. The scaled problem set in the fixed domain Ω. In the curvilinear coordinate setting, the equilibrium problem now reads:

$$\left\{\begin{array}{l} \text{Find } \boldsymbol{u}(\varepsilon) \in \tilde{\boldsymbol{K}}(\varepsilon)(\Omega) \text{ such that } \forall\ \boldsymbol{v}(\varepsilon) = (\varepsilon^2 v_1, \varepsilon^2 v_2, \varepsilon v_3) \in \tilde{\boldsymbol{K}}(\varepsilon)(\Omega), \\ \displaystyle\int_\Omega g^{ijkl}(\varepsilon) e_{k||l}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{i||j}(\varepsilon)(\boldsymbol{v}(\varepsilon) - \boldsymbol{u}(\varepsilon))\sqrt{g(\varepsilon)}dx \\ \displaystyle\geq \int_\Omega f^i(\varepsilon)(v_i(\varepsilon) - u_i(\varepsilon))\sqrt{g(\varepsilon)}dx + \int_{\mathcal{B}_+} l^i(\varepsilon)(v_i(\varepsilon) - u_i(\varepsilon))\sqrt{g(\varepsilon)}d\mathcal{B} \end{array}\right. \tag{20}$$

where the set of scaled test functions is again a convex cone given by

$$\begin{aligned} \tilde{\boldsymbol{K}}(\varepsilon)(\Omega) = \{&\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega),\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \mathcal{B}_0, \\ &\varepsilon v_\alpha \boldsymbol{g}^\alpha(\varepsilon)|_3 + v_3 \boldsymbol{a}^3(\varepsilon)|_3 \geq -\varphi_3 + \boldsymbol{a}_3(\varepsilon)|_3 - 1 \text{ on } \mathcal{B}_-\}. \end{aligned}$$

Remark 3. *The formulation of the problem in terms of curvilinear coordinates introduces a coupling between the in-plane and transverse covariant components of the displacement fields in the cone $\tilde{\boldsymbol{K}}(\varepsilon)(\Omega)$. It is worth noting that these components will now be uncoupled, so that the non-penetrability condition will still concern only the transverse component, as occurs in the case of the Cartesian framework.*

As in the previous sections, classical tools can be used to show that problem (20) possesses a unique solution for any fixed $\varepsilon > 0$.

5. Asymptotic analysis

The aim of this section is to transform problem (20) into a singular perturbations problem for which it will be easy to find a limit. The asymptotic analysis involves two steps: expanding all the data around the middle surface $\overline{\omega}$ and then computing the associated strain tensor. We will deal with each of these steps separately.

5.1. Taylor expansion of the data associated with the mapping $\boldsymbol{\varphi}(\varepsilon)$. Any function $v(\varepsilon)$ given in Ω can be expanded into Taylor series with respect to ε

$$v(\varepsilon) = v + \varepsilon v_1 + \varepsilon^2 v_2 + ...$$

In particular, the first terms in the expansion of the covariant and contravariant basis vectors and in the Christoffell symbols are obtained as follows. By definition we have:

$$\boldsymbol{a}_\alpha(\varepsilon) = \partial_\alpha \boldsymbol{\varphi}(\varepsilon) = \begin{pmatrix} \partial_\alpha\varphi_1 \\ \partial_\alpha\varphi_2 \\ \varepsilon\partial_\alpha\varphi_3 \end{pmatrix}, \qquad \boldsymbol{a}_3(\varepsilon) = \frac{1}{\sqrt{d(\varepsilon)}} \begin{pmatrix} \varepsilon(\partial_1\varphi_2\partial_2\varphi_3 - \partial_1\varphi_3\partial_2\varphi_2) \\ \varepsilon(\partial_1\varphi_3\partial_2\varphi_1 - \partial_1\varphi_1\partial_2\varphi_3) \\ \partial_1\varphi_1\partial_2\varphi_2 - \partial_1\varphi_2\partial_2\varphi_1 \end{pmatrix},$$

with

$$d(\varepsilon) = (\partial_1\varphi_1\partial_2\varphi_2 - \partial_1\varphi_2\partial_2\varphi_1)^2 + \varepsilon^2(\partial_1\varphi_2\partial_2\varphi_3 - \partial_1\varphi_3\partial_2\varphi_2)^2 + \varepsilon^2(\partial_1\varphi_3\partial_2\varphi_1 - \partial_1\varphi_1\partial_2\varphi_3)^2.$$

Hence, there exist a covariant basis $(\boldsymbol{a}_i)$ and a contravariant basis $(\boldsymbol{a}^i)$, where $\boldsymbol{a}_i \cdot \boldsymbol{a}^j = \delta_i^j$, *independent* of ε and associated with the "limit" middle surface of the shell defined by the mapping $(\varphi_1, \varphi_2, 0)$, which read:

$$\begin{cases} \boldsymbol{a}_\alpha = \begin{pmatrix} \partial_\alpha\varphi_1 \\ \partial_\alpha\varphi_2 \\ 0 \end{pmatrix}, \boldsymbol{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\[2ex] \boldsymbol{a}^1 = \dfrac{1}{a}\begin{pmatrix} \partial_2\varphi_2 \\ -\partial_2\varphi_1 \\ 0 \end{pmatrix}, \boldsymbol{a}^2 = \dfrac{1}{a}\begin{pmatrix} -\partial_1\varphi_2 \\ \partial_1\varphi_1 \\ 0 \end{pmatrix}, \boldsymbol{a}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\[2ex] \text{with } a = \partial_1\varphi_1\partial_2\varphi_2 - \partial_1\varphi_2\partial_2\varphi_1. \end{cases}$$

The limit two-dimensional metric tensor $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ and $a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$ can also be defined. The above expressions give the following expansions:

$$\begin{cases} \boldsymbol{a}_\alpha(\varepsilon) = \begin{pmatrix} \partial_\alpha\varphi_1 \\ \partial_\alpha\varphi_2 \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ 0 \\ \partial_\alpha\varphi_3 \end{pmatrix} + .., \\[2ex] \boldsymbol{a}_3(\varepsilon) = \boldsymbol{a}^3(\varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \dfrac{\varepsilon}{a} \begin{pmatrix} \partial_1\varphi_2\partial_2\varphi_3 - \partial_1\varphi_3\partial_2\varphi_2 \\ \partial_1\varphi_3\partial_2\varphi_1 - \partial_1\varphi_1\partial_2\varphi_3 \\ 0 \end{pmatrix} + \cdots, \end{cases} \tag{21}$$

which lead to the expansions of the metric tensor and of the elasticity tensor

$$\left\{\begin{array}{l} \boldsymbol{g}_\alpha(\varepsilon)=\boldsymbol{a}_\alpha+\varepsilon\cdots, \quad \boldsymbol{g}^\alpha(\varepsilon)=\boldsymbol{a}^\alpha+\varepsilon\cdots \quad \boldsymbol{g}_3(\varepsilon)=\boldsymbol{g}^3(\varepsilon)=\boldsymbol{a}_3(\varepsilon)=\boldsymbol{a}^3(\varepsilon)=\boldsymbol{a}^3+\varepsilon\cdots, \\ g_{ij}(\varepsilon)=a_{ij}+\varepsilon\cdots, \quad g^{ij}(\varepsilon)=a^{ij}+\varepsilon\cdots, \quad det(g_{ij}(\varepsilon))=det(a_{ij})+\varepsilon\cdots=a+\varepsilon\cdots, \\ g^{ijkl}(\varepsilon)=a^{ijkl}+\varepsilon\cdots, \text{ with } a^{\alpha 3}=0,\ a^{33}=1. \end{array}\right.$$

Since $a^{\alpha 3}=0, a^{33}=1$ the leading terms a^{ijkl} in the expression of the limit two-dimensional elasticity tensor are

$$\left\{\begin{array}{ll} a^{\alpha\beta\sigma\tau} & =\lambda a^{\alpha\beta}a^{\sigma\tau}+\mu\Big(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}\Big), \\ a^{\alpha\beta 33} & =\lambda a^{\alpha\beta}, \\ a^{\alpha 3\beta 3} & =\mu a^{\alpha\beta}, \\ a^{3333} & =\lambda+2\mu. \end{array}\right.$$

