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FINITE VOLUME ELEMENT METHODS: AN OVERVIEW ON RECENT DEVELOPMENTS

YANPING LIN, JIANGGUO LIU, AND MIN YANG

Abstract. In this paper, we present an overview of the progress of the finite volume element (FVE) methods. We show that the linear FVE methods are quite mature due to their close relationship to the linear finite element methods, while development of higher order finite volume methods remains a difficult and promising research front. Theoretical analysis, as well as the algorithms and applications of these methods, are reviewed.

Key words. elliptic equations, finite element methods, finite volume element (FVE) methods, higher order FVE, parabolic problems, Stokes problems

1. Introduction

Finite volume methods have been widely used in sciences and engineering, e.g., computational fluid mechanics and petroleum reservoir simulations. Compared to finite difference (FD) and finite element (FE) methods, finite volume methods are usually easier to implement and offer flexibility in handling complicated domain geometries. More importantly, the methods ensure local mass conservation, a highly desirable property in many applications.

The construction of finite volume methods is based on a balance approach: a local balance is written on each cell which is usually called a control volume; By the divergence theorem, an integral formulation of the fluxes on the boundary of a control volume is obtained; the integral formulation is then discretized with respect to the discrete unknowns.

Finite volume methods have been developed along two directions. First, finite volume methods can be viewed as an extension of finite difference methods on irregular meshes. It is then called cell centered methods or finite difference methods [50]. Such methods usually satisfy the maximum principle and maintain flux consistency. The higher order formulations of cell centered methods need to use a large stencils of neighboring cells in polynomial reconstruction. Second, finite volume methods can be developed in a Petrov-Galerkin form by using two types of meshes: a primal one and its dual, where the primal mesh allows to approximate the exact solution, while the dual mesh allows to discretize the equation. Such finite volume methods are relatively close to finite element methods and are called finite volume element (FVE) methods. FVE methods have the following advantages: 1). the accuracy of FVE methods solely depends on the exact solution and can be obtained arbitrarily by suitably choosing the degree of the approximation polynomials; 2). FVE methods are well suited for complicated domain and require simple treatment to handle boundary conditions. This overview will concentrate on the methodological issues that arise in FVE methods.

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We first use an abstract equation to illustrate the idea of FVE methods. Consider the following equation

$$\mathcal{L}u = f \quad \text{on} \quad \Omega, \tag{1}$$

where $\mathcal{L} : X \longrightarrow Y$ is an operator. Let $\Omega_h = \{K\}$ denote a primal partition of Ω with elements K. Each element is associated with a number of nodes. Nodes are points on the elements K at which linearly independent functionals are prescribed. For each node, we shall associate a domain K^* with it, which is usually called a control volume. All of the control volumes form a dual partition $\Omega_h^* = \{K^*\}$ of Ω. Denote by S_h^* the piecewise test space on Ω_h^*, which is constructed by generalized characteristic functions [70, 71] of control volumes. We recap the definition of generalized characteristic functions here: Let x_0 be a node and D a domain containing x_0. A generalized characteristic functions of D at x_0 comprises the functions which are the polynomial basis functions in the Taylor expansion of a fixed order of a function at x_0 within D , and zero outside D.

Then a variational FVE form for equation (1) is established as

$$(\mathcal{L}u, v_h^*) = (f, v_h^*), \quad \text{for all } v_h^* \in S_h^*. \tag{2}$$

Note that S_h^* contains piecewise constants. Hence, conservation is locally preserved by applying the divergence theorem. The discrete FVE form is to seek an approximation of u in S_h for (2), where S_h is a finite element space defined on Ω_h. Different choices for dual partitions, solution spaces, and test spaces lead to different FVE methods. If piecewise polynomials of degrees k and k' are used for the solution space and the test space, respectively, the corresponding FVE scheme is called $k-k'$ dual grid scheme [23].

2. Linear FVE methods

Linear $(1-0)$ FVE methods have been extensively studied and their theories and algorithms are relatively mature now.

2.1. FVE methods for elliptic problems.

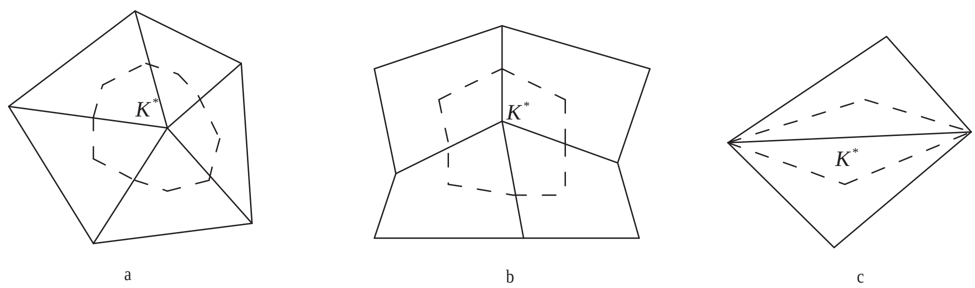

FIGURE 1. A control volume

2.1.1. Conforming, nonconforming, and discontinuous methods. A primal partition for 1D domain (a, b) is denoted by $\Omega_h : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, while its dual is denoted by $\Omega_h^* : a = x_0 < x_{1/2} < \ldots < x_{n-1/2} < x_n = b$ with $(x_{i-1/2}, x_{i+1/2})$ being control volumes. A primal partition Ω_h for 2D linear FVE problems can be constructed using triangles and quadrilaterals. If the solution space S_h is constructed using conforming linear elements, then an associate control volume

K^* for a vertex is obtained by connecting midpoints of edges and an arbitrary interior point in the element. See, e.g., Figure 1 (a)(b), where the control volume is the dotted polygonal domain around the vertex. If the interior point is chosen as the barycenter or circumcenter, the corresponding dual partition is named as barycenter or circumcenter (Voronoi) dual partitions, which are widely used in engineering. If the solution space S_h is constructed using nonconforming Crouzeix-Raviart elements, then the control volume for each midpoint of an edge is obtained by connecting the vertices and an arbitrary interior point in the element (Figure 1 (c)). Dual partition for 3D problems can be constructed in a similar way, see e.g. [81, 101]

In 1982, the original FVE methods (called generalized difference methods) were developed by Li for 1D and 2D elliptic problems [67, 72, 120]. The approximate solutions were sought in the conforming linear (bilinear) finite elements defined on a primal partition. The test space S_h^* consists of characteristic functions defined on a dual partition. The ellipticity of the auxiliary schemes, which were generated by use of some quadrature formulas, was first investigated. Then the ellipticity and H^1 errors of the schemes were proved. Since then, many Chinese researchers have contributed to the development of the generalized difference methods. Most of their results, till to 1994, are collected in the monograph [70, 71].

