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MULTISCALE ANALYSIS AND COMPUTATION FOR PARABOLIC EQUATIONS WITH RAPIDLY OSCILLATING COEFFICIENTS IN GENERAL DOMAINS

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Abstract. This paper presents the multiscale analysis and computation for parabolic equations with rapidly oscillating coefficients in general domains. The major contributions of this study are twofold. First, we define the boundary layer solution and the convergence rate with $\varepsilon^{1/2}$ for the multiscale asymptotic solutions in general domains. Secondly, a highly accurate computational algorithm is developed. Numerical simulations are then carried out to validate the theoretical results.

Key words. Parabolic equation with rapidly oscillating coefficients, homogenization, multiscale analysis, finite element method.

1. Introduction

We consider the initial-boundary value problems for second order parabolic equations with rapidly oscillating coefficients as follows:

$$(1)\qquad \begin{cases} \dfrac{\partial u^\varepsilon(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\big(a^\varepsilon_{ij}(x,t)\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\big) = f(x,t), \quad (x,t)\in\Omega\times(0,T) \\ u^\varepsilon(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times(0,T) \\ u^\varepsilon(x,0) = \bar{u}_0(x), \end{cases}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded convex polygonal domain with the boundary $\partial\Omega$. $f(x,t)$, $g(x,t)$, $\bar{u}_0(x)$ are known functions. In this study, we consider the following specific cases for the coefficients $a^\varepsilon_{ij}(x,t)$: i.e. $a^\varepsilon_{ij}(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, and $k = 0,1,2,3$.

Let $\xi = \varepsilon^{-1}x,\ \tau = \varepsilon^{-k}t,\ k = 0,1,2,3$. We make the following assumptions:

$(\mathbf{A_1})$ For $k = 1,2,3,\ a_{ij}(\xi,\tau)$ are 1-periodic and τ_0-periodic in ξ,τ, respectively. For $k=0,\ a_{ij}(\xi,t)$ are 1-periodic in ξ.

$(\mathbf{A_2})$ $a_{ij} = a_{ji}$, $\gamma_0|\eta|^2 \le a_{ij}(\xi,\tau)\eta_i\eta_j \le \gamma_1|\eta|^2$, $\gamma_0,\gamma_1>0$, $\forall(\eta_1,\cdots,\eta_n)\in\mathbb{R}^n$, where γ_0, γ_1 are constants independent of ε.

$(\mathbf{A_3})$ Let $Q=(0,1)^n$ be the reference cell of composite materials with a periodic microstructure, $Q\subset\subset Q'$ and $Q' = (\bigcup_{m=1}^{L}\overline{D}_m)\setminus\partial Q'$. Suppose that the boundaries ∂D_m are $C^{1,\gamma}$ for some $0<\gamma<1$. $a^\varepsilon_{ij}(x,t)\in C^{\mu,\infty}(\overline{D}_m\times(0,T))$, $i,j=1,2,\cdots,n$ for some constants $0<\mu<1$.

$(\mathbf{A_4})$ $f\in L^2(0,T;L^2(\Omega))$, $g\in L^2(0,T;H^{1/2}(\partial\Omega))$, $\bar{u}_0\in H^1(\Omega)$.

Problem (1) arises frequently in modeling the heat and mass transfer problem in composite materials or porous media (see, e.g., [11]). It involves materials with

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a large number of heterogeneities (inclusions or holes). For homogenization results concerning linear parabolic equations with rapidly oscillating coefficients which depend on the spatial and time variables, we refer to Bensoussan, Lions and Papanicolaou [2] for periodic cases and to Colombini and Spagnolo [7] for the general non-periodic case. For a type of nonlinear parabolic partial differential operators, Pankov [23] and Svanstedt [24] derived the G-convergence and the homogenization results. Zhikov, Kozlov and Oleinik [26] investigated parabolic operators with almost periodic coefficients and presented convergence results for the asymptotic homogenization.

By introducing the cutoff function, Bensoussan, Lions and Papanicolaou (cf. [2]) obtained the strong convergence result without an explicit rate for the first-order corrector of the solution of linear parabolic equations in $L^2(0,T;H^1(\Omega))$. Brahim-Otsmane, Francfort and Murat (cf. [3]) extended this result to $L^2(0,T;W^{1,1}(\Omega))$. Ming and Zhang (cf.[21]) derived the convergence result with an explicit rate $\varepsilon^{1/2}$ for the case $k=0$ under the assumption $u^0\in H^{3,1}(\Omega\times(0,T))$, where $u^0(x,t)$ is the solution of the linear homogenized parabolic equation. Allegretto, Cao and Lin (cf. [1]) investigated the higher-order multiscale method for linear parabolic equations in four specific cases $k=0,1,2,3$, and derived the convergence results with an explicit rate $\varepsilon^{1/2}$ under the assumption $u^0\in H^{s+2,1}(\Omega\times(0,T))$, $s=1,2$. It is well known that, for a bounded polygonal Lipschitz domain Ω, the assumptions $u^0\in H^{s+2,1}(\Omega\times(0,T))$, $s=1,2$ may be invalid. Thus the error estimates in [1] fail. In this study, we present the following two major contributions. First, we define the boundary layer solution and derive the convergence results with an explicit rate $\varepsilon^{1/2}$ for the multiscale asymptotic solutions in a bounded polygonal Lipschitz domain Ω. Secondly, we present a highly accurate computational algorithm.

The remainder of this paper is organized as follows. Section 2 is devoted to the proofs of the main convergence results for the multiscale asymptotic method. In Section 3, we discuss finite element computations and the error estimates for the related problems. In particular, a new computational scheme is proposed to solve the boundary layer solutions numerically. In Section 4, a finite element post-processing technique and a numerical method with high accuracy are presented. Finally, numerical simulations are carried out to validate the theoretical results reported in this paper.

Throughout the paper the Einstein summation convention on repeated indices is adopted. By C we shall denote a positive constant independent of ε.

2. Multiscale Asymptotic Expansions and the Convergence Results

In this section, we first introduce the multiscale asymptotic expansions for problem (1) which has been investigated in [1]. Then we define the boundary layer solutions and derive the convergence results for the modified multiscale asymptotic solutions.

Let $\xi=\varepsilon^{-1}x$, $\tau=\varepsilon^{-k}t$, $k=0,1,2,3$. For the four specific cases $k=0,1,2,3$, following the idea of [1], we define the formal multiscale asymptotic expansions of the solution for problem (1) given by

$$
\begin{aligned}
u_1^\varepsilon(x,t)&=u^0(x,t)+\varepsilon N_{\alpha_1}(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_{\alpha_1}},\\
u_2^\varepsilon(x,t)&=u^0(x,t)+\varepsilon N_{\alpha_1}(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_{\alpha_1}}+\varepsilon^2 N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2 u^0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}},
\end{aligned}
\tag{2}
$$

where the cell functions $N_{\alpha_1}(\xi,\tau)$, $N_{\alpha_1\alpha_2}(\xi,\tau)$, $\alpha_1,\alpha_2 = 1,2,\cdots,n$ are given in [1]. The function $u^0(x,t)$ is the solution of the homogenized parabolic equation and can be computed from:

$$\text{(3)}\qquad \begin{cases} \dfrac{\partial u^0(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\big(\hat{a}_{ij}\dfrac{\partial u^0(x,t)}{\partial x_j}\big) = f(x,t), \quad (x,t)\in\Omega\times(0,T) \\ u^0(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times(0,T) \\ u^0(x,0) = \bar{u}_0(x), \end{cases}$$

where $(\hat{a}_{ij})$ is the homogenized coefficients tensor given in [1]. It can be proved that $(\hat{a}_{ij})$ is a symmetric and positive-definite matrix.

As already mentioned in Section 1, in order to derive the convergence results with an explicit rate $\varepsilon^{1/2}$ for the multiscale asymptotic solutions defined in (2), we must assume that $u^0 \in H^{s+2,1}(\Omega\times(0,T))$. However, for a bounded polygonal Lipschitz domain Ω, generally speaking, the assumptions $u^0 \in H^{s+2,1}(\Omega\times(0,T))$, $s=1,2$ may be invalid. Moreover, the multiscale asymptotic solutions $u_s^\varepsilon(x,t)$, $s=1,2$ do not satisfy the boundary conditions on $\partial\Omega$ in a general domain. To overcome these difficulties, we define the boundary layer solutions. Now we introduce the notation, let Ω_0 be a subdomain of the whole domain Ω consisting of the union of periodic cells, i.e. $\Omega_0 = \bigcup_{z\in I_\varepsilon}\varepsilon(z+\overline{Q})$, where $I_\varepsilon = \{z\in Z^n, \varepsilon(z+\overline{Q}) \subset\subset \Omega\}$, $dist(\partial\Omega_0,\partial\Omega) > 2\varepsilon$, and $\Omega_1 = \Omega\setminus\overline{\Omega}_0$. They are illustrated as in Fig.1 (a) and (b).

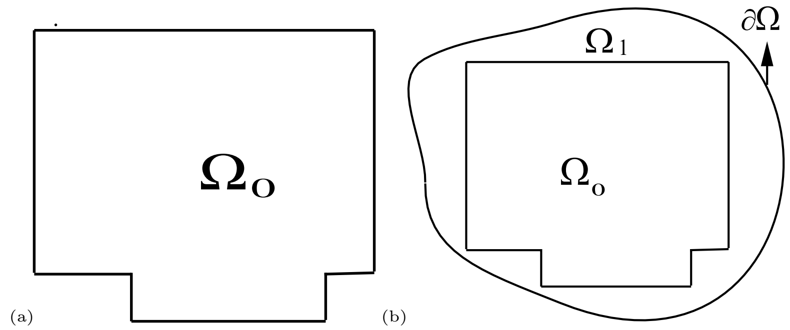

FIGURE 1. (a) Interior subdomain Ω_0, (b) the boundary layer Ω_1.

We define the boundary layer solutions given by

$$\text{(4)}\qquad \begin{cases} \dfrac{\partial u_s^{\varepsilon,b}(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\big(a_{ij}^\varepsilon(x,t)\dfrac{\partial u_s^{\varepsilon,b}(x,t)}{\partial x_j}\big) = f(x,t), \quad (x,t)\in\Omega_1\times(0,T) \\ u_s^{\varepsilon,b}(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times(0,T), \\ u_s^{\varepsilon,b}(x,t) = u_s^\varepsilon(x,t), \quad (x,t)\in(\partial\Omega_0\cap\partial\Omega_1)\times(0,T), \\ u_s^{\varepsilon,b}(x,0) = \bar{u}_0(x), \end{cases}$$

where $u_s^\varepsilon(x,t)$, $s=1,2$ are defined in (2).

Remark 2.1 Existence and uniqueness of the boundary layer solutions can be established under the assumptions $(A_2)-(A_4)$. Hence we define the multiscale asymptotic solutions for problem (1) as follows:

$$\text{(5)}\qquad U_s^\varepsilon(x,t) = \begin{cases} u_s^\varepsilon(x,t), \quad (x,t)\in\overline{\Omega}_0\times(0,T), \\ u_s^{\varepsilon,b}(x,t), \quad (x,t)\in\Omega_1\times(0,T), \end{cases}$$

where $s=1,2$.

Next, we derive the interior error estimates for the multiscale asymptotic solutions $u_s^\varepsilon(x,t)$, $s=1,2$, and then employ the boundary layer solutions to obtain the error estimates in the whole domain Ω.

Theorem 2.1 Suppose that $\Omega\subset\mathbb{R}^n$, $n\geq 1$ is a bounded convex polygonal domain. Let $u^\varepsilon(x,t)$ be a weak solution of problem (1), and let $u^0(x,t)$ be the solution of the homogenized parabolic equation (3), and $u_1^\varepsilon(x,t)$, $u_2^\varepsilon(x,t)$ be the first-order and the second-order multiscale asymptotic solutions defined in (2), respectively. Here, $f\in L^2(0,T;L^2(\Omega))\cap H^1(0,T;H^s(\Omega''))$, $g\in L^2(0,T;H^{1/2}(\partial\Omega))$, $\bar{u}_0\in H^1(\Omega)\cap H^{s+1}(\Omega'')$, where $\Omega_0\subset\subset\Omega''\subset\subset\Omega$. For the specified case $k=0$, we assume that $a_{ij}(\frac{x}{\varepsilon},t)\in C^1(0,T)$ for any fixed $x\in\Omega$. For $k=2$, we assume that $u^0\in H^{3,1}(\Omega\times(0,T))$. Under the assumptions $(A_1)-(A_4)$, then it holds

$$(6)\qquad \begin{aligned}&\sup_{0\leq t\leq T}\int_{\Omega_0}(u^\varepsilon(x,t)-u_s^\varepsilon(x,t))^2\mathrm{d}x+\int_0^T\|u^\varepsilon(x,t)-u_s^\varepsilon(x,t)\|^2_{H^1(\Omega_0)}\mathrm{d}t\\ &\leq\begin{cases}C(T)\varepsilon, & \text{for}\quad k=0;\quad s=1,2\\ C(T)\varepsilon, & \text{for}\quad k=1,3;\quad s=2\\ C(T)\varepsilon^2, & \text{for}\quad k=2;\quad s=2\end{cases}\end{aligned}$$

where $C(T)$ is a positive constant independent of ε but dependent of T.

Proof. First, we introduce the following subdomains:

$$\begin{aligned}\Omega'&=\{x\in\Omega:\quad \text{if}\quad \mathrm{dist}(x,\partial\Omega)\geq\varepsilon/2\},\\ K_\varepsilon&=\{x\in\Omega:\quad \text{if}\quad \mathrm{dist}(x,\partial\Omega)\leq 2\varepsilon\},\\ K'_\varepsilon&=\{x\in\Omega:\quad \text{if}\quad \varepsilon\leq\mathrm{dist}(x,\partial\Omega)\leq 2\varepsilon\}.\end{aligned}$$

It is obvious that $\Omega_0\subset\subset\Omega'\subset\subset\Omega$. Under the assumptions of the theorem and using the interior regularity of linear parabolic equations, we can conclude that $u^0\in H^1(0,T;H^{s+2}(\Omega'))$, $s=1,2$. Let introduce the cutoff function $m_\varepsilon(x)$ given by

$$(7)\qquad \begin{aligned}&m_\varepsilon\in\mathcal{D}(\Omega)\\ &m_\varepsilon=0,\quad\text{if}\quad \mathrm{dist}(x,\partial\Omega)\leq\varepsilon\\ &m_\varepsilon=1,\quad\text{if}\quad \mathrm{dist}(x,\partial\Omega)\geq 2\varepsilon\\ &\varepsilon|\frac{\partial m_\varepsilon}{\partial x_i}|\leq C,\quad i=1,2,\cdots,n.\end{aligned}$$

For cases $k=0,1,3$, define

$$(8)\qquad \begin{aligned}\theta_1^\varepsilon(x,t)&=u^0(x,t)+\varepsilon m_\varepsilon(x)N_{\alpha_1}(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_{\alpha_1}}\\ \theta_2^\varepsilon(x,t)&=u^0(x,t)+m_\varepsilon(x)\big[\varepsilon N_{\alpha_1}(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_{\alpha_1}}+\varepsilon^2N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2u^0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\big].\end{aligned}$$

For $k=2$, define

$$(9)\qquad \theta_2^\varepsilon(x,t)=u^0(x,t)+\varepsilon N_{\alpha_1}(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_{\alpha_1}}+\varepsilon^2m_\varepsilon(x)N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2u^0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}.$$

Let $\Omega_t=\Omega\times(0,t)$ and define

$$(u,v)_{\Omega_t}=\int_0^t\int_\Omega uv\mathrm{d}x\mathrm{d}t,\quad \mathcal{L}_\varepsilon w\equiv-\frac{\partial}{\partial x_i}\big(a_{ij}^\varepsilon(x,t)\frac{\partial w}{\partial x_j}\big).$$