Similar computations yield

$$\left\{\begin{array}{ll} \mathcal{G}^\sigma_{\alpha\beta}(\varepsilon) & =\partial_\alpha\boldsymbol{g}_\beta(\varepsilon)\cdot\boldsymbol{g}^\sigma(\varepsilon)=\partial_\alpha\Big(\boldsymbol{a}_\beta(\varepsilon)+\varepsilon x_3\partial_\beta\boldsymbol{a}_3(\varepsilon)\Big)\cdot\boldsymbol{g}^\sigma(\varepsilon), \\ & =\partial_\alpha\Big(\boldsymbol{a}_\beta+\varepsilon\boldsymbol{a}_{\beta,1}+...\Big)\cdot(\boldsymbol{a}^\sigma+\varepsilon\boldsymbol{a}^{\sigma,1}+...), \\ & =\Gamma^\sigma_{\alpha\beta}+\varepsilon\mathcal{G}^{\sigma,\sharp}_{\alpha\beta}(\varepsilon,\boldsymbol{\varphi}) \quad \text{ with } \Gamma^\sigma_{\alpha\beta}=\partial_\alpha\boldsymbol{a}_\beta\cdot\boldsymbol{a}^\sigma, \\ & \\ \mathcal{G}^3_{\alpha\beta}(\varepsilon) & =\partial_\alpha\boldsymbol{g}_\beta(\varepsilon)\cdot\boldsymbol{g}^3(\varepsilon)=\partial_\alpha\Big(\boldsymbol{a}_\beta(\varepsilon)+\varepsilon x_3\partial_\beta\boldsymbol{a}_3(\varepsilon)\Big)\cdot\boldsymbol{a}^3(\varepsilon), \\ & =\partial_\alpha\Big(\boldsymbol{a}_\beta+\varepsilon x_3\partial_\beta\boldsymbol{a}_3+\varepsilon\boldsymbol{a}_{\beta,1}+\varepsilon^2x_3\partial_\beta\boldsymbol{a}_{3,1}+...\Big)\cdot(\boldsymbol{a}_3+\varepsilon\boldsymbol{a}_{3,1}+\varepsilon^2\boldsymbol{a}_{3,2}+...), \\ & =\varepsilon b_{\alpha\beta}+\varepsilon^2\mathcal{G}^{3,\sharp}_{\alpha\beta}(\varepsilon,\boldsymbol{\varphi}) \quad \text{ with } b_{\alpha\beta}=\partial_\alpha\boldsymbol{a}_\beta\cdot\boldsymbol{a}_{3,1}, \\ & \\ \mathcal{G}^\sigma_{\alpha 3}(\varepsilon) & =\partial_\alpha\boldsymbol{g}_3(\varepsilon)\cdot\boldsymbol{g}^\sigma(\varepsilon)=\partial_\alpha\boldsymbol{a}_3(\varepsilon)\cdot\boldsymbol{g}^\sigma(\varepsilon), \\ & =\partial_\alpha\Big(\boldsymbol{a}_3+\varepsilon\boldsymbol{a}_{3,1}+...\Big)\cdot(\boldsymbol{a}^\sigma+\varepsilon\boldsymbol{g}^\sigma_1+\varepsilon^2...), \\ & =\varepsilon b^\sigma_\alpha+\varepsilon^2\mathcal{G}^{\sigma,\sharp}_{\alpha 3}(\varepsilon,\boldsymbol{\varphi}) \quad \text{ with } b^\sigma_\alpha=-\partial_\alpha\boldsymbol{a}_{3,1}\cdot\boldsymbol{a}^\sigma, \\ & \\ \mathcal{G}^3_{\alpha 3}(\varepsilon) & =0, \quad \mathcal{G}^3_{33}(\varepsilon)=0. \end{array}\right.$$

where the regularity of the remainders $\mathcal{G}^{k,\sharp}_{ij}(\varepsilon,\boldsymbol{\varphi})$ depends on the regularity of the mapping $\boldsymbol{\varphi}$. If $\boldsymbol{\varphi}\in\boldsymbol{C}^3(\overline{\omega})$, then there exists $\varepsilon_0>0$ such that

$$\max_{i,j,k}\sup_{0\le\varepsilon\le\varepsilon_0}\max_{x\in\Omega}|\mathcal{G}^{k,\sharp}_{ij}(\varepsilon,\boldsymbol{\varphi})(x)|\le C,$$

where C is a positive constant.
A similar approach was used in [13] to obtain the associated expansions in the case of general shells.

5.2. Influence of the scaling of the unknowns. We recall the definition of the strain tensor in terms of covariant components

$$e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon)=\frac{1}{2}\Big(\frac{\partial v^\varepsilon_j}{\partial x^\varepsilon_i}+\frac{\partial v^\varepsilon_i}{\partial x^\varepsilon_j}\Big)-\mathcal{G}^{k,\varepsilon}_{ij}v^\varepsilon_k \quad \text{ with } \quad \frac{\partial\cdot}{\partial x^\varepsilon_\alpha}=\frac{\partial\cdot}{\partial x_\alpha},\ \frac{\partial\cdot}{\partial x^\varepsilon_3}=\frac{1}{\varepsilon}\frac{\partial\cdot}{\partial x_3}.$$

So that the first significant terms of the scaled strain tensor are:

$$\left\{\begin{array}{ll} e_{\alpha||\beta}(\varepsilon)(\boldsymbol{v}(\varepsilon)) & =\varepsilon^2\Big(e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{v})+\varepsilon e^\sharp_{\alpha||\beta}(\varepsilon,\boldsymbol{\varphi},\boldsymbol{v})\Big), \\ & \\ e_{\alpha||3}(\varepsilon)(\boldsymbol{v}(\varepsilon)) & =\varepsilon\Big(e^{\boldsymbol{\varphi}}_{\alpha||3}(\boldsymbol{v})+\varepsilon e^\sharp_{\alpha||3}(\varepsilon,\boldsymbol{\varphi},\boldsymbol{v})\Big), \\ & \\ e_{3||3}(\varepsilon)(\boldsymbol{v}(\varepsilon)) & =e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{v}), \end{array}\right. \tag{22}$$

where $\boldsymbol{v}(\varepsilon) = (\varepsilon^2 v_1, \varepsilon^2 v_2, \varepsilon v_3)$, $\boldsymbol{v} = (v_i)$ and

$$\begin{cases} e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{v}) = \dfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta} v_\sigma - b_{\alpha\beta} v_3, \\ \\ e^{\boldsymbol{\varphi}}_{\alpha||3}(\boldsymbol{v}) = \dfrac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha), \quad e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{v}) = \partial_3 v_3. \end{cases}$$

For all fixed ε, the quantities $e^\sharp_{i||j}(\varepsilon, \boldsymbol{\varphi}; \boldsymbol{v})$ are bounded in $\boldsymbol{L}^2(\Omega)$ when $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{C}^3(\overline{\omega})$, i.e., there exists a constant $C > 0$ such that

$$\sup_{0 \le \varepsilon \le \varepsilon_0} \sum_{i,j} ||e^\sharp_{i||j}(\varepsilon, \boldsymbol{\varphi}; \boldsymbol{v})||_{0,\Omega} \le C ||\boldsymbol{v}||_{1,\Omega}.$$

Similar computations give the tensor $(e_{\iota||j}(\varepsilon)(\boldsymbol{u}(\varepsilon)))$ and

$$\begin{cases} e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{u}(\varepsilon)) & = \dfrac{1}{2}\Big(\partial_\alpha u_\beta(\varepsilon) + \partial_\beta u_\alpha(\varepsilon)\Big) - \Gamma^\sigma_{\alpha\beta} u_\sigma(\varepsilon) - b_{\alpha\beta} u_3(\varepsilon), \\ \\ e^{\boldsymbol{\varphi}}_{\alpha||3}(\boldsymbol{u}(\varepsilon)) & = \dfrac{1}{2}\Big(\partial_\alpha u_3(\varepsilon) + \partial_3 u_\alpha(\varepsilon)\Big), \\ \\ e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{u}(\varepsilon)) & = \partial_3 u_3(\varepsilon). \end{cases} \tag{23}$$

As we will see, only the first order terms $e^{\boldsymbol{\varphi}}_{\alpha||\beta}$ will occur in the limit problem, and the remainders $e^\sharp_{i||j}$ will be neglected.

5.3. Another way of formulating the problem. By replacing the scaled strain tensor $e_{i||j}(\varepsilon)$ by its expression (22-23) in problem (20), we obtain a new formulation with remainders, namely:

$$\begin{cases} \text{Find } \boldsymbol{u}(\varepsilon) \in \tilde{\boldsymbol{K}}(\varepsilon)(\Omega) \text{ such that for all } \boldsymbol{v} \in \tilde{\boldsymbol{K}}(\varepsilon)(\Omega), \\ \\ \varepsilon^4 \displaystyle\int_\Omega g^{\alpha\beta\sigma\tau}(\varepsilon)\Big(e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{\alpha\beta}(\boldsymbol{u}(\varepsilon))\Big)\Big(e^{\varphi}_{\sigma||\tau}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{\sigma\tau}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\Big)\sqrt{g(\varepsilon)} \\ \\ +\varepsilon^2 \displaystyle\int_\Omega g^{\alpha\beta 33}(\varepsilon)\Big(e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{\alpha\beta}(\boldsymbol{u}(\varepsilon))\Big)\Big(e^{\varphi}_{3||3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{33}(\boldsymbol{v})\Big)\sqrt{g(\varepsilon)} \\ \\ +\varepsilon^2 \displaystyle\int_\Omega g^{\alpha\beta 33}(\varepsilon)\Big(e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{33}(\boldsymbol{u}(\varepsilon))\Big)\Big(e^{\varphi}_{\alpha||\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\Big)\sqrt{g(\varepsilon)} \\ \\ +\varepsilon^2 \displaystyle\int_\Omega g^{\alpha 3\beta 3}(\varepsilon)\Big(e^{\boldsymbol{\varphi}}_{\alpha||3}(\boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{\alpha 3}(\boldsymbol{u}(\varepsilon))\Big)\Big(e^{\varphi}_{\beta||3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{\beta 3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\Big)\sqrt{g(\varepsilon)} \\ \\ + \displaystyle\int_\Omega g^{3333}(\varepsilon)\Big(e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{33}(\boldsymbol{u}(\varepsilon))\Big)\Big(e^{\varphi}_{3||3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) + \varepsilon e^\sharp_{33}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\Big)\sqrt{g(\varepsilon)} \\ \\ \ge \;\; \varepsilon^4 \displaystyle\int_\Omega f^i (v_i - u_i(\varepsilon))\sqrt{g(\varepsilon)} dx + \varepsilon^4 \int_{\mathcal{B}_+} l^i (v_i - u_i(\varepsilon))\sqrt{g(\varepsilon)} d\mathcal{B}, \end{cases}$$