Another approach to analyze linear FVE methods on triangular meshes is to compare FVE scheme with the corresponding FE scheme. Bank and Rose [2] found that the difference of stiffness matrices between FVE (called box method) method and FE method is $O(h)$ and is the same if the diffusion matrix is elementary. Their observation can be expressed as

$$|a_{FV}(u_h, I_h^* v_h) - a_{FE}(u_h, v_h)| \le Ch|u_h|_1|v_h|_1, \quad u_h, v_h \in S_h, \tag{3}$$

where $I_h^* : S_h \longrightarrow S_h^*$ is a linear transfer operator,

$$a_{FV}(u_h, I_h^* v_h) = -\sum_{K^* \in \Omega_h^*} I_h^* v_h \int_{\partial K^*} a\nabla u \cdot \mathbf{n} ds$$

is the FVE bilinear form of the elliptic term $-\nabla \cdot (a\nabla u)$, and

$$a_{FE}(u_h, v_h) = \int_\Omega a\nabla u \cdot \nabla u dx$$

is the standard FE bilinear form. By the closeness relationship (3), the following error estimates hold

$$|||u - u_h||| \le \inf_{v \in S_h} (|||u - v||| + \|u - \bar{v}\|_0),$$

$$|||u - u_L||| \le |||u - u_h||| \le C(|||u - u_L||| + \|u - \bar{u_L}\|_0),$$

where $|||\cdot|||$ denotes the energy norm, $\bar{v} = I_h^* v$ a piecewise projection of v, and u_L the linear FE approximation. From then on, treating linear FVE methods as "a perturbation" of the corresponding FE methods becomes an efficient and popular tool in the analysis of linear FVE methods. Properties of FVE methods could be derived from the standard FE results, see e.g. [39, 46, 56, 61, 100].

Early on, the FVE methods mainly used conforming linear elements to construct solution spaces. Nonconforming elements have good stability properties and parallelability. A dual partition for the FVE method based on the Crouzeix-Raviart element can be constructed as in Figure 1 (c). Thus, for each edge, there exists an associated control volume K^*, which consists of the union of the sub-triangular sharing the edge. In 1999, Chatzipantelidis [15] studied such FVE scheme for definite elliptic problems. Optimal order errors in the L^2- norm and a mesh-dependent H^1-norm were proved by a direct examination of coercivity of the scheme and a duality argument. It is worth noticing that the interior point in Figure 1 (c) should be chosen as the barycenter of the element to obtain the optimal order L^2 error estimate. This is also the case for the conforming FVE methods. FVE methods based on other nonconforming elements can be constructed similarly, see e.g. [80], where a nonparametric P1-nonconforming quadrilateral FVE method is introduced.

For triangular meshes, linear FVE methods (conforming or nonconforming) and the corresponding FE methods will reduce to the same stiffness matrix when the diffusion coefficient is constant; they differ only in the right-hand side terms. But this difference can be controlled as follows when the barycenter dual partition is used [16, 33, 100]

$$|(f, v_h^*) - (f, v_h)| \leq C \sum_{K\in\Omega_h} h_K \|f\|_{0,K} |v_h|_{1,K}, \tag{4}$$

$$|(f, v_h^*) - (f, v_h)| \leq C \sum_{K\in\Omega_h} h_K^2 \|f\|_{W^{1,p}(K)} |v_h|_{W^{1,q}(K)}, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{5}$$

where $v_h^* \in S_h^*$ is a piecewise constant interpolation of $v_h \in S_h$. Based on this observation, a unified analysis for conforming and nonconforming linear FVE methods was performed in [16]. The main idea there is to write the FVE methods as

$$\sum_{K\in\Omega_h} (A\nabla u_h \cdot n, Q_2^K \chi)_{\partial K} + (\mathcal{L}u_h, Q_1\chi)_K = (f, Q_1\chi), \quad \forall \chi \in S_h. \tag{6}$$

where $\mathcal{L}u = -\nabla \cdot (a\nabla u)$, and $Q_1\chi$ and $Q_2^K\chi$ are defined as an appropriate linear combination of point values of χ. Compared with the corresponding FE methods, the following optimal order errors were obtained

$$\|u - u_h\|_1 \leq Ch\|u\|_2,$$

$$\|u - u_h\|_0 \leq Ch^2(\|u\|_2 + \|f\|_{1,p}), \quad 1 < p \leq 2.$$

In particular, the discrete method (6) led to some new overlapping FVE schemes.

Discretization methods utilizing discontinuous elements possess the advantages of high parallelizability, localizability, and easy handling of complicated geometries. In 2004, Ye [113] developed a FVE method based on discontinuous P1 elements (DFVEM) for second order elliptic problems. The primal partition Ω_h can be nonconforming, allowing hanging nodes. The dual partition is constructed by connecting the barycenter of each primal element with line segments to the vertices. The dual partition of DFVEM looks like that of nonconforming FVE method (see Figure 1 (c)). But for DFVEM, one sub-triangle forms one control volume. Hence, there exist two control volumes for an interior edge. A discrete form for diffusion term $-\nabla \cdot (a\nabla u)$ is defined as: for $u_h, v_h \in S_h$,

$$a_{FV}(u_h, \gamma v_h)$$

$$= -\sum_{K\in\Omega_h} \sum_{i=1}^{3} \int_{P_{i+1}QP_i} (a\nabla u_h \cdot \mathbf{n})\gamma v_h ds - \sum_{e\in\mathcal{E}_h} \int_e [\gamma v_h] \cdot \{a\nabla u_h\} + \alpha \sum_{e\in\mathcal{E}_h} [\gamma u_h][\gamma v_h],$$

where Q is the barycenter of the element K, P_i three vertices, $\mathcal{E}_h$ edges of the primal partition, α penalty factor, γ a transfer operator from S_h to S_h^*. A unified way to analyze linear FVE methods, including DFVE methods, was investigated by exploring their natural relation to the FE methods [39].