Here we prove Theorem 2.1 only for the case $k=2$, $s=2$. The other cases can be shown similarly. Without loss of generality, we assume that $g(x,t)\equiv 0$. For $\forall v\in L^2(0,t;H_0^1(\Omega))$, recalling (2.2c)-(2.2d) of [1], (1),(8) and by complex computations, we obtain the following equation in the sense of distributions:

$$(10)\qquad (\frac{\partial(u^\varepsilon-\theta_2^\varepsilon)}{\partial t},v)_{\Omega_t}+(\mathcal{L}_\varepsilon(u^\varepsilon-\theta_2^\varepsilon),v)_{\Omega_t}=J_2^\varepsilon(v),$$

where

(11)

$$
\begin{aligned}
&J_2^\varepsilon(v)\\
&= -\big(\varepsilon N_{\alpha_1}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial t}, v\big)_{\Omega_t} - \big(\varepsilon^2 m_\varepsilon(x) N_{\alpha_1\alpha_2}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial t}, v\big)_{\Omega_t}\\
&\quad -\big((m_\varepsilon(x)-1)\frac{\partial N_{\alpha_1\alpha_2}}{\partial \tau}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, v\big)_{\Omega_t} - \big((m_\varepsilon(x)-1)\frac{\partial}{\partial \xi_i}(a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j})\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, v\big)_{\Omega_t}\\
&\quad -\big(\varepsilon a_{ij} N_{\alpha_1}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_j\partial x_i}, v\big)_{\Omega_t} - \big(\varepsilon \frac{\partial m_\varepsilon(x)}{\partial x_i} a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, v\big)_{\Omega_t}\\
&\quad -\big(\varepsilon \frac{\partial m_\varepsilon(x)}{\partial x_j}\frac{\partial}{\partial \xi_i}(a_{ij}N_{\alpha_1\alpha_2})\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, v\big)_{\Omega_t} - \big(\varepsilon^2 \frac{\partial m_\varepsilon(x)}{\partial x_j} a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i}, v\big)_{\Omega_t}\\
&\quad -\big(\varepsilon^2 \frac{\partial^2 m_\varepsilon(x)}{\partial x_i\partial x_j} a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, v\big)_{\Omega_t} - \big(\varepsilon m_\varepsilon(x) a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i}, v\big)_{\Omega_t}\\
&\quad -\big(\varepsilon^2 \frac{\partial m_\varepsilon(x)}{\partial x_i} a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j}, v\big)_{\Omega_t} - \big(\varepsilon m_\varepsilon(x)\frac{\partial}{\partial \xi_i}(a_{ij}N_{\alpha_1\alpha_2})\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j}, v\big)_{\Omega_t}\\
&\quad -\big(\varepsilon^2 m_\varepsilon(x) a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^4 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j\partial x_i}, v\big)_{\Omega_t}.
\end{aligned}
$$

Under the assumptions $(A_1)-(A_4)$, it follows from Theorem 1.2 of [16] that $N_{\alpha_1}, N_{\alpha_1\alpha_2}, \dfrac{\partial N_{\alpha_1}}{\partial\tau}, \dfrac{\partial N_{\alpha_1\alpha_2}}{\partial\tau} \in L^2(0,\tau_0;W^{1,\infty}(Q))$. If we assume that $u^0\in H^{3,1}(\Omega\times(0,T))$, then we can verify that

$$
\begin{aligned}
&|\big(\varepsilon N_{\alpha_1}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial t}, v\big)_{\Omega_t}| \le C\varepsilon\|\frac{\partial u^0}{\partial t}\|_{L^2(0,t;H^1(\Omega))}\|v\|_{L^2(0,t;H^1(\Omega))},\\
&|\big(\varepsilon a_{ij}N_{\alpha_1}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_j\partial x_i}, v\big)_{\Omega_t}| \le C\varepsilon\|u^0\|_{L^2(0,t;H^3(\Omega))}\|v\|_{L^2(0,t;H^1(\Omega))}.
\end{aligned}
\tag{12}
$$

On the other hand, under the assumptions of this theorem, we apply the interior regularity for linear parabolic equations to obtain $u^0\in H^{s+2,1}(\Omega'\times(0,T))$, $s=1,2$, where $\Omega'\subset\subset\Omega$, and u^0 is the solution of the homogenized parabolic equation (3). From (7) and $(A_2)-(A_3)$, we can verify that

(13)

$$
\begin{aligned}
&|\big(\varepsilon^2 m_\varepsilon(x)N_{\alpha_1\alpha_2}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial t}, v\big)_{\Omega_t}| \le C\varepsilon^2\|\frac{\partial u^0}{\partial t}\|_{L^2(0,t;H^2(\Omega'))}\|v\|_{L^2(0,t;H^1(\Omega))},\\
&|\big(\varepsilon^2 m_\varepsilon(x)a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^4 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j\partial x_i}, v\big)_{\Omega_t}| \le C\varepsilon^2\|u^0\|_{L^2(0,t;H^4(\Omega'))}\|v\|_{L^2(0,t;H^1(\Omega))}.
\end{aligned}
$$

Thanks to Theorem 1.2 of [16] and using (7) and Lemma 1.5 of [22], we obtain

(14)

$$
\begin{aligned}
&|\big(\varepsilon\frac{\partial m_\varepsilon(x)}{\partial x_i}a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, v\big)_{\Omega_t}| \le C\varepsilon\varepsilon^{-1}\|u^0\|_{L^2(0,t;H^2(K'_\varepsilon))}\|v\|_{L^2(0,t;L^2(K'_\varepsilon))}\\
&\le C\varepsilon\|u^0\|_{L^2(0,t;H^3(\Omega'))}\|v\|_{L^2(0,t;H^1(\Omega))},
\end{aligned}
$$

(15)

$$
\begin{aligned}
&|\big((m_\varepsilon(x)-1)\frac{\partial N_{\alpha_1\alpha_2}}{\partial\tau}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, v\big)_{\Omega_t}| \le C\|u^0\|_{L^2(0,t;H^2(K_\varepsilon))}\|v\|_{L^2(0,t;L^2(K_\varepsilon))}\\
&\le C\varepsilon\|u^0\|_{L^2(0,t;H^3(\Omega))}\|v\|_{L^2(0,t;H^1(\Omega))},
\end{aligned}
$$

(16)

$$
\begin{aligned}
&|\big(\varepsilon^2\frac{\partial m_\varepsilon(x)}{\partial x_j}a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i}, v\big)_{\Omega_t}| \le C\varepsilon\|u^0\|_{L^2(0,t;H^3(K'_\varepsilon))}\|v\|_{L^2(0,t;L^2(K'_\varepsilon))}\\
&\le C\varepsilon^2\|u^0\|_{L^2(0,t;H^4(\Omega'))}\|v\|_{L^2(0,t;H^1(\Omega))},
\end{aligned}
$$

(17)

$$
\begin{aligned}
&|\big(\varepsilon^2\frac{\partial^2 m_\varepsilon(x)}{\partial x_i\partial x_j}a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, v\big)_{\Omega_t}| \le C\|u^0\|_{L^2(0,t;H^2(K'_\varepsilon))}\|v\|_{L^2(0,t;L^2(K'_\varepsilon))}\\
&\le C\varepsilon\|u^0\|_{L^2(0,t;H^3(\Omega'))}\|v\|_{L^2(0,t;H^1(\Omega))}.
\end{aligned}
$$

Similarly, we have
(18)
$$|\big(\varepsilon m_\varepsilon(x)a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i},v\big)_{\Omega_t}| \le C\varepsilon\|u^0\|_{L^2(0,t;H^3(\Omega'))}\|v\|_{L^2(0,t;L^2(\Omega'))}$$
$$\le C\varepsilon\|u^0\|_{L^2(0,t;H^3(\Omega))}\|v\|_{L^2(0,t;H^1(\Omega))},$$
(19)
$$|\big(\varepsilon^2\frac{\partial m_\varepsilon(x)}{\partial x_i}a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j},v\big)_{\Omega_t}| \le C\varepsilon\|u^0\|_{L^2(0,t;H^3(K'_\varepsilon))}\|v\|_{L^2(0,t;L^2(K'_\varepsilon))}$$
$$\le C\varepsilon^2\|u^0\|_{L^2(0,t;H^4(\Omega'))}\|v\|_{L^2(0,t;H^1(\Omega))}.$$
We observe that
(20)
$$(1-m_\varepsilon(x))\frac{\partial}{\partial \xi_i}(a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j})\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} = \varepsilon(1-m_\varepsilon(x))\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j})\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}$$
$$= \varepsilon\frac{\partial}{\partial x_i}\big[(1-m_\varepsilon(x))a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\big] - \varepsilon\frac{\partial(1-m_\varepsilon(x))}{\partial x_i}a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}$$
$$-\varepsilon(1-m_\varepsilon(x))a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i}.$$
By integration by parts, we get
$$\begin{aligned}&|\big(\varepsilon\frac{\partial}{\partial x_i}\big[(1-m_\varepsilon(x))a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\big],v\big)_{\Omega_t}|\\ &= |\big(\varepsilon\big[(1-m_\varepsilon(x))a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\big],\frac{\partial v}{\partial x_i}\big)_{\Omega_t}|\\ &\le C\varepsilon\|u^0\|_{L^2(0,t;H^2(K_\varepsilon))}\|v\|_{L^2(0,t;H^1(K_\varepsilon))}\\ &\le C\varepsilon^{3/2}\|u^0\|_{L^2(0,t;H^3(\Omega))}\|v\|_{L^2(0,t;H^1(\Omega))}.\end{aligned} \tag{21}$$
Using Theorem 1.2 of [16], we have
$$\begin{aligned}&|\big(\varepsilon\frac{\partial(1-m_\varepsilon(x))}{\partial x_i}a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}},v\big)_{\Omega_t}|\\ &\le C\|u^0\|_{L^2(0,t;H^2(K'_\varepsilon))}\|v\|_{L^2(0,t;L^2(K'_\varepsilon))}\\ &\le C\varepsilon\|u^0\|_{L^2(0,t;H^3(\Omega'))}\|v\|_{L^2(0,t;H^1(\Omega))},\end{aligned} \tag{22}$$
$$\begin{aligned}&|\big(\varepsilon(1-m_\varepsilon(x))a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i},v\big)_{\Omega_t}|\\ &\le C\varepsilon\|u^0\|_{L^2(0,t;H^3(K_\varepsilon))}\|v\|_{L^2(0,t;L^2(K_\varepsilon))}\\ &\le C\varepsilon^{3/2}\|u^0\|_{L^2(0,t;H^3(\Omega'))}\|v\|_{L^2(0,t;H^1(\Omega))}.\end{aligned} \tag{23}$$
Combining (21)-(23), it gives
$$|\big((1-m_\varepsilon(x))\frac{\partial}{\partial \xi_i}(a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial \xi_j})\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}},v\big)_{\Omega_t}| \le C\varepsilon\|v\|_{L^2(0,t;H^1(\Omega))}, \tag{24}$$
where C is a positive constant independent of ε. Hence,
$$\begin{aligned}\varepsilon\frac{\partial m_\varepsilon(x)}{\partial x_j}\frac{\partial}{\partial \xi_i}(a_{ij}N_{\alpha_1\alpha_2})\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} &= \varepsilon^2\frac{\partial}{\partial x_i}\big[\frac{\partial m_\varepsilon(x)}{\partial x_j}a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\big]\\ -\varepsilon^2\frac{\partial^2 m_\varepsilon(x)}{\partial x_i\partial x_j}a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} &- \varepsilon^2\frac{\partial m_\varepsilon(x)}{\partial x_j}a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i}.\end{aligned} \tag{25}$$
Similarly to (20)-(24), we have
$$|\big(\varepsilon\frac{\partial m_\varepsilon(x)}{\partial x_j}\frac{\partial}{\partial \xi_i}(a_{ij}N_{\alpha_1\alpha_2})\frac{\partial^2 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}},v\big)_{\Omega_t} \le C\varepsilon\|v\|_{L^2(0,t;H^1(\Omega))}, \tag{26}$$

$$|(\varepsilon m_\varepsilon(x)\frac{\partial}{\partial \xi_i}(a_{ij}N_{\alpha_1\alpha_2})\frac{\partial^3 u^0}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_j}, v)_{\Omega_t} \leq C\varepsilon\|v\|_{L^2(0,t;H^1(\Omega))}, \tag{27}$$

where C is a positive constant independent of ε. From (11)-(19), (24), and (26)-(27), we obtain

$$|J_2^\varepsilon(v)| \leq C\varepsilon\|v\|_{L^2(0,t;H^1(\Omega))}, \tag{28}$$

where C is a positive constant independent of ε. From the initial conditions, we have

$$u^\varepsilon(x,0) - \theta_2^\varepsilon(x,0) = \varepsilon\varphi(x,\xi), \tag{29}$$

where $\varphi(x,\xi) = \big[N_{\alpha_1}(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_{\alpha_1}} + \varepsilon m_\varepsilon(x)N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2 u^0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\big]|_{t=0}$. It is not difficult to show that $\|\varphi\|_{L^2(\Omega)} \leq C\|u^0\|_{H^2(\Omega')} \leq C$. By using the Gronwall's inequality, we complete the proof of Theorem 2.1.

Theorem 2.2 Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded convex polygonal domain. Let $u^\varepsilon(x,t)$ be a unique weak solution of problem (1), and let $U_1^\varepsilon(x,t)$, $U_2^\varepsilon(x,t)$ be the first-order and the second-order multiscale solutions defined in (5), respectively. Under the hypotheses of Theorem 2.1, we obtain the following estimates:

$$\begin{aligned}&\sup_{0\leq t\leq T}\int_\Omega (u^\varepsilon(x,t) - U_s^\varepsilon(x,t))^2 dx + \int_0^T \|u^\varepsilon - U_s^\varepsilon\|_{H^1(\Omega)}^2 \mathrm{d}t \\ &\leq \begin{cases} C(T)\varepsilon, & \text{for } \ k=0; \quad s=1,2 \\ C(T)\varepsilon, & \text{for } \ k=1,3; \quad s=2 \\ C(T)\varepsilon^2, & \text{for } \ k=2; \quad s=2\end{cases}\end{aligned} \tag{30}$$

where $C(T)$ is a positive constant independent of ε but dependent of T.

Proof. We prove Theorem 2.2 only for the case $k=2$, $s=2$. Recall that the boundary layer solutions $u_s^{\varepsilon,b}(x,t)$, $s=2$ defined in (4), using (6) and the trace theorem, we have

$$\begin{aligned}&\|u^\varepsilon - u_s^{\varepsilon,b}\|_{L^2(0,T;H^1(\Omega_1))} \leq C\|u^\varepsilon - u_s^\varepsilon\|_{L^2(0,T;H^{1/2}(\partial\Omega_0\cap\partial\Omega_1))} \\ &\leq C\|u^\varepsilon - u_s^\varepsilon\|_{L^2(0,T;H^1(\Omega_0))} \leq C(T)\varepsilon,\end{aligned} \tag{31}$$

where C is a positive constant independent of ε, but dependent of T.

From (6) and (31), using the triangle inequality, we have

$$\|u^\varepsilon - U_s^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq \|u^\varepsilon - u_s^\varepsilon\|_{L^2(0,T;H^1(\Omega_0))} + \|u^\varepsilon - u_s^{\varepsilon,b}\|_{L^2(0,T;H^1(\Omega_1))} \leq C(T)\varepsilon. \tag{32}$$

The other terms are similar. Thus, we complete the proof of Theorem 2.2.

Remark 2.2 It should be emphasized that, in order to obtain the convergence results with an explicit rate $\varepsilon^{1/2}$ for cases $k=1,2,3$, we need to apply the second-order multiscale asymptotic expansion defined in $(8)_2$ and (9). Here, $(8)_2$ denotes the second equation in (8) .The numerical results presented in Section 5 show that the second-order correctors are necessary.

3. Numerical Algorithms for Related Problems

3.1. Adaptive backward Euler-Galerkin method for cell problems. We recall the definitions of the cell functions for the four specific cases $k=0,1,2,3$. Observe that the cell problems with $k=0,1,3$ are all elliptic equations, where t (or τ) plays the role of a parameter. Hence, the standard finite element method can be applied. However, for the case $k=2$, the cell problems are of second order parabolic equations with respect to the scales $(\xi,\tau) \in Q \times (0,\tau_0)$, where $\xi = \varepsilon^{-1}x$,

$\tau = \varepsilon^{-2}t$ and τ_0 is a time period. Now, consider solving the cell problems for $k = 2$, in which the cell functions $N_p(\xi, \tau)$, $N_{pq}(\xi, \tau)$, $p, q = 1, 2, \cdots, n$ is defined in turn

(33)
$$\begin{cases} \dfrac{\partial N_p(\xi,\tau)}{\partial \tau} - \dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,\tau)\dfrac{\partial N_p(\xi,\tau)}{\partial \xi_j}\big) = \dfrac{\partial}{\partial \xi_i}(a_{ip}(\xi,\tau)), \quad (\xi,\tau) \in Q \times (0,\tau_0) \\ N_p(\xi,\tau) \quad \text{is 1-periodic in} \quad \xi, \quad \int\limits_Q N_p(\xi,\tau)\mathrm{d}\xi = 0 \\ N_p(\xi,0) = N_p(\xi,\tau_0), \end{cases}$$

where $Q = (0,1)^n$, τ_0 is a time period.

$$\begin{cases} \dfrac{\partial N_{pq}(\xi,\tau)}{\partial \tau} - \dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,\tau)\dfrac{\partial N_{pq}(\xi,\tau)}{\partial \xi_j}\big) = \dfrac{\partial}{\partial \xi_i}(a_{ip}(\xi,\tau)N_q(\xi,\tau)) \\ \quad +a_{pj}(\xi,\tau)\dfrac{\partial N_q(\xi,\tau)}{\partial \xi_j} + a_{pq}(\xi,\tau) - \hat{a}_{pq}, \quad (\xi,\tau) \in Q \times (0,\tau_0) \\ N_{pq}(\xi,\tau) \quad \text{is 1-periodic in} \quad \xi, \quad \int\limits_0^{\tau_0}\int\limits_Q N_{pq}(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau = 0 \\ N_{pq}(\xi,0) = N_{pq}(\xi,\tau_0), \end{cases} \tag{34}$$

where $\hat{a}_{pq} = \dfrac{1}{\tau_0}\int\limits_0^{\tau_0}\int\limits_Q \big[a_{pq}(\xi,\tau) + a_{pj}(\xi,\tau)\dfrac{\partial N_q(\xi,\tau)}{\partial \xi_j}\big]\mathrm{d}\xi\mathrm{d}\tau, \quad p, q = 1, 2, \cdots, n.$

In order to solve the time periodic boundary value problem, we refer the interested reader to Thomée's classical book (see [25], p.21). The basic idea is to employ an iterative method to compute the sequence $\{v_m\}_{m\geq 0}$ defined by

$$v_{m+1}(0) = v_m(\tau_0) = E(\tau_0)v_m(0), \quad m = 0, 1, \cdots \tag{35}$$

where $E(\tau_0)v_m(0)$ denotes the value of the solution for the linear parabolic equation at time $\tau = \tau_0$ for a given initial value $v_m(0)$ and $v_0(0)$ chosen arbitrarily. It can be proved that this equation has a unique solution v. Once v is known, the time periodic boundary-value problem may be solved as an initial boundary-value problem.