where $e^\sharp_{\alpha\beta}(\cdot)$, $e^\sharp_{\alpha 3}(\cdot)$, $e^\sharp_{33}(\cdot)$ should be taken to mean $e^\sharp_{\alpha\beta}(\varepsilon, \boldsymbol{\varphi}; \cdot)$, $e^\sharp_{\alpha 3}(\varepsilon, \boldsymbol{\varphi}; \cdot)$, $e^\sharp_{33}(\varepsilon, \boldsymbol{\varphi}; \cdot)$ respectively.

Before studying the limit problem, we combine all the remainders together which leads to problem (24)

$$
(24)\qquad \begin{cases}
\text{Find } \boldsymbol{u}(\varepsilon) \in \tilde{\boldsymbol{K}}(\varepsilon)(\Omega) \text{ such that for all } \boldsymbol{v} \in \tilde{\boldsymbol{K}}(\varepsilon)(\Omega),\\
\varepsilon^4 \displaystyle\int_\Omega a^{\alpha\beta\sigma\tau} e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{\sigma||\tau}(\boldsymbol{v}-\boldsymbol{u}(\varepsilon))\sqrt{a}\\
+\ \varepsilon^2 \displaystyle\int_\Omega a^{\alpha\beta 33}(\varepsilon) e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{3||3}(\boldsymbol{v}-\boldsymbol{u}(\varepsilon))\sqrt{a}\\
+\ \varepsilon^2 \displaystyle\int_\Omega a^{\alpha\beta 33} e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{\alpha||\beta}(\boldsymbol{v}-\boldsymbol{u}(\varepsilon))\sqrt{a}\\
+\ \varepsilon^2 \displaystyle\int_\Omega a^{\alpha 3\beta 3} e^{\boldsymbol{\varphi}}_{\alpha||3}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{\beta||3}(\boldsymbol{v}-\boldsymbol{u}(\varepsilon))\sqrt{a}\\
+ \displaystyle\int_\Omega a^{3333} e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{3||3}(\boldsymbol{v}-\boldsymbol{u}(\varepsilon))\sqrt{a} + B^\flat(\varepsilon,\boldsymbol{\varphi},\boldsymbol{u}(\varepsilon),\boldsymbol{v}-\boldsymbol{u}(\varepsilon))\\
\geq\ \ \varepsilon^4 \displaystyle\int_\Omega f^i(v_i-u_i(\varepsilon))\sqrt{a}dx + \varepsilon^4\int_{\mathcal{B}_+} l^i(v_i-u_i(\varepsilon))\sqrt{a}d\mathcal{B}\\
+\varepsilon^2 L^\flat(\varepsilon,\boldsymbol{\varphi},\boldsymbol{v}-\boldsymbol{u}(\varepsilon)),
\end{cases}
$$

where the quantities $B^\flat$ and $L^\flat$ stand for bounded remainders, i.e., there exists a positive constant C independent of ε such that

$$|B^\flat(\varepsilon,\boldsymbol{\varphi},\boldsymbol{u}(\varepsilon),\boldsymbol{v}-\boldsymbol{u}(\varepsilon))| + |L^\flat(\varepsilon,\boldsymbol{\varphi},\boldsymbol{v}-\boldsymbol{u}(\varepsilon))| \leq C||\boldsymbol{u}(\varepsilon)||_{1,\Omega}||\boldsymbol{v}||_{1,\Omega}.$$

The next important step consists in decoupling the covariant components of the test displacement field which occurs in the definition of the cone $\tilde{\boldsymbol{K}}(\varepsilon)(\Omega)$. This is done by means of the following lemma.

Lemma 5.1. *For all $\varepsilon > 0$, we take the new test function $\boldsymbol{w}(\varepsilon) \in \boldsymbol{H}^1(\Omega)$ associated with $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$ by*

$$(25)\qquad w_\alpha(\varepsilon) = v_\alpha, \quad w_3(\varepsilon) = \varepsilon v_\alpha \boldsymbol{g}^\alpha(\varepsilon)|_3 + v_3 \boldsymbol{g}^3(\varepsilon)|_3.$$

(i) The non-penetrability condition $\varepsilon v_\alpha \boldsymbol{g}^\alpha(\varepsilon)|_3 + v_3\boldsymbol{g}^3(\varepsilon)|_3 \geq -\varphi_3 + \boldsymbol{a}_3(\varepsilon)|_3 - 1$ now reads $w_3(\varepsilon) \geq -\varphi_3 + \boldsymbol{a}_3(\varepsilon)|_3 - 1$ on $\mathcal{B}_-$.
(ii) Let $z \in L^2(\Omega)$, then every integral involving v_3 can be replaced as follows

$$
\begin{cases}
\displaystyle\int_\Omega z v_3 dx &= \displaystyle\int_\Omega z w_3(\varepsilon)dx + \varepsilon^2\int_\Omega z w_i(\varepsilon) g^{\sharp,i}(\varepsilon,\boldsymbol{\varphi})dxdx,\\
\displaystyle\int_\Omega z\partial_k v_3 dx &= \displaystyle\int_\Omega z\partial_k w_3(\varepsilon)dx + \varepsilon^2\int_\Omega z\partial_k w_i(\varepsilon) g^{\sharp,i}(\varepsilon,\boldsymbol{\varphi})dx\\
&\quad + \varepsilon^2 \displaystyle\int_\Omega z w_i(\varepsilon) g^{\sharp,i}_k(\varepsilon,\boldsymbol{\varphi})dx,
\end{cases}
$$

where the remainders $g^{\sharp,i}$ and $g^{\sharp,i}_k$ are bounded in $L^2(\Omega)$, i.e., there exists a positive constant C independent of ε such that

$$\sup_{0\leq\varepsilon\leq\varepsilon_0}\max_{x\in\Omega}\Big(\sum_i |g^{\sharp,i}(\varepsilon,\boldsymbol{\varphi})(x)| + \sum_{i,k}|g^{\sharp,i}_k(\varepsilon,\boldsymbol{\varphi})(x)|\Big) \leq C.$$

Proof. The expansions (21) entail that $\boldsymbol{g}^3(\varepsilon)|_3 = \boldsymbol{a}^3(\varepsilon)|_3 = 1 + \varepsilon^2 \boldsymbol{g}^{3,2}|_3 + \cdots$ and $\boldsymbol{g}^\alpha(\varepsilon)|_3 = \varepsilon \boldsymbol{g}^{\alpha,1}|_3 + \cdots$, and we therefore have $v_3 = w_3(\varepsilon) + \varepsilon^2 w_i(\varepsilon) g^{\sharp,i}(\varepsilon, \boldsymbol{\varphi})$ and we obtain the first approximation. Next we note that

$$\begin{aligned} \partial_k v_3(\varepsilon) &= \partial_k w_3(\varepsilon) + \varepsilon^2 \partial_k w_i(\varepsilon) g^{\sharp,i}(\varepsilon, \boldsymbol{\varphi}) + \varepsilon^2 w_i(\varepsilon) \partial_k g^{\sharp,i}(\varepsilon, \boldsymbol{\varphi}), \\ &= \partial_k w_3(\varepsilon) + \varepsilon^2 \partial_k w_i(\varepsilon) g^{\sharp,i}(\varepsilon, \boldsymbol{\varphi}) + \varepsilon^2 w_i(\varepsilon) g_k^{\sharp,i}(\varepsilon, \boldsymbol{\varphi}) \end{aligned}$$

and the second approximation is obtained. $\square$

One immediate consequence of Lemma 5.1 is that the convex cone $\tilde{\boldsymbol{K}}(\varepsilon)(\Omega)$ can now be written as $\boldsymbol{K}(\varepsilon)(\Omega)$, involving only the third covariant component of the displacement field:

$$\boldsymbol{K}(\varepsilon)(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \mathcal{B}_0,\ v_3 \geq -\varphi_3 + \boldsymbol{a}_3(\varepsilon)|_3 - 1 \text{ on } \mathcal{B}_- \right\},$$

and problem (24) becomes:

$$(26)\quad \begin{cases} \text{Find } \boldsymbol{u}(\varepsilon) \in \boldsymbol{K}(\varepsilon)(\Omega) \text{ such that for all } \boldsymbol{v} \in \boldsymbol{K}(\varepsilon)(\Omega), \\ \varepsilon^4 \displaystyle\int_\Omega a^{\alpha\beta\sigma\tau} e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{\sigma||\tau}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \sqrt{a} \\ +\varepsilon^2 \displaystyle\int_\Omega a^{\alpha\beta 33}(\varepsilon) e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{3||3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \sqrt{a} \\ +\varepsilon^2 \displaystyle\int_\Omega a^{\alpha\beta 33} e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{\alpha||\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \sqrt{a} \\ +\varepsilon^2 \displaystyle\int_\Omega a^{\alpha 3\beta 3} e^{\boldsymbol{\varphi}}_{\alpha||3}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{\beta||3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \sqrt{a} \\ + \displaystyle\int_\Omega a^{3333} e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{u}(\varepsilon)) e^{\varphi}_{3||3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \sqrt{a}\ + B^\sharp(\varepsilon, \boldsymbol{\varphi}, \boldsymbol{u}(\varepsilon), \boldsymbol{v}) \\ \geq\ \varepsilon^4 \displaystyle\int_\Omega f^i (v_i - u_i(\varepsilon)) \sqrt{a} dx \\ +\varepsilon^4 \displaystyle\int_{\mathcal{B}_+} l^i (v_i - u_i(\varepsilon)) \sqrt{a} d\mathcal{B} + \varepsilon^2 L^\sharp(\varepsilon, \boldsymbol{\varphi}, \boldsymbol{v} - \boldsymbol{u}(\varepsilon)), \end{cases}$$

where quantities $B^\sharp$ and $L^\sharp$ stand for remainders which satisfy the uniform bounds: there exists a positive constant C independent of ε such that

$$|B^\sharp(\varepsilon, \boldsymbol{\varphi}, \boldsymbol{u}(\varepsilon), \boldsymbol{v} - \boldsymbol{u}(\varepsilon))| + |L^\sharp(\varepsilon, \boldsymbol{\varphi}, \boldsymbol{v} - \boldsymbol{u}(\varepsilon))| \leq C ||\boldsymbol{u}(\varepsilon)||_{1,\Omega} ||\boldsymbol{v}||_{1,\Omega}.$$

We are now ready to compute the limit bidimensional contact problem set in ω.

6. The limit problem

The main result of this study establishes that the three-dimensional equilibrium problem converges, when the thickness tends to zero, towards a model of shallow shell in unilateral contact with an obstacle, written in curvilinear coordinates. After the convergence result itself, we give the associated strong formulation of the limit problem, and it will be interesting both to come back to the physical domain for a better interpretation of the model and to verify that removing the unilateral conditions we recover a shallow shell model in curvilinear coordinates subjected to bilateral boundary conditions.

6.1. A convergence result. As noticed previously, we restrict our attention to the case where the three-dimensional domain is subjected to homogeneous Dirichlet boundary conditions everywhere on $\mathcal{B}_0$.

Theorem 6.1. *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{C}^3(\overline{\omega})$, $\boldsymbol{\varphi} : \overline{\omega} \longrightarrow \mathbb{R}^3$* with $\varphi_3 > 0$ in $\overline{\omega}$. *Then*
i) As ε tends to 0, the family $\{\boldsymbol{u}(\varepsilon)\}_{\varepsilon>0}$, which is the solution to problem (26), converges strongly in the cone:

$$\boldsymbol{K}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \boldsymbol{v} = \boldsymbol{0} \text{ on } \mathcal{B}_0,\ v_3 \geq -\varphi_3 \text{ on } \mathcal{B}_-\}.$$

ii) The space of in-plane displacements $\boldsymbol{V}_H(\omega)$ and the convex cone $K_3(\omega)$ are given by:

$$\begin{cases} \boldsymbol{V}_H(\omega) = \boldsymbol{H}_0^1(\omega), \\ \\ K_3(\omega) = \{\eta_3 \in H_0^2(\omega),\ \eta_3 \geq -\varphi_3 \text{ in } \omega\}. \end{cases}$$

Then, as ε tends to 0, the limit of $\mathbf{u}(\varepsilon)$ is a Kirchhoff-Love displacement field, namely, there exist $\boldsymbol{\zeta}_H = (\zeta_\alpha) \in \boldsymbol{V}_H(\omega)$ and $\zeta_3 \in K_3(\omega)$ such that

$$u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3,\ u_3 = \zeta_3.$$

iii) The function $\boldsymbol{\zeta}$ is the unique solution to the following problem:

$$(27)\quad \begin{cases} \text{Find } \boldsymbol{\zeta} = (\boldsymbol{\zeta}_H, \zeta_3) \in \boldsymbol{V}_H(\omega) \times K_3(\omega) \text{ such that} \\ \\ \dfrac{2}{3}\displaystyle\int_\omega b^{\alpha\beta\sigma\tau}\Big(\partial_{\alpha\beta}\zeta_3 - \Gamma^\kappa_{\alpha\beta}\partial_\kappa\zeta_3\Big)\Big(\partial_{\sigma\tau}(\eta_3 - \zeta_3) - \Gamma^\kappa_{\sigma\tau}\partial_\kappa(\eta_3 - \zeta_3)\Big)\sqrt{a} \\ \\ +2\displaystyle\int_\omega b^{\alpha\beta\sigma\tau} e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{\zeta}) e^{\boldsymbol{\varphi}}_{\sigma||\tau}(\boldsymbol{\eta} - \boldsymbol{\zeta})\sqrt{a} \\ \\ \geq \displaystyle\int_\omega p^i(\eta_i - \zeta_i)\sqrt{a}d\omega + \int_\omega s^\alpha\partial_\alpha(\eta_3 - \zeta_3)\sqrt{a}d\omega, \\ \\ \text{for all } \boldsymbol{\eta} = (\boldsymbol{\eta}_H, \eta_3) \in \boldsymbol{V}_H(\omega) \times K_3(\omega) \end{cases}$$

with $b^{\alpha\beta\gamma\delta} = \dfrac{2\lambda\mu}{\lambda+\mu}a^{\alpha\beta}a^{\gamma\delta} + \mu(a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma})$
and $p^i(x_1, x_2) = \displaystyle\int_{-1}^{1} f^i dx_3 + l^i(x_1, x_2, 1), \quad s^\alpha(x_1, x_2) = \int_{-1}^{1} x_3 f^\alpha dx_3 + l^\alpha(x_1, x_2, 1).$

6.2. Strong formulation. The strong formulation will be obtained from problem (27) after making two assumptions.

- *Assumption H1.* There exists a contact zone, which means that the domain ω is divided into two parts: a free part, say ω_f, which corresponds to points strictly above the obstacle, and a part corresponding to points in contact with the obstacle, say ω_c.
- *Assumption H2.* The boundary of the contact zone (the so-called *free boundary*) is sufficiently smooth to have a normal derivative everywhere. For a structure clamped at the boundary and loaded by a smooth enough force transverse to its middle surface, we can easily think of these two parts with ω_c homeomorphic to a disc and ω_f homeomorphic to an annulus.

Remark 4. *Assumption H1 is not restrictive since if it were not satisfied the problem would be bilateral, but there remain difficult problems concerning assumption H2. As a matter of fact we cannot say in general that ω_c is a smooth subdomain even if the domain ω and the loading $\boldsymbol{f}$ are $\boldsymbol{C}^\infty$, so that the smoothness of the free boundary has only been conjectured (see* [18]*). We shall comment about this hereafter.*

Let us introduce the internal resultants and momentum $\mathcal{N}^{\sigma\tau}$ and $\mathcal{M}^{\sigma\tau}$. They are defined as follows

$$\begin{cases} \mathcal{M}^{\sigma\tau} = b^{\alpha\beta\sigma\tau}\Big(\partial_{\alpha\beta}\zeta_3 - \Gamma^\kappa_{\alpha\beta}\partial_\kappa\zeta_3\Big), \\ \\ \mathcal{N}^{\sigma\tau} = b^{\alpha\beta\sigma\tau}e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{\zeta}) = b^{\alpha\beta\sigma\tau}\Big(\frac{1}{2}\Big(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha\Big) - \Gamma^\sigma_{\alpha\beta}\zeta_\sigma - b_{\alpha\beta}\zeta_3\Big). \end{cases}$$

Using these quantities, variational calculus starting from problem (27) and using assumption *H2* lead to the following result:

Theorem 6.2. *The strong formulation of the equilibrium problem of the shallow shell in unilateral contact is the following system, which should be understood in the sense of distributions:*

$$\begin{cases} -2\partial_\sigma\mathcal{N}^{\sigma\kappa} - 2\mathcal{N}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau} = p^\kappa, \\ \\ \frac{2}{3}\partial_{\sigma\tau}\mathcal{M}^{\sigma\tau} + \frac{2}{3}\partial_\kappa\mathcal{M}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau} - 2\mathcal{N}^{\sigma\tau}b_{\sigma\tau} \ \geq \ p^3 - \partial_\alpha s^\alpha, \\ \\ \zeta_3 \geq -\varphi_3, \\ \\ \left[\frac{2}{3}\partial_{\sigma\tau}\mathcal{M}^{\sigma\tau} + \frac{2}{3}\partial_\kappa\mathcal{M}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau} - 2\mathcal{N}^{\sigma\tau}b_{\sigma\tau} \ - p^3 + \partial_\alpha s^\alpha\right](\zeta_3 - \varphi_3) = 0, \end{cases} \tag{28}$$

or in another form:

$$\begin{cases} -2\partial_\sigma\mathcal{N}^{\sigma\kappa} - 2\mathcal{N}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau} = p^\kappa, \\ \\ \frac{2}{3}\partial_{\sigma\tau}\mathcal{M}^{\sigma\tau} + \frac{2}{3}\partial_\kappa\mathcal{M}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau} - 2\mathcal{N}^{\sigma\tau}b_{\sigma\tau} \ = \ p^3 - \partial_\alpha s^\alpha + \mu \\ \\ \zeta_3 \geq -\varphi_3, \\ \\ \mu \geq 0, \\ \\ \mu(\zeta_+\varphi_3) = 0. \end{cases} \tag{29}$$

Systems (28) or (29) must be satisfied in ω with the following boundary conditions, which should be understood in the sense of traces[3]

$$\zeta_\alpha = \zeta_3 = \partial\zeta_3 = 0 \ \textit{on} \ \ \partial\omega. \tag{30}$$

The proof of this result involves relatively intricate calculations: it is given in appendix. Here we will simply make a few comments.

