

GAUGE-UZAWA METHODS FOR THE NAVIER-STOKES EQUATIONS WITH NONLINEAR SLIP BOUNDARY CONDITIONS

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Abstract. In this paper, the Gauge-Uzawa method is applied to solve the Navier-Stokes equations with nonlinear slip boundary conditions whose variational formulation is a variational inequality of the second kind with the Navier-Stokes operator. In [1], a multiplier was introduced such that the variational inequality is equivalent to the variational identity. We give the Gauge-Uzawa scheme to compute this variational identity and provide a finite element approximation for the Gauge-Uzawa scheme. The stability of the Gauge-Uzawa scheme is showed. Finally, numerical experiments are given, which confirm the theoretical analysis and demonstrate the efficiency of the new method.

Key words. Navier-Stokes equations, nonlinear slip boundary, variational inequality, Gauge-Uzawa method, finite element approximation.

1. Introduction

Numerical simulation for incompressible flow is a fundamental and significant problem in computational mathematics and computational fluid mechanics. It is well known that a mathematical model for a viscous incompressible fluid with homogeneous boundary conditions involves the Navier-Stokes equations.

In this paper, we will consider the following Navier-Stokes equations

$$(1) \quad \begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \operatorname{div} u = 0. \end{cases}$$

It is obvious that (1) is a coupled system with a first-order nonlinear evolution equation and an imposed incompressible constrain so that the numerical simulation for the Navier-Stokes equations is very difficult. A popular technique to overcome this difficulty is to relax the solenoidal condition in an appropriate method to result in a pseudo-compressible system, such as a penalty method or a artificial compressible method. An operator splitting method is also very useful to overcome this shortage. The main advantage of the operator splitting method is that it can decouple the difficulties associated to the nonlinear property with those associated to the incompressible condition. For more details, see [2].

The Gauge-Uzawa method has been a popular tool for the numerical simulation of incompressible viscous flow. The purpose of this paper is to propose two new Gauge-Uzawa schemes for incompressible flows with nonlinear slip boundary conditions. This class of boundary conditions are introduced by Fujita in [3, 4, 5]. The first scheme will be based on a system in convected form [6] while the second scheme will be based on the stabilized Gauge-Uzawa method [7]. We recall that the

Received by the editors February 5, 2014 and, in revised form, March 20, 2014.

2000 *Mathematics Subject Classification.* 65K15, 65N30.

This research was supported by the Natural Science Foundation of China under Grant No. (11371289) and China Scholarship Council.

Gauge-Uzawa method is introduced in [6, 8] to overcome some implementation difficulties associated with the Gauge method introduced in [9]. It has been shown in [6, 10, 11, 12] that the Gauge-Uzawa method has many advantages over the original Gauge method and the pressure-correction method. We will show that a proper Gauge-Uzawa formulation is well suitable for problems with variable density. More precisely, our two new schemes will only involve one projection step and will be proved unconditionally stable.

The paper is organized as follows. In the next two sections, we present the two Gauge-Uzawa schemes and show that they are unconditionally stable, respectively. In Section 4, we describe the finite element approximation of the two Gauge-Uzawa schemes. In Section 5, we present some numerical results which reveal the convergence rate of our schemes for each of the three unknown functions and some concluding remarks are given.

2. Navier-Stokes Equations with Nonlinear Slip Boundary Conditions

Consider the Navier-Stokes equations:

$$(2) \quad \begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla)u + \nabla p = f & \text{in } Q_T, \\ \operatorname{div} u = 0 & \text{in } Q_T, \end{cases}$$

where $Q_T = \Omega \times [0, T]$ for some $T > 0$, $u(t, x)$ denotes velocity, $p(t, x)$ denotes pressure, and $f(t, x)$ denotes the external force. The domain $\Omega \subset \mathbb{R}^2$ is a bounded domain. Given the initial value $u(0, x) = u_0(x)$ in Ω , we consider the following nonlinear slip boundary conditions:

$$(3) \quad \begin{cases} u = 0, & \text{on } \Gamma, \\ u_n = 0, \quad -\sigma_\tau(u) \in g\partial|u_\tau| & \text{on } S, \end{cases}$$

where $\Gamma \cap S = \emptyset$, $\overline{\Gamma \cup S} = \partial\Omega$ with $|\Gamma| \neq 0$, $|S| \neq 0$. The viscous coefficient $\mu > 0$ is a positive constant, g is a scalar function, and $u_n = u \cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, where n stands for the unit vector of the external normal to S . $\sigma_\tau(u) = \sigma - \sigma_n n$, independent of p , is the tangential components of the stress vector σ which is defined by $\sigma_i = \sigma_i(u, p) = (\mu e_{ij}(u) - p\delta_{ij})n_j$, where $e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$, $i, j = 1, 2$. The set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a :

$$\partial\psi(a) = \{b \in \mathbb{R}^2 : \psi(h) - \psi(a) \geq b \cdot (h - a) \quad \forall h \in \mathbb{R}^2\}.$$