It should be emphasized that the optimal order errors of FVE methods need more regular assumptions on the exact solution than FE methods. The regularities in both the exact solution and the source term will affect the accuracy of FVE methods. Ewing et. al. [46] confirmed that the conforming linear FVE method cannot have the standard $O(h^2)$ convergence rate in the L^2-norm when the source term has minimum regularity, only being in L^2, even if the exact solution is in H^2. The L^2-error estimate in such case is

$$\|u - u_h\|_0 \leq C(h^2\|u\|_2 + h^{1+\beta}\|f\|_\beta), \quad 0 \leq \beta \leq 1. \tag{7}$$

It was also shown in [21] that when the source term $f \in W^{1,p}$, the L^2-error should be

$$\|u - u_h\|_0 \leq Ch^2|\ln h|^{\delta_{1,p}/2}\|f\|_{W^{1,p}}, \quad p \geq 1, \tag{8}$$

where δ is the Kronecker symbol. Unlike FE methods, the estimate $\|u - u_h\|_0 \leq Ch^2\|f\|_{0,p}$, $p \geq 2$ does not exist.

2.1.2. Mixed FVE methods. Another type of FVE methods are based on mixed forms, which are also called as covolume methods [27, 28, 29, 33]. The main idea of such methods is to use two partitions of the domain to find approximations of the state and flux variables simultaneously. A conservation law on the primal volumes is used for the state variable and a constitutive law on the dual volumes or covolumes is used for the flux variable. For example, we depict the nonoverlapping covolumes in Figure 2, where a typical interior covolume (conrol volume) in the dual partition is the dashed quadrilateral, the closure of the union of the two sub-triangles sharing the common edge E. The two interior points in the primal elements are usually chosen as the barycenters. Thus each edge E of the primal element corresponds to a covolume. On the boundary, the covolume reduce to a sub-triangle. In Figure 3, the dashed covolumes are overlapping. This type of staggered grid is also adopted in the MAC method [26] and is particularly of interest in oil recovery simulations. The right quadrilateral case in Figure 3 can be viewed as a distorted figure of left rectangles.

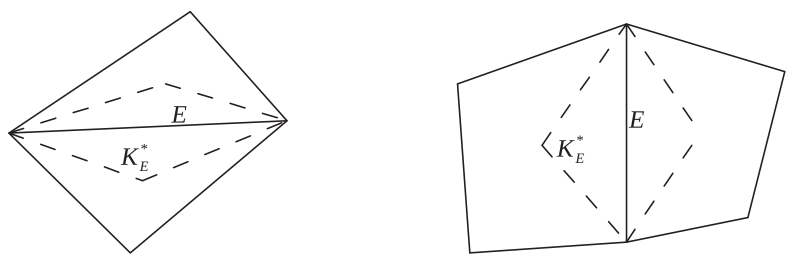

FIGURE 2. Primal elements and nonoverlapping control volumes

The significance of the mixed FVE methods is that, unlike lower order mixed finite element methods, mixed finite volume methods can decouple the pressure from the flux and compute it basically cost-free. Thus they require fewer degrees of freedom than the mixed finite element methods.

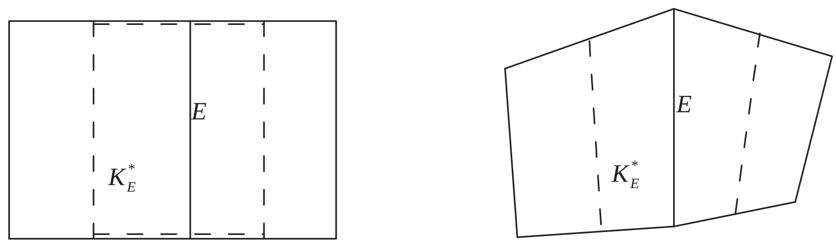

FIGURE 3. Primal elements and overlapping control volumes

To illustrate the basic idea of the mixed FVE methods, let us consider a model elliptic problem

$$\begin{aligned} -\nabla\cdot(\mathcal{K}\nabla p) &= f, \quad \text{in } \Omega, \\ \mathcal{K}\nabla p\cdot n &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\mathcal{K}$ is a diffusion matrix. The above equation can be written as a system of a first order equations

$$\begin{aligned} \mathcal{K}^{-1}\mathbf{u} &= -\nabla p, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= f, \quad \text{in } \Omega, \\ \mathbf{u}\cdot n &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

Let Ω_h be a triangular or quadrilateral partition of Ω. To define a mixed FVE scheme, a dual partition Ω_h^* can be constructed as in Figure 2 and Figure 3, where a control volume for an edge E is the dashed line quadrilateral K_E^*. The solution space $\mathbf{H}_h$ with velocity $\mathbf{u}$ can be taken as piecewise constant vector functions on Ω_h^* that have continuous normal traces across the interior edges, or lowest-order Raviat-Thomas space with respect to Ω_h. The solution space with pressure p can be chosen as the piecewise constant space L_h on Ω_h. The test space $\mathbf{Y}_h$ is usually built by piecewise constant vectors, which have continuous normal traces, but take on different constant vector values on the left and right pieces of an interior dual element and are zero on boundary dual elements. The mixed FVE scheme for the model elliptic problem is to find $\mathbf{u}_h \times p_h \in \mathbf{H}_h \times L_h$ such that

$$a(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p_h) = (f,\mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{Y}_h \tag{9}$$

$$c(\mathbf{u}_h,q_h) = (f,q_h), \quad q_h \in L_h, \tag{10}$$

where

$$\begin{aligned} a(\mathbf{u}_h,\mathbf{v}_h) &= \int_\Omega \mathcal{K}^{-1}\mathbf{u}_h\cdot\mathbf{v}_h dx, \\ b(\mathbf{v}_h,q_h) &= \sum_{K_E^*\in\Omega_h^*}\int_E \mathbf{v}_h\cdot n[q_h]_E ds, \\ c(u_h,q_h) &= \begin{cases} -\displaystyle\sum_{K_E^*\in\Omega_h^*}\int_E \mathbf{u}_h\cdot n[q_h]_E ds, & \\ \displaystyle\int_\Omega q_h\nabla\cdot\mathbf{u}_h dx, & \text{if } \mathbf{u}_h \text{ is linear.} \end{cases} \end{aligned}$$

By introducing a suitable transfer operator γ_h from the solution space $\mathbf{H}_h$ to the corresponding test spaces $\mathbf{Y}_h$ (e.g., piecewise edge averaging vectors), the above scheme can be rewritten as

$$a(\mathbf{u}_h, \gamma_h \mathbf{v}_h) + b(\gamma_h \mathbf{v}_h, p_h) = (f, \gamma_h \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{H}_h \tag{11}$$

$$c(\mathbf{u}_h, q_h) = (f, q_h), \quad q_h \in L_h. \tag{12}$$

The scheme is now well connected to the standard mixed FVE method and they share some common properties. In fact, the most important issue in the covolume framework is the construction of the operator γ_h and test spaces that will maintain optimal order convergence rates, and superconvergence results, when compared with the corresponding mixed FE methods.

