Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. 3, No. 2, pp. 181-203

DOI: 10.4208/aamm.10-m1018
April 2011

Finite Element θ-Schemes for the Acoustic Wave Equation

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Received 11 March 2010; Accepted (in revised version) 2 June 2010
Available online 5 January 2011

Abstract. In this paper, we investigate the stability and convergence of a family of implicit finite difference schemes in time and Galerkin finite element methods in space for the numerical solution of the acoustic wave equation. The schemes cover the classical explicit second-order leapfrog scheme and the fourth-order accurate scheme in time obtained by the modified equation method. We derive general stability conditions for the family of implicit schemes covering some well-known CFL conditions. Optimal error estimates are obtained. For sufficiently smooth solutions, we demonstrate that the maximal error in the L^2-norm error over a finite time interval converges optimally as $\mathcal{O}(h^{p+1}+\Delta t^s)$, where p denotes the polynomial degree, s=2 or 4, h the mesh size, and Δt the time step.

AMS subject classifications: 65M60, 65M12, 65M15

Key words: Finite element methods, discontinuous Galerkin methods, wave equation, implicit methods, energy method, stability condition, optimal error estimates.

1 Introduction

The