On the other hand, since the elements $a_{ij}(\xi, \tau)$ of the coefficients matrix of (33) and (34) are discontinuous, and the solutions of (33) and (34) admit singularities. Thus the computational error is likely concentrated around the singularities. We now propose an adaptive algorithm to solve the cell problems (33) and (34). For the solution of (34), we apply the same mesh as for solving (33). For the *a-posteriori* error estimates for the finite element method and the adaptive algorithm for solving linear parabolic problems, a comprehensive survey is given in [6], p.86-96. Following the terminology and the notation of [6], we propose the adaptive algorithm for problem (33). We introduce the following notation: Let $A = (a_{ij}(\xi, \tau))$ be the coefficients matrix and A_p be the p-th column of A, $p = 1, 2, \cdots, n$. Then, (33) can be rewritten as follows:

$$\partial_\tau N_p - \nabla \cdot \big[A\nabla N_p + A_p\big] = 0, \tag{36}$$

where $\nabla\cdot$ is a divergence operator. Here, $\nabla = \nabla_\xi$ and $\xi = \varepsilon^{-1}x$. For simplicity and without confusion, we continue to use ∇ instead of ∇_ξ.

The weak formulation of (36) can be expressed as:

$$\langle \partial_\tau N_p, \varphi\rangle - \big([A\nabla N_p + A_p], \nabla\varphi\big) = 0, \quad \forall \varphi \in H^1_{per}(Q), \tag{37}$$

where $H^1_{per}(Q) = \{v \in H^1(Q), \quad v \text{ is 1-periodic function}\}$, $Q = (0,1)^n$.

We now consider the backward Euler fully discrete approximation with variable time steps for (37). Let τ_m be the step size at the $m-$ th time-step and set

$\tau^m = \sum_{j=1}^{m} \tau_j$, and N be the total number of steps, that is $\tau^N \geq \tau_0$. At each time-step m, $m = 1, 2, \cdots, N$, denote by $\mathcal{M}^m$ a uniformly regular partition of Q which is obtained from $\mathcal{M}^{m-1}$ by using refinement/coarsening procedures. Note that the elements must be aligned with the boundary of Q to employ the periodic boundary conditions on ∂Q. Let $V_{per}^m \subset H_{per}^1(Q)$ be the usual space of conforming linear finite elements over $\mathcal{M}^m$. Then the fully discrete scheme for problem (33) at the $m-$ th time-step is given as follows:

$$\langle \bar{\partial}^m N_{p,h}^m, v_h \rangle + \big([A\nabla N_p + A_p], \nabla v_h\big) = 0, \quad \forall v_h \in V_{per}^m, \tag{38}$$

where $\bar{\partial}^m N_{p,h}^m = \big(N_{p,h}^m - N_{p,h}^{m-1}\big)/\tau_m$. Denote by $\mathcal{B}^m$ the collection of interior inter-element sides e of $\mathcal{M}^m$ in Q. h_K is the diameter of $K \in \mathcal{M}^m$ and h_e is the size of $e \in \mathcal{B}^m$. Define the jump residual across $e \in \mathcal{B}^m$ as follows:

(39)
$$\begin{aligned} J_e^m \overset{\text{def}}{=} & \Big[A(\cdot,\tau^m)\nabla N_{p,h}^m + A_p(\cdot,\tau^m)\Big]_e \cdot \nu_e = \big\{\big(A(\cdot,\tau^m)\nabla N_{p,h}^m + A_p(\cdot,\tau^m)\big)|_{K_1} \\ & -\big(A(\cdot,\tau^m)\nabla N_{p,h}^m + A_p(\cdot,\tau^m)\big)|_{K_2}\big\} \cdot \nu_e, \quad e = \partial K_1 \cap \partial K_2, \end{aligned}$$

where ν_e is the unit normal to e from K_2 to K_1.

By introducing the energy norm $\|u\|_{E,Q} = (A\nabla u, \nabla u)^{1/2}$, we have the following upper bound estimate.

Proposition 3.1 Let $N_p(\xi,\tau)$, $p = 1, 2, \cdots, n$ be the solutions of the cell problems (33), and let $N_{p,h}^m$ be the fully discrete finite element approximations at the $m-$th time-step. For any integer $1 \leq m \leq N$, we have the following *a posteriori* error estimate:

(40)
$$\begin{aligned} & \|N_p^m - N_{p,h}^m\|_{L^2(Q)} + \sum_{j=1}^{m} \int_{\tau^{j-1}}^{\tau^j} \|N_p - N_{p,h}^j\|_{E,Q}^2 dt \leq \sum_{j=1}^{m} \tau_j \eta_{time}^j + C \sum_{j=1}^{m} \tau_j \eta_{space}^j, \\ & + C\Big(\sum_{j=1}^{m} \int_{\tau^{j-1}}^{\tau^j} \|(A(\tau) - A(\tau^j))\nabla N_{p,h}^j + (A_p(\tau) - A_p(\tau^j))\|_{L^2(Q)}^2 \mathrm{d}\tau\Big)^2, \end{aligned}$$

where C is a positive constant independent of ε, h_K, h_e, but dependent on the minimum angle of the meshes $\mathcal{M}^j$, $j = 1, 2, \cdots, m$. The time error indicator η_{time}^j and the space error indicator η_{space}^j are given by

$$\eta_{time}^j = \frac{1}{3}\|N_{p,h}^j - N_{p,h}^{j-1}\|_{E,Q}^2, \quad \eta_{space}^j = \sum_{e \in \mathcal{B}^j} \eta_e^j, \quad j = 1, 2, \cdots, m$$

with the local error indicator η_e^j defined as

$$\eta_e^j = \frac{1}{2} \sum_{K \in Q_e} h_K^2 \|\bar{\partial}^j N_{p,h}^j\|_{L^2(K)}^2 + h_e \|J_e^j\|_{L^2(e)}^2.$$

Here, Q_e is the collection of two elements sharing the common side $e \in \mathcal{B}^j$.

Proof. From (37) and (38), for $\forall \varphi \in H_{per}^1(Q)$, $\forall v_h \in V^j$, we get

$$\begin{aligned} & \langle \bar{\partial}^j N_{p,h}^j, \varphi \rangle + \big(A(\tau^j)\nabla N_{p,h}^j, \nabla \varphi\big) = \langle \bar{\partial}^j N_{p,h}^j, \varphi - v_h \rangle \\ & \big(A(\tau^j)\nabla N_{p,h}^j, \nabla(\varphi - v_h)\big) - \big(A_p(\tau^j), \nabla v_h\big), \end{aligned} \tag{41}$$

where $A(\tau^j) = A(\cdot,\tau^j)$, $A_p(\tau^j) = A_p(\cdot,\tau^j)$. For $\tau \in (\tau^{j-1}, \tau^j]$, set $N_{p,h}(\tau) = l(\tau)N_{p,h}^j + (1 - l(\tau))N_{p,h}^{j-1}$, where $l(\tau) = (\tau - \tau^{j-1})/\tau_j$. It is not difficult to verify that

$$
\begin{aligned}
&\langle \frac{\partial(N_p - N_{p,h})}{\partial \tau}, \varphi\rangle + \big(A(\tau)\nabla(N_p - N_{p,h}^j), \nabla\varphi\big) \\
&= -\langle \bar{\partial}^j N_{p,h}^j, \varphi - v_h\rangle - \big([A(\tau^j)\nabla N_{p,h}^j + A_p(\tau^j)], \nabla(\varphi - v_h)\big) \\
&-\big([(A(\tau) - A(\tau^j))\nabla N_{p,h}^j + (A_p(\tau) - A_p(\tau^j))], \nabla\varphi\big).
\end{aligned} \tag{42}
$$

Let $\varphi = N_p - N_{p,h}$ and $v_h = r_h(N_p - N_{p,h})$, where $r_h : H^1_{per}(Q) \to V^j$ is the Clément interpolation operator (see [6], p.36). We can show that

$$
\begin{aligned}
&\big(A\nabla(N_p - N_{p,h}^j), \nabla(N_p - N_{p,h})\big) = \frac{1}{2}\|N_p - N_{p,h}^j\|^2_{E,Q} \\
&+\frac{1}{2}\|N_p - N_{p,h}\|^2_{E,Q} - \frac{1}{2}\|N_{p,h} - N_{p,h}^j\|^2_{E,Q}.
\end{aligned} \tag{43}
$$

Assuming that the elements of $A(\tau^j)$ and $A_p(\tau^j)$ are piecewise constants in each $K \in \mathcal{M}^j$, and using the fact that $\nabla N_{p,h}^j$ is piecewise constant in K, we get

$$
\begin{aligned}
&\big([A(\tau^j)\nabla N_{p,h}^j + A_p(\tau^j)], \nabla[(N_p - N_{p,h}) - r_h(N_p - N_{p,h})]\big) \\
&= -\sum_{e\in\mathcal{B}^j} \int_e J_e^j\big[(N_p - N_{p,h}) - r_h(N_p - N_{p,h})\big]ds.
\end{aligned} \tag{44}
$$

From (42), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|N_p - N_{p,h}\|^2_{L^2(Q)} + \frac{1}{2}\|N_p - N_{p,h}^j\|^2_{E,Q} + \frac{1}{2}\|N_p - N_{p,h}\|^2_{E,Q} \\
&= \frac{1}{2}\|N_{p,h} - N_{p,h}^j\|^2_{E,Q} - \langle \bar{\partial}^j N_{p,h}^j, (N_p - N_{p,h}) - r_h(N_p - N_{p,h})\rangle \\
&+ \sum_{e\in\mathcal{B}^j} \int_e J_e^j\big[(N_p - N_{p,h}) - r_h(N_p - N_{p,h})\big]ds. \\
&-\big((A(\tau) - A(\tau^j))\nabla N_{p,h}^j + (A_p(\tau) - A_p(\tau^j)), \nabla(N_p - N_{p,h})\big).
\end{aligned} \tag{45}
$$

For any $\tau^* \in (\tau^{m-1}, \tau^m]$, by integrating (45) in time from 0 to τ^* and using (A_2) and the error estimate for the Clément interpolation operator, we obtain

(46)

$$
\begin{aligned}
&\frac{1}{2}\|(N_p - N_{p,h})(\tau^*)\|^2_{L^2(Q)} + \frac{1}{2}\sum_{j=1}^m \int_{\tau^{j-1}}^{\tau^j\wedge\tau^*} \Big[\|N_p - N_{p,h}^j\|^2_{E,Q} + \|N_p - N_{p,h}\|^2_{E,Q}\Big]\mathrm{d}\tau \\
&\le \frac{1}{2}\|(N_p - N_{p,h})(0)\|^2_{L^2(Q)} + \frac{1}{2}\sum_{j=1}^m \int_{\tau^{j-1}}^{\tau^j} \|N_{p,h} - N_{p,h}^j\|^2_{E,Q}\mathrm{d}\tau \\
&+C\sum_{j=1}^m \int_{\tau^{j-1}}^{\tau^j} \big(\eta^j_{space}\big)^{1/2}\|N_p - N_{p,h}\|_{E,Q}\mathrm{d}\tau \\
&+\sum_{j=1}^m \int_{\tau^{j-1}}^{\tau^j} \|(A(\tau) - A(\tau^j))\nabla N_{p,h}^j + (A_p(\tau) - A_p(\tau^j))\|_{L^2(Q)}\|N_p - N_{p,h}\|_{E,Q}\mathrm{d}\tau.
\end{aligned}
$$

From (33) and (35), we have $(N_p - N_{p,h})(0) \equiv 0$. It is obvious that

$$
\begin{aligned}
\int_{\tau^{j-1}}^{\tau^j} \|N_{p,h} - N_{p,h}^j\|^2_{E,Q}\mathrm{d}\tau &= \int_{\tau^{j-1}}^{\tau^j} (1 - l(\tau))^2\|N_{p,h}^j - N_{p,h}^{j-1}\|^2_{E,Q}\mathrm{d}\tau \\
&= \frac{1}{3}\tau_j\|N_{p,h}^j - N_{p,h}^{j-1}\|^2_{E,Q}.
\end{aligned} \tag{47}
$$

Using the Young inequality $ab \le \lambda a^2 + \frac{1}{4\lambda}b^2$, we compete the proof of Proposition 3.1.

Remark 3.1 Based on the local error indicators η^j_{time} and η^j_{space} in Proposition 3.1, one can present a time and space adaptive algorithm. We refer the interested reader to Algorithm 7.1 of [6], p.93.

Repeating the procedure of Theorem 7.7 of ([6], p.83), we obtain the following proposition:

Proposition 3.2 Let N_p, N_{pq}, $p,q=1,2,\cdots,n$, be the solutions of the cell problems (33) and (34), respectively. Suppose that $N^m_{p,h}$, $N^m_{pq,h}$ are the corresponding fully discrete solutions at time $\tau=\tau^m=\sum\limits_{j=1}^{m}\tau_j$ by using the *backward Euler-Galerkin method*, where h and τ_m are the size of the final mesh and the time-step size, respectively. Under the assumptions $(A_1)-(A_4)$, if $N_p\,N_{pq}\in H^{2,1}(Q\times(0,\tau_0))\cap H^{0,2}(Q\times(0,\tau_0))$, where τ_0 is a time period, then we have

(48)
$$\|N^m_{p,h}-N_p(\tau^m)\|_{L^2(Q)}\le C\big\{h^2+\tau_m\big\},\quad \|N^m_{pq,h}-N_{pq}(\tau^m)\|_{L^2(Q)}\le C\big\{h^2+\tau_m\big\},$$
$$\|N^m_{p,h}-N_p(\tau^m)\|_{H^1(Q)}\le C\big\{h+\tau_m\big\},\quad \|N^m_{pq,h}-N_{pq}(\tau^m)\|_{H^1(Q)}\le C\big\{h+\tau_m\big\},$$

where $1\le m\le N$, $\tau^N\ge\tau_0$, and C is a positive constant independent of h, τ_m.