The covolume methods using nonoverlapping dual partitions were analyzed in [27] and finally presented in a unified manner in [36]. For nonoverlapping cases, Chou and Kwak [29] proved the first order of convergence for the approximate velocities as well as for the approximate pressures on rectangular meshes. Chou, Kwak, and Kim [31] extended the analysis to the general second-order elliptic problems on quadrilateral grids. They formulated a new framework where the locally supported test functions are images of the natural unit coordinate vectors under the Piola transformation. Under the assumption that the quadrilateral mesh satisfies the "almost parallelogram" condition, the optimal order error estimates were obtained.

There also exist FVE methods that use only a single nonstaggered grid system to achieve stability. An approximate flux can be sought in the lowest-order Raviart-Thomas space $\mathbf{H}_h$, while the solution space L_h for pressure is chosen as the nonconforming $P1$ elements (triangular meshes) [40] or the rotated nonconforming $P1$ space (quadrialteral meshes) [28]. Then the mixed FVE scheme is to find $\mathbf{u}_h \times p_h \in \mathbf{H}_h \times L_h$ such that for any $Q \in \Omega_h$,

$$\int_Q (\mathbf{u}_h + \mathcal{K}\nabla p_h) \cdot \nabla \chi dx = 0, \quad \chi \in N_h(Q), \tag{13}$$

$$\int_Q \nabla \cdot \mathbf{u}_h dx = \int_Q f dx. \tag{14}$$

2.1.3. Superconvergence. Study of superconvergence of FVE methods includes two aspects. One is to treat the linear FVE methods as a perturbation of the FE methods and prove that the differences between the FVE and FE solutions are higher order terms. This phenomenon is also named as "supercloseness". In 1988, Hackbusch [56] proved that the error between the FEM solution and the FVM solution for two-dimensional elliptic problems is of first order in general case and is of second order for barycenter dual partition. As a consequence of Hackbush's result, some superconvergence of FE methods are also naturally valid for the FVE methods, see, e.g., [4, 5, 17, 33, 38, 87, 100].

Another aspect is to obtain superconvergence when the meshes satisfy some special properties. In 1991, Cai, Mandel and McCormick [9, 11] established improved $O(h^{3/2})$ and $O(h^2)$ H^1 errors for triangular meshes when the control volumes are symmetric or essentially symmetric. For quadrilateral meshes, if the meshes are h^2-uniform, i.e., each two neighbor quadrilaterals is close to a parallelogram, superconvergence was derived in an average gradient norm in [78].

2.2. Extensions. Due to the nonsymmetry of the schemes, coercivity of FVE methods for nonlinear problems is difficult to verify. In 1987, Li [68] considered an elliptic problem with nonlinear diffusion $a(x, u)$. A circumcenter dual partition was adapted to ensure the symmetry of the scheme and then the Brouwer fixed point theorem gave the existence of the approximate solution. The first order error in H^1-norm was obtained. In 2005, a barycenter type FVE method was analyzed for an elliptic problem with the diffusion coefficient being a tensor [17]. Compared with the corresponding FE method, optimal order errors in general norms were derived. Recently, Bi and Ginting [8] extended the analysis in [17] to a more general quasilinear elliptic problem

$$-\nabla \cdot F(x, \nabla u) + g(x, u, \nabla u) = 0, \quad \text{in} \quad \Omega,$$
$$u = 0, \quad \text{on} \quad \partial\Omega.$$

It was proved that the approximations are convergent with $O(h)$, $O(h^{1-2/r}|\ln h|)$, $r > 2$, and $O(h^2|\ln h|)$ in the H^1-, $W^{1,\infty}$- and L^2-norm when $u \in W^{2,r}(\Omega)$ and $u \in W^{2,\infty}(\Omega) \cap W^{3,p}(\Omega), p > 1$, respectively. Moreover, the optimal order error estimates in the $W^{1,\infty}$- and L^2-norm, and an $\mathcal{O}(h^2|\ln h|)$ estimate in the L^∞-norm are derived under the assumption $u \in W^{2,\infty}(\Omega) \cap H^3(\Omega)$. For the following quasilinear elliptic problem

$$-\nabla \cdot (a(p) + b(p))) + c(p) = f, \quad \text{in } \Omega,$$
$$p = 0, \quad \text{on } \partial\Omega.$$

Kwak and Kim [63] used an overlapping mixed FVE method [29] to discretize it and obtained the first order errors for both the state and the flux variables.

Since the linear FVE methods are close to the corresponding FE methods, extension of linear FVE methods from elliptic problems to time-dependent problems is usually straightforward by a perturbation argument. A unified approach and general error estimates for parabolic problems were presented in [33] by a perturbation analysis. FVE methods for parabolic integro-differential problems, which arise in modeling reactive flows or material with memory effects, have been intensively studied by using the Petrov-Volterra projection [48, 49, 93]. Symmetric FVE schemes can also be developed by using the "lumped mass" technique to solve the discrete equations more efficiently [79, 85, 86, 91]. FVE methods combined with the upwind or characteristic techniques can well handle convection-dominated problems [35, 88, 92, 95].

Remark. We should notice that until now, little progress has be made on FVE methods for nonlinear time-dependent problems. The existence of the approximations for such problems is still not well-known.

Navier-Stokes equations are of crucial importance in fluid dynamics. Covolume methods can be applied to discretize the generalized Stokes equations, where the velocity might be approximated by conforming and nonconforming linear elements or rotated bilinear elements, and the pressure by piecewise constants [26, 27, 28]. In 2001, Ye [112] revealed a close relationship between the FVE and FE approximations for lower-order elements for Stokes equations. The solution spaces used there include conforming, linear velocity-constant pressure on triangles, conforming bilinear velocity-constant pressure on rectangles and their macro-element versions, and a nonconforming linear velocity-constant pressure on triangles and nonconforming rotated bilinear velocity-constant pressure on rectangles. FVE methods based on other nonconforming elements [19, 43, 99], discontinuous $P1$ elements [41, 114], stabilized elements [57, 58, 59, 65, 66], can also be used to discretize the Stokes and Navier-Stokes problems. A critical step in the analysis of FVE methods for Stokes (Navier-Stokes) problems is to verify the inf-sup condition, which can be successfully obtained from the corresponding FE methods by a perturbation argument.

