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The Existence of Coupled Solutions for a Kind of Nonlinear Operator Equations in Partial Ordered Linear Topology Space*

Wu Yue-xiang[1], Huo Yan-mei[2] and Wu Ya-kun[1]

(*1. College of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan, 030006*)

(*2. College of Economics, Shanxi University of Finance and Economics, Taiyuan, 030006*)

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Abstract: The main purpose of this paper is to examine the existence of coupled solutions and coupled minimal-maximal solutions for a kind of nonlinear operator equations in partial ordered linear topology spaces by employing the semi-order method. Some new existence results are obtained.

Key words: partial order, mixed monotone operator, coupled solution, existence

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1 Introduction

The techniques of partial order theory are used to discuss the existence of coupled solutions and coupled minimal-maximal solutions for a kind of nonlinear operator equation in a partial ordered linear topology space as follows:

$$Nx = A(x, x), \tag{1.1}$$

where N is an increasing operator and A is a mixed monotone operator.

In 1987, Guo and Lakshmikantham[1] studied a nonlinear operator equation in a Banach space as

$$x = A(x, x), \tag{1.2}$$

where A is a mixed monotone operator. They obtained some existence results of coupled solution for this operator equation. In 2005, Liu and Feng[2] considered the following operator equation:

$$Nx = Ax \tag{1.3}$$

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in a complete metric space and a Banach space, respectively, and by using the technique of partial order theory they obtained some existence results of solution. Very recently, He[3] has dealt with the operator equation (1.1) in Banach spaces and have given some solvability results for this kind of equations by using the concept of ϕ concave-ψ convex operator (see [4]).

Motivated and inspired by the above works, the main purpose of this paper is to further study the solvability of the equation (1.1). Under some suitable conditions, we give some new existence theorems for this kind of equations. To the knowledge of the author, there are very few works on the existence of coupled solutions and coupled minimal-maximal solutions for the equation (1.1) in partial ordered linear topology space, and therefore, our results generalize and improve some corresponding results.

2 Preliminaries

In this section, we give some concepts and lemmas which are necessary for proving the main results of this paper, and the other unstated concepts can be seen in [5–8].

Let E be a real linear topology space, P be a cone of E and " $\leq$ " be a partial order induced by the cone P, i.e., for any x, $y \in E$, $x \leq y$ (or alternatively, denoted as $y \geq x$) if and only if $y - x \in P$. We write $x < y$, if $x \leq y$ and $x \neq y$.

Let x, $y \in E$, $x < y$. The set defined by $[x, y] = \{z \,|\, x \leq z \leq y\}$ is called an ordered interval in E. For any subset $D \subset E \times E$, we denote by $\bar{D}^w$, $\overline{\mathrm{co}}(D)$ and CD the weak closure of D, the closed convex hull of D and the complement of D, respectively.

Let

$$P_1 = \{(x, y) \in E \times E \mid x \geq \theta,\ y \leq \theta\},$$

where θ denotes the zero element of E. It is easy to see that P_1 is a cone of the product space $E \times E$, and P_1 defines a partial order in $E \times E$ as follows (denoted as $\prec$):

$(x,\, y) \prec (u,\, v)$ (or alternatively, denoted as $(u,\, v) \succ (x,\, y)$) if and only if $x \leq u$ and $y \geq v$.

Definition 2.1[9−10] *Let D be a nonempty subset of a real partial order linear topology space $(E, \leq)$.*