Introduce

$$V = \{u \in H^1(\Omega)^2, u|_\Gamma = 0, u \cdot n|_S = 0\}, \quad V_0 = H_0^1(\Omega)^2,$$

$$V_\sigma = \{u \in V, \operatorname{div} u = 0\}, \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega), (1, q)_{L^2(\Omega)} = 0\}.$$

Let $\|\cdot\|_k$ be the norm in the Hilbert space $H^k(\Omega)^2$, and (\cdot, \cdot) and $\|\cdot\|$ be the inner product and the norm in $L^2(\Omega)^2$, respectively. Then we can equip the inner product and the norm in V by $(\nabla \cdot, \nabla \cdot)$ and $\|\cdot\|_V = \|\nabla \cdot\|$, respectively, because $\|\nabla \cdot\|$ is equivalent to $\|\cdot\|_1$. Let \mathbb{X} be a Banach space. Denote by \mathbb{X}' the dual space of \mathbb{X} and $\langle \cdot, \cdot \rangle$ be the dual pairing in $\mathbb{X} \times \mathbb{X}'$. Also we will use δ as a difference of two functions, for example, for any sequence function z^{n+1} ,

$$\delta z^{n+1} = z^{n+1} - z^n, \quad \delta\delta z^{n+1} = \delta(\delta z^{n+1}) = z^{n+1} - 2z^n + z^{n-1}, \dots$$

Introduce the following bilinear forms and trilinear form:

$$\begin{cases} a(u, v) = \mu(\nabla u, \nabla v) & \forall u, v \in V, \\ d(v, p) = (p, \operatorname{div} v) & \forall v \in V, p \in M, \\ b(u, v, w) = \int_{\Omega} (u \cdot \nabla) v \cdot w dx & \forall u, v, w \in V. \end{cases}$$

Moreover, if $\operatorname{div} u = 0$, then trilinear form $b(\cdot, \cdot, \cdot)$ satisfies

$$\begin{aligned} b(u, v, w) &= ((u \cdot \nabla) v, w) + \frac{1}{2}((\operatorname{div} u) v, w) \\ &= \frac{1}{2}((u \cdot \nabla) v, w) - \frac{1}{2}((u \cdot \nabla) w, v) \quad \forall u, v, w \in V. \end{aligned}$$

Thus we have

$$b(u, v, w) = -b(u, w, v) \quad \forall u, v, w \in V.$$

The weak formulation associated with the problem (1) and (2) is the following variational inequality problem of the second kind with the Navier-Stokes operator:

$$(4) \quad \begin{cases} \text{Find } (u, p) \in V \times M \text{ such that} \\ \langle u', v - u \rangle + a(u, v - u) + b(u, u, v - u) + j(v_{\tau}) - j(u_{\tau}) \\ \quad - d(v - u, p) \geq (f, v - u) \quad \forall v \in V, \\ d(u, q) = 0 \quad \forall q \in M. \end{cases}$$

Using a regularized method in [13, 14], we can show the following theorem about the existence and uniqueness of a solution to (4):

Theorem 2.1. *Given $u_0 \in V_{\sigma}$, $f \in L^2(0, T, H)$ and $g \in L^2(0, T, L^{\infty}(S))$, there exists a unique solution $u \in L^{\infty}(0, T, H) \cap L^2(0, T, V)$ with $u' \in L^2(0, T, V')$ and $p \in L^2(0, T, M)$ of the variational inequality (4). Moreover, the following energy inequality holds:*

$$\sup_{0 \leq t \leq T} \|u(t)\|^2 + \mu \int_0^T \|u(\xi)\|_V^2 d\xi \leq \frac{4}{\mu} \int_0^T (\|f(\xi)\|_{V'}^2 + \|g(\xi)\|_S^2) d\xi + 2 \|u_0\|^2.$$

Theorem 2.2. *For almost everywhere $t \in (0, T]$, that $u \in L^{\infty}(0, T, H) \cap L^2(0, T, V)$ with $u' \in L^2(0, T, V')$ and $p \in L^2(0, T, M)$ is the solution of the variational inequality (4) if and only if there exists a $\lambda(t) \in \Lambda$ such that*

$$(5) \quad \begin{cases} \langle u', v \rangle + a(u, v) + b(u, u, v) - d(v, p) + \int_S \lambda g v_{\tau} ds = (f, v) & \forall v \in V, \\ d(u, q) = 0 & \forall q \in M, \\ \lambda u_{\tau} = |u_{\tau}| & \text{a.e. on } S. \end{cases}$$

3. Gauge-Uzawa Method

In this section, we will state the Gauge-Uzawa method to solve the variational problem (5).