[3]This clamping condition everywhere on $\partial\omega$ obviously results from the assumption $\boldsymbol{v} = \boldsymbol{0}$ on $\mathcal{B}_0$. If the three-dimensional domain was clamped only on a nonzero measure part of $\mathcal{B}_0$, we would have obtained a clamping boundary condition only on the corresponding part of $\partial\omega$, where the correspondence results from the mapping $\boldsymbol{\Phi}^\varepsilon$ and the scaling (15), and the complementary part will be stress free for instance.

(1) We verify that as long as the domain is three-dimensional, a clamping conditions means that the displacement must be equal to zero at the boundary, while the displacement and the normal derivative of its tranverse component must be equal to zero at the boundary in the case of a structure.

(2) It is important to observe that the inequality in the second line of system (28) has been changed into and equality in system (29). The quantity $\mu = \dfrac{2}{3}\partial_{\sigma\tau}\mathcal{M}^{\sigma\tau} + \dfrac{2}{3}\partial_\kappa \mathcal{M}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau} - 2\mathcal{N}^{\sigma\tau}b_{\sigma\tau} \; - p^3 + \partial_\alpha s^\alpha$ is a positive measure which can be interpreted as the reaction of the obstacle. In the present case, where the obstacle is smooth, and assuming that the external forces are smooth, this measure is smooth in the interior of nonzero measure parts of the contact zone, and singular both at the boundary of smooth parts of the contact zone and on zero measure parts (isolated point or lines) of the contact zone (see [12]). This singular part is specific to structures having a nonzero bending stiffness and would disappear in the case of the obstacle problem for a membrane.

(3) The strong formulation (28) is obtained formally, since it is based on a regularity result which is specific to obstacle problems: the solution is assumed to be such that any connected subset of the contact zone is limited by a smooth Jordan curve, or in other words, that the contact zone is smooth if all the data are smooth. This is the meaning of assumption *H2*. In the case of a $\boldsymbol{C}^\infty$ laoding, this regularity has been addressed as a conjecture in [18], which has been partly proved to be true only in the case of a particular membrane problem [16], and still remains to be proved in the case of plates or shells.

(4) The next comment which requires to be made is very important since it focuses on the difference between the meaning of the present strong formulation and the formulation resulting from usual asumptions in the case of bilateral problems. It has been proved in [10] that the solution to problems of this kind (obstacle problems with a linear fourth order operator and a flat obstacle) is in the Sobolev space H^3, even if the loading is very smooth. This means that contrary to bilateral problems, where we usually obtain a strong formulation by assuming that the solution is smooth enough to be able to perform all the derivatives required, we cannot assume here in general that the solution is smoother than $H^3(\omega)$, and so problems (28) or (29) must be understood in the sense of distributions.

(5) In the case of a one-dimensional structure (a beam), the H^3-regularity means that there is a match of the curvature at the boundary of the contact zone since H^3 is embedded into C^2. But this is false in the case of a two-dimensional domain. Nevertheless, it has been proved in [4] in the case of the biharmonic operator for a scalar problem (i.e. the transverse displacement of a linearly elastic plate at small strains) that the solution belongs to C^2, but no such result have been obtained for a shell.

6.3. Back to the physical domain Ω^ε. We return to the scaling of the unknowns (18) and the assumptions made in the data, and introduce a new limit displacement field $\boldsymbol{\zeta}^\varepsilon = (\zeta^\varepsilon_i)$ and test functions $\boldsymbol{\eta}^\varepsilon = (\zeta^\varepsilon_i)$, which are given by their covariant components

$$\begin{cases} \zeta_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2\zeta_\alpha(\varepsilon)(x), & \zeta_3^\varepsilon(x^\varepsilon) = \varepsilon\zeta_3(\varepsilon)(x), \\ \eta_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2\eta_\alpha(x), & \eta_3^\varepsilon(x^\varepsilon) = \varepsilon\eta_3(x). \end{cases}$$

We take $\boldsymbol{f}^\varepsilon(x^\varepsilon) = \varepsilon^2 f^\alpha \boldsymbol{g}_\alpha(\varepsilon) + \varepsilon^3 f^3 \boldsymbol{g}_3(\varepsilon)$ to denote the volume force and $\boldsymbol{l}^\varepsilon(x^\varepsilon) = \varepsilon^3 l^\alpha \boldsymbol{g}_\alpha(\varepsilon) + \varepsilon^4 l^3 \boldsymbol{g}_3(\varepsilon)$ to denote the surface force. Therefore the limit $\boldsymbol{\zeta}^\varepsilon$ is the solution to the following problem

$$\begin{cases} \text{Find } \boldsymbol{\zeta}^\varepsilon \in \boldsymbol{H}_0^1(\omega) \times K_3(\omega) \text{ such that for all } \boldsymbol{\eta}^\varepsilon \in H_0^1(\omega) \times H_0^1(\omega) \times K_3(\omega) \\ \dfrac{2}{3}\displaystyle\int_\omega b^{\alpha\beta\sigma\tau}\Big(\partial_{\alpha\beta}\zeta_3^\varepsilon - \Gamma_{\alpha\beta}^\kappa\partial_\kappa\zeta_3^\varepsilon\Big)\Big(\partial_{\sigma\tau}(\eta_3 - \zeta_3^\varepsilon) - \Gamma_{\sigma\tau}^\kappa\partial_\kappa(\eta_3 - \zeta_3^\varepsilon)\Big)\sqrt{a} \\ +2\displaystyle\int_\omega b^{\alpha\beta\sigma\tau} e_{\alpha||\beta}^{\boldsymbol{\varphi}}(\boldsymbol{\zeta}^\varepsilon) e_{\sigma||\tau}^{\boldsymbol{\varphi}}(\boldsymbol{\eta} - \boldsymbol{\zeta}^\varepsilon)\sqrt{a} \\ \geq \displaystyle\int_\omega p^{i,\varepsilon}(\eta_i - \zeta_i^\varepsilon)\sqrt{a}d\omega + \int_\omega s^{\alpha,\varepsilon}\partial_\alpha(\eta_3 - \zeta_3^\varepsilon)\sqrt{a}d\omega, \end{cases}$$

with $p^{i,\varepsilon} = \displaystyle\int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}dx_3^\varepsilon + \varepsilon l^{i,\varepsilon}(x_1, x_2, \varepsilon), \quad s^{\alpha,\varepsilon} = \int_{-\varepsilon}^{\varepsilon} x_3^\varepsilon f^{\alpha,\varepsilon}dx_3^\varepsilon + \varepsilon l^{\alpha,\varepsilon}(x_1, x_2, \varepsilon).$

6.4. The bilateral problem. All that has just been done can be used to find again the model of a shallow shell in curvilinear coordinates with bilateral boundary conditions, which means that the unilateral contact conditions are removed and replaced by usual Neumann boundary conditions. We start from the three-dimensional problem where the unilateral boundary conditions on $\widehat{\mathcal{B}}_-$ are replaced by homogeneous Neumann boundary conditions. This changes the cone $K_3(\omega)$ into a vector space

$$V_3(\omega) = \{\eta_3 \in H^2(\omega), \eta_3 = \partial_\nu\eta_3 = 0 \text{ on } \partial\omega\} \equiv H_0^2(\omega)$$

and not any inequalities remain in the equilibrium problem which turns to be the following variational equality:

$$\begin{cases} \text{Find } \boldsymbol{\zeta} = (\boldsymbol{\zeta}_H, \zeta_3) \in \boldsymbol{H}_0^1(\omega) \times H_0^2(\omega) \text{ such that} \\ \dfrac{2}{3}\displaystyle\int_\omega b^{\alpha\beta\sigma\tau}\Big(\partial_{\alpha\beta}\zeta_3 - \Gamma_{\alpha\beta}^\kappa\partial_\kappa\zeta_3\Big)\Big(\partial_{\sigma\tau}\eta_3 - \Gamma_{\sigma\tau}^\kappa\partial_\kappa\eta_3\Big)\sqrt{a} \\ +2\displaystyle\int_\omega b^{\alpha\beta\sigma\tau} e_{\alpha||\beta}^{\boldsymbol{\varphi}}(\boldsymbol{\zeta}) e_{\sigma||\tau}^{\boldsymbol{\varphi}}(\boldsymbol{\eta})\sqrt{a} \\ = \displaystyle\int_\omega p^i\eta_i\sqrt{a}d\omega + \int_\omega s^\alpha\partial_\alpha\eta_3\sqrt{a}d\omega \\ \text{for all } \boldsymbol{\eta} = (\boldsymbol{\eta}_H, \eta_3) \in \boldsymbol{V}_H(\omega) \times V_3(\omega) \end{cases} \tag{31}$$

with the same definition of p^i, s^α as above.