(i) *The operator $A : D \times D \to E$ is said to be mixed monotone if $A(x, y)$ is both nondecreasing in x and nonincreasing in y, i.e., if $u_1 \leq u_2$, $v_2 \leq v_1$, u_i, $v_i \in D$ $(i = 1, 2)$ imply*

$$A(u_1, v_1) \leq A(u_2, v_2).$$

(ii) *A point $(x^*, y^*) \in D \times D$, $x^* \leq y^*$ is called a coupled solution of the nonlinear operator equation* (1.1) *if*

$$Nx^* = A(x^*, y^*), \qquad A(y^*, x^*) = Ny^*.$$

(iii) *A point $(x^*, y^*) \in D \times D$, $x^* \leq y^*$ is called a coupled minimal-maximal solution of the nonlinear operator equation* (1.1), *if (x^*, y^*) is a coupled solution of the nonlinear* *operator equation* (1.1) *such that for any coupled solution* (u^*, v^*) *of* (1.1), *we have*

$$x^* \leq u^*, \qquad y^* \geq v^*.$$

Lemma 2.1 *Assume that* $G : D \times D \to E$ *is a mixed monotone operator and* N *is a nonlinear operator. Let*

$$H(x,y) \doteq (G(x,y),\, G(y,x)), \quad B(x,y) \doteq (Nx, Ny), \qquad (x,y) \in D \times D.$$

Then the following conclusions hold:

(i) H *is an increasing operator on the partial order deduced by* P_1;

(ii) $H(x,y) = B(x,y)$ *has a solution* (x^*, y^*) *if and only if* (x^*, y^*) *is a coupled solution of*

$$Nx = G(x,x);$$

(iii) *A minimal solution of*

$$H(x,y) = B(x,y)$$

is a coupled minimal-maximal solution of

$$Nx = G(x,x).$$

Proof. (i) For any (u_1, v_1), $(u_2, v_2) \in D \times D$, if $(u_1, v_1) \prec (u_2, v_2)$, then it follows from the definition of $\prec$ that

$$u_1 \leq u_2, \qquad v_1 \geq v_2.$$

The mixed monotonicity of G implies that

$$G(u_1, v_1) \leq G(u_2, v_2), \qquad G(v_2, u_2) \leq G(v_1, u_1).$$

Therefore, by the definition of $\prec$ again, we have

$$(G(u_1, v_1), G(v_1, u_1)) \prec (G(u_2, v_2), G(v_2, u_2)),$$

i.e.,

$$H(u_1, v_1) \prec H(u_2, v_2).$$

Thus, H is an increasing operator on the partial order deduced by P_1.

(ii) (x^*, y^*) is a solution of

$$H(x,y) = B(x,y)$$

if and only if (x^*, y^*) is a solution of

$$(G(x,y), G(y,x)) = (Nx, Ny),$$

i.e.,

$$Nx^* = G(x^*, y^*), \qquad Ny^* = G(y^*, x^*).$$

Thus,

$$H(x,y) = B(x,y)$$

has a solution (x^*, y^*) if and only if (x^*, y^*) is a coupled solution of

$$Nx = G(x,x).$$

(iii) Suppose that (u^*, v^*) is a minimal solution of

$$H(x,y) = B(x,y).$$

For any solution (u, v) of

$$H(x, y) = B(x, y),$$

by the minimal quality, we have

$$(u^*, v^*) \prec (u, v).$$

Therefore,

$$u^* \le u, \qquad v \le v^*.$$

By (ii) and Definition 2.1, it is easy to see that (u^*, v^*) is a coupled minimal-maximal solution of

$$Nx = G(x, x).$$

This completes the proof.

We also need the following lemmas.

Lemma 2.2 [8] *Assume that (E, P) is a partially ordered space, D is a nonempty subset of E and $y \in E$. If $z \le y$ (or $y \le z$) for all $z \in D$, then $z \le y$ (corresponding $y \le z$) for all $z \in \overline{\mathrm{co}}(D)$.*

Let $L(E)$ be the space of all linear operators on E. We give the following lemma on an operator, whose proof is omitted, due to it is easy to prove.