In many applications, a simulation domain is often formed by several materials separated by curves or surfaces from each other, and this often leads to interface problems. In the immersed element method, standard FE functions are used in elements occupied by one of the materials, but piecewise polynomials patched by interface jump conditions are employed in elements formed by multiple materials. Particularly, the meshes used can be independent of the interface. In 1999, a FVE method based on immersed elements on triangular meshes were presented in [47]. Optimal error estimates in an energy norm are obtained by a perturbation argument. A bilinear immersed FVE method was constructed in [60], and the optimal order L^2- and H^1- errors were confirmed by numerical examples.

FVE methods are also well adapted to discretize systems of equations coupled with elliptic, parabolic and hyperbolic types. Examples of performance of FVE methods combined with multiscale methods for two-phase flows in oil reservoirs can be found in [44, 51]. Level set methods are widely used for predicting evolutions of complex free surface topologies. [83] presented a characteristic level set equation derived by using the characteristic-based scheme. An explicit FVE method was developed to discretize the equation on triangular grids.

2.3. Algorithms.

2.3.1. Adaptive algorithms. A posteriori error estimation is particularly useful and successful in designing efficient adaptive algorithms for numerical schemes. Developing the theory of a posteriori error estimates for finite volume methods has attracted much attention. In 2000, Agouzal and Oudin [1] compared finite volume methods with some well-known FE methods, namely the dual mixed methods and nonconforming primal methods, for elliptic equations. Equivalences were exploited to give a posteriori error estimators for finite volume methods. In 2002, Lazarov and Tomov [64] adopted the finite element local error estimation techniques to the case of conforming linear FVE approximations for 2D and 3D indefinite elliptic problems, and a residual type error estimator and its practical performance were investigated. New residual-type error estimators and averaging techniques were given in [14]. In 2003, Bergam, Mghazli, and Verfürth [3] obtained a residual-type a posteriori error estimates of a FVE method for a quasi-linear elliptic problem of nonmonotone type using the Kirchhoff transformation. The approach in [3] can not be used to derive a posteriori error estimates of the FVE method when the diffusion coefficient is a matrix. Bi and Ginting [7] overcame this difficulty and derived a residual-type a posterior error estimate of FVE method when diffusion is a nonlinear matrix function. A posteriori error estimates for nonconforming FVE methods for steady Stokes problems and indefinite elliptic problems were obtained in [19, 104], respectively. Residual type estimators, which could be applied to different finite volume methods associated with different trial functions including conforming, nonconforming and totally discontinuous trial functions, were established in a systematic way [116]. All the results above have demonstrated that the a posteriori error estimates for the FVE method are quite close to those for the FE methods, and the mathematical tools from FE theory can be successfully applied for the analysis.

For the discontinuous finite volume methods (DFVM), a posteriori error estimation and adaptive schemes have been investigated in [76, 116].

2.3.2. Multigrid algorithms. Multigrid methods have proven to be robust and effective in conjunction with the FE methods for elliptic problems. Since linear FVE methods can be obtained from related FE methods by adding small perturbation terms to the bilinear forms (on the left-hand side) and the linear functionals (on the right-hand side) corresponding to the FE formulations, this gives possibility for designing multigrid algorithms for FVE methods in a similar way for the FE methods.

In 2002, Chou and Kwak [30] analyzed V-cycle multigrid algorithms for a class of perturbed problems whose perturbation in the bilinear form preserves the convergence properties of the multigrid algorithm of the original problem. The requirement for small coarse grid was necessary for the covolume method to make sense. As an application, they studied the convergence of multigrid algorithms for a FVE method for variable coefficient elliptic problems on polygonal domains. Similar to the FE methods, the V-cycle algorithm with one pre-smoothing converges with a rate independent of the number of levels. Two most important classes of smoothers, Jacobi type and Gauss-Seidel type, were analyzed. Cascadic multigrid method, which requires no coarse grid corrections and can be viewed as a "one-way" multigrid method, is effective for solving large-scale problems. The cascadic multigrid algorithms have been proposed for solving the algebraic systems arising from the conforming and nonconforming FVE methods [80, 90]. It was shown that these algorithms are optimal in both accuracy and computational complexity. In 2007, Bi and Ginting [6] studied two-grid FVE algorithms for linear and nonlinear elliptic problems, which involves a nonlinear solve on the coarse grid with size H and a linear solve on the fine grid with size $h \ll H$, In 2009, postprocessing FVE procedures were developed for the time-dependent Stokes problem and the semilinear parabolic problem [106, 110]. The postprocessing technique can be seen as a novel two-grid method, which involves an additional solution on a finer grid after the time evolution is finished. Unlike the traditional two-grid algorithm, there is no communication from fine to coarse meshes until the end of time-marching. This implies that the extra cost of the postprocessing is relatively negligible when compared with the cost of computations from $t = 0$ to $t = T$ on the coarser mesh.

2.3.3. Other algorithms. The elliptic boundary value problems require a large number of unknowns and the use of parallel computers. Under the assumption that the coefficients are of two scales and periodic in the small scale, Ginting [55] analyzed numerical methods based on finite volumes that can capture the small scale effect on the large scale solution without resolving the small scale details and thus reduce the number of degrees of freedom. The convergence analysis was based on estimating the perturbation of the two-scale finite volumes with respect to finite elements. Moreover, the author presented an application of the method to flows in porous media. The FVE methods for solving differential equations have a shortcoming that the matrices are not well-conditioned. With the help of an interpolation operator from a trial space to a test space, [74] showed that both wavelet preconditioners and multilevel preconditioners designed originally for the FE method can be used to precondition the FVE matrices. These preconditoners lead to matrices with uniformly bounded condition numbers.

3. Higher order FVE methods

To design a higher order FVE method, the first requirement is that the degrees of freedom of the test space and the solution space should be the same, and the second requirement is that the test space should include the characteristic functions of the control volumes so that local conservation will be preserved. Compared to linear FVE methods, higher order FVE methods differ considerably from the corresponding FE methods. The perturbation argument is no longer applicable for analyzing higher order schemes. Nonconforming and nonsymmetric discretizations result in difficulty of establishing a general framework for analysis. The main idea for analyzing higher order FVE methods is to first derive the elementary matrix forms, and then manage to obtain positiveness and boundness of the resulted matrices under certain geometric assumptions.

3.1. One-dimensional case. In 1982, Li [67] introduced higher-order FVE methods for two-point boundary value problem. The solution space S_h used usual finite elements on Ω_h of a domain (a,b). The dual partition Ω_h^* was built by the midpoints of the primal elements. The basis functions for the test space S_h^* were chosen from the following set of piecewise polynomials,

$$\psi_j^{(r)}(x) = \begin{cases} \dfrac{(x-x_i)^r}{r!}, & x_{i-1/2} \le x \le x_{i+1/2}, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

where $x_{i-1/2} = \frac{1}{2}(x_i + x_{i-1})$ and $r = 0, 1, \cdots$.