3.1. A first-order version. The first-order semi-discrete Gauge-Uzawa method based on the conserved system (5) reads as follows:

Algorithm 3.1. *Set $u^0 = u_0 \in V_{\sigma}$, $s^0 = 0$, and $\lambda^0 \in \Lambda$ is given; repeat for $1 \leq n \leq N \leq T/\tau - 1$.*

Step I: *Find \hat{u}^{n+1} as the solution of*

$$(6) \quad \begin{aligned} \langle \frac{\hat{u}^{n+1} - u^n}{\tau}, v \rangle + a(\hat{u}^{n+1}, v) + b(u^n, \hat{u}^{n+1}, v) - \mu(s^n, \nabla \cdot v) \\ + \int_S \lambda^n g^{n+1} v_{\tau} ds = (f^{n+1}, v) \quad \forall v \in V. \end{aligned}$$

Step II: Find ϕ^{n+1} as the solution of

$$(7) \quad \langle \nabla \phi^{n+1}, \nabla \psi \rangle = \langle \operatorname{div} \hat{u}^{n+1}, \psi \rangle \quad \forall \psi \in H^1(\Omega).$$

Step III: Update u^{n+1} and s^{n+1} as the solution of

$$(8) \quad \begin{cases} u^{n+1} = \hat{u}^{n+1} + \nabla \phi^{n+1}, \\ s^{n+1} = s^n - \nabla \cdot \hat{u}^{n+1}. \end{cases}$$

Step IV: Update λ^{n+1} as the solution of

$$(9) \quad \begin{cases} \lambda^{n+1} = p_\Lambda(\lambda^n + \rho g^{n+1} u_\tau^{n+1}), & \rho > 0, \\ p_\Lambda(\mu) = \sup(-1, \inf(1, \mu)) & \forall \mu \in L^2(S). \end{cases}$$

Remark 3.1. In practice, we derive immediately from (7) and (8) that

$$(10) \quad \langle u^{n+1}, \nabla q \rangle = 0 \quad \forall q \in H^1(\Omega),$$

which implies that in the space continuous case, we have

$$(11) \quad \nabla \cdot u^{n+1} = 0.$$

However, in the space discrete case, only a discrete version of (10) will be satisfied so the discrete velocity field will generally not be divergence free.

Remark 3.2. Note that the pressure does not appear in the above algorithm. However, a proper approximation of the pressure can be constructed [6, 11]. In addition, the pressure $p^{n+1} \in \mathbb{P}$ can be computed via

$$(12) \quad p^{n+1} = \mu s^{n+1} - \tau^{-1} \phi^{n+1}.$$

Theorem 3.1. The Gauge-Uzawa Algorithm 3.1 is unconditionally stable in the sense that, for all $\tau > 0$ and $0 \leq N \leq T/\tau - 1$, the following a priori bound holds:

$$(13) \quad \begin{aligned} & \|\hat{u}^{n+1}\|_0^2 + \sum_{n=1}^N (\|\hat{u}^{n+1} - u^n\|_0^2 + \|\nabla \phi^n\|_0^2) + \mu\tau \|s^{N+1}\|_0^2 + \frac{\mu}{2}\tau \sum_{n=1}^N \|\nabla \hat{u}^{n+1}\|_0^2 \\ & \leq \|\hat{u}^0\|_0^2 + C\mu\tau \sum_{n=1}^N \|f^{n+1}\|_{-1}^2 + C\mu\tau \sum_{n=1}^N \|g^{n+1}\|_{L^\infty(S)}^2. \end{aligned}$$

Proof: We take $v = 2\tau \hat{u}^{n+1}$ in (6), and get

$$(14) \quad \begin{aligned} & \|\hat{u}^{n+1}\|_0^2 + \|\hat{u}^{n+1} - u^n\|_0^2 - \|\hat{u}\|_0^2 + 2\tau\mu \|\nabla \hat{u}^n\|_0^2 + 2\mu\tau \|s^{N+1}\|_0^2 \\ & + 2\mu\tau \langle \nabla s^n, \hat{u}^{n+1} \rangle - 2\mu\tau \int_S \lambda^n g^n \hat{u}_\tau^{n+1} ds = 2\mu\tau \langle f^{n+1}, \hat{u}^{n+1} \rangle. \end{aligned}$$

The next task is to derive a suitable relation between $\|u^n\|_0^2$ and $\|\hat{u}^n\|_0^2$ so that we can sum over n the relation (14). To this end, we derive from (7) and (10) that

$$(15) \quad \begin{aligned} \|u^n\|_0^2 &= \langle u^n, u^n \rangle \\ &= \langle \hat{u}^n + \nabla \phi^n, u^n \rangle \\ &= \langle \hat{u}^n, u^n \rangle \\ &= \langle \hat{u}^n, \hat{u}^n + \nabla \phi^n \rangle \\ &= \|\hat{u}^n\|_0^2 + \langle u^n - \nabla \phi^n, \nabla \phi^n \rangle \\ &= \|\hat{u}^n\|_0^2 - \|\nabla \phi^n\|_0^2. \end{aligned}$$

We now sum up (14) and (15) to get

$$(16) \quad \|\hat{u}^{n+1}\|_0^2 - \|\hat{u}^n\|_0^2 + \|\hat{u}^{n+1} - u^n\|_0^2 - \|\nabla \phi^n\|_0^2 + 2\mu\tau \|\nabla \hat{u}^{n+1}\|_0^2 = A_1 + A_2 + A_3$$

with

$$(17) \quad \begin{cases} A_1 = 2\mu\tau \langle s^n, \nabla \cdot \hat{u}^{n+1} \rangle, \\ A_2 = -2\mu\tau \int_S \lambda^n g^n \hat{u}_\tau^{n+1} ds, \\ A_3 = 2\mu\tau \langle f^{n+1}, \hat{u}^{n+1} \rangle. \end{cases}$$