7. Conclusions

It is proposed to conclude this study by making some comments.

It is particularly worth noting that the present asymptotic analysis, yielding a convergence result, was carried out in the case of a problem associated with variational inequalities. After previously approaching this problem in the Cartesian framework, it was addressed here in terms of curvilinear coordinates. It is worth mentioning that the difference between a three-dimensional Signorini problem and a two-dimensional obstacle problem had to be clearly understood. More specifically, the local form of unilateral contact conditions in the three-dimensional domain results in a variational inequality on a convex cone. The two-dimensional limit is also characterized by a variational inequality on a convex cone, but in the latter case, the convex cone involves a positivity condition in the domain instead of at the boundary. This is how we approached the so-called obstacle problem. But the difference between a Signorini problem and an obstacle problem is probably more interesting from the point of view of the strong formulation. The strong three-dimensional Signorini problem involves an equation in the domain which is the same as in the usual bilateral case, and the inequalities hold on the boundary. But the two-dimensional obstacle problem involves an inequality in the domain, while the boundary conditions are the same as those of the usual bilateral problem. It is essential to keep in mind that, in addition to the external forces, the right hand side of the equilibrium equations involves the reaction of the obstacle, which is an unknown positive measure. Upon removing this positive quantity we obtain an inequality.

Analyses of this kind are still difficult to perform in the nonlinear case; this problem could be approached only formally in the framework of nonlinear strains, and is probably out of reach for the moment in the case of general nonlinear materials. But some interesting remaining problems could be completed at relatively short term in the linear case.

(1) The general case where the shell is not shallow: this subject should certainly be approched with care, but we are confident that the present study constitutes an important first step.
(2) The case of mappings φ with a lower regularity, giving rise in particular to discontinuities of curvature or folds, which have been dealt with in the bilateral case: coupling the non smoothness with unilateral contact conditions might be a very interesting problem.
(3) The regularity of the solution: we mentionned is subsection 6.2 that the solution of the obstacle problem is in H^3 in general, but we also mentionned that in fact the solution is in C^2 in the case of a linear plate, which means that the solution actually belongs to the Sobolev space $H^{3+\varepsilon}$, due to classical Sobolev embedding theorems [1]. It would be very interesting to get the same result in the case of a shallow shell, since the match of the curvature means the match of the momentum, so that this would mean on the one hand that there is no localized momentum at the boundary of the contact zone but only a localized resultant (in the form of Dirac measures) and on the other hand that the shell begins to come off the obstacle by the third derivative of the displacement.

8. Appendix

In this section we shall give the proof of theorems 1 et 2. This will require some lemmas obtained in, or simply deduced from, previous studies, which are recalled in a preliminary subsection.

8.1. Some lemmas. The first lemma is an important tool for the calculus of variations in variational inequalities. It was drawn up in [11].

Lemma 8.1. *Let K be defined as*

$$K \overset{def}{\equiv} \left\{v \in H^1(\Omega),\ v = 0 \text{ on } \mathcal{B}_0,\ v \geq -\varphi_3 \text{ on } \mathcal{B}_-\right\}$$

and let $u \in L^2(\Omega)$, v and $w \in K$ be such that

$$\int_\Omega u\partial_3(v-w)dx \geq 0 \quad \forall v \in K$$

Then $u = 0$.

The two following lemmas follow from [11] and [5]. They are necessary to establish the existence of a solution respectively to problem (26) and (27).

Lemma 8.2. *Let $\boldsymbol{\varphi} \in \boldsymbol{C}^3(\overline{\omega})$ be a given function. For any ε, the mapping*

$$\boldsymbol{v} \longrightarrow \left\{\sum_{i,j} \left|e^{\boldsymbol{\varphi}}_{i||j}(\boldsymbol{v})\right|^2_{0,\Omega}\right\}^{1/2}$$

is a norm over the cone $\boldsymbol{K}(\varepsilon)(\Omega)$, which is equivalent to the norm induced by $\|.\|_{1,\Omega}$.

Lemma 8.3. *The mapping*

$$\boldsymbol{\eta} \longrightarrow \left\{\sum_{\alpha\beta} \left|e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{\eta})\right|^2_{0,\Omega} + \sum_{\alpha\beta} \left|\partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3\right|^2_{0,\Omega}\right\}^{1/2}$$

is a norm over the cone $\boldsymbol{H}^1_0(\omega)\times K_3(\omega)$ which is equivalent to the norm of $\boldsymbol{H}^1(\omega)\times H^2(\omega)$.

8.2. An intermediate formulation. Taking $\boldsymbol{u}(\varepsilon) \in \boldsymbol{H}^1(\Omega)$, let us now introduce the following scaled symmetric tensor $\boldsymbol{R}(\varepsilon) = (R_{ij}(\varepsilon)) \in \boldsymbol{L}^2(\Omega)$ given by:

$$R_{\alpha\beta}(\varepsilon) = e^{\boldsymbol{\varphi}}_{\alpha\beta}(\boldsymbol{u}(\varepsilon)), \quad R_{\alpha 3}(\varepsilon) = \frac{1}{\varepsilon}e^{\boldsymbol{\varphi}}_{\alpha 3}(\boldsymbol{u}(\varepsilon)), \quad R_{33}(\varepsilon) = \frac{1}{\varepsilon^2}e^{\boldsymbol{\varphi}}_{33}\boldsymbol{u}(\varepsilon), \tag{32}$$

By substituting tensor $\boldsymbol{R}(\varepsilon)$ into problem (26), we obtain the variational inequality:

$$\begin{cases} \text{For all } \boldsymbol{v} \in \boldsymbol{K}(\varepsilon)(\Omega) \\ \displaystyle\int_\Omega \Big(a^{\alpha\beta\sigma\tau}R_{\alpha\beta}(\varepsilon) + a^{\alpha\beta 33}R_{33}(\varepsilon)\Big)e^{\boldsymbol{\varphi}}_{\sigma||\tau}(\boldsymbol{v}-\boldsymbol{u}(\varepsilon)) \\ \displaystyle+\frac{1}{\varepsilon^2}\int_\Omega \Big(a^{\alpha\beta 33}R_{\alpha\beta}(\varepsilon) + a^{3333}R_{33}(\varepsilon)\Big)e^{\boldsymbol{\varphi}}_{3||3}(\boldsymbol{v}-\boldsymbol{u}(\varepsilon)) \\ \displaystyle+\frac{4}{\varepsilon}\int_\Omega a^{\alpha 3\beta 3}R_{\alpha 3}(\varepsilon)e^{\boldsymbol{\varphi}}_{\beta||3}(\boldsymbol{v}-\boldsymbol{u}(\varepsilon)) + B^\sharp\Big(\varepsilon,\boldsymbol{\varphi},\boldsymbol{R}(\varepsilon),\boldsymbol{v}-\boldsymbol{u}(\varepsilon)\Big) \\ \geq \quad L(\boldsymbol{v}-\boldsymbol{u}(\varepsilon)) + \varepsilon L^\sharp\Big(\varepsilon,\boldsymbol{\varphi};\boldsymbol{v}-\boldsymbol{u}(\varepsilon)\Big), \end{cases} \tag{33}$$

where, due to the contact conditions, $\boldsymbol{K}(\varepsilon)(\Omega)$ is the convex cone

$$\boldsymbol{K}(\varepsilon)(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \mathcal{B}_0,\ v_3 \geq -\varphi_3 + \boldsymbol{a}_3(\varepsilon)|_3 - 1 \text{ on } \mathcal{B}_-\right\}$$

and where the work of the external loads $L(\boldsymbol{z})$ is given by $L(\boldsymbol{z}) = \displaystyle\int_\Omega f^i z_i\sqrt{a}dx + \int_{\mathcal{B}_+} l^i z_i\sqrt{a}d\mathcal{B}$. The quantities $B^\sharp$ and $L^\sharp$ are bounded remainders i.e. there exists a positive constant C independent of ε such that

$$\begin{cases} |B^\sharp(\varepsilon,\boldsymbol{\varphi},\boldsymbol{R}(\varepsilon),\boldsymbol{v}-\boldsymbol{u}(\varepsilon))| \le C\Big(||\boldsymbol{R}(\varepsilon)||_{0,\Omega}+||\boldsymbol{u}(\varepsilon)||_{1,\Omega}\Big)||\boldsymbol{v}||_{1,\Omega}. \\ |L^\sharp(\varepsilon,\boldsymbol{\varphi},\boldsymbol{v}-\boldsymbol{u}(\varepsilon))| \le C||\boldsymbol{u}(\varepsilon)||_{1,\Omega}||\boldsymbol{v}||_{1,\Omega}. \end{cases}$$

8.3. Proof of the convergence theorem. We adapt the proof given in [11] to deal with this proof, which is broken down into 5 steps.