For example, for a quadratic FVE method, the primal and dual partitions could be

$$\begin{aligned} \Omega_h :& a = x_0 < x_{1/2} < x_1 < x_{3/2} < \cdots < x_{n-1/2} < x_n = b, \\ \Omega_h^* :& a = x_0 < x_{1/4} < x_{3/4} < \cdots < x_{n-1/4} < x_n = b. \end{aligned}$$

The solution space S_h is chosen as piecewise Lagrange quadratic elements, whereas the test space S_h^* is chosen as piecewise characteristic functions defined on Ω_h^*;

For a cubic FVE method, the primal and dual partitions could be

$$\begin{aligned} \Omega_h :& a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \\ \Omega_h^* :& a = x_0 < x_{1/2} < x_{3/2} < \ldots < x_{n-1/2} < x_n = b. \end{aligned}$$

The solution space S_h is chosen as piecewise Hermit elements on Ω_h, the test space S_h^* is spanned by the following basis functions

$$\psi_i^{(0)} = \begin{cases} 1, & x \in [x_{i-1/2}, x_{i+1/2}], \\ 0, & \text{otherwise}; \end{cases}$$

$$\psi_i^{(1)} = \begin{cases} x - x_i, & x \in [x_{i-1/2}, x_{i+1/2}], \\ 0, & \text{otherwise}. \end{cases}$$

Optimal order error estimates in the H^1-norm for these methods can be derived by investigating directly positiveness and boundness of the resulted matrices, see, e.g., [22, 67].

The shape of the dual partition could affect the accuracy of the corresponding higher order FVE methods. Choosing certain special points as the nodes of the dual partition may help derive sharp error estimates. The set of Barlow points (optimal stress points) is a good choice for this purpose. The derivative of a rth Lagrange interpolation I_h at Barlow points $\{b_i\}_{i=1}^r$ satisfies

$$(u - I_h u)'|_{b_i} = 0, \quad \forall u \in \mathbb{P}^{r+1}, \quad i = 1, \ldots, r, \tag{16}$$

which means that the accuracy of the derivative at Barlow points is one order higher when compared with the finite element derivative field. The analysis for quadratic (2-0) and cubic (3-0) FVE methods based on Barlow points could be found in [53, 54], where superconvergence and optimal order L^2-error were obtained.

Borrowing the idea from [16] (see (6)), another type of higher order FVE methods were constructed for one dimensional indefinite elliptic problems in [84]. Compared

TABLE 1. Barlow points in the reference element $[-1, 1]$

r	2	3	4	5
Barlow points	$\pm\frac{1}{\sqrt{3}}$	$0, \pm\frac{\sqrt{5}}{3}$	$\pm\frac{\sqrt{3\pm\sqrt{29/5}}}{2\sqrt{2}}$	$0, \pm\frac{\sqrt{35\pm 8\sqrt{7}}}{5\sqrt{3}}$

with FE methods, a systematic way is presented to analyze the FVE schemes. The main results obtained there include

- If $r = 2, 4, 6$, where r denotes the degree of the approximation piecewise Lagrange polynomials, then the finite volume methods, with control volumes based on the roots of r-th Legendre polynomial, have optimal order of convergence in the H^1- and L^2-norms;
- If $r \geq 3$, then the finite volume methods, with control volumes based on the arbitrary internal nodes of $[0, 1]$, have only optimal order of convergence in the H^1-norm.

In [89] a class of high-order FVE schemes with spectral-like spatial resolution characteristics is developed. An implicit reduction of the number of unknowns was obtained by a local implicit mapping of some degrees of freedom.

Recently, a family of arbitrary order FVM schemes, with control volumes based on the roots Gauss points, were constructed and analyzed in a unified approach [13]. The solution space there was chosen as the Lagrange finite element with the interpolation points being the Lobatto points, significantly different from other FVE methods. With help of the inf-sup condition, the optimal order convergence in the H^1-and L^2-norms, and superconvergence, some of which was much better than that of the counterpart finite element method, were derived.

3.2. Multidimensional case. A difficult task in multidimensional higher order FVE methods is to construct a suitable dual partition to ensure the solvability of the scheme. This is quite complicated compared to those in linear FVE methods. Moreover, certain geometric requirements have to be specified for different higher order methods.

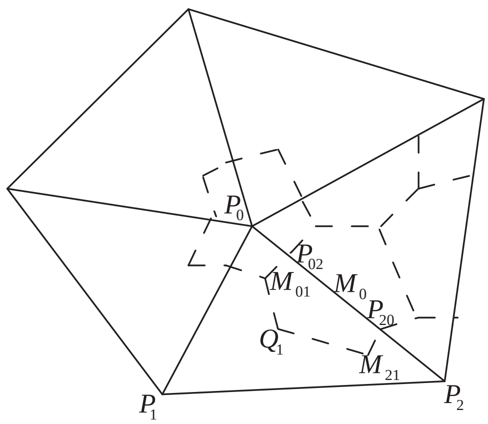

FIGURE 4. A dual partition for the quadratic FVE methods on a triangular mesh

3.2.1. Triangular meshes. As an example, we first consider a quadratic (2-0) FVE method. Let Ω_h be a triangular partition in $\mathbb{R}^2$. Let N_h and M_h be respectively the sets of vertices and edge midpoints. The dual partition Ω_h^* is built from the control volumes of all points in N_h and M_h. Its construction is specificized as follows:

(1) Let P_0 be a vertex in N_h. Let P_i $(i = 1, 2, \ldots, m)$ be the adjacent vertices of P_0 , Q_i the barycenter of the triangle of $\triangle P_0 P_i P_{i+1}$ and P_{0i} , M_{0i} given points on the segment P_0P_i and P_0Q_i, respectively such that

$$|P_0P_{0i}| = \alpha|P_0P_i|, |P_0M_{0i}| = \frac{3}{2}\beta|P_0Q_i|, 1 \le i \le m,$$

where $|PQ|$ denotes the length of the line segment joining points P and Q and $0 < \alpha < 1/2$, $0 < \beta < 2/3$ are two given parameters. We connect P_{0i}, M_{0i} as in Figure 4 to obtain a control volume D_{P_0} surrounding P_0.