We derive from the well-known inequality

$$(18) \quad \|\nabla \cdot v\|_0 \leq \|\nabla v\|_0 \quad \forall v \in H_0^1(\Omega),$$

and (8) that

$$(19) \quad \begin{aligned} A_1 &= -2\mu\tau \langle s^n, s^{n+1} - s^n \rangle \\ &= -\mu\tau(\|s^{n+1}\|_0^2 - \|s^{n+1} - s^n\|_0^2 - \|s^n\|_0^2) \\ &= -\mu\tau(\|s^{n+1}\|_0^2 - \|s^n\|_0^2) + \mu\tau\|\nabla \cdot \hat{u}^{n+1}\|_0^2 \\ &\leq -\mu\tau(\|s^{n+1}\|_0^2 - \|s^n\|_0^2) + \mu\tau\|\nabla \hat{u}^{n+1}\|_0^2. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we find

$$(20) \quad \begin{aligned} A_2 &\leq 2\mu\tau\|g^n\|_{L^\infty(S)}\|\hat{u}^{n+1}\| \\ &\leq C\mu\tau\|g^{n+1}\|_{L^\infty(S)}^2 + \frac{\mu}{4}\tau\|\nabla \cdot \hat{u}^{n+1}\|_0^2, \end{aligned}$$

$$(21) \quad A_3 \leq C\mu\tau\|f^{n+1}\|_{-1}^2 + \frac{\mu}{4}\tau\|\nabla \cdot \hat{u}^{n+1}\|_0^2.$$

Inserting the above two results into (16) leads to

$$(22) \quad \begin{aligned} \|\hat{u}^{n+1}\|_0^2 &- \|\hat{u}^n\|_0^2 + \mu\tau(\|s^{n+1}\|_0^2 - \|s^n\|_0^2)\|\hat{u}^{n+1} - u^n\|_0^2 - \|\nabla\phi^n\|_0^2 \\ &+ \frac{\mu}{2}\tau\|\nabla\hat{u}^{n+1}\|_0^2 \leq C\mu\tau\|f^{n+1}\|_{-1}^2 + C\mu\tau\|g^{n+1}\|_{L^\infty(S)}^2. \end{aligned}$$

Summing the above over n from 1 to N yields (13). □

3.2. A second-order version. Algorithm 3.1 is only first-order accurate. However, a second-order version with essentially the same computational procedures can be constructed as follows. For simplicity, we denote, for any function a , its second-order extrapolation by $\bar{a}^{n+1} = 2a^n - a^{n-1}$.

Algorithm 3.2. (*The stabilized Gauge-Uzawa Method*) Set $u^0 = u_0 \in V_\sigma$, $s^0 = 0$, and $\lambda^0 \in \Lambda$ is given; compute u^1, ϕ^1, s^1, p^1 with Algorithm 3.1 and set $\phi^1 = -\frac{2\tau}{3}p^1$: repeat for $2 \leq n \leq N \leq T/\tau - 1$.

Step I: Find \hat{u}^{n+1} as the solution of

$$(23) \quad \begin{aligned} &\langle \frac{3\hat{u}^{n+1} - 4u^n + u^{n-1}}{2\tau}, v \rangle + a(\hat{u}^{n+1}, v) + b(\bar{a}^{n+1}, \hat{u}^{n+1}, v) - (p^n, \nabla \cdot v) \\ &\quad + \int_S \lambda^n g^{n+1} v_\tau ds = (f^{n+1}, v) \quad \forall v \in V. \end{aligned}$$

Step II: Find ϕ^{n+1} as the solution of

$$(24) \quad \langle \nabla\phi^{n+1}, \nabla\psi \rangle = \langle \nabla\phi^n, \nabla\psi \rangle + \langle \text{div}\hat{u}^{n+1}, \psi \rangle \quad \forall \psi \in H^1(\Omega).$$

Step III: Update u^{n+1} and s^{n+1} as the solution of

$$(25) \quad \begin{cases} u^{n+1} = \hat{u}^{n+1} + \nabla(\phi^{n+1} - \phi^n), \\ s^{n+1} = s^n - \nabla \cdot \hat{u}^{n+1}, \\ p^{n+1} = \mu s^{n+1} - \frac{3\phi^{n+1}}{2\tau}. \end{cases}$$

Step IV: Update λ^{n+1} as the solution of

$$(26) \quad \begin{cases} \lambda^{n+1} = p_\Lambda(\lambda^n + \rho g^{n+1} u_\tau^{n+1}), & \rho > 0, \\ p_\Lambda(\mu) = \sup(-1, \inf(1, \mu)) & \forall \mu \in L^2(S). \end{cases}$$

To see that the above scheme is indeed (formally) second-order accurate, we drop the nonlinear terms (note that it is obvious that the approximation for the nonlinear terms is second-order); after eliminating \hat{u}^{n+1} , we find

$$\begin{aligned} &\langle \frac{3\hat{u}^{n+1} - 4u^n + u^{n-1}}{2\tau}, v \rangle + a(\hat{u}^{n+1}, v) - (p^{n+1}, \nabla \cdot v) + \int_S \lambda^n g^{n+1} v_\tau ds = (f^{n+1}, v) \\ &\quad \forall v \in V. \end{aligned}$$

Hence the scheme is formally second-order accurate.