3.2. Crank-Nicolson Galerkin method for the homogenized parabolic equation.

Suppose that $(\hat a_{ij})$ is a homogenized coefficients matrix and $(\hat a^h_{ij})$ is the approximation of $(\hat a_{ij})$. For example, in the case $k=2$, we know

$$\hat a_{ij}=\frac{1}{\tau_0}\int_0^{\tau_0}\int_Q\big(a_{ij}(\xi,\tau)+a_{il}(\xi,\tau)\frac{\partial N_j(\xi,\tau)}{\partial\xi_l}\big)\mathrm{d}\xi\mathrm{d}\tau. \tag{49}$$

For the Crank-Nicolson-Galerkin method, we have

$$\hat a^h_{ij}=\frac{1}{\tau_0}\sum_{m=1}^{N}\tau_m\int_Q\big[a_{ij}(\xi,\tau^{m-1/2})+a_{il}(\xi,\tau^{m-1/2})\frac{\partial N^{m-1/2}_{j,h}}{\partial\xi_l}\big]\mathrm{d}\xi, \tag{50}$$

where $\tau^{m-1/2}=(\tau^{m-1}+\tau^m)/2$, $N^{m-1/2}_{j,h}=(N^m_{j,h}+N^{m-1}_{j,h})/2$. For the backward Euler-Galerkin method, we have

$$\hat a^h_{ij}=\frac{1}{\tau_0}\sum_{m=1}^{N}\tau_m\int_Q\big[a_{ij}(\xi,\tau^m)+a_{il}(\xi,\tau^m)\frac{\partial N^m_{j,h}}{\partial\xi_l}\big]\mathrm{d}\xi. \tag{51}$$

For $\tau\in(\tau^{m-1},\tau^m]$, $m=1,2,\cdots,N$, $\tau^N\ge\tau_0$, $\tau_m=\tau^m-\tau^{m-1}$, where τ_0 is a time period, $N^m_{j,h}$ is the fully discrete approximation of N_j. Therefore, in practice, we need to solve the following modified homogenized parabolic equation:

$$\begin{cases}\dfrac{\partial\tilde u^0(x,t)}{\partial t}-\dfrac{\partial}{\partial x_i}\big(\hat a^h_{ij}\dfrac{\partial\tilde u^0(x,t)}{\partial x_j}\big)=f(x,t), & (x,t)\in\Omega\times(0,T)\\ \tilde u^0(x,t)=g(x,t), \quad (x,t)\in\partial\Omega\times(0,T) & \\ \tilde u^0(x,0)=\overline{u}_0(x), \quad x\in\Omega. & \end{cases} \tag{52}$$

Remark 3.2 For cases $k=0,1,3$, it is not difficult to verify that $(\hat a^h_{ij})$ is a symmetric, positive-definite matrix. Therefore, the modified homogenized parabolic equation has a unique weak solution in $L^2(0,T;H^1(\Omega))$.

Proposition 3.3 For the case $k = 2$, we now prove that the modified homogenized coefficients matrix $(\hat{a}_{ij}^h)$ satisfies the following conditions:

(53)
$$\begin{cases} \hat{a}_{ij}^h = \hat{a}_{ji}^h, & \text{for the Crank-Nicolson-Galerkin method,} \\ |\hat{a}_{ij}^h - \hat{a}_{ji}^h| \leq C\Big(\tau_{max}^{-1}h^4 + h^2 + \tau_{max}\Big), & \text{for the backward Euler-Galerkin method,} \end{cases}$$

(54)
$$\hat{a}_{ij}^h \eta_i \eta_j \geq \bar{\mu}|\eta|^2, \quad |\eta|^2 = \eta_i\eta_i, \quad \forall\,(\eta_1, \cdots, \eta_n) \in \mathbb{R}^n,$$

where C and $\bar{\mu} > 0$ are positive constants independent of h and τ_{max}; h is the mesh size and $\tau_{max} = \max\limits_{1\leq m\leq N} \tau_m$ is the largest time-step size.

Proof. For the Crank-Nicolson-Galerkin method, we get

(55)
$$\langle \bar{\partial}^m N_{j,h}^m, v_h\rangle + \big(A(\tau^{m-1/2})\nabla N_{j,h}^{m-1/2} + A_j(\tau^{m-1/2}), \nabla v_h\big) = 0, \quad \forall v_h \in V_{per}^m,$$

where $A(\tau^{m-1/2}) = (a_{ij}(\xi, \tau^{m-1/2}))$ and $N_{j,h}^{m-1/2} = (N_{j,h}^m + N_{j,h}^{m-1})/2$. By setting $v_h = N_{i,h}^{m-1/2} \in V_{per}^m$, we get

(56)
$$\int_Q \frac{(N_{j,h}^m - N_{j,h}^{m-1})}{\tau_m} N_{i,h}^{m-1/2}\mathrm{d}\xi + \int_Q a_{pl}(\xi, \tau^{m-1/2}) \frac{\partial(N_{j,h}^{m-1/2} + \xi_j)}{\partial\xi_l} \frac{\partial N_{i,h}^{m-1/2}}{\partial\xi_p}\mathrm{d}\xi = 0.$$

We observe that

(57)
$$\begin{aligned} &\int_Q a_{pl}(\xi, \tau^{m-1/2}) \frac{\partial(N_{j,h}^{m-1/2} + \xi_j)}{\partial\xi_l} \frac{\partial(N_{i,h}^{m-1/2} + \xi_i)}{\partial\xi_p}\mathrm{d}\xi \\ &= \int_Q a_{pl}(\xi, \tau^{m-1/2}) \frac{\partial(N_{j,h}^{m-1/2} + \xi_j)}{\partial\xi_l} \frac{\partial N_{i,h}^{m-1/2}}{\partial\xi_p}\mathrm{d}\xi + \int_Q a_{il}(\xi, \tau^{m-1/2}) \frac{\partial(N_{j,h}^{m-1/2} + \xi_j)}{\partial\xi_l}\mathrm{d}\xi \\ &= -\int_Q \frac{(N_{j,h}^m - N_{j,h}^{m-1})}{\tau_m} N_{i,h}^{m-1/2}\mathrm{d}\xi + \int_Q a_{il}(\xi, \tau^{m-1/2}) \frac{\partial(N_{j,h}^{m-1/2} + \xi_j)}{\partial\xi_l}. \end{aligned}$$

Hence, we have

(58)
$$\begin{aligned} \hat{a}_{ij}^h &= \frac{1}{\tau_0}\sum_{m=1}^N \tau_m \int_Q a_{il}(\xi, \tau^{m-1/2}) \frac{\partial(N_{j,h}^{m-1/2} + \xi_j)}{\partial\xi_l}\mathrm{d}\xi \\ &= \frac{1}{\tau_0}\sum_{m=1}^N \tau_m \int_Q a_{pl}(\xi, \tau^{m-1/2}) \frac{\partial(N_{j,h}^{m-1/2} + \xi_j)}{\partial\xi_l} \frac{\partial(N_{i,h}^{m-1/2} + \xi_i)}{\partial\xi_p}\mathrm{d}\xi \\ &+ \frac{1}{\tau_0}\sum_{m=1}^N \int_Q (N_{j,h}^m - N_{j,h}^{m-1}) N_{i,h}^{m-1/2}\mathrm{d}\xi. \end{aligned}$$

Let $l(\tau) = (\tau - \tau^{m-1})/\tau_m$ and $N_{j,h}(\xi, \tau) = l(\tau)N_{j,h}^m + (1 - l(\tau))N_{j,h}^{m-1}$. One can directly verify that $\dfrac{\partial N_{j,h}(\xi,\tau)}{\partial\tau} = \dfrac{(N_{j,h}^m - N_{j,h}^{m-1})}{\tau_m}$. Furthermore, we get

(59)
$$\frac{1}{\tau_0}\sum_{m=1}^N \int_Q (N_{j,h}^m - N_{j,h}^{m-1}) N_{i,h}^{m-1/2}\mathrm{d}\xi = \frac{1}{\tau_0}\int_0^{\tau_0}\int_Q \frac{\partial N_{j,h}(\xi,\tau)}{\partial\tau} N_{i,h}(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau.$$

Since

$$
\begin{aligned}
(60)\quad &\sum_{i,j=1}^{n}\frac{1}{\tau_0}\Big[\int_0^{\tau_0}\int_Q\frac{\partial N_{j,h}(\xi,\tau)}{\partial\tau}N_{i,h}(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau\Big]\frac{\partial^2\tilde{u}^0}{\partial x_i\partial x_j}\\
&=\sum_{i,j=1}^{n}\frac{1}{\tau_0}\frac{\partial^2\tilde{u}^0}{\partial x_i\partial x_j}\int_Q\mathrm{d}\xi\Big\{\frac{1}{2}\int_0^{\tau_0}\frac{\partial N_{j,h}}{\partial\tau}N_{i,h}\mathrm{d}\tau+\frac{1}{2}\int_0^{\tau_0}\frac{\partial N_{i,h}}{\partial\tau}N_{j,h}\mathrm{d}\tau\Big\}\\
&=\sum_{i,j=1}^{n}\frac{1}{2\tau_0}\frac{\partial^2\tilde{u}^0}{\partial x_i\partial x_j}\int_Q\mathrm{d}\xi\int_0^{\tau_0}\frac{\partial N_{j,h}N_{i,h}}{\partial\tau}\mathrm{d}\tau\\
&=\sum_{i,j=1}^{n}\frac{1}{2\tau_0}\frac{\partial^2\tilde{u}^0}{\partial x_i\partial x_j}\int_Q\mathrm{d}\xi\big(N_{j,h}N_{i,h}(\tau_0)-N_{j,h}N_{i,h}(0)\big)=0,
\end{aligned}
$$

we have

$$
(61)\qquad \hat{a}^h_{ij}=\frac{1}{\tau_0}\sum_{m=1}^{N}\tau_m\int_Q a_{pl}(\xi,\tau^{m-1/2})\frac{\partial(N^{m-1/2}_{j,h}+\xi_j)}{\partial\xi_l}\frac{\partial(N^{m-1/2}_{i,h}+\xi_i)}{\partial\xi_p}\mathrm{d}\xi.
$$

The fact that $a_{pl}=a_{lp}$ implies that $\hat{a}^h_{ij}=\hat{a}^h_{ji}$.

For the backward Euler-Galerkin method, by setting $v_h=N^m_{i,h}\in V^m_{per}$ in (38), we get

$$
(62)\qquad \int_Q\frac{(N^m_{j,h}-N^{m-1}_{j,h})}{\tau_m}N^m_{i,h}\mathrm{d}\xi+\int_Q a_{pl}(\xi,\tau^m)\frac{\partial(N^m_{j,h}+\xi_j)}{\partial\xi_l}\frac{\partial N^m_{i,h}}{\partial\xi_p}\mathrm{d}\xi=0.
$$

Similarly to (57) and (58), we have

(63)

$$
\begin{aligned}
\hat{a}^h_{ij}&=\frac{1}{\tau_0}\sum_{m=1}^{N}\tau_m\int_Q a_{il}(\xi,\tau^m)\frac{\partial(N^m_{j,h}+\xi_j)}{\partial\xi_l}\mathrm{d}\xi\\
&=\frac{1}{\tau_0}\sum_{m=1}^{N}\tau_m\int_Q a_{pl}(\xi,\tau^m)\frac{\partial(N^m_{j,h}+\xi_j)}{\partial\xi_l}\frac{\partial(N^m_{i,h}+\xi_i)}{\partial\xi_p}\mathrm{d}\xi.\\
&+\frac{1}{\tau_0}\sum_{m=1}^{N}\int_Q(N^m_{j,h}-N^{m-1}_{j,h})N^m_{i,h}\mathrm{d}\xi\\
&=\frac{1}{\tau_0}\sum_{m=1}^{N}\tau_m\int_Q a_{pl}(\xi,\tau^m)\frac{\partial(N^m_{j,h}+\xi_j)}{\partial\xi_l}\frac{\partial(N^m_{i,h}+\xi_i)}{\partial\xi_p}\mathrm{d}\xi\\
&+\frac{1}{\tau_0}\sum_{m=1}^{N}\int_Q(N^m_{j,h}-N^{m-1}_{j,h})\frac{(N^m_{i,h}+N^{m-1}_{i,h})}{2}\mathrm{d}\xi\\
&+\frac{1}{\tau_0}\sum_{m=1}^{N}\int_Q(N^m_{j,h}-N^{m-1}_{j,h})\frac{(N^m_{i,h}-N^{m-1}_{i,h})}{2}\mathrm{d}\xi\\
&=\frac{1}{\tau_0}\sum_{m=1}^{N}\tau_m\int_Q a_{pl}(\xi,\tau^m)\frac{\partial(N^m_{j,h}+\xi_j)}{\partial\xi_l}\frac{\partial(N^m_{i,h}+\xi_i)}{\partial\xi_p}\mathrm{d}\xi\\
&+\frac{1}{\tau_0}\int_0^{\tau_0}\int_Q\frac{\partial N_{j,h}}{\partial\tau}N_{i,h}\mathrm{d}\xi\mathrm{d}\tau+\frac{1}{2\tau_0}\sum_{m=1}^{N}\int_Q(N^m_{j,h}-N^{m-1}_{j,h})(N^m_{i,h}-N^{m-1}_{i,h})\mathrm{d}\xi.
\end{aligned}
$$

It is obvious that

$$
N^m_{j,h}-N^{m-1}_{j,h}=(N^m_{j,h}-N_j(\xi,\tau^m))+(N_j(\xi,\tau^m)-N_j(\xi,\tau^{m-1}))+(N_j(\xi,\tau^{m-1})-N^{m-1}_{j,h}).
$$

It follows from Proposition 3.2 that

$$
\begin{aligned}
\|(N_{j,h}^m - N_j(\xi,\tau^m))\|_{L^2(Q)} &\leq Ch^2 \int_0^{\tau^m} \|\partial_\tau N_j(\cdot,\tau)\|_{H^2(Q)} \mathrm{d}\tau \\
&\quad + C\tau_{max} \int_0^{\tau^m} \|\partial_{\tau\tau} N_j(\cdot,\tau)\|_{L^2(Q)} \mathrm{d}\tau, \\
\|(N_{j,h}^{m-1} - N_j(\xi,\tau^{m-1}))\|_{L^2(Q)} &\leq Ch^2 \int_0^{\tau^{m-1}} \|\partial_\tau N_j(\cdot,\tau)\|_{H^2(Q)} \mathrm{d}\tau \\
&\quad + C\tau_{max} \int_0^{\tau^{m-1}} \|\partial_{\tau\tau} N_j(\cdot,\tau)\|_{L^2(Q)} \mathrm{d}\tau,
\end{aligned}
$$

where C is a positive constant independent of h and τ_{max}; h and τ_{max} are the mesh size and the largest time-step size, respectively. We observe that

$$
\|N_j(\xi,\tau^m) - N_j(\xi,\tau^{m-1})\|_{L^2(Q)} = \tau_m \Big(\int_Q (\partial_\tau N_j(\tilde{\tau}^m))^2 \mathrm{d}\xi \Big)^{1/2} \leq C\tau_{max}.
$$

We thus have

$$
\begin{aligned}
&\frac{1}{2\tau_0} \sum_{m=1}^{N} \Big| \int_Q (N_{j,h}^m - N_{j,h}^{m-1})(N_{i,h}^m - N_{i,h}^{m-1}) \mathrm{d}\xi \Big| \\
&\leq \frac{1}{2\tau_0} \sum_{m=1}^{N} \|N_{j,h}^m - N_{j,h}^{m-1}\|_{L^2(Q)} \|N_{i,h}^m - N_{i,h}^{m-1}\|_{L^2(Q)} \\
&\leq C \sum_{m=1}^{N} \big(h^2 + \tau_{max}\big)^2 = CN\big(h^2 + \tau_{max}\big)^2 \\
&\leq C\big\{\tau_{max}^{-1} h^4 + h^2 + \tau_{max}\big\}.
\end{aligned} \tag{64}
$$

Similarly to (60) and (61), using (64), we obtain

$$
|\hat{a}_{ij}^h - \hat{a}_{ji}^h| \leq C\Big(\tau_{max}^{-1} h^4 + h^2 + \tau_{max}\Big). \tag{65}
$$

It remains to prove (54). Here, we only consider the backward Euler-Galerkin method, since the Crank-Nicolson Galerkin method is similar. We observe that

$$
\begin{aligned}
|\hat{a}_{ij} - \hat{a}_{ij}^h| &= \frac{1}{\tau_0} \sum_{m=1}^{N} \int_{\tau^{m-1}}^{\tau^m} \int_Q \big[a_{ij}(\xi,\tau) - a_{ij}(\xi,\tau^m) \\
&\quad + a_{il}(\xi,\tau) \frac{\partial N_j(\xi,\tau)}{\partial \xi_l} - a_{il}(\xi,\tau^m) \frac{\partial N_{j,h}^m}{\partial \xi_l} \big] \mathrm{d}\xi \mathrm{d}\tau \\
&= \frac{1}{\tau_0} \sum_{m=1}^{N} \int_{\tau^{m-1}}^{\tau^m} \int_Q \big[(a_{ij}(\xi,\tau) - a_{ij}(\xi,\tau^m))(1 + \frac{\partial N_j(\xi,\tau)}{\partial \xi_l}) \\
&\quad + a_{il}(\xi,\tau^m) \frac{\partial (N_j(\xi,\tau) - N_{j,h}^m)}{\partial \xi_l} \big] \mathrm{d}\xi \mathrm{d}\tau.
\end{aligned} \tag{66}
$$

It follows from Proposition 3.2 that

$$
\begin{aligned}
|\hat{a}_{ij} - \hat{a}_{ij}^h| &\leq C\tau_{max} \int_0^{\tau_0} \|\partial_\tau a_{ij}\|_{L^2(Q)} \mathrm{d}\tau \|N_j\|_{L^2(0,\tau_0;H_{per}^1(Q))} \\
&\quad + Ch \int_0^{\tau_0} \|\partial_\tau N_j(\cdot,\tau)\|_{H^2(Q)} \mathrm{d}\tau \\
&\quad + C\tau_{max} \int_0^{\tau_0} \|\partial_{\tau\tau} N_j(\cdot,\tau)\|_{L^2(Q)} \mathrm{d}\tau.
\end{aligned} \tag{67}
$$

Since $(\hat{a}_{ij})$ is a symmetric and positive-definite matrix, from (67), we complete the proof of (54).