(2) Let $M_0 \in M_h$ be a midpoint of the common side of two adjacent elements $K_1 = \triangle P_0P_1P_2$ and $K_2 = \triangle P_0P_2P_3$. We denote by Q_1 and Q_2 the barycenter of K_1 and K_2 respectively. Let $M_{01}, M_{21}, M_{02}, M_22$ be the points respectively on the segments such that

$$|P_0M_{0i}| = \frac{3}{2}\beta|P_0Q_i|, |P_2M_{1i}| = \frac{3}{2}\beta|P_2Q_i|.$$

A control volume D_{M_0} surrounding M_0 is obtained by connecting successively P_{01}, M_{01}, Q_1, M_{11}, P_{10}, M_{12}, Q_2, M_{02} and P_{01}. Then all these control volumes form the dual partition Ω_h^*. The solution space S_h is chosen as the quadratic Lagrange elements on Ω_h, while the test space S_h^* is chosen as the piecewise constants on Ω_h^*. Different choices of α and β lead to different quadratic FVE schemes.

In order to extend the 2-0 FVE method to general $r - 0$ $(r > 2)$ methods, a crucial step is to construct a corresponding dual partition. This might be realized in a systematic way by using several techniques based on the Voronoi diagram and its variants [25]. In 2010, a $r - 0$, $r > 2$ higher order FVE method [96] was proposed for elliptic problems in 2D. Numerical tests demonstrated optimal errors in the H^1-norm. While the errors in the L^2-norm were one order below optimal for even polynomial degrees and optimal for odd degrees.

Next we consider a sample cubic Hermit type (3-1) FVE method. The dual partition is constructed as follows (see Figure 5). Suppose that P_0 is a vertex of a triangular element in the primal partition Ω_h, The control volume of P_0 is the polygonal domain $M_1M_2 \cdots M_6$, where $M_i(1 \le i \le 6)$ are midpoints of the corresponding sides. Suppose that Q is a barycenter of an element $\triangle P_iP_jP_k$, The control volume of Q is the triangle $\triangle M_iM_jM_k$. All these control volumes make dual partition Ω_h^*. The solution space S_h is chosen as the piecewise cubic Hermit polynomials on Ω_h, while the test space S_h^* is spanned by the following basis

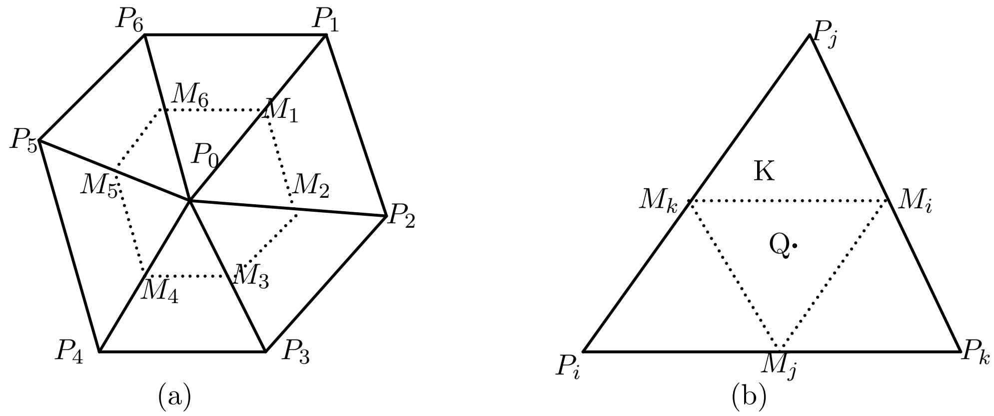

FIGURE 5. A dual partition for the Hermit 3-1 FVE method on a triangular mesh

functions:

$$\psi_P^{(0)} = \begin{cases} 1, & (x,y) \in K_P^*, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_P^{(x)} = \begin{cases} x - x_P, & (x,y) \in K_P^*, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_P^{(y)} = \begin{cases} y - y_P, & (x,y) \in K_P^*, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_Q = \begin{cases} 1, & (x,y) \in K_Q^*, \\ 0, & \text{otherwise.} \end{cases}$$

Here P, Q denote respectively the vertex and barycenter of an arbitrary primal element.

In 1991, Tian and Chen [94] considered a quadratic FVM scheme with $\alpha = \beta = 1/3$. Optimal order H^1- error was obtained under the assumption that the maximum angle of each element of the triangulation is not greater than $\pi/2$ and the ratio of the lengths of the two sides of the maximal angle is within the interval $[(2/3)^{1/2}, (3/2)^{1/2}]$. In 1996, Liebau [75] considered the case in which $\alpha = 1/4, \beta = 1/3$. In 2009, Xu and Zou [101] proved that when α, β varies from 0.18979 to 0.18991, the mass matrix was positive definitive for any $\theta_0 \geq 2.99°$, where θ_0 was the minimal angle of the triangulations. In 2011, Ding and Li [42] studied a Lagrangian cubic FVE method. In 1994, Chen [20] presented the cubic Hermite type FVE method described above. Under the same mesh assumption as that in [94], a third error in the H^1-norm was obtained when the exact solution $u \in H^4(\Omega)$. The main idea in all analysis is to rewrite the scheme into a summation of local bilinear forms on the primal elements by a transfer operator from S_h to S_h^*; Then map the local bilinear forms into the reference element and examine positiveness of the local matrix directly.

The above higher order FVE methods require a complicated construction of control volumes. FVE methods mixing the discretization of a linear FVE and a higher order FE formulation can avoid such complication. Given a triangulation Ω_h, the solution space is chosen as a kth-order finite element space V_{k,Ω_h}. There exists a hierarchical decomposition

$$V_k = V_1 + W_k,$$

where V_1 is the linear finite element space, and W_k is spanned by the hierarchical basis function up to order k excluding linear basis. The test function space is chosen as

$$V_k := V_{0;B} \oplus W_k,$$

where $V_{0,B}$ is a piecewise constants defined on a dual mesh B, which arises from linear FVE methods. The kth-order order FVEM reads as: Given $f \in L^2(\Omega)$, find $u \in V_k$ such that

$$\bar{a}(u; v) = (f; v) \quad \forall v \in V_{0;B}; \tag{17}$$

$$a(u; v) = (f; v) \quad \forall v \in W_k, \tag{18}$$

where $\bar{a}(\cdot, \cdot)$ is the linear FVE bilinear form and $a(\cdot, \cdot)$ the FE counterpart. Such hybrid FVE methods were developed by Chen [24]. Ellipticity and optimal order H^1- errors for quadratic cases on triangular and rectangular meshes have been obtained and verified by numerical experiments.