Lemma 2.3 *Assume that $\Lambda \in (0, 1]$, $T \in L(E)$, and $(\Lambda I + T)^{-1} \in L(E)$. Then*

$$(\Lambda I + T)^{-1}[\Lambda A(x, y) + Tu] = u$$

if and only if

$$A(x, y) = u.$$

3 Main Results and Their Proofs

Our main results are the following two theorems:

Theorem 3.1 *Let E be a real linear topology space, P be a cone of E, u_0, $v_0 \in E$, $u_0 < v_0$, $D_0 = [u_0, v_0]$ be an ordered interval in E and N be an increasing operator with $N(D_0) = D_0$. Assume that $A : D \doteq [(u_0, v_0), (v_0, u_0)] \to E$ is a mixed monotone operator, $\Lambda \in (0, 1]$, $T \in L(E)$ and $(\Lambda I + T)^{-1} \in L(E)$ are positive operators. If the following conditions are satisfied:*

(i) $Nu_0 \le A(u_0, v_0)$, $A(v_0, u_0) \le Nv_0$;

(ii) *for any x_1, $x_2 \in D_0$, $Nx_1 \le Nx_2$ implies $x_1 \le x_2$;*

(iii) *any totally ordered subset of $G(D)$ is relatively compact with weak topology, where*

$$G(x, y) \doteq (\Lambda I + T)^{-1}[\Lambda A(x, y) + TNx], \qquad (x, y) \in D,$$

then the nonlinear operator equation (1.1) *has a coupled solution $(x^*, y^*) \in D$.*

Proof. First, we verify that the following conclusions hold:

$$G : D \to [u_0, v_0]$$

is a mixed monotone operator and

$$Nu_0 \le G(u_0, v_0), \qquad G(v_0, u_0) \le Nv_0.$$

In fact, if $(x, y) \in D$, then

$$u_0 \le x, \qquad y \le v_0.$$

Since N is an increasing operator with $N(D_0) = D_0$, we can get

$$u_0 \le Nu_0 \le Nx \le Nv_0 \le v_0.$$

Since $T \in L(E)$ is a positive operator, we have

$$Tu_0 \le TNx \le Tv_0.$$

On the other hand, by the mixed monotonicity of A and the condition (i), we have

$$A(x, y) \le A(v_0, u_0) \le Nv_0 \le v_0,$$

$$A(x, y) \ge A(u_0, v_0) \ge Nu_0 \ge u_0.$$

Therefore, we can get

$$\Lambda u_0 + Tu_0 \le \Lambda A(x, y) + TNx \le \Lambda v_0 + Tv_0,$$

i.e.,

$$(\Lambda I + T)u_0 \le \Lambda A(x, y) + TNx \le (\Lambda I + T)v_0.$$

Since $(\Lambda I + T)^{-1} \in L(E)$ is a positive operator, we have

$$u_0 \le (\Lambda I + T)^{-1}[\Lambda A(x, y) + TNx] \le v_0,$$

i.e.,

$$u_0 \le G(x, y) \le v_0.$$

If (x_1, y_1), $(x_2, y_2) \in D$, and $(x_1, y_1) \prec (x_2, y_2)$, then

$$x_1 \le x_2, \qquad y_1 \ge y_2.$$

Hence

$$Nx_1 \le Nx_2, \qquad Nx_2 - Nx_1 \in P.$$

Since $T \in L(E)$ is a positive operator, by the mixed monotonicity of A, we have $T(Nx_2 - Nx_1) \in P$, i.e.,

$$TNx_2 \ge TNx_1, \qquad A(x_1, y_1) \le A(x_2, y_2).$$

Therefore,

$$\Lambda A(x_1, y_1) + TNx_1 \le \Lambda A(x_2, y_2) + TNx_2.$$

Since $(\Lambda I + T)^{-1} \in L(E)$ is a positive operator, we have

$$(\Lambda I + T)^{-1}[\Lambda A(x_1, y_1) + TNx_1] \le (\Lambda I + T)^{-1}[\Lambda A(x_2, y_2) + TNx_2],$$

i.e.,

$$G(x_1, y_1) \le G(x_2, y_2).$$

Therefore, G is a mixed monotone operator.