Next, we present the error estimates between $u^0(x,t)$ and $\tilde{u}^0(x,t)$, where $u^0(x,t)$ and $\tilde{u}^0(x,t)$ are the solutions of (3) and (52), respectively.

Proposition 3.4 For $k=2$, let $u^0(x,t)$ and $\tilde{u}^0(x,t)$ be the weak solutions of (3) and (52), respectively. Under the assumptions $(A_1)-(A_4)$, if $f\in L^2(\Omega\times(0,T))$, $g\in L^2(0,T;H^{1/2}(\partial\Omega))$, $\bar{u}_0\in H^1(\Omega)$, then we have the following error estimate:

$$\begin{aligned}(68)\qquad &\sup_{0\le t\le T}\int_\Omega (u^0(x,t)-\tilde{u}^0(x,t))^2\mathrm{d}x+\int_0^T\|u^0(\cdot,t)-\tilde{u}^0(\cdot,t)\|^2_{H^1(\Omega)}\mathrm{d}t\\ &\le C(h^2+\tau^{2q}_{max})\Big\{\|f\|^2_{L^2(\Omega\times(0,T))}+\|g\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))}+\|\bar{u}_0\|^2_{H^1(\Omega)}\Big\},\end{aligned}$$

where C is a positive constant independent of h and τ_{max}; h and τ_{max} are the mesh size and the largest time-step size, respectively. Note that $q=1$ for the backward Euler-Galerkin method; and $q=2$ for the Crank-Nicolson Galerkin method.

Proof. By setting $w(x,t)=\tilde{u}^0(x,t)-u^0(x,t)$ and $\hat{r}_{ij}=\hat{a}_{ij}-\hat{a}^h_{ij}$, we have

$$(69)\qquad \begin{cases}\dfrac{\partial w(x,t)}{\partial t}-\dfrac{\partial}{\partial x_i}\big(\hat{a}^h_{ij}\dfrac{\partial w(x,t)}{\partial x_j}\big)=\dfrac{\partial}{\partial x_i}\big(\hat{r}_{ij}\dfrac{\partial u^0(x,t)}{\partial x_j}\big), & (x,t)\in\Omega\times(0,T)\\ w(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T)\\ w(x,0)=0, \quad x\in\Omega.\end{cases}$$

Multiplying by $w(x,t)$ to both sides of (69) and by integration by parts in $\Omega\times(0,t)$ with $t\in(0,T)$, it gives

$$\begin{aligned}&\frac{1}{2}\int_0^t\int_\Omega\frac{\partial w^2(x,t)}{\partial t}\mathrm{d}x\mathrm{d}t+\int_0^t\int_\Omega\hat{a}^h_{ij}\frac{\partial w(x,t)}{\partial x_j}\frac{\partial w(x,t)}{\partial x_i}\mathrm{d}x\mathrm{d}t\\ &=\int_0^t\int_\Omega\hat{r}_{ij}\frac{\partial u^0(x,t)}{\partial x_j}\frac{\partial w(x,t)}{\partial x_i}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By using Propositions 3.2, 3.3 and the trace theorem, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega w^2(x,t)\mathrm{d}x+\bar{\mu}\int_0^t\|w(\cdot,t)\|^2_{H^1(\Omega)}\mathrm{d}t\\ &\le C\int_0^t\big(h+\tau^q_{max}\big)\|u^0(\cdot,t)\|_{H^1(\Omega)}\|w(\cdot,t)\|_{H^1(\Omega)}\mathrm{d}t.\end{aligned}$$

By means of the Young's inequality, we get

$$\begin{aligned}\frac{1}{2}\int_\Omega w^2(x,t)\mathrm{d}x+\bar{\mu}\int_0^t\|w(\cdot,t)\|^2_{H^1(\Omega)}\mathrm{d}t\le C\big(h^2+\tau^{2q}_{max}\big)\int_0^t\frac{1}{4\lambda}\|u^0(\cdot,t)\|^2_{H^1(\Omega)}\mathrm{d}t\\ +\lambda\int_0^t\|w(\cdot,t)\|^2_{H^1(\Omega)}\mathrm{d}t.\end{aligned}$$

Choosing a sufficiently small $\lambda>0$ such that $\lambda<\frac{\bar{\mu}}{2}$, it yields

$$\frac{1}{2}\int_\Omega w^2(x,t)\mathrm{d}x+\frac{\bar{\mu}}{2}\int_0^t\|w(\cdot,t)\|^2_{H^1(\Omega)}\mathrm{d}t\le C\big(h^2+\tau^{2q}_{max}\big)\int_0^t\|u^0(\cdot,t)\|^2_{H^1(\Omega)}\mathrm{d}t.$$

Using the Gronwall's inequality and the regular estimates of the solution $u^0(x,t)$ of the homogenized parabolic equation (3), we complete the proof of Proposition 3.4.

Remark 3.3 For cases $k=0,1,3$, one can derive similar results to those of Proposition 3.4.

By the virtue of the interior estimates for second order parabolic equations (see [14], p.351), we can prove the following proposition without any difficulty.

Proposition 3.5 Let $u^0(x,t)$ and $\tilde{u}^0(x,t)$ be the weak solutions of the homogenized equation (3) and the modified homogenized equation (52), respectively.

Under the assumptions $(A_1)-(A_4)$, if $f\in L^2(\Omega\times(0,T))\bigcap H^1(0,T;H^s(\Omega''))$, $g\in L^2(0,T;H^{1/2}(\partial\Omega))$, $\bar{u}_0\in H^1(\Omega)\bigcap H^{s+1}(\Omega'')$, $s=1,2$, where $\Omega_0\subset\subset\Omega''\subset\subset\Omega$, then it holds

$$\begin{aligned}&\sup_{0\le t\le T}\|u^0(x,t)-\tilde{u}^0(x,t))\|^2_{s+1,\Omega_0}+\int_0^T\|u^0(\cdot,t)-\tilde{u}^0(\cdot,t)\|^2_{s+2,\Omega_0}\mathrm{d}t\\&\le C\big(h^2+\tau^{2q}_{max}\big)\Big\{\|f\|^2_{H^1(0,T;H^s(\Omega''))}+\|\bar{u}_0\|^2_{H^{s+1}(\Omega'')}\\&+\|f\|^2_{L^2(\Omega\times(0,T))}+\|g\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))}+\|\bar{u}_0\|^2_{H^1(\Omega)}\Big\},\end{aligned}\tag{70}$$

where C is a positive constant independent of h and tau_{max}; h and τ_{max} are respectively the mesh size and the largest time-step size. Note that $q=1$ and 2 for the backward Euler-Galerkin method and the Crank-Nicolson Galerkin method, respectively.

Next we discuss the finite element computation for the modified homogenized parabolic equation (52). Based on the numerical results by the standard finite element method, we introduce the finite element post-processing technique presented in [18, 19]. Let $\mathcal{J}^{h_1}$ be a regular family of subdivisions of Ω, where h_1 is the mesh size, and satisfy the following properties:

(1) The elements are uniform hexahedrons in the interior subdomain $\Omega_0\subset\subset\Omega$.

(2) The elements are regular tetrahedron in region $\Omega_1=\Omega\backslash\overline{\Omega}_0$, and the elements are tetrahedrons near the boundary $\partial\Omega$.

(3) Any face of the element K_1 is either a subset of the boundary $\partial\Omega$, or a face of another element K_2 in the subdivision.

Define a $r-$ th finite element space:

$$S^{h_1}(\Omega)=\{v\in C(\overline{\Omega}):\quad v|_K\in\overline{P}_r(K),\quad v|_{\partial\Omega}=0\}\subset H^1_0(\Omega),\tag{71}$$

where

$$\overline{P}_r=\begin{cases}Q_r, & K\quad\text{is a hexahedron}\\ P_r, & K\quad\text{is a tetrahedron,}\end{cases}$$

and Q_r and P_r are bi r-th and r-th finite elements, respectively.

The semi-discrete scheme for solving problem (52) is given as follows:

$$\langle\frac{d}{dt}\tilde{u}^0_{h_1},v_{h_1}\rangle+a_\Omega(\tilde{u}^0_{h_1},v_{h_1})=\langle f,v_{h_1}\rangle,\quad\forall v_{h_1}\in S^{h_1}(\Omega),\quad t\in(0,T),\tag{72}$$

where $\langle u,v\rangle=\int_\Omega uv\mathrm{d}x$, $a_\Omega(u,v)=\int_\Omega\hat{a}^h_{ij}\dfrac{\partial u}{\partial x_j}\dfrac{\partial u}{\partial x_i}\mathrm{d}x$.

We consider the Crank-Nicolson fully discrete approximation with variable time steps for (72). Let κ_m be the step size for the $m-$th time-step and set $t_m=\sum\limits_{i=1}^m\kappa_i$. We define $U^m_{h_1}\in S^{h_1}(\Omega)$ recursively for $m\ge1$ by

$$\begin{cases}\langle\bar{\partial}_tU^m_{h_1},v_{h_1}\rangle+a_\Omega(\frac{1}{2}(U^m_{h_1}+U^{m-1}_{h_1}),v_{h_1})=\langle f(t_{m-1/2}),v_{h_1}\rangle,\quad\forall v_{h_1}\in S^{h_1}(\Omega),\\ U^0_{h_1}=\bar{u}_{0,h_1},\end{cases}\tag{73}$$

where $\bar{\partial}_tU^m_{h_1}=(U^m_{h_1}-U^{m-1}_{h_1})/\kappa_m$, $f(t_{m-1/2})=f(x,(t_m+t_{m-1})/2)$, $\bar{u}_{0,h_1}\in S^{h_1}(\Omega)$ is some approximation of $\bar{u}_0(x)$.

Remark 3.4 Under some regularity hypotheses, one can derive the error estimates of the Crank-Nicolson Galerkin method for problem (52). We refer the reader to additional references, see, e.g., [6, 15, 25].

In order to improve the numerical accuracy of approximate solutions, we introduce the finite element post-processing technique presented in [18, 19]. The key

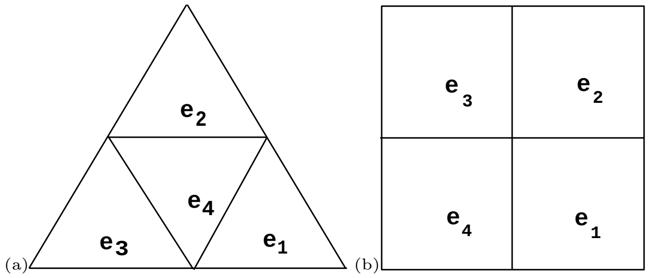

FIGURE 2. (a) Triangular mesh ; (b) rectangular mesh

step of the method is to construct a bi-2r-th (2r-th) interpolation operator at a new larger element with respect to a coarse mesh as shown in Fig.2 (a), (b), by using the nodal values of the bi-r-th (r-th)finite element solution. Denote by $\mathcal{I}_{2h_1}^{(2r)}$ the bi-2r-th (2r-th) order interpolation operator, where h_1 is the mesh size to solve problem (52).

Lemma 3.1 ([19]) Let $\mathcal{I}_{h_1} : H^1(\Omega) \to S^{h_1}(\Omega)$ be the usual Lagrange's interpolation operator. Then the interpolation operators $\mathcal{I}_{h_1}$ and $\mathcal{I}_{2h_1}^{(2r)}$ satisfy the following properties:

$$\|\mathcal{I}_{2h_1}^{(2r)} u\|_{\sigma,p} \le C\|u\|_{\sigma,p} \quad 1 \le p \le \infty,\ \sigma = 0, 1, \quad \forall\, u \in S^{h_1}(\Omega), \tag{74}$$

$$\begin{cases} (\mathcal{I}_{2h_1}^{(2r)})^2 = \mathcal{I}_{2h_1}^{(2r)}, \quad \mathcal{I}_{2h_1}^{(2r)}\mathcal{I}_{h_1} = \mathcal{I}_{2h_1}^{(2r)}, \\ \mathcal{I}_{2h_1}^{(2r)} u(P_i) = \mathcal{I}_{h_1} u(P_i) = u(P_i), \quad \forall\, P_i \in T^{h_1}, \quad u \in C(\overline{\Omega}), \end{cases} \tag{75}$$

$$\begin{cases} \|u - \mathcal{I}_{2h_1}^{(2r)} u\|_{\sigma,p,E} \le C h_1^{2r+1-\sigma} \|u\|_{2r+1,p,E}, \\ \forall u \in W^{2r+1,p}(E), \quad \sigma = 0, 1, \quad 1 \le p \le +\infty, \quad \forall E \in J^{2h_1}|_{\Omega}, \end{cases} \tag{76}$$

where $C > 0$ is a positive constant independent of h_1, T^{h_1} denotes the set of nodal points of $\mathcal{J}^{h_1}$ of a domain Ω.

After the fully discrete approximate solution $U_{h_1}^m$ for problem (52) is computed, we then employ the post-processing technique to the solution. Next we show the following convergence results for this method.

Theorem 3.1 Let $\tilde{u}^0(x, t)$ be the unique weak solution of problem (52), and let $U_{h_1}^m \in S^{h_1}(\Omega)$ be the fully discrete approximation of $\tilde{u}^0(x, t)$ at time $t = t_m$ by using the Crank-Nicolson Galerkin method. Suppose that $\Omega_0 \subset\subset \Omega'' \subset\subset \Omega$, and Ω_0, Ω'' are covered by a uniform mesh. Under the assumptions of Proposition 3.5, we have the following error estimates:

(77)

$$\begin{aligned} &\|\mathcal{I}_{2h_1}^{(2r)} U_{h_1}^m - \tilde{u}^0(x, t_m)\|_{H^1(\Omega_0)} \le C h_1^{r+1}\Big(\|\bar{u}_0(x)\|_{H^{r+1}(\Omega'')} + \|\tilde{u}^0(x, t_m)\|_{H^{r+2}(\Omega'')} \\ &\quad + \int_0^{t_m} \|\partial_t \tilde{u}^0\|_{H^{r+1}(\Omega)} \mathrm{d}t\Big) + C\kappa^2 \int_0^{t_m} \big(\|\partial_{ttt}\tilde{u}^0\|_{L^2(\Omega)} + \|\partial_{tt}\tilde{u}^0\|_{H^2(\Omega)}\big)\mathrm{d}t, \end{aligned}$$

where C is a positive constant independent of h_1 and κ; h_1 and κ are the mesh size and the largest time-step size, respectively. $\partial_t \tilde{u}^0$ denotes the derivative of $\tilde{u}^0$ with respect to time t. $r \ge 1$ is the degree of piecewise polynomials in the finite element space $S^{h_1}(\Omega)$, $\mathcal{I}_{2h_1}^{(2r)}$ is the higher order interpolation operator.