Recently, Chen, Wu, and Xu [23] provided a systematic study of the geometric requirements for various higher order FVE methods on triangular meshes. Their study was carried out in two steps. The 1st step is to parameterize the mesh geometric requirements by using a linear combination of the mesh feature matrices obtained from the reference element. The 2nd step is to analyze the linear combination. Edge-length parametrization was used to measure the shape of triangles. Necessary and sufficient conditions for the uniformly local ellipticity condition of higher order FVE schemes are introduced, some of of which can be verified easily by a computer program.

3.2.2. Tensor-product and quadrilateral meshes. In 2003, Cai, Douglas and Park [10] presented a systematic way to derive higher order finite volume schemes from higher order mixed finite element methods in two and three dimensions. The procedure starts from hybridization of the mixed method. Then a localized (hybridized) mixed finite element approximation is formulated. Use a quadrature rule to make one matrix in the discrete form diagonal, instead of block diagonal. Finally, the elimination of the flux and Lagrange multiplier yields equations in the scalar variable, which gives the higher order finite volume method. Higher order finite volume methods derived from BDM2 elements and Raviart–Thomas elements have been studied as well.

In 2005, Kim [62] extended the mixed FVE method in [32] to arbitrary orders, and applied the new method to an elliptic problem with nonlinear diffusion $a(x, |\nabla p|)$. They used $H(div; \Omega)$-conforming RT_k elements for the vector variable, completely discontinuous $k+1$ polynomials for the scalar variable. If one eliminates the flux variable in a local manner, then the method will reduce to a discontinuous Galerkin method for the scalar variable.

Higher order FVE methods can also be constructed from the one-dimensional methods presented in Section 3.1 by using tensor products, see, e.g., [98, 117, 119].

Quadrilateral meshes can be regarded as mappings from rectangular ones. Measuring the effect of the distortion becomes a critical task. For an affine biquadratic FVE method, the primal partition Ω_h could be a conforming quadrilateral mesh, whereas the dual partition Ω_h^* can be constructed as follows. As shown in Figure 6, each edge of $Q \in \Omega_h$ is partitioned into three segments so that the ratio of these segments is $1 : n : 1$. We connect these partition points with line segments to the corresponding points on the opposite edge. This way, each quadrilateral of Ω_h is divided into nine sub-quadrilaterals Q_z, $z \in \mathcal{Z}_h(Q)$, where $\mathcal{Z}_h(Q)$ is the set of the vertices, the midpoints of edges, and the center of Q. For each node $z \in \mathcal{Z}_h = \cup_{Q\in\Omega_h}\mathcal{Z}_h(Q)$, we associate a control volume V_z, which is the union of the subregions Q_z containing the node z. Therefore, we obtain a collection of control volumes covering the domain Ω. This is the dual partition Ω_h^* of the primal partition Ω_h. The solution space S_h is chosen as the affine biquadratic Lagrange elements on Ω_h, while the test space S_h^* is chosen as the piecewise constants on Ω_h^*. In 2006, Yang [102] studied a quadratic FVE method with $n = 2$. Optimal or-

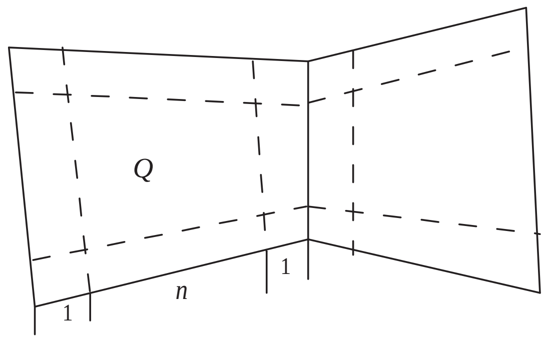

FIGURE 6. A generic quadrilateral Q is partitioned into nine subregions

der H^1 error was proved under the "almost parallelogram" mesh assumption. The method was extended to three-dimensional problems on right quadrangular prism grids [109], where the ratio of the segments for control volumes is $1 : 4 : 1$ to ensure the symmetry.

3.3. Extensions to time-dependent problems. Up to now, little progress has been made on higher order FVE methods for time-dependent problems, due to the nonsymmetry of the schemes. Higher order FVE methods differ greatly from the corresponding FE methods, so that the perturbation argument is no longer applicable for analysis. How to control and measure the nonsymmetry of the related schemes is still an open problem.

In 1984, Li and Wu studied a cubic Hermit FVE method for 1D parabolic problems. The bilinear form associated with the diffusion term is divided into symmetry and nonsymmetry parts and was proved to be not too far away from being symmetric, i.e., only having an $O(h)$ discrepency. It is found in [111] that if the dual partition for a biquadratic FVE method is constructed by the points related to the Simpson quadrature, then the scheme will be symmetric for constant coefficients problems. Hence, such a quadratic method can be successfully applied to many time-dependent problems, see, e.g., [103, 107]. Quadratic (biquadratic) FVE methods whose dual partition is based on Barlow points can have optimal order L^2- error and superconvergence. But such dual partition will destroy the symmetry of the scheme. Special test functions should be used to symmetrize the scheme [105, 118]. More specifically, for any $u_h, v_h \in S_h$, there exist corresponding $\tilde{u}_h, \tilde{v}_h \in S_h$ such that $(u_h, \Pi_h^*\tilde{v}_h) = (v_h, \Pi_h^*\tilde{u}_h)$ and $a_h(u_h, \Pi_h^*\tilde{v}_h) = a_h(v_h, \Pi_h^*\tilde{u}_h)$.

For the quadratic FVE scheme on quadrilateral meshes, the analysis is much more complicated. In 2011, Yang and Liu [108] managed to measure the nonsymmetry of $(\cdot, I_h^* \cdot)$ for a FVE method whose dual partition was based on the Simpson quadrature. Then an optimal convergence rate in the $L^2(H^1)$-norm was proved. However, there are technical difficulties in deriving error estimates in $L^\infty(0,T;H^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ norms.

We conclude the overview with few remarks on challenging research fronts in FVEM.

(i) For general unstructured meshes, how should we construct higher order FVE methods that possess optimal order errors in various norms?
(ii) Nonsymmetry of higher order FVE schemes is difficult to measure and become a fetter in the development for time-dependent problems.
(iii) Efficient algorithms need to be developed.

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Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Hong Kong
E-mail: malin@polyu.edu.hk

Department of Mathematics, Colorado State University, Fort Collins, CO 80523-1874, USA
E-mail: liu@math.colostate.edu

Department of Mathematics, Yantai University, Yantai, Shandong, China
E-mail: yang@ytu.edu.cn