And then we show that

$$Nu_0 \le G(u_0, v_0), \qquad G(v_0, u_0) \le Nv_0.$$

In fact, by the condition (i), we have

$$\Lambda A(v_0, u_0) \le \Lambda Nv_0, \qquad \Lambda Nu_0 \le \Lambda A(u_0, v_0).$$

Hence,

$$\varLambda A(v_0,u_0)+TNv_0\le \varLambda Nv_0+TNv_0=(\varLambda I+T)Nv_0,$$

$$(\varLambda I+T)Nu_0=\varLambda Nu_0+TNu_0\le \varLambda A(u_0,v_0)+TNu_0.$$

Notice that $(\varLambda I+T)^{-1}\in L(E)$ is a positive operator. Thus we have

$$Nv_0\ge(\varLambda I+T)^{-1}[\varLambda A(v_0,u_0)+TNv_0]=G(v_0,u_0),$$

$$Nu_0\le(\varLambda I+T)^{-1}[\varLambda A(u_0,v_0)+TNu_0]=G(u_0,v_0).$$

Next, we show that the nonlinear operator equation

$$B(x,y)=H(x,y) \tag{$*$}$$

has a solution in D, where

$$H(x,y)\doteq(G(x,y),G(y,x)),\qquad B(x,y)\doteq(Nx,\ Ny).$$

Step 1. By Lemma 2.1, H is an increasing operator. Let

$$M_1=\{(x,y)\in D|B(x,y)\prec H(x,y)\},$$

$$M_2=\{(y,x)|(x,y)\in M_1\}.$$

Then $M_1\ne\emptyset$ (since $(u_0,v_0)\in M_1$).

Suppose that K_1 is a totally ordered subset of M_1. Then $K_2=\{(y,x)|(x,y)\in K_1\}$ is a totally ordered subset of M_2. For any $q_1\in G(K_1)$, $q_2\in G(K_2)$, let

$$R_1(q_1)=\{z\in D_0|q_1\le z\},$$

$$R_2(q_2)=\{z\in D_0|z\le q_2\},$$

$$S_1(q_1)=\overline{\mathrm{co}}(G(K_1))\cap R_1(q_1),$$

$$S_2(q_2)=\overline{\mathrm{co}}(G(K_2))\cap R_2(q_2).$$

It is easy to see that $R_1(q_1)$, $R_2(q_2)$, $S_1(q_1)$ and $S_2(q_2)$ are all convex and closed sets.

The mixed monotonicity for G implies that $G(K_i)$ $(i=1,\,2)$ are totally ordered subsets of $G(D)$. From the condition (iii), we know that $\overline{G(K_i)}^{w}$ $(i=1,\,2)$ are weakly compact sets in $G(D)$. Hence $\overline{\mathrm{co}}\big(\overline{G(K_i)}\big)^{w}$ $(i=1,\,2)$ are also weakly compact sets due to the Krein-Smulian theorem.

Since $\overline{\mathrm{co}}(G(K_i))\subset\overline{\mathrm{co}}\big(\overline{G(K_i)}\big)^{w}$ $(i=1,2)$, we know that $\overline{\mathrm{co}}(G(K_i))$ $(i=1,2)$ are weakly compact.

Step 2. Notice that $S_i(q_i)\ne\emptyset$ (since for any $q_i\in G(K_i)$, $q_i\in S_i(q_i)$, $i=1,2$). For any $q_1',\,q_2',\,\cdots,\,q_n'\in G(K_1)$ and $q_1'',\,q_2'',\,\cdots,\,q_n''\in G(K_2)$, without loss of generality, we suppose that $q_1'\le q_2'\le\cdots\le q_n'$ and $q_1''\le q_2''\le\cdots\le q_n''$. Then $S_1(q_1')\supset S_1(q_2')\supset\cdots\supset S_1(q_n')$ and $S_2(q_1'')\subset S_2(q_2'')\subset\cdots\subset S_2(q_n'')$. It is obvious that

$$\bigcap_{i=1}^{n}S_1(q_i')\supset S_1(q_n')\ne\emptyset,\qquad \bigcap_{i=1}^{n}S_2(q_i'')\supset S_2(q_1'')\ne\emptyset. \tag{3.1}$$