Proof. Observe that

$$\tilde{u}^0(x,t_m) - \mathcal{I}_{2h_1}^{(2r)} U_{h_1}^m = \tilde{u}^0(x,t_m) - \mathcal{I}_{2h_1}^{(2r)} R_{h_1}\tilde{u}^0(x,t_m)$$
$$+\mathcal{I}_{2h_1}^{(2r)} R_{h_1}\tilde{u}^0(x,t_m) - \mathcal{I}_{2h_1}^{(2r)} U_{h_1}^m = \theta^m + \eta^m,$$

where R_{h_1} is an elliptic type projection operator, see [6, 15, 25]. It follows from Lemma 3.1 that

$$\begin{aligned}\|\theta^m\|_{H^1(\Omega_0)} &\le Ch_1^{r+1}\|\tilde{u}^0(x,t_m)\|_{H^{r+2}(\Omega'')} + \|\tilde{u}^0(x,t_m) - R_{h_1}\tilde{u}^0(x,t_m)\|_{H^{-\gamma}(\Omega'')}\\ &\le Ch_1^{r+1}\big(\|\bar{u}_0(x)\|_{H^{r+1}(\Omega'')} + \|\tilde{u}^0(x,t_m)\|_{H^{r+2}(\Omega'')}\big) + Ch_1^{r+2+\gamma}\|\tilde{u}^0(x,t_m)\|_{H^{r+2}(\Omega'')}\\ &\le Ch_1^{r+1}\big(\|\bar{u}_0(x)\|_{H^{r+1}(\Omega'')} + \|\tilde{u}^0(x,t_m)\|_{H^{r+2}(\Omega'')}\big), \quad \gamma \ge 0.\end{aligned}$$

On the other hand,

$$\eta^m = \mathcal{I}_{2h_1}^{(2r)} R_{h_1}\tilde{u}^0(x,t_m) - \mathcal{I}_{2h_1}^{(2r)} U_{h_1}^m = \mathcal{I}_{2h_1}^{(2r)}\big(R_{h_1}\tilde{u}^0(x,t_m) - U_{h_1}^m\big).$$

Thanks to $(R_{h_1}\tilde{u}^0(x,t_m) - U_{h_1}^m) \in S^{h_1}(\Omega)$, it follows from Lemma 3.1 that

$$\|\mathcal{I}_{2h_1}^{(2r)}\big(R_{h_1}\tilde{u}^0(x,t_m) - U_{h_1}^m\big)\|_{H^1(\Omega_0)} \le C\|R_{h_1}\tilde{u}^0(x,t_m) - U_{h_1}^m\|_{H^1(\Omega_0)}$$

We employ the superconvergence estimates (see, e.g., Theorem 13.3.4 of [5], also see [25]), and obtain

$$\|\eta^m\|_{H^1(\Omega_0)} \le C\|R_{h_1}\tilde{u}^0(x,t_m) - U_{h_1}^m\|_{H^1(\Omega_0)} \le Ch_1^{r+1}\Big(\int_0^{t_m}\|\partial_t\tilde{u}^0\|^2_{H^{r+1}(\Omega'')}dt\Big)^{\frac{1}{2}}$$
$$+C\kappa^2\Big(\int_0^{t_m}\|\partial_{tt}\tilde{u}^0\|_{H^2(\Omega'')} + \|\partial_{ttt}\tilde{u}^0\|^2_{L^2(\Omega'')}\Big)^{\frac{1}{2}}.$$

Therefore, we complete the proof of Theorem 3.1.

3.3. Novel algorithm for boundary layer solutions. In this section, we present the numerical algorithm for solving the boundary layer equations (4). For the case $k = 0$, we employ the backward Euler-Galerkin method or the Crank-Nicolson Galerkin method. Next, we consider when $k = 1, 2, 3$. In practice, we need to solve the following modified boundary value problems:

(78)
$$\begin{cases} \dfrac{\partial \tilde{u}_s^{\varepsilon,b}(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\big(a_{ij}(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^k})\dfrac{\partial \tilde{u}_s^{\varepsilon,b}(x,t)}{\partial x_j}\big) = f(x,t), \quad (x,t)\in\Omega_1\times(0,T)\\ \tilde{u}_s^{\varepsilon,b}(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T)\\ \tilde{u}_s^{\varepsilon,b}(x,t) = \tilde{U}_{h_1}^0(x,t) \quad (x,t)\in(\partial\Omega_0\cap\partial\Omega_1)\times(0,T)\\ \tilde{u}_s^{\varepsilon,b}(x,0) = \bar{u}_0(x), \quad x\in\Omega_1,\end{cases}$$

where $\tilde{U}_{h_1}^0(x,t) = \bar{l}(t)\mathcal{I}_{2h_1}^{(2r)}U_{h_1}^m + (1-\bar{l}(t))\mathcal{I}_{2h_1}^{(2r)}U_{h_1}^{m-1}$ for $t\in(t_{m-1},t_m]$, $\bar{l}(t) = (t-t_{m-1})/\kappa_m$, $\kappa_m = (t_m - t_{m-1})$, $U_{h_1}^m, U_{h_1}^{m-1} \in S^{h_1}(\Omega)$ are the fully discrete approximations for the modified homogenized parabolic equation (52), the interpolation operator $\mathcal{I}_{2h_1}^{(2r)}$ has been defined in (74), also see [18, 19].

Let $\mathcal{T}_{h_2}$ be a regular family of tetrahedron for subdomain $\Omega_1 = \Omega\setminus\overline{\Omega}_0$, where h_2 is the mesh size for Ω_1. Define a linear finite element space

(79) $$W_{h_2}(\Omega_1) = \{v\in C(\overline{\Omega}_1): \quad v|_{\partial\Omega_0\cap\partial\Omega_1} = 0,\, v|_K\in P_1(K),\ v|_{\partial\Omega} = 0\}.$$

The semi-discrete problem of (79) is to find $\tilde{u}^{\varepsilon,b}_{s,h_2} - \tilde{U}^0_{h_1} \in L^2(0,T; W_{h_2}(\Omega_1))$ with $\tilde{u}^{\varepsilon,b}_{s,h_2}(0) = \bar{u}_{0,h_2}$ and

(80)
$$\langle \frac{d}{dt}\tilde{u}^{\varepsilon,b}_{s,h_2}, v_{h_2}\rangle_{\Omega_1} + a^{\varepsilon}_{\Omega_1}(\tilde{u}^{\varepsilon,b}_{s,h_2}, v_{h_2}) = \langle f, v_{h_2}\rangle_{\Omega_1}, \quad \forall v_{h_2} \in W_{h_2}(\Omega_1), \quad t \in (0,T),$$

where $\langle u, w\rangle_{\Omega_1} = \int_{\Omega_1} uw\mathrm{d}x$, $\quad a^{\varepsilon}_{\Omega_1}(u,w) = \int_{\Omega_1} a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})\frac{\partial u}{\partial x_j}\frac{\partial w}{\partial x_i}\mathrm{d}x$. Let $\{\phi_j(x)\}_{i=1}^{n_2}$ be a basis of $W_{h_2}(\Omega_1)$, and $\tilde{u}^{\varepsilon,b}_{s,h_2} = \sum\limits_{i=1}^{n_2} \zeta_i(t)\phi_i(x)$. Then for any fixed $t \in (0,T)$, (80) is equivalent to

$$\sum_{j=1}^{n_2}\langle \phi_j, \phi_i\rangle_{\Omega_1}\frac{d\zeta_j(t)}{dt} + \sum_{j=1}^{n_2} a^{\varepsilon}_{\Omega_1}(\phi_j, \phi_i)\zeta_j(t) = \langle f(x,t), \phi_i\rangle_{\Omega_1}, \quad i = 1, 2, \cdots, n_2. \tag{81}$$

Denoting by $\hat{A}^{\varepsilon}_{h_2} := (a^{\varepsilon}_{\Omega_1}(\phi_j, \phi_i))$ the stiffness matrix, by $B^{\varepsilon}_{h_2} := (\langle \phi_j, \phi_i\rangle_{\Omega_1})$ the mass matrix, and by $\beta_{h_2}(t) := (\langle f(x,t), \phi_i\rangle_{\Omega_1})$, we obtain for $\zeta_{h_2} := (\zeta_i(t))$ the following system of linear ordinary differential equations:

$$\begin{cases} B^{\varepsilon}_{h_2}\dfrac{d}{dt}\zeta_{h_2}(t) + \hat{A}^{\varepsilon}_{h_2}\zeta_{h_2}(t) = \beta_{h_2}(t), \quad t \in (0,T) \\ \zeta_{h_2}(0) = \zeta_{0h_2}. \end{cases} \tag{82}$$

Since the matrix $B^{\varepsilon}_{h_2}$ is symmetric and positive definite, it can be factored as $B^{\varepsilon}_{h_2} = (E^{\varepsilon}_{h_2})^T E^{\varepsilon}_{h_2}$. Introducing the new variable $w^{\varepsilon}_{h_2} := E^{\varepsilon}_{h_2}\zeta_{h_2}(t)$, system (82) can be rewritten as follows:

$$\begin{cases} \dfrac{d}{dt}w^{\varepsilon}_{h_2} + A^{\varepsilon}_{h_2}w^{\varepsilon}_{h_2} = q_{h_2}(t), \quad t \in (0,T) \\ w^{\varepsilon}_{h_2}(0) = E^{\varepsilon}_{h_2}\zeta_{0h_2}, \end{cases} \tag{83}$$

where $A^{\varepsilon}_{h_2} := (E^{\varepsilon}_{h_2})^{-T}\hat{A}^{\varepsilon}_{h_2}(E^{\varepsilon}_{h_2})^{-1}$ is an $\mathbb{R}^{n_2\times n_2}$-elliptic matrix and $q_{h_2} := (E^{\varepsilon}_{h_2})^{-T}\beta_{h_2}$.

It should be emphasized that since the elements of the matrix $A^{\varepsilon}_{h_2}(t)$ are rapidly oscillating with respect to time t , it is difficult to directly solve the linear differential system (83) by the standard numerical methods such as Runge-Kutta methods, since it will require a large computational scaling.

Let $\tau = \varepsilon^{-k}t$, $k = 1, 2, 3$ and $A_{h_2}(\frac{t}{\varepsilon^k}) = A_{h_2}(\tau)$ be τ_0-periodic in τ, i.e. $A_{h_2}(\tau + \tau_0) = A_{h_2}(\tau)$. Then $A^{\varepsilon}_{h_2}(t)$ is $t^0_{\varepsilon} = \varepsilon^k\tau_0$ -periodic in t, but $q_{h_2}(t)$ is not a periodic function in t. Therefore, problem (83) is not a usual non-homogeneous linear system of differential equations with real periodic coefficients(see, e.g., [9, 13]). Suppose that $y^{\varepsilon}_{h_2}(t)$ is the solution of the following homogeneous linear system with real periodic coefficients:

$$\begin{cases} \dfrac{d}{dt}y^{\varepsilon}_{h_2}(t) = -A^{\varepsilon}_{h_2}(t)y^{\varepsilon}_{h_2}(t) \\ y^{\varepsilon}_{h_2}(0) = w^{\varepsilon}_{h_2}(t^0_{\varepsilon}) - w^{\varepsilon}_{h_2}(0), \quad t \in (0,T). \end{cases} \tag{84}$$

Lemma 3.2 ([1]) If $\|\partial_t q_{h_2}\|_{0,\infty} \le K$, where K is a constant, then it holds

$$\|w^{\varepsilon}_{h_2}(t+mt^0_{\varepsilon}) - w^{\varepsilon}_{h_2}(t+(m-1)t^0_{\varepsilon}) - y^{\varepsilon}_{h_2}(t+mt^0_{\varepsilon})\|_{0,\infty} \le C(t^0_{\varepsilon})^2, \quad t \in (0, t^0_{\varepsilon}), \tag{85}$$

where C is a constant independent of t^0_{ε}; $m = 1, 2, \cdots$.

Lemma 3.3 ([1]) Let $t = N_\varepsilon t_\varepsilon^0 + t'$, $N_\varepsilon = 0, 1, 2, \cdots$, $0 \le t' < t_\varepsilon^0$, $t_\varepsilon^0 = \varepsilon^k \tau_0$. Under the assumptions of Lemma 3.2, we have

(86)
$$\begin{array}{rcl} w_{h_2}^\varepsilon(t) - w_{h_2}^\varepsilon(0) & = & \int_0^t q_{h_2}(\sigma)\mathrm{d}\sigma - \sum\limits_{m=1}^{N_\varepsilon - 1}\sum\limits_{j=1}^{m} \int_0^{t_\varepsilon^0} A_{h_2}^\varepsilon(\sigma')y_{h_2}^\varepsilon(\sigma' + jt_\varepsilon^0)\mathrm{d}\sigma' \\ & - & \int_0^{t'} A_{h_2}^\varepsilon(\sigma')w_{h_2}^\varepsilon(\sigma')\mathrm{d}\sigma' - \int_0^{t'} A_{h_2}^\varepsilon(\sigma')y_{h_2}^\varepsilon(\sigma' + (N_\varepsilon - 1)t_\varepsilon^0)\mathrm{d}\sigma' \\ & + & O(\varepsilon^k), \end{array}$$

where $k = 1, 2, 3$.

From Lemma 3.3, in order to compute $w_{h_2}^\varepsilon(t)$, $t \in (0, T)$ in (83), we present a numerical algorithm as follows:

Step I: Compute $w_{h_2}^\varepsilon(t)$ of (83) in a time period $[0, t_\varepsilon^0)$.

Step II: Solve the homogeneous linear system (84) of the differential equations with real periodic coefficients.

The computation on Step I is similar to that of the function $\tilde{u}^0(x,t)$, see (52). However, the error estimates are different, due to the following lemma.

Lemma 3.4 ([1]) Suppose that $\Omega_1 = \Omega \setminus \overline{\Omega}_0 \subset \mathbb{R}^n$ is illustrated in Fig.1:(b). Let $u_s^{\varepsilon,b}(x,t)$, $s = 1, 2$ be the weak solutions of the boundary layer equations (4) with pure Dirichlet boundary conditions. Under the assumptions $(A_1) - (A_4)$, if $a_{ij}(\xi,\tau) \in C^1(\overline{Q} \times [0, \tau_0])$, $\nabla_p a_{ij}(\xi,\tau) \in L^2(0, \tau_0; L^\infty(Q))$, $p = \xi, \tau, \nabla_{\xi\tau}^2 a_{ij} \in L^2(0, \tau_0; L^\infty(Q))$, $\xi = \varepsilon^{-1}x$, $\tau = \varepsilon^{-k}t$, τ_0 is a time period, $f \equiv 0$ for $t < t_0$, where $t_0 = const > 0$, and $f, \partial^k f/\partial t^k \in L^2(\Omega_1 \times (0,T))$, and $g(x,t) \equiv 0$, $\bar{u}_0(x) \equiv 0$, then we conclude that $u_s^{\varepsilon,b} \in C^2(0, T; H^1(\Omega_1))$, and

$$(87) \quad \begin{array}{rcl} \|\partial_{tt} u_s^{\varepsilon,b}\|_{L^2(0,T;L^2(\Omega_1))} & \le & C\varepsilon^{-k-1}\{\|f\|_{C^1(t_0,T;L^2(\Omega))} + \|u^0\|_{L^2(0,T;H^3(\Omega))} \\ & + & \|\partial_t u^0\|_{L^2(0,T;H^1(\Omega))}\}, \quad k = 1, 2, 3, \end{array}$$

where C is a constant independent of ε, $u^0(x,t)$ is the unique weak solution of the homogenized parabolic equation (3).

Remark 3.5 We would like to state that the regularity assumptions for the coefficients $a_{ij}(\xi,\tau)$ in Lemma 3.4 can be extended into the following case: $a_{ij}(\xi,\tau) \in C^1(0, \tau_0; L^\infty(Q))$. It is well known that $C^\infty(Q)$ are not dense in $L^\infty(Q)$. To this end, we introduce q such that $1/q + 1/r = 1/2$. For any fixed $\tau \in (0, \tau_0]$, we can find a sequence of smooth functions $a_{ij}^{(\beta)}(\xi,\tau) \in C^\infty(Q)$ such that (see [2], p.104)

$$(88) \quad \|a_{ij}^{(\beta)}(\xi,\tau) - a_{ij}(\xi,\tau)\|_{L^q(Q)} \to 0, \text{ as } \beta \to +\infty.$$

Let $u_{s,(\beta)}^{\varepsilon,b}(x,t)$ be the solutions of the boundary layer equations (4) by replacing $a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$ with $a_{ij}^{(\beta)}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$. One can directly verify that

$$(89) \quad \begin{array}{l} \sup\limits_{0 \le t \le T} \int\limits_{\Omega_1} (u_{s,(\beta)}^{\varepsilon,b}(x,t) - u_s^{\varepsilon,b}(x,t))^2 \mathrm{d}x + \int\limits_0^T \|u_{s,(\beta)}^{\varepsilon,b}(x,t) - u_s^{\varepsilon,b}(x,t)\|_{H^1(\Omega_1)}^2 \mathrm{d}t \\ \le C(T) \int\limits_0^{\tau_0} \|a_{ij}^{(\beta)}(\xi,\tau) - a_{ij}(\xi,\tau)\|_{L^q(Q)}^2 \mathrm{d}\tau \to 0, \quad \text{as} \quad \beta \to +\infty. \end{array}$$

Following the lines of proof of Theorem 7.31 in ([12], p.338) and using Lemma 3.4, we have the following error estimates for the boundary layer solutions.