We remark that Algorithm 3.2 consists with (5), like the classical Gauge-Uzawa Method. In order to derive the rotational form of the pressure correction projection method which is studied in [15], we denote

$$\xi^{n+1} := -\frac{3(\phi^{n+1} - \phi^n)}{2\tau}$$

and we subtract the three consecutive equations of (25) to get

$$p^{n+1} = p^n + \xi^{n+1} - \mu \nabla \cdot \hat{u}^{n+1}.$$

Then we arrive at the rotational form of the pressure correction projection method in [15].

Algorithm 3.3. (*The rotational form of the pressure correction projection method*) Set $u^0 = u_0 \in V_\sigma$, $s^0 = 0$, and $\lambda^0 \in \Lambda$ is given; compute u^1, ϕ^1, s^1, p^1 with Algorithm 3.1 and set $\phi^1 = -\frac{2\tau}{3}p^1$: repeat for $2 \leq n \leq N \leq T/\tau - 1$.

Step I: Set $\bar{u} = 2u^n - u^{n-1}$ and find \hat{u}^{n+1} as the solution of (23).

Step II: Find ξ^{n+1} as the solution of

$$(27) \quad \langle \nabla \xi^{n+1}, \nabla \psi \rangle = \langle \frac{3}{2\tau} \operatorname{div} \hat{u}^{n+1}, \psi \rangle \quad \forall \psi \in H^1(\Omega).$$

Step III: Update u^{n+1} and p^{n+1} as the solution of

$$(28) \quad \begin{cases} u^{n+1} = \hat{u}^{n+1} - \nabla \xi^{n+1}, \\ p^{n+1} = p^n + \xi^{n+1} - \mu \nabla \cdot \hat{u}^{n+1}. \end{cases}$$

Step IV: Update λ^{n+1} as the solution of

$$(29) \quad \begin{cases} \lambda^{n+1} = p_\Lambda(\lambda^n + \rho g^{n+1} u_\tau^{n+1}) & \rho > 0, \\ p_\Lambda(\mu) = \sup(-1, \inf(1, \mu)) & \forall \mu \in L^2(S). \end{cases}$$

Remark 3.3. Algorithms 3.2 and 3.3 are basically equivalent at the semi-discrete level. The only difference is the representation of pressure between (25) and (28). The pressure p^{n+1} in (25) is designed by addition of two functions of ϕ and s . Both of them can be expressed by $\nabla \cdot \hat{u}$, so we can replace p^n in momentum equation (23) to the terms of $\nabla \cdot \hat{u}$.

We will prove that the following stability of Algorithms 3.2. Because Algorithms 3.2 and 3.3 are equivalent, we conclude that Algorithm 3.3 is also unconditionally stable.

Lemma 3.1. For any sequence $\{z\}_{n=0}^N$, we have:

$$(30) \quad 2 \langle 3z^{n+1} - 4z^n + z^{n+1}, z^{n+1} \rangle = \|z^{n+1}\|_0^2 + \|2z^{n+1} - z^n\|_0^2 + \|\delta \delta z^{n+1}\|_0^2 - \|z^n\|_0^2 - \|2z^{n+1} - z^n\|_0^2,$$

$$(31) \quad 2 \langle z^{n+1} - z^n, z^{n+1} \rangle = \|z^{n+1}\|_0^2 + \|2z^{n+1} - z^n\|_0^2 - \|z^n\|_0^2,$$

$$(32) \quad 2 \langle z^{n+1} - z^n, z^n \rangle = \|z^{n+1}\|_0^2 - \|2z^{n+1} - z^n\|_0^2 - \|z^n\|_0^2.$$

Theorem 3.2. *The Gauge-Uzawa Algorithm 3.2 is unconditionally stable in the sense that, for all $\tau > 0$ and $0 \leq N \leq T/\tau - 1$, the following a priori bound holds:*

$$\begin{aligned}
(33) \quad & \|\widehat{u}^{n+1}\|_0^2 + \|u^{N+1}\|_0^2 + \|2u^{N+1} - u^N\|_0^2 + 3\|\nabla\phi^{N+1}\|_0^2 \\
& + 2\tau\mu\|s^{N+1}\|_0^2 + \sum_{n=1}^N (\|\delta\delta u^{N+1}\|_0^2 + 3\|\nabla\delta\phi^{n+1}\|_0^2 + \tau\mu\|\nabla\widehat{u}^{n+1}\|_0^2) \\
& \leq \|2u^1 - u^0\|_0^2 + \|u^0\|_0^2 + 3\|\nabla\phi^1\|_0^2 + 2\tau\mu\|s^1\|_0^2\|\widehat{u}^0\|_0^2 \\
& + C\mu\tau \sum_{n=1}^N \|f^{n+1}\|_{-1}^2 + C\mu\tau \sum_{n=1}^N \|g^{n+1}\|_{L^\infty(S)}^2.
\end{aligned}$$