It is easy to prove that

$$\bigcap_{q_j\in G(K_i)}S_i(q_j)\ne\emptyset,\qquad i=1,2,\quad j=1,2,\cdots,n.$$

Step 3. There exist $q_i^*\in\bigcap\limits_{q_j\in G(K_i)}S_i(q_j)$ $(i=1,2)$ such that $q_i^*\in S_i(q_j)$ for all $q_j\in G(K_i)$. Thus $q_i^*\in R_i(q_j)$ for all $q_j\in G(K_i)$. By the construction of $R_i(q_j)$, we have

$$q_1\le q_1^*,\quad q_1\in G(K_1),\qquad q_2\ge q_2^*,\quad q_2\in G(K_2).$$

Since $N(D_0) = D_0$, we know that there exist w_1, $w_2 \in D_0$ such that

$$Nw_1 = q_1^*, \qquad Nw_2 = q_2^*.$$

Now for any $(x, y) \in K_1$, we have $(y, x) \in K_2$. Hence

$$G(x, y) \leq q_1^* = Nw_1, \qquad G(y, x) \geq q_2^* = Nw_2.$$

Therefore,

$$(Nx, Ny) \prec H(x, y) = (G(x, y), G(y, x)),$$

i.e.,

$$Nx \leq G(x, y), \qquad G(y, x) \leq Ny.$$

Thus

$$Nx \leq Nw_1, \qquad Nw_2 \leq Ny.$$

From the condition (ii), we have

$$x \leq w_1, \qquad w_2 \leq y.$$

Therefore

$$(x, y) \prec (w_1, w_2), \qquad (y, x) \succ (w_2, w_1). \tag{3.2}$$

This indicates that (w_1, w_2) is an upper bound of K_1 and $(w_1, w_2) \in M_1$. From Zorn's Lemma we know that M_1 contains a maximal element (x^*, y^*).

Step 4. Finally we prove that the maximal element (x^*, y^*) is the solution of the nonlinear operator equation $(*)$.

By the definition of B, the condition (ii) and N being an increasing operator, it is not difficult to check that B is also an increasing operator and if

$$B(x_1, y_1) \prec B(x_2, y_2), \qquad (x_i, y_i) \in D \ (i = 1, 2),$$

then

$$(x_1, y_1) \prec (x_2, y_2).$$

Since $(x^*, y^*) \in M_1$, we have

$$B(x^*, y^*) \prec H(x^*, y^*) = B(B^{-1}H(x^*, y^*)),$$

and hence

$$(x^*, y^*) \prec B^{-1}H(x^*, y^*).$$

Since H is increasing, we have

$$B(B^{-1}H(x^*, y^*)) = H(x^*, y^*)) \prec H(B^{-1}H(x^*, y^*)),$$

and hence $B^{-1}H(x^*, y^*) \in M_1$.

Since (x^*, y^*) is the maximal element of M_1, we have

$$B(B^{-1}H(x^*, y^*)) = H(x^*, y^*) \prec B(x^*, y^*),$$

and therefore

$$H(x^*, y^*) = B(x^*, y^*),$$

i.e., (x^*, y^*) is a solution of the nonlinear operator equation $(*)$.

By Lemma 2.1, that is,

$$Nx^* = G(x^*, y^*), \qquad Ny^* = G(y^*, x^*).$$

It follows from Lemma 2.3 that

$$Nx^* = A(x^*, y^*), \qquad Ny^* = A(y^*, x^*).$$

Therefore, (x^*, y^*) is a coupled solution of the equation (1.1). The proof is completed.

Theorem 3.2 *Assume that all conditions of Theorem* 3.1 *are satisfied. Then the nonlinear operator equation* (1.1) *has a coupled minimal-maximal solution* $(x^*, y^*) \in D$.