Proposition 3.6 Let $\tilde{u}_s^{\varepsilon,b}(x,t)$, $s = 1, 2$ be the weak solutions of the boundary layer equations (78), and let $U_{s,h_2}^{\varepsilon,b,m}$ be the fully discrete solutions using the one-step Theta method for problem (78)(see, e.g., [18]). Under the assumptions of Lemma

3.4, we have

$$\|\tilde{u}_s^{\varepsilon,b}(x,t'_m) - U_{s,h_2}^{\varepsilon,b,m}\|_{0,p,\Omega_1} \le C(p)\Big\{\varepsilon^{-2}h_2^{2-\delta_0} + \varepsilon^{k-1}\frac{\tau_0^2}{N^2}\Big\}, \tag{90}$$

where $\delta_0 = \dfrac{(q-2)n}{2q}$, $q > 2$, $1/p + 1/q = 1$, $1 \le p < 2$, $t_\varepsilon^0 = \varepsilon^k\tau_0$, $k = 1,2,3$, τ_0 is the time period with respect to $\tau = \varepsilon^{-k}t$. The time interval $(0, t_\varepsilon^0)$ is subdivided into N subintervals, the time-step size $\Delta t' = t_\varepsilon^0/N$, h_2 is the mesh size in Ω_1 for the boundary layer equations (78), $C(p)$ is a constant independent of ε, h_2 and $\Delta t'$, but dependent of p.

In the linear system (84), set $\tau = \varepsilon^{-k}t$, $k = 1,2,3$ and $A_{h_2}^\varepsilon(t) = A_{h_2}(\frac{t}{\varepsilon^k}) = A_{h_2}(\tau)$. Assume that the $n_2 \times n_2$ matrix $A_{h_2}(\tau+\tau_0) = A_{h_2}(\tau)$, i.e. $A_{h_2}^\varepsilon(t+t_\varepsilon^0) = A_{h_2}^\varepsilon(t)$. Denote by $Y_{h_2}^\varepsilon(t)$ the matrix which satisfies the equation given by

$$\begin{cases} \dfrac{dY_h^\varepsilon(t)}{dt} = -A_h^\varepsilon(t)Y_h^\varepsilon(t) \\ Y_h^\varepsilon(0) = I, \end{cases} \tag{91}$$

where I is the unit matrix. The n_2 columns of this matrix represents n_2 linearly independent solutions of equation (84). The solution $y_h^\varepsilon(t)$ of equation (84) provided with the initial value $y_{h_2}^\varepsilon(0) = w_{h_2}^\varepsilon(t_\varepsilon^0) - w_{h_2}^\varepsilon(0)$ can be expressed by means of the matrix $Y_{h_2}^\varepsilon(t)$ in the form $y_{h_2}^\varepsilon(t) = Y_{h_2}^\varepsilon(t)y_{h_2}^\varepsilon(0)$. For convenience, we rewrite the linear system as follows:

$$\begin{cases} \dfrac{dY(t)}{dt} = A(t)Y(t) \\ Y(0) = I, \end{cases} \tag{92}$$

where $Y(t) = Y_{h_2}^\varepsilon(t)$, $A(t) = -A_{h_2}^\varepsilon(t)$, $A(t+t_\varepsilon^0) = A(t)$, $t_\varepsilon^0 = \varepsilon^k\tau_0$.

Lemma 3.5 (see Theorem 6.1.3 of [13], p.136) Let $t_\varepsilon^0 > 0$. If the function $A : \mathbb{R} \to M_{n_2}(\mathbb{C})$ is continuous and satisfies $A(t+t_\varepsilon^0) = A(t)$ for $t \in \mathbb{R}$, then a function $P : \mathbb{R} \to M_{n_2}(\mathbb{C})$ and the matrix $W \in M_{n_2}(\mathbb{C})$ exists such that

$$P(t+t_\varepsilon^0) = P(t), \quad \text{for} \quad t \in \mathbb{R},$$

$$Y(t) = exp(Wt)P(t), \quad \text{for} \quad t \in \mathbb{R}.$$

It should be emphasized that Lemma 3.5 does not hold in the real domain, i.e. if we replace the set $\mathbb{C}$ by the set $\mathbb{R}$ in general cases. But in the real domain the following lemma holds which is obtained from Lemma 3.5 by a slight modification:

Lemma 3.6 (see Theorem 6.1.5 of [13], p.138) Let $t_\varepsilon^0 > 0$. If the function $A : \mathbb{R} \to M_{n_2}(\mathbb{R})$ is continuous and satisfies $A(t+t_\varepsilon^0) = A(t)$ for $t \in \mathbb{R}$, then a function $P : \mathbb{R} \to M_{n_2}(\mathbb{R})$ and a matrix $W \in M_{n_2}(\mathbb{R})$ exist such that

$$P(t+2t_\varepsilon^0) = P(t), \quad \text{for} \quad t \in \mathbb{R},$$

$$Y(t) = P(t)exp(Wt), \quad \text{for} \quad t \in \mathbb{R}.$$

The proof of Lemma 3.6 and the detailed constructions of $W \in M_n(\mathbb{R})$ in the special cases can be found in ([13], p.142). Set $P(t) = Y(t)exp(-Wt)$, then $P(t) \in M_{n_2}(\mathbb{R})$, due to $Y(t) \in M_{n_2}(\mathbb{R})$ in [13]. Note that only the existence of $P(t) \in M_{n_2}(\mathbb{R})$ was stated, but no detail was given on its construction. In this study, we present the constructions of $P(t)$, $W \in M_{n_2}(\mathbb{R})$ for specific cases. How to construct $P(t)$ in general cases remains an open question.

Let $A(t)$ in (92) be a continuous periodic matrix with period t_ε^0 in the interval $0 \le t < T$. Then we obtain the matrix $Y(t)$ that is normalized at the point $t = 0$

in the form

$$Y = \sum_{k=0}^{\infty} Y_k(t), \quad Y_0(t) = I,$$

where I is the unit matrix, and

$$Y_k(t) = \int_0^t A(t) Y_{k-1}(t) dt. \tag{93}$$

It can be proved that the series converges uniformly in the interval $0 \leq t < T$ (see, [9], p.45). From a property of fundamental systems of solutions of linear differential equations, $Y(t + t_\varepsilon^0)$ can be expressed in terms of $Y(t)$ by the equation

$$Y(t + t_\varepsilon^0) = VY(t),$$

where V is a constant matrix with nonzero determinant for $t = 0$. From this, we have $Y(t_\varepsilon^0) = V$ and

$$V = \sum_{k=0}^{\infty} Y_k(t_\varepsilon^0), \quad Y_0(t) = I,$$

where $Y_k(t)$ has been defined in (93). Consider the system with a real parameter λ as follows:

$$\frac{dY(t)}{dt} = A(t) Y(t) \lambda. \tag{94}$$

Now, we seek a solution of the system (94) in the following form:

$$Y(t) = P(t) exp(Wt), \tag{95}$$

where W is a real constant matrix and $P(t)$ is a real periodic matrix with a period t_ε^0. Assuming the characteristic numbers of the integral substitution V are all positive, then we have

$$W = \frac{1}{t_\varepsilon^0} ln V = \sum_{k=1}^{\infty} W_k \lambda^k, \tag{96}$$

where W_k will be defined shortly.

It can be shown that the series converges for $|\lambda| \leq 1$ (see Theorem 2.1 of [9], p.22).

$$P(t) = (\sum_{k=0}^{\infty} Y_k(t) \lambda^k) exp\Big(-\sum_{k=1}^{\infty} W_k \lambda^k t\Big), \quad Y_0 = I.$$

Set

$$P(t) = I + \sum_{k=1}^{\infty} P_k(t) \lambda^k. \tag{97}$$

Substituting (95) into (94) and multiplying on the right by $exp(-Wt)$, then we get

$$\frac{dP(t)}{dt} = A(t) P(t) \lambda - P(t) W. \tag{98}$$

Putting (96) and (97) into (98), and equating coefficients of the same power of λ, it gives

$$\frac{dP_k(t)}{dt} = A(t) P_{k-1}(t) - W_k - \sum_{l=1}^{k-1} P_{k-l}(t) W_l, \tag{99}$$

$$\frac{dP_1(t)}{dt} = A(t) - W_1. \tag{100}$$

From (99) and (100), we obtain

$$W_k = \frac{1}{t_\varepsilon^0}\int_0^{t_\varepsilon^0}\Big[A(t)P_{k-1}(t) - \sum_{l=1}^{k-1} P_{k-l}(t)W_l\Big]dt, \tag{101}$$

and

$$P_k(t) = \int_0^t \Big[A(t)P_{k-1}(t) - \sum_{l=1}^{k-1} P_{k-l}(t)\cdot W_l\Big]dt - W_k t.$$

In particular, from (101), it gives $W_1 = \frac{1}{t_\varepsilon^0}\int_0^{t_\varepsilon^0} A(t)dt$. Hence, we have a solution of the system (93) by setting $\lambda = 1$ in (96) and (97). Therefore, $Y(t) = P(t)exp(Wt)$, where W is a real constant matrix, and $P(t)$ is a real periodic matrix with a period t_ε^0. In real applications, we need to replace $P(t)$ and W by using the truncated solutions $P_m(t)$ and W_m, $m \geq 1$, respectively.

4. Multiscale Numerical Algorithm

To summarize the above results, we conclude that the multiscale method for parabolic equations with rapidly oscillating coefficients consists of the following parts:

Part I: Compute the cell functions N_{α_1} and $N_{\alpha_1\alpha_2}$, $\alpha_1, \alpha_2 = 1, 2, \cdots, n$.

Part II: Solve the modified homogenized parabolic equation (52) with constant coefficients in the whole domain $\Omega\times(0,T)$ using coarse mesh and with a larger time-step.

Part III: Compute the boundary layer solutions in a subdomain $\Omega_1 \times (0,T)$ with a fine mesh, see (78).

Part IV: Compute the derivatives $\frac{\partial \tilde{u}^0(x,t)}{\partial x_{\alpha_1}}$ and $\frac{\partial^2 \tilde{u}^0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}$ by means of the finite difference method. In this paper the first-order difference quotients are defined as

$$\delta_{x_j} U_{h_1}^m(N_p,t_m) = \frac{1}{\tau(N_p)}\sum_{K\in\sigma(N_p)} [\frac{\partial U_{h_1}^m}{\partial x_j}]_K(N_p,t_m), \tag{102}$$

where $\sigma(N_p)$ is the set of elements with node N_p; $\tau(N_p)$ is the number of elements of $\sigma(N_p)$; $[\frac{\partial U_{h_1}^m}{\partial x_j}]_K(N_p,t_m)$ is the value of the derivative $\frac{\partial U_{h_1}^m}{\partial x_j}$ at node N_p associated with element K at time $t = t_m$. Similarly, the second-order difference quotients are defined as follows:

$$\delta^2_{x_l x_k} U_{h_1}^m(N_p,t_m) = \frac{1}{\tau(N_p)}\sum_{K\in\sigma(N_p)} [\sum_{j=1}^{d} \delta_{x_l} U_{h_1}^m(P_j,t_m)\frac{\partial \chi_j}{\partial x_k}]_K(N_p,t_m), \tag{103}$$

where d is the number of nodes on K, P_j are the nodes of K, $\chi_j(x)$ are Lagrange's type shape functions.

For simplicity, we present a unified multiscale numerical scheme only for the case $k = 2$. Similar schemes can be developed for other cases.

(104)

$$\mathcal{U}_{1,h_2}^{\varepsilon,h,h_1}(N_p,t_m) = \begin{cases} U_{h_1}^m(N_p,t_m) + \varepsilon\sum\limits_{\alpha_1=1}^{n} N_{\alpha_1,h}^m \delta_{x_{\alpha_1}} U_{h_1}^m(N_p,t_m), & (N_p,t_m)\in\overline{\Omega}_0\times(0,T) \\ U_{1,h_2}^{\varepsilon,b,m}(N_p,t_m), \quad (N_p,t_m)\in\Omega_1\times(0,T), & \end{cases}$$

(105)

$$\mathcal{U}_{2,h_2}^{\varepsilon,h,h_1}(N_p,t_m)=\begin{cases} U_{h_1}^m(N_p,t_m)+\varepsilon\sum\limits_{\alpha_1=1}^{n}N_{\alpha_1,h}^m\delta_{x_{\alpha_1}}U_{h_1}^m(N_p,t_m) \\ \quad +\varepsilon^2\sum\limits_{\alpha_1,\alpha_2=1}^{n}N_{\alpha_1\alpha_2,h}^m\delta_{x_{\alpha_1}x_{\alpha_2}}^2U_{h_1}^m(N_p,t_m), \quad (N_p,t_m)\in\overline{\Omega}_0\times(0,T) \\ U_{2,h_2}^{\varepsilon,b,m}(N_p,t_m), \quad (N_p,t_m)\in\Omega_1\times(0,T), \end{cases}$$

where $U_{h_1}^m(N_p,t_m)$, $U_{1,h_2}^{\varepsilon,b,m}(N_p,t_m)$, $U_{2,h_2}^{\varepsilon,b,m}(N_p,t_m)$ are the fully discrete approximations associated with problems (52), (78) at the nodal point N_p and at time $t=t_m$, respectively. $N_{\alpha_1,h}^m$ and $N_{\alpha_1\alpha_2,h}^m$ are the fully discrete approximations associated with the cell problems (33) and (34), respectively. Note that h, h_1, h_2 are respectively the mesh sizes of domains Q, Ω and Ω_1.

In order to improve the numerical accuracy, we introduce the post-processing technique given by

(106)

$$\mathcal{P}\mathcal{U}_{1,h_2}^{\varepsilon,h,h_1}(x,t)=\begin{cases} \mathcal{P}U_{h_1}(x,t)+\varepsilon\sum\limits_{\alpha_1=1}^{n}N_{\alpha_1,h}\delta_{x_{\alpha_1}}\mathcal{P}U_{h_1}(x,t), \quad (x,t)\in\overline{\Omega}_0\times(0,T) \\ U_{1,h_2}^{\varepsilon,b}(x,t), \quad (x,t)\in\Omega_1\times(0,T), \end{cases}$$

(107)

$$\mathcal{P}\mathcal{U}_{2,h_2}^{\varepsilon,h,h_1}(x,t)=\begin{cases} \mathcal{P}U_{h_1}(x,t)+\varepsilon\sum\limits_{\alpha_1=1}^{n}N_{\alpha_1,h}\delta_{x_{\alpha_1}}\mathcal{P}U_{h_1}(x,t) \\ \quad +\varepsilon^2\sum\limits_{\alpha_1,\alpha_2=1}^{n}N_{\alpha_1\alpha_2,h}\delta_{x_{\alpha_1}x_{\alpha_2}}^2\mathcal{P}U_{h_1}(x,t), \quad (x,t)\in\overline{\Omega}_0\times(0,T) \\ U_{2,h_2}^{\varepsilon,b}(x,t), \quad (x,t)\in\Omega_1\times(0,T), \end{cases}$$

where

$$\begin{aligned} &\mathcal{P}U_{h_1}(x,t)=\bar{l}(t)\mathcal{I}_{2h_1}^{(2r)}U_{h_1}^m(x,t_m)+(1-\bar{l}(t))\mathcal{I}_{2h_1}^{(2r)}U_{h_1}^{m-1}(x,t_{m-1}),\\ &U_{s,h_2}^{\varepsilon,b}(x,t)=\bar{l}(t)U_{s,h_2}^{\varepsilon,b}(x,t_m)+(1-\bar{l}(t))U_{s,h_2}^{\varepsilon,b}(x,t_{m-1}), \quad s=1,2,\\ &N_{\alpha_1,h}(\xi,\tau)=l(\tau)N_{\alpha_1,h}^m(\xi,\tau^m)+(1-l(\tau))N_{\alpha_1,h}^{m-1}(\xi,\tau^{m-1}),\\ &N_{\alpha_1\alpha_2,h}(\xi,\tau)=l(\tau)N_{\alpha_1\alpha_2,h}^m(\xi,\tau^m)+(1-l(\tau))N_{\alpha_1\alpha_2,h}^{m-1}(\xi,\tau^{m-1}),\\ &\bar{l}(t)=(t-t_{m-1})/\kappa_m, \quad l(\tau)=(\tau-\tau^{m-1})/\tau_m, \quad \xi=\varepsilon^{-1}x, \quad \tau=\varepsilon^{-2}t, \end{aligned}$$

κ_m, τ_m are the time-step sizes for solving the modified homogenized parabolic equation (52) and the cell problems (14) and (15), respectively.

Remark 4.1 Combining Theorems 2.1, 2.2, Propositions 3.2,3.4,3.5 and Theorem 3.1, we obtain the final error estimates for the first-order and the second-order multiscale approximate solutions $\mathcal{U}_{s,h_2}^{\varepsilon,h,h_1}(x,t)$, $\mathcal{P}\mathcal{U}_{s,h_2}^{\varepsilon,h,h_1}(x,t)$, $s=1,2$. Due to space limitation, we omit their proofs.

5. Numerical Examples

To validate the convergence results presented in this paper, we consider the following test cases.