Proof: We first rewrite the momentum equation (23) by using (24) and (25) as follows:

$$\begin{aligned}
(34) \quad & \left\langle \frac{3\widehat{u}^{n+1} - 4u^n + u^{n-1}}{3\phi^{n+1}}, v \right\rangle + a(\widehat{u}^{n+1}, v) + b(\bar{a}^{n+1}, \widehat{u}^{n+1}, v) \\
& - \left(\frac{3\phi^{n+1}}{2\tau} - \mu q^n, \nabla \cdot v \right) + \int_S \lambda^n g^{n+1} v_\tau ds = (f^{n+1}, v) \quad \forall v \in V.
\end{aligned}$$

We now take $v = 4\tau\widehat{u}^{n+1} \in H_0^1(\Omega)$ and use (30) to get

$$\begin{aligned}
(35) \quad & \|u^{n+1}\|_0^2 + \|2u^{n+1} - u^n\|_0^2 + \|\delta\delta u^{n+1}\|_0^2 - \|u^n\|_0^2 \\
& - \|2u^n - u^{n-1}\|_0^2 + 4\tau\mu\|\nabla\widehat{u}^{n+1}\|_0^2 = \sum_{i=1}^4 A_i,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= 6 \langle \nabla\phi^{n+1}, \widehat{u}^{n+1} \rangle, & A_2 &= 4\tau \langle f^{n+1}, \widehat{u}^{n+1} \rangle, \\
A_3 &= 4\tau\mu \langle s^n, \nabla \cdot \widehat{u}^{n+1} \rangle, & A_4 &= 4\tau \int_S \lambda^n g^{n+1} \widehat{u}_\tau^{n+1} ds.
\end{aligned}$$

We note here that the convection term is vanished. In conjunction with $\widehat{u}^{n+1} = u^{n+1} - \nabla\delta\phi^{n+1}$ and (31) yields

$$A_1 = -6 \langle \nabla\phi^{n+1}, \nabla\delta\phi^{n+1} \rangle = -3(\|\nabla\phi^{n+1}\|_0^2 - \|\nabla\phi^n\|_0^2 + \|\nabla\delta\phi^{n+1}\|_0^2).$$

Clearly, we have

$$A_2 \leq C\frac{\tau}{\mu}\|f^{n+1}\|_{-1}^2 + \frac{\tau\mu}{2}\|\nabla\widehat{u}^{n+1}\|_0^2.$$

In the view of (25) and (18), we have $\|\delta q^{n+1}\|_0^2 = \|\nabla \cdot \widehat{u}^{n+1}\|_0^2 \leq \|\nabla\widehat{u}^{n+1}\|_0^2$. Hence

$$\begin{aligned}
A_3 &= -4\mu\tau \langle s^n, \delta q^{n+1} \rangle \\
&= -2\mu\tau (\|s^{n+1}\|_0^2 - \|s^n\|_0^2 - \|\delta q^{n+1}\|_0^2) \\
&\leq -2\mu\tau (\|s^{n+1}\|_0^2 - \|s^n\|_0^2 + 2\mu\tau\|\nabla\widehat{u}^{n+1}\|_0^2), \\
A_4 &\leq 2\mu\tau \|g^n\|_{L^\infty(S)} \|\widehat{u}^{n+1}\|_0 \\
&\leq C\mu\tau \|g^{n+1}\|_{L^\infty(S)}^2 + \frac{\mu}{2}\tau \|\nabla \cdot \widehat{u}^{n+1}\|_0^2.
\end{aligned}$$

Inserting A1-A4 back into (35) and summing over n from 1 to N lead to (33) by using $\|\widehat{u}^{n+1}\|_0^2 = \|u^{n+1}\|_0^2 + \|\nabla\delta\phi^{n+1}\|_0^2$.

The proof of Theorem 3.2 is complete. \square

4. Finite Element Approximation

We now describe, as an example of space discretizations, a finite element method for Algorithm 3.1. Let T_h be a family of regular triangular partitions of Ω into triangles of diameter not greater than $0 < h < 1$ [13]. Let $V_h \subset V$ and $M_h \subset V$ be conforming finite element subspaces, which satisfy the discrete inf-sup condition, i.e., there exists a positive constant $\beta > 0$, independent of h , such that

$$\beta \|p_h\| \leq \sup_{v_h \in V_h} \frac{d(v_h, p_h)}{\|v_h\|_V}.$$

Denote $V_{\sigma h}$ the discretized solenoidal subspace of V_h . According to the definition of $b(\cdot, \cdot, \cdot)$, we have

$$b(u_h, v_h, v_h) \equiv 0 \quad \forall u_h, v_h \in V_h.$$

Denote $\Lambda_h = \{w_h : |w_h(x_S)| \leq 1 \forall x_S \in N_S\}$, where N_S is the set of all nodes on S . For every $\lambda_h \in \Lambda_h$ and $v_h \in V_h$, we have

$$c(\lambda_h, gv_h) \leq C \|g\|_{L^\infty(S)} \|v_h\|_V.$$

For initial value $u_0 \in V\sigma$, the discretized initial value $u_{0h} \in V\sigma h$ is defined as follows:

$$a(u_{0h}, v_h) = a(u_0, v_h) \quad \forall v_h \in V_{\sigma h}.$$

The finite element approximation of Algorithm 3.1 in (6)-(9) is:

Algorithm 4.1. Set $u_h^0 = u_{0h} \in V_{\sigma h}$, $s_h^0 = 0$, and $\lambda_h^0 \in \Lambda_h$ is given; repeat for $1 \leq n \leq N \leq T/\tau - 1$.

Step I: Find \widehat{u}_h^{n+1} as the solution of

$$(36) \quad \left\langle \frac{\widehat{u}_h^{n+1} - u_h^n}{\tau}, v_h \right\rangle + a(\widehat{u}_h^{n+1}, v_h) + b(u_h^n, \widehat{u}_h^{n+1}, v_h) - \mu(s_h^n, \nabla \cdot v_h) \\ + \int_S \lambda_h^n g^{n+1} v_{h\tau} ds = (f^{n+1}, v_h) \forall v_h \in V_h.$$

Step II: Find ϕ_h^{n+1} as the solution of

$$(37) \quad \langle \nabla \phi_h^{n+1}, \nabla \psi_h \rangle = \langle \operatorname{div} \widehat{u}_h^{n+1}, \psi_h \rangle \quad \forall \psi_h \in H_h^1(\Omega).$$

Step III: Update u_h^{n+1} and s_h^{n+1} as the solution of

$$(38) \quad \begin{cases} u_h^{n+1} = \widehat{u}_h^{n+1} + \nabla \phi_h^{n+1}, \\ s_h^{n+1} = s_h^n - \nabla \cdot \widehat{u}_h^{n+1}. \end{cases}$$

Step IV: Update λ_h^{n+1} as the solution of

$$(39) \quad \begin{cases} \lambda_h^{n+1} = p_{\Lambda_h}(\lambda_h^n + \rho g^{n+1} u_{h\tau}^{n+1}), & \rho > 0, \\ p_\Lambda(\mu) = \sup(-1, \inf(1, \mu)) & \forall \mu \in L^2(S). \end{cases}$$

Algorithm 4.2. (The stabilized Gauge-Uzawa Method) Set $u_h^0 = u_{0h} \in V_\sigma$, $s_h^0 = 0$, and $\lambda^0 \in \Lambda$ is given; compute $u_h^1, \phi_h^1, s_h^1, p_h^1$ with Algorithm 3.1 and set $\phi^1 = -\frac{2\tau}{3} p_h^1$; repeat for $2 \leq n \leq N \leq T/\tau - 1$.

Step I: Find \widehat{u}_h^{n+1} as the solution of

$$(40) \quad \left\langle \frac{3\widehat{u}_h^{n+1} - 4u_h^n + u_h^{n-1}}{2\tau}, v \right\rangle + a(\widehat{u}_h^{n+1}, v) + b(\overline{a}_h^{n+1}, \widehat{u}_h^{n+1}, v) - (p_h^n, \nabla \cdot v) \\ + \int_S \lambda_h^n g_h^{n+1} v_\tau ds = (f_h^{n+1}, v) \quad \forall v \in V_h.$$

Step II: Find ϕ_h^{n+1} as the solution of

$$(41) \quad \langle \nabla \phi_h^{n+1}, \nabla \psi \rangle = \langle \nabla \phi_h^n, \nabla \psi \rangle + \langle \operatorname{div} \widehat{u}_h^{n+1}, \psi \rangle \quad \forall \psi \in H_h^1(\Omega).$$

Step III: Update u_h^{n+1} and s_h^{n+1} as the solution of

$$(42) \quad \begin{cases} u_h^{n+1} = \widehat{u}_h^{n+1} + \nabla(\phi_h^{n+1} - \phi_h^n), \\ s_h^{n+1} = s_h^n - \nabla \cdot \widehat{u}_h^{n+1}, \\ p_h^{n+1} = \mu s_h^{n+1} - \frac{3\phi_h^{n+1}}{2\tau}. \end{cases}$$

Step IV: Update λ_h^{n+1} as the solution of

$$(43) \quad \begin{cases} \lambda_h^{n+1} = p_\Lambda(\lambda_h^n + \rho g_h^{n+1} u_{h\tau}^{n+1}), & \rho > 0, \\ p_\Lambda(\mu) = \sup(-1, \inf(1, \mu)) & \forall \mu \in L^2(S). \end{cases}$$

5. Numerical Results

In this section, we present some computational experiments using the Gauge-Uzawa methods.