- **Step 1.** Since $\boldsymbol{a}_3(\varepsilon)$ is a unit vector we have $-\varphi_3+\Big(\boldsymbol{a}_3(\varepsilon)|_3-1\Big)\le 0$. We can therefore take $\boldsymbol{v}=\boldsymbol{0}$ in Ω in problem (26), and it emerges that the sequence of solutions $\{\boldsymbol{u}(\varepsilon)\}_\varepsilon$ is bounded uniformly in ε in the space $\boldsymbol{H}^1(\Omega)$.
  Next we observe that the sequence of symmetric tensors $\{\boldsymbol{R}(\varepsilon)=(R_{ij}(\varepsilon))\}_\varepsilon$ given by (33) is bounded uniformly in ε in the space $\boldsymbol{L}^2(\Omega)$. Therefore there exist two subsequences, still indexed by ε, a limit displacement field $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$ and a limit scaled strain tensor $\boldsymbol{R}\in\boldsymbol{L}^2(\Omega)$ such that we have the weak convergence
  $$\boldsymbol{u}(\varepsilon)\rightharpoonup\boldsymbol{u}\text{ in }\boldsymbol{H}^1(\Omega)\quad\text{and}\quad\boldsymbol{R}(\varepsilon)\rightharpoonup\boldsymbol{R}\text{ in }\boldsymbol{L}^2(\Omega).$$
  In addition, based on the definition of $\boldsymbol{R}(\varepsilon)$, we can easily show that the weak limit $\boldsymbol{u}$ satisfies $e^{\boldsymbol{\varphi}}_{i3}(\boldsymbol{u})=\boldsymbol{0}$, which in turn means that $\boldsymbol{u}$ is a Kirchhoff-Love field, i.e., that there exists $\zeta_\alpha\in H^1(\omega)$ and $\zeta_3\in H^2(\omega)$ such that:
  $$u_\alpha=\zeta_\alpha-x_3\partial_\alpha\zeta_3,\quad u_3=\zeta_3.$$
  Once again, since $\boldsymbol{a}_3(\varepsilon)$ is a unit vector, we have $\boldsymbol{a}_3(\varepsilon)|_3-1<0$, and hence
  $$\boldsymbol{K}(\Omega)\overset{def}{\equiv}\big\{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega),\ \boldsymbol{v}=\boldsymbol{0}\text{ on }\mathcal{B}_0, v_3\ge-\varphi_3\text{ on }\mathcal{B}_-\big\}\subset\boldsymbol{K}(\varepsilon)(\Omega)$$
  and the limit $\boldsymbol{u}$ belongs to $\boldsymbol{K}(\Omega)$.
- **Step 2.** Let $v_3=u_3(\varepsilon)$, multiply inequality (33) by ε, let ε tends to zero and recall that v_α belongs to a vector space $\{v_\alpha\in H^1(\Omega), v_\alpha=0\text{ on }\partial\Omega\}$,
  $$\int_\Omega a^{\alpha3\beta3}R_{\alpha3}\partial_3(v_\beta-u_\beta)\sqrt{a}\ge0\quad\forall\ v_\alpha\in H^1(\Omega), v_\alpha=0\text{ on }\mathcal{B}_0.$$
  We note that $a^{\alpha3\beta3}=\mu a^{\alpha\beta}$. Based on Lemma 8.1, we therefore obtain $R_{\alpha3}=0$.
- **Step 3.** Now let $v_\alpha=u_\alpha(\varepsilon)$, multiply inequality (33) by ε^2 and let ε tends to zero,
  $$\begin{cases}\displaystyle\int_\Omega\Big(a^{\alpha\beta33}R_{\alpha\beta}+a^{3333}R_{33}\Big)\partial_3(v_3-u_3)\sqrt{a}\ge0\\ \forall v_3\in H^1(\Omega), v_\alpha=0\text{ on }\mathcal{B}_0, v_3\ge-\varphi_3\text{ on }\mathcal{B}_-.\end{cases}$$
  We observe that we have $a^{\alpha\beta33}=\lambda a^{\alpha\beta}, a^{3333}=\lambda+2\mu$, and hence that $\lambda a^{\alpha\beta}R_{\alpha\beta}+(\lambda+2\mu)R_{33}=0$.
- **Step 4.** Strong convergence. Since this step closely follows the similar step in [11], it is omitted here.
- **Step 5.** Let $\boldsymbol{v}\in\boldsymbol{K}(\Omega)$ be a Kirchhoff-Love test field, i.e. $e^\varphi_{i||3}(\boldsymbol{v})=0$ or $\partial_iv_3+\partial_3v_i=0, v_\alpha=\eta_\alpha-x_3\partial_\alpha\eta_3, v_3=\eta_3$ with $\boldsymbol{\eta}\in\boldsymbol{V}_H(\omega)\times K_3(\omega)$, and let ε tends to zero, then
  $$\int_\Omega\Big(a^{\alpha\beta\sigma\tau}R_{\alpha\beta}+a^{\sigma\tau33}R_{33}\Big)e^\varphi_{\sigma||\tau}(\boldsymbol{v}-\boldsymbol{u})\sqrt{a}\ge L(\boldsymbol{v}-\boldsymbol{u})\quad\forall\boldsymbol{v}\in\boldsymbol{K}(\Omega).$$

We note that $a^{\alpha\beta\sigma\tau}R_{\alpha\beta} + a^{\sigma\tau 33}R_{33} = b^{\alpha\beta\sigma\tau}R_{\alpha\beta}$, hence

$$\int_\Omega b^{\alpha\beta\sigma\tau}R_{\alpha\beta}e^{\varphi}_{\sigma||\tau}(\boldsymbol{v}-\boldsymbol{u})\sqrt{a} \geq L(\boldsymbol{v}-\boldsymbol{u}) \quad \forall \boldsymbol{v} \in \boldsymbol{K}(\Omega). \tag{34}$$

Replacing $\boldsymbol{u}$ and $\boldsymbol{v}$ by their Kirchhoff-Love fields $(\boldsymbol{\zeta},\boldsymbol{\eta})$ in the expression of $e^{\varphi}_{\sigma||\tau}$ yields

$$\begin{cases} e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{u}) = R_{\alpha\beta} = e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{\zeta}) - x_3\Big(\partial_{\alpha\beta}\zeta_3 - \Gamma^{\kappa}_{\alpha\beta}\partial_\kappa\zeta_3\Big), \\ e^{\boldsymbol{\varphi}}_{\sigma||\tau}(\boldsymbol{v}) = e^{\boldsymbol{\varphi}}_{\sigma||\tau}(\boldsymbol{\eta}) - x_3\Big(\partial_{\sigma\tau}\eta_3 - \Gamma^{\kappa}_{\sigma\tau}\partial_\kappa\eta_3\Big). \end{cases}$$

The left-hand side of (34) can therefore be changed as follows

$$\begin{aligned} &\int_\Omega b^{\alpha\beta\sigma\tau}R_{\alpha\beta}e^{\varphi}_{\sigma||\tau}(\boldsymbol{v}-\boldsymbol{u})\sqrt{a} \\ =\; &\int_\Omega b^{\alpha\beta\sigma\tau}\Big(e^{\varphi_3}_{\alpha||\beta}(\boldsymbol{\zeta}) - x_3\Big(\partial_{\alpha\beta}\zeta_3 - \Gamma^{\sigma}_{\alpha\beta}\partial_\sigma\zeta_3\Big)\Big(e^{\varphi_3}_{\sigma||\tau}(\boldsymbol{\eta}-\boldsymbol{\zeta}) - x_3\Big(\partial_{\sigma\tau}\eta_3 - \Gamma^{\kappa}_{\sigma\tau}\partial_\kappa\eta_3\Big)\sqrt{a} \\ =\; &\frac{2}{3}\int_\omega b^{\alpha\beta\sigma\tau}\Big(\partial_{\alpha\beta}\zeta_3 - \Gamma^{\sigma}_{\alpha\beta}\partial_\sigma\zeta_3\Big)\Big(\partial_{\sigma\tau}\eta_3 - \Gamma^{\kappa}_{\sigma\tau}\partial_\kappa\eta_3\Big)\sqrt{a} + 2\int_\omega b^{\alpha\beta\sigma\tau}e^{\boldsymbol{\varphi}}_{\sigma||\tau}(\boldsymbol{\zeta})e^{\boldsymbol{\varphi}}_{\sigma||\tau}(\boldsymbol{\eta}-\boldsymbol{\zeta})\sqrt{a}. \end{aligned}$$