Example 5.1. Consider the second order parabolic equations with rapidly oscillating coefficients given by

$$\begin{cases} \dfrac{\partial u^\varepsilon(x,t)}{\partial t}-\dfrac{\partial}{\partial x_i}\big(a_{ij}^\varepsilon(x,t)\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\big)=f(x,t), \quad (x,t)\in\Omega\times(0,T) \\ u^\varepsilon(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T) \\ u^\varepsilon(x,0)=0, \quad x\in\Omega \end{cases} \tag{108}$$

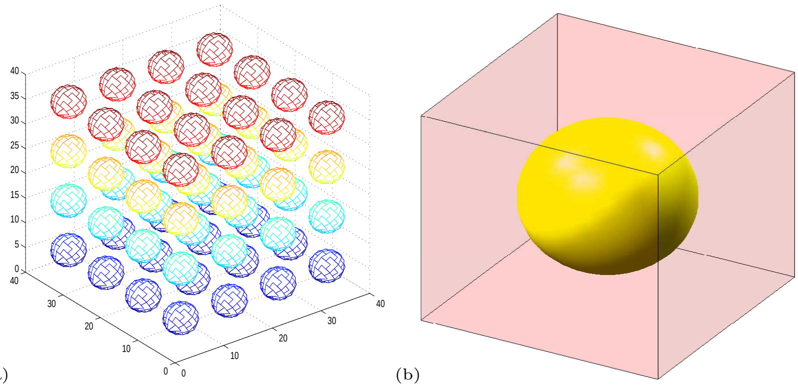

FIGURE 3. (a) The whole domain Ω; (b) the reference cell Q.

where the domain Ω is shown in Fig.3 (a), which is the union of periodic cells and the reference cell $Q = (0,1)^3$ as illustrated in Fig.3 (b). Here, $a_{ij}^{\varepsilon}(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $k = 0,1,2,3$, and $\varepsilon = \frac{1}{4}$.

Numerical simulations are carried out for the following cases:

Case 1: $k = 0$, $a_{ij0} = (200 + 100\sin(2\pi t))\delta_{ij}$, $a_{ij1} = (2 + \sin(2\pi t))\delta_{ij}$, $T = 1.0$, $f(x,t) = 100$.

Case 2: $k = 1$, $a_{ij0} = (200 + 100\sin(2\pi t/\varepsilon))\delta_{ij}$, $a_{ij1} = (2 + \sin(2\pi t/\varepsilon))\delta_{ij}$, $T = 1.0$, $f(x,t) = 100(2 + \sin(2\pi t))(x^2 + y^2)$.

Case 3: $k = 2$, $a_{ij0} = (200 + 100\sin(2\pi t/\varepsilon^2))\delta_{ij}$, $a_{ij1} = (2 + \sin(2\pi t/\varepsilon^2))\delta_{ij}$, $T = 1.0$, $f(x,t) = 100(2 + \cos(2\pi t))$.

Case 4: $k = 3$, $a_{ij0} = (200 + 100\sin(2\pi t/\varepsilon^3))\delta_{ij}$, $a_{ij1} = (2 + \sin(2\pi t/\varepsilon^3))\delta_{ij}$, $T = 1.0$, $f(x,t) = 100(2 + \sin(2\pi t))$.

Note that δ_{ij} is the Kronecker symbol, a_{ij1} denote the coefficients inside the ellipsoids, and a_{ij0} represent the coefficients on the other part.

Since it is difficult to derive the exact solution of (108), $u^{\varepsilon}(x,t)$ is taken from the numerical solution computed using a fine mesh. We implement the tetrahedron partition for Ω in a fine mesh, in which the discontinuities of the coefficients $a_{ij}^{\varepsilon}(x,t)$ approximately coincide with the faces of the tetrahedron, and linear Lagrangian elements are used to solve problem (108). In order to solve the cell problems and the homogenized equation (52) numerically, we implement respectively the tetrahedron partitions for Q and Ω, and then employ linear Lagrangian elements. Since the whole domain Ω is the union of entire unit cells, we do not need to define the boundary layer solutions. Comparisons of the computational costs are listed in Tables 1 and 2.

TABLE 1. Comparison of the numbers of elements and nodes

	original equation	cell problems	homogenized equation
number of elements	129002	2006	24576
number of nodes	25044	461	4913

For simplicity and without confusion, let $\tilde{u}^0(x,t)$ denote the finite element solution for the modified homogenized equation (52) in a coarse mesh, and $U_1^{\varepsilon}(x,t), U_2^{\varepsilon}(x,t)$ are the first-order and the second-order multiscale numerical solutions based on

TABLE 2. Comparison of the time-step size for cases k=0,1,2,3

	original equation	homogenized equation	cell problem
k=0	0.001	0.001	
k=1	0.001	0.001	
k=2	0.00025	0.00625	0.001
k=3	0.000125	0.003125	

(104)-(105). Set $e_0 = u^\varepsilon(x,t) - \tilde{u}^0(x,t)$, $e_1 = u^\varepsilon(x,t) - U_1^\varepsilon(x,t)$, $e_2 = u^\varepsilon(x,t) - U_2^\varepsilon(x,t)$.

The numerical results are reported in Table 3. For convenience, we introduce the notation: $\|u\|_{(0)} = \left(\int_0^T \|u\|^2_{L^2(\Omega)} dt\right)^{1/2}$, and $\|u\|_{(1)} = \left(\int_0^T \|u\|^2_{H^1(\Omega)} dt\right)^{1/2}$. The

TABLE 3. Comparison of computational results

	$\frac{\|e_0\|_{(0)}}{\|u^0\|_{(0)}}$	$\frac{\|e_1\|_{(0)}}{\|U_1^\varepsilon\|_{(0)}}$	$\frac{\|e_2\|_{(0)}}{\|U_2^\varepsilon\|_{(0)}}$	$\frac{\|e_0\|_{(1)}}{\|u^0\|_{(1)}}$	$\frac{\|e_1\|_{(1)}}{\|U_1^\varepsilon\|_{(1)}}$	$\frac{\|e_2\|_{(1)}}{\|U_2^\varepsilon\|_{(1)}}$
Case 1	0.356047	0.358712	0.0259064	3.12531	3.03234	0.191436
Case 2	0.777564	0.781666	0.370077	4.55738	4.45790	0.221588
Case 3	0.703128	0.705362	0.392032	3.91941	3.81342	0.225665
Case 4	0.703245	0.705473	0.394334	3.90867	3.80290	0.232405

numerical results for Case 3 (i.e. $k = 2$) at the section $z = 0.625$ at time $T = 1.0$ are shown in Fig4: (a)-(d).

Remark 5.1 From the computed solutions illustrated in Table 3, we conclude that if the contrast between different parts of the reference cell Q for coefficients $a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $k = 0, 1, 2, 3$ are large, then the numerical results obtained by the homogenization method and the first-order multiscale method are not satisfactory. However, the second-order multiscale method is capable of producing more accurate solutions.

In order to demonstrate the efficiency of the proposed method for treating multiple time scales, for simplicity, we focus numerical simulations for the 1-D case. The case $k = 0$ is routine, so we will consider other cases $k = 1, 2, 3$. We select the reference cell $Q = (0, 1)$, a_0, a_1 are the coefficients on two different parts,respectively,i.e. $a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}) = a_0$, $x \in (0, \frac{1}{3}) \cup (\frac{2}{3}, 1)$; $a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}) = a_1$, $x \in (\frac{1}{3}, \frac{2}{3})$.

Example 5.2. For the initial-boundary value problem (108), we consider the following numerical experiments:

Case 5: $k = 1$, $a_0 = (200 + 100\sin(2\pi t/\varepsilon))\delta_{ij}$, $a_1 = (2 + \sin(2\pi t/\varepsilon))\delta_{ij}$, $T = 0.1$, $f(x,t) = 100(2 + \sin(2\pi t))x$, $\varepsilon = 10^{-2}$.

Case 6: $k = 2$, $a_0 = (200 + 100\sin(2\pi t/\varepsilon^2))\delta_{ij}$, $a_1 = (2 + \sin(2\pi t/\varepsilon^2))\delta_{ij}$, $T = 0.1$, $f(x,t) = 100(2 + \cos(2\pi t))x^2$, $\varepsilon = 10^{-2}$.

Case 7: $k = 3$, $a_0 = (200 + 100\sin(2\pi t/\varepsilon^3))\delta_{ij}$, $a_1 = (2 + \sin(2\pi t/\varepsilon^3))\delta_{ij}$, $T = 0.1$, $f(x,t) = 100(2 + \sin(2\pi t))$, $\varepsilon = 10^{-2}$.

Tables 4 and 5 compare the computational costs.

Symbols $\tilde{u}^0$, U_1^ε, U_2^ε, e_0, e_1, e_2 have been defined in Example 5.1. Let $\|v\|_0 = \left(\int_0^1 |v(x,t)|^2 dx\right)^{1/2}$, $\|v\|_1 = \left(\int_0^1 (|v(x,t)|^2 + |\nabla v(x,t)|^2) dx\right)^{1/2}$ at time $t = T$. The computational results are reported in Table 6.

Remark 5.2 Table 6 clearly confirms that the first-order and the second-order multiscale methods produce more accurate numerical solutions, whereas the homogenization method may not be satisfactory for some cases.

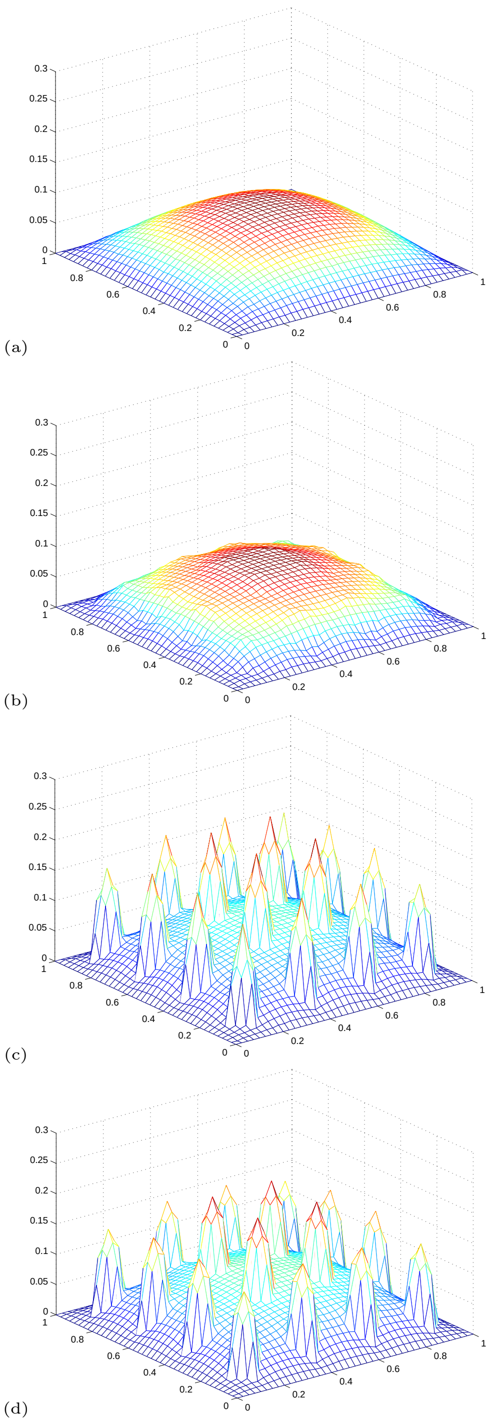

FIGURE 4. (a)Case 3: $\tilde{u}^0$; (b)Case 3: U_1^ε; (c) Case 3: U_2^ε; (d)Case 3: u^ε.

TABLE 4. Comparison of the numbers of elements and nodes

	original equation	cell problems	homogenized equation
number of elements	1200	36	600
number of nodes	1201	37	601

TABLE 5. Comparison of the time-step sizes for cases $k = 1, 2, 3$

	original equation	homogenized equation	cell problems
k=1	0.0001	0.001	
k=2	0.00001	0.001	0.001
k=3	0.0000001	0.001	

TABLE 6. Comparison of computational results

	$\frac{\|e_0\|_{(0)}}{\|\tilde{u}^0\|_{(0)}}$	$\frac{\|e_1\|_{(0)}}{\|U_1^\varepsilon\|_{(0)}}$	$\frac{\|e_2\|_{(0)}}{\|U_2^\varepsilon\|_{(0)}}$	$\frac{\|e_0\|_{(1)}}{\|\tilde{u}^0\|_{(1)}}$	$\frac{\|e_1\|_{(1)}}{\|U_1^\varepsilon\|_{(1)}}$	$\frac{\|e_2\|_{(1)}}{\|U_2^\varepsilon\|_{(1)}}$
Case 5	0.0398570	0.0393782	0.0393853	1.32722	0.0652034	0.0650771
Case 6	0.0101555	0.00779333	0.00782594	1.32668	0.0115531	0.0102391
Case 7	0.00824699	0.00571813	0.00574223	1.31402	0.00715503	0.00606501

Remark 5.3 The relative errors of the approximate solutions versus time t in Examples 5.2 for $k = 1, 2, 3$ are plotted in Fig.5 with the x- axis in time t and the y- axis for the relative error. Fig.5 shows that the present multiscale method is stable and efficient for solving problems (108).

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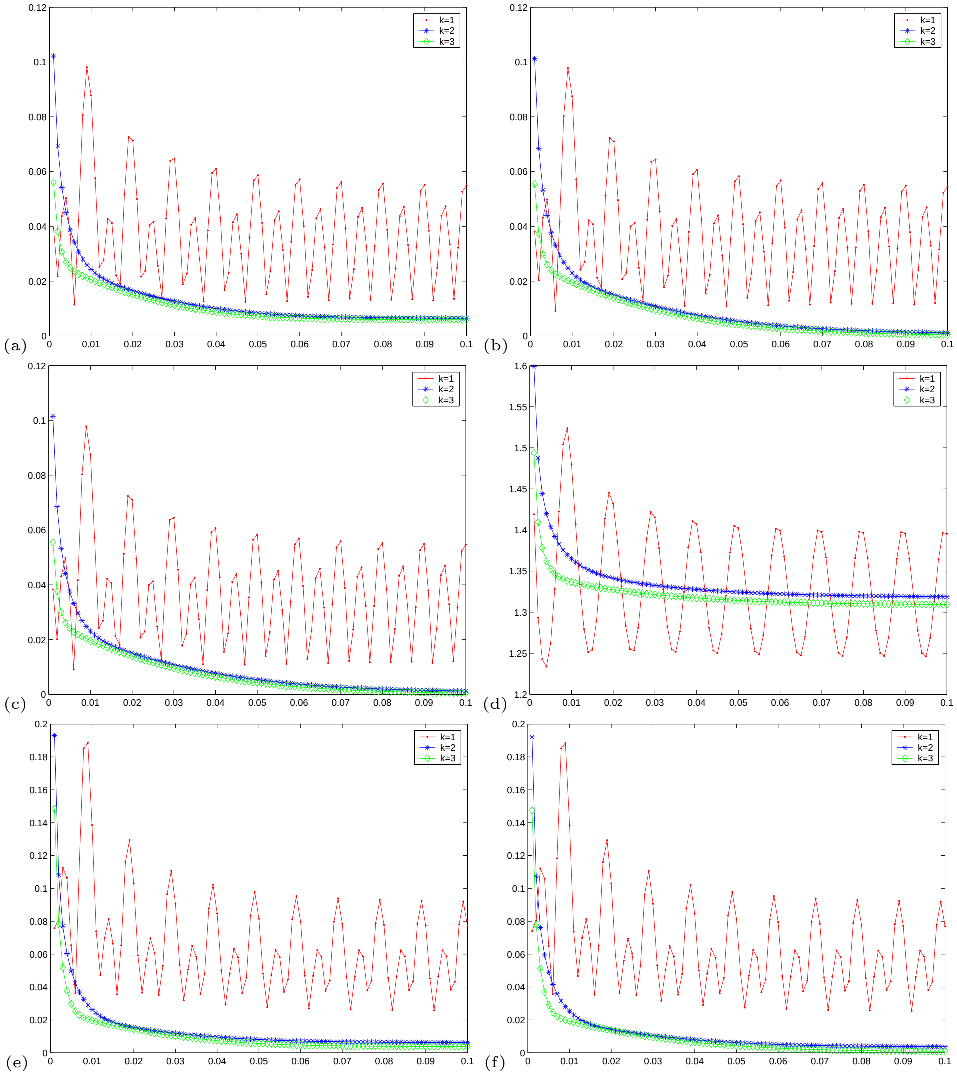

FIGURE 5. In Example 5.2: (a) $\|e_0\|_0/\|\tilde{u}^0\|_0$; (b) $\|e_1\|_0/\|U_1^\varepsilon\|_0$; (c) $\|e_2\|_0/\|U_2^\varepsilon\|_0$; (d) $\|e_0\|_1/\|u^0\|_1$; (e) $\|e_1\|_1/\|U_1^\varepsilon\|_1$; (f) $\|e_2\|_1/\|U_2^\varepsilon\|_1$

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