Assume that the domain is the standard square domain $([1, 13])$ i.e., $\Omega = [0, 1] \times [0, 1]$. The exact solutions u and p are

$$\begin{aligned} u(x, y) &= (u_1(t, x, y), u_2(t, x, y)), & p(x, y) &= t(2x - 1)(2y - 1), \\ u_1(x, y) &= -tx^2y(x - 1)(3y - 2), & u_2(x, y) &= txy^2(y - 1)(3x - 2), \end{aligned}$$

and f is defined.

It is easy to verify that the exact solution u satisfies $u = 0$ on Γ , $u \cdot \vec{n} = u_1 = 0$, $u_2 \neq 0$ on S_1 and $u_1 \neq 0$, $u \cdot \vec{n} = u_2 = 0$ on S_2 . Moreover, the tangential vectors τ on S_1 and S_2 are $(0, 1)$ and $(-1, 0)$, so

$$\begin{cases} \sigma_\tau = 4\mu ty^2(y - 1) & \text{on } S_1, \\ \sigma_\tau = 4\mu tx^2(x - 1) & \text{on } S_2. \end{cases}$$

On the other hand, from the nonlinear boundary conditions (2), we have

$$|\sigma_\tau| \leq g, \quad \sigma_\tau u_\tau + g|u_\tau| = 0,$$

on $S = S_1 \cup S_2$. Then the function g can be chosen such that $g = -\sigma_\tau \geq 0$ on S_2 .

Let $\mu = 0.1$. The external force f can be determined by the first equation of (2). Since the finite element space (V_h, M_h) must satisfy the discretized inf-sup condition, we use the Taylor-Hood element ($P_2 - P_1$ element). Take the initial value $u_0 = 0$, $\lambda_0 = 1$, the space step $h = \tau$, and the parameter $\varrho = 0.5\mu$.

TABLE 1. The error and the rates of convergence for Algorithm 4.1

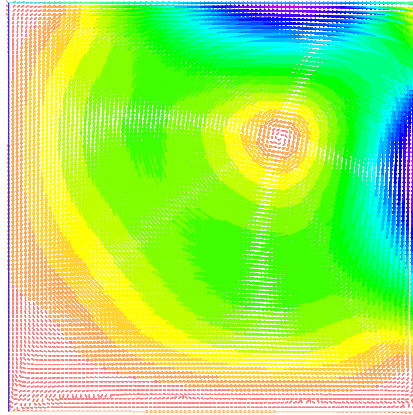
h	$\frac{\ u - u_h\ _{L^2}}{\ u\ _{L^2}}$	Order	$\frac{\ u - u_h\ _{H^1}}{\ u\ _{H^1}}$	Order	$\frac{\ p - p_h\ _{L^2}}{\ p\ _{L^2}}$	Order
1/8	0.00290906	/	0.0256771	/	0.033258	/
1/16	0.00150963	0.9464	0.0138472	0.8909	0.0174686	0.9290
1/32	0.000768923	0.9733	0.00678532	0.9487	0.00869924	1.0058
1/64	0.000389941	0.9796	0.00346521	0.9695	0.00437643	0.9911

TABLE 2. The error and the rates of convergence for Algorithm 4.2

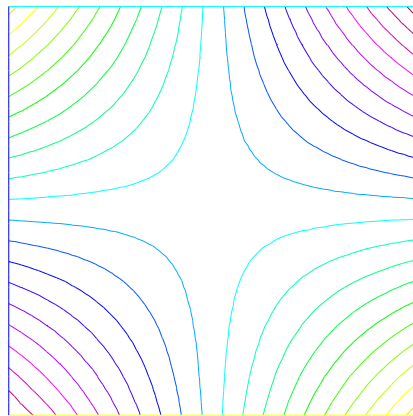
h	$\frac{\ u - u_h\ _{L^2}}{\ u\ _{L^2}}$	Order	$\frac{\ u - u_h\ _{H^1}}{\ u\ _{H^1}}$	Order	$\frac{\ p - p_h\ _{L^2}}{\ p\ _{L^2}}$	Order
1/8	0.000370236	/	0.00240039	/	0.00487653	/
1/16	9.42152e-5	1.9744	0.000674267	1.8320	0.001346908	1.8562
1/32	2.46783e-5	1.9327	0.000170945	1.9800	0.0003477467	1.9535
1/64	6.35522e-6	1.9573	4.25791e-5	2.0049	8.75396e-5	1.9900

Next, we give the results of Algorithm 4.1 in Table 1 and the result of Algorithm 4.2 in Table 2 if the Gauge-Uzawa method is used, such as the usual Galerkin finite element method with ($P_2 - P_1$ element). The L^2 and H^1 error of the velocity field and the pressure isovalue are displayed. Figure 1 shows the velocity field and the pressure isovalue at $T = 1$ as the space step $h = 1/32$, the time step 0.01 and the parameter $\varrho = 0.5\mu$.

In summary, the Gauge-Uzawa method is very valid for the Navier-Stokes equation with nonlinear slip boundary and the numerical results are consistent with the



(a)



(b)

FIGURE 1. The velocity field and the pressure isovalue.

theoretical analysis.

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