8.4. Proof of the strong formulation. The following notations are classical:

$$\begin{cases} \mathcal{M}^{\sigma\tau} = b^{\alpha\beta\sigma\tau}\Big(\partial_{\alpha\beta}\zeta_3 - \Gamma^{\kappa}_{\alpha\beta}\partial_\kappa\zeta_3\Big), \\ \mathcal{N}^{\sigma\tau} = b^{\alpha\beta\sigma\tau}e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{\zeta}) = b^{\alpha\beta\sigma\tau}\Big(\frac{1}{2}\Big(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha\Big) - \Gamma^{\sigma}_{\alpha\beta}\zeta_\sigma - b_{\alpha\beta}\zeta_3\Big). \end{cases}$$

We then compute the two integrals of the left hand side of the two-dimensional variational inequality (27) by repeatedly integrating by parts

$$\begin{cases} \displaystyle\int_\omega \mathcal{M}^{\sigma\tau}\Big(\partial_{\sigma\tau}(\eta_3-\zeta_3) - \Gamma^{\kappa}_{\sigma\tau}\partial_\kappa(\eta_3-\zeta_3)\Big)\sqrt{a} \\ \quad = \displaystyle\int_\omega \Big(\partial_{\sigma\tau}\mathcal{M}^{\sigma\tau} + \partial_\kappa(\mathcal{M}^{\sigma\tau}\Gamma^{\kappa}_{\sigma\tau})\Big)(\eta_3-\zeta_3)\sqrt{a} \\ \quad\quad + \displaystyle\int_{\partial\omega} \mathcal{M}^{\sigma\tau}\partial_\tau(\eta_3-\zeta_3)n_\sigma\sqrt{a} - \int_{\partial\omega}\partial_\sigma\mathcal{M}^{\sigma\tau}(\eta_3-\zeta_3)n_\tau\sqrt{a} \\ \quad\quad - \displaystyle\int_{\partial\omega}\mathcal{M}^{\sigma\tau}\Gamma^{\kappa}_{\sigma\tau}(\eta_3-\zeta_3)n_\kappa\sqrt{a}, \end{cases}$$

$$\left\{\begin{aligned}
&\int_\omega \mathcal{N}^{\sigma\tau} e^{\boldsymbol{\varphi}}_{\sigma||\tau}(\boldsymbol{\eta}-\boldsymbol{\zeta})\sqrt{a}\\
&\quad = \int_\omega \Big(\mathcal{N}^{\sigma\tau}\partial_\sigma(\eta_\tau-\zeta_\tau) - \Gamma^\kappa_{\sigma\tau}(\eta_\kappa-\zeta_\kappa) - b_{\sigma\tau}(\eta_3-\zeta_3)\Big)\sqrt{a}\\
&\quad = \int_\omega -\partial_\sigma\mathcal{N}^{\sigma\tau}(\eta_\tau-\zeta_\tau)\sqrt{a} \;+\; \int_{\partial\omega}\mathcal{N}^{\sigma\tau}(\eta_\tau-\zeta_\tau)n_\sigma\sqrt{a}\\
&\qquad - \int_\omega \Big(\mathcal{N}^{\sigma\tau}b_{\sigma\tau}(\eta_3-\zeta_3) + \mathcal{N}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau}(\eta_\kappa-\zeta_\kappa)\Big)\sqrt{a}\\
&\quad = \int_\omega \Big(-\partial_\sigma\mathcal{N}^{\sigma\kappa} - \mathcal{N}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau}\Big)(\eta_\kappa-\zeta_\kappa)\sqrt{a}\\
&\qquad - \int_\omega \mathcal{N}^{\sigma\tau}b_{\sigma\tau}(\eta_3-\zeta_3)\sqrt{a} + \int_{\partial\omega}\mathcal{N}^{\sigma\tau}(\eta_\tau-\zeta_\tau)n_\sigma\sqrt{a}.
\end{aligned}\right.$$

From these calculations, the left hand side of (27) reads:

$$\tag{35}\left\{\begin{aligned}
&\frac{2}{3}\int_\omega b^{\alpha\beta\sigma\tau}\Big(\partial_{\alpha\beta}\zeta_3 - \Gamma^\kappa_{\alpha\beta}\partial_\kappa\zeta_3\Big)\Big(\partial_{\sigma\tau}(\eta_3-\zeta_3) - \Gamma^\kappa_{\sigma\tau}\partial_\kappa(\eta_3-\zeta_3)\Big)\sqrt{a}\\
&+ 2\int_\omega b^{\alpha\beta\sigma\tau} e^{\boldsymbol{\varphi}}_{\alpha||\beta}(\boldsymbol{\zeta}) e^{\boldsymbol{\varphi}}_{\sigma||\tau}(\boldsymbol{\eta}-\boldsymbol{\zeta})\sqrt{a}\\
&= \int_\omega \Big(\frac{2}{3}\partial_{\sigma\tau}\mathcal{M}^{\sigma\tau} + \frac{2}{3}\partial_\kappa(\mathcal{M}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau}) - 2\mathcal{N}^{\sigma\tau}b_{\sigma\tau}\Big)(\eta_3-\zeta_3)\sqrt{a}\\
&+ \int_\omega \Big(-2\partial_\sigma\mathcal{N}^{\sigma\tau} - 2\mathcal{N}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau}\Big)(\eta_\kappa-\zeta_\kappa)\sqrt{a}\\
&+ \frac{2}{3}\int_{\partial\omega}\mathcal{M}^{\sigma\tau}\partial_\tau(\eta_3-\zeta_3)n_\sigma\sqrt{a} - \frac{2}{3}\int_{\partial\omega}\partial_\sigma\mathcal{M}^{\sigma\tau}(\eta_3-\zeta_3)n_\tau\sqrt{a}\\
&-\frac{2}{3}\int_{\partial\omega}\mathcal{M}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau}(\eta_3-\zeta_3)n_\kappa\sqrt{a} \;+2\int_{\partial\omega}\mathcal{N}^{\sigma\tau}(\eta_\tau-\zeta_\tau)n_\sigma\sqrt{a}.
\end{aligned}\right.$$

According to assumptions *H1* and *H2* of subsection 6.2, all the integrals over ω must be understood as integrals over $\omega_c \cup \omega_f$. Moreover, performing the calculations separately in ω_f and ω_c, with two successive integrations by parts in each subdomain, results in jumps of the displacement and of its first order normal derivative across the boundary between the subdomains (i.e. accross $\partial\omega_c \equiv \partial\omega_c \cap \partial\omega_f$ due to assumption *H2*). But it is known that the functional framework, involving H^2 for the normal component of the displacement, allows to cancel the jumps of the displacement. Moreover, the H^3-regularity result obtained in [10] allows in addition to remove the jumps of the first order normal derivative on $\partial\omega_c$, so that only remain as boundary terms the integrals over $\partial\omega$, as written in equation (35). Now we get the strong formulation, using remark 4 and assumptions *H1* and *H2*.

Let us first choose test functions such that $\eta_3 = -\varphi_3$ in ω_c, we obtain in ω_f:

$$\tag{36}\left\{\begin{aligned}
&-2\partial_\sigma\mathcal{N}^{\sigma\tau} - 2\mathcal{N}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau} \;=\; p^\kappa,\\
&\frac{2}{3}\partial_{\sigma\tau}\mathcal{M}^{\sigma\tau} + \frac{2}{3}\partial_\kappa(\mathcal{M}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau}) - 2\mathcal{N}^{\sigma\tau}b_{\sigma\tau} = p^3 - \partial_\alpha s^\alpha,
\end{aligned}\right.$$

Then we go back to (27) with (35) and we observe that the difference $\eta_3 - \zeta_3$ is positive in ω_c by definition, so that, using (36), we obtain in ω_c

$$(37) \qquad \begin{cases} -2\partial_\sigma \mathcal{N}^{\sigma\tau} - 2\mathcal{N}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau} \;=\; p^\kappa, \\ \\ \dfrac{2}{3}\partial_{\sigma\tau}\mathcal{M}^{\sigma\tau} + \dfrac{2}{3}\partial_\kappa(\mathcal{M}^{\sigma\tau}\Gamma^\kappa_{\sigma\tau}) - 2\mathcal{N}^{\sigma\tau} b_{\sigma\tau} \geq p^3 - \partial_\alpha s^\alpha. \end{cases}$$

Formula (36) and (37) imply that formula (28) holds in $\omega \setminus \partial\omega_c$. Due to the fact that there does not remain any jump on $\partial\omega_c$ and that, from assumption *H2*, the contact zone ω_c is a subdomain equal to the adherence of its interior, the boundary terms are only on $\partial\omega$, which is the classical bilateral situation. They read simply:

$$\zeta_\alpha = \zeta_3 = \partial\zeta_3 = 0 \text{ on } \;\partial\omega$$

Of course, if part of $\mathcal{B}_0$ is not clamped but stress free or subjected to external loading, then we would have boundary conditions on this part involving $\mathcal{N}^{\sigma\tau}$ or $\mathcal{M}^{\sigma\tau}$, but these conditions would be the same as those occuring in the bilateral problem, and have been given in [3].

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