Proof. Let

$$F(H) = \{(x, y) \in D | H(x, y) = B(x, y)\}.$$

Theorem 3.1 implies that $F(H)$ is nonempty. Let

$$S \doteq \{[(u, v), (v, u)] | B(u, v) \prec H(u, v), (u, v) \in D, F(H) \subset [(u, v), (v, u)]\},$$

where $[(u, v), (v, u)]$ is an ordered interval in $E \times E$. Then $S \neq \emptyset$ (since $D \in S$). Define the relation " $\leq_1$ " in S as follows:

$$I_1,\, I_2 \in S, \qquad I_1 \leq_1 I_2 \Longleftrightarrow I_1 \subset I_2.$$

It is easy to see that " $\leq_1$ " is a partial order in S.

Next we show that S has a minimal element.

Step 1. Suppose that $\varGamma = \{[(u_\alpha, v_\alpha), (v_\alpha, u_\alpha)] | \alpha \in \varLambda\}$ is any totally order subset of S, where $\varLambda$ is an index set. Let

$$R_1 = \{(u_\alpha, v_\alpha) | \alpha \in \varLambda\}, \qquad R_2 = \{(v_\alpha, u_\alpha) | \alpha \in \varLambda\}.$$

Then R_1 and R_2 are totally ordered subsets of D. It follows from the mixed monotonicity of G that $G(R_i)$ $(i = 1, 2)$ are totally ordered subsets of $G(D)$.

Let $K_1 = R_1$ and $K_2 = R_2$ be the same as in Theorem 3.1. Then by similar proofs of Steps 1–3 of Theorem 3.1, we know that there exist $\bar{q}_i \in \overline{\text{co}}(G(R_i))$ $(i = 1, 2)$ with $N\bar{w}_i = \bar{q}_i$ such that

$$(u_\alpha, v_\alpha) \prec (\bar{w}_1, \bar{w}_2), \qquad \alpha \in \varLambda.$$

On the other hand, for any $(u_\alpha, v_\alpha) \in R_1$, we have $(v_\alpha, u_\alpha) \in R_2$. Thus

$$(u_\alpha, v_\alpha) \prec (\bar{w}_1, \bar{w}_2), \qquad (v_\alpha, u_\alpha) \succ (\bar{w}_2, \bar{w}_1). \tag{3.3}$$

It follows from the mixed monotonicity for G that

$$G(u_\alpha, v_\alpha) \leq G(\bar{w}_1, \bar{w}_2), \qquad G(v_\alpha, u_\alpha) \geq G(\bar{w}_2, \bar{w}_1).$$

By Lemma 2.2, for $N\bar{w}_i = \bar{q}_i \in \overline{\text{co}}(G(R_i))$ $(i = 1, 2)$, we have

$$N\bar{w}_1 \leq G(\bar{w}_1, \bar{w}_2), \qquad G(\bar{w}_2, \bar{w}_1) \leq N\bar{w}_2,$$

i.e.,

$$B(\bar{w}_1, \bar{w}_2) = (N\bar{w}_1, N\bar{w}_2) \prec (G(\bar{w}_1, \bar{w}_2), \qquad G(\bar{w}_2, \bar{w}_1)) = H(\bar{w}_1, \bar{w}_2). \tag{3.4}$$

Step 2. Given any $(u_\alpha, v_\alpha) \in R_1$, we have $(v_\alpha, u_\alpha) \in R_2$. Let $(x, y) \in F(H)$. By the definition of S, one has

$$(u_\alpha, v_\alpha) \prec (x, y) \prec (v_\alpha, u_\alpha).$$

The mixed monotonicity for G implies that

$$G(u_\alpha, v_\alpha) \leq G(x, y) \leq G(v_\alpha, u_\alpha).$$

Since $\bar{q}_i \in \overline{\text{co}}(G(R_i))$ $(i = 1, 2)$, by Lemma 2.2, we have

$$N\bar{w}_1 \leq G(x, y) \leq N\bar{w}_2, \tag{3.5}$$

Similarly to the proof of (3.5), we also get

$$N\bar{w}_1 \leq G(y,x) \leq N\bar{w}_2. \tag{3.6}$$

Thus,

$$(N\bar{w}_1, N\bar{w}_2) \prec (G(x,y), G(y,x)) = H(x,y) = (Nx, Ny)$$

and

$$(N\bar{w}_2, N\bar{w}_1) \succ (G(x,y), G(y,x)) = H(x,y) = (Nx, Ny).$$

Therefore

$$(\bar{w}_1, \bar{w}_2) \prec (x,y) \prec (\bar{w}_2, \bar{w}_1), \qquad (x,y) \in F(H). \tag{3.7}$$

Let $I = [(\bar{w}_1, \bar{w}_2), (\bar{w}_2, \bar{w}_1)]$. Then it follows from (3.4) and (3.7) that $I \in S$. (3.3) shows that if $(x,y) \in I$, then $(x,y) \in [(u_\alpha, v_\alpha), (v_\alpha, u_\alpha)]$, i.e., $I \subset [(u_\alpha, v_\alpha), (v_\alpha, u_\alpha)]$. Hence,

$$I \leq_1 [(u_\alpha, v_\alpha), (v_\alpha, u_\alpha)], \qquad \alpha \in \Lambda.$$

I is a lower bound of Γ in S. By Zorn' Lemma, S contains a minimal element denoted as

$$I^* = [(x^*, y^*), (y^*, x^*)].$$

Step 3. By the definition of S, we have

$$B(x^*, y^*) \prec H(x^*, y^*) = B(B^{-1}H(x^*, y^*)),$$

i.e.,

$$(x^*, y^*) \prec B^{-1}H(x^*, y^*). \tag{3.8}$$

The monotonicity of H implies that

$$H(x^*, y^*) = B(B^{-1}H(x^*, y^*)) \prec H(B^{-1}H(x^*, y^*)). \tag{3.9}$$

For any $(x,y) \in F(H)$, the monotonicity of H and the definition of S show that

$$H(x^*, y^*) \prec H(x,y) = B(x,y) \prec H(y^*, x^*).$$

Hence,

$$B(B^{-1}H(x^*, y^*)) \prec B(B^{-1}H(x,y)) = B(x,y) \prec B(B^{-1}H(y^*, x^*)).$$

Therefore,

$$B^{-1}H(x^*, y^*) \prec (x,y) \prec B^{-1}H(y^*, x^*),$$

i.e.,

$$F(H) \subset [B^{-1}H(x^*, y^*),\ B^{-1}H(y^*, x^*)]. \tag{3.10}$$

From (3.9) and (3.10) we know that $[B^{-1}H(x^*, y^*),\ B^{-1}H(y^*, x^*)] \in S$.

By virtue of the minimality of I^*, we get

$$I^* \leq_1 [B^{-1}H(x^*, y^*),\ B^{-1}H(y^*, x^*)],$$

i.e.,

$$(x^*, y^*) \succ B^{-1}H(x^*, y^*). \tag{3.11}$$

(3.8) and (3.11) indicate that

$$(x^*, y^*) = B^{-1}H(x^*, y^*),$$

i.e.,

$$B(x^*, y^*) = H(x^*, y^*).$$

On the other hand, for any $(x, y) \in F(H) \subset I^*$, it is easy to see that

$$(x^*, y^*) \prec (x, y) \prec (y^*, x^*).$$

This shows that (x^*, y^*) is a minimal solution of the equation $(*)$.

By Lemma 2.1, (x^*, y^*) is a coupled minimal-maximal solution of

$$Nx = G(x, x).$$

It follows from Lemma 2.3 that

$$Nx^* = A(x^*, y^*), \qquad Ny^* = A(y^*, x^*).$$

Therefore, (x^*, y^*) is a coupled minimal-maximal solution of the equation (1.1). The proof is completed.

Remark 3.1 In Theorems 3.1 and 3.2, we do not assume that the operators are continuous or compact, and the results hold in partial ordered linear topology space. Therefore our conclusions generalize or improve some corresponding results of [3, 5, 8, 11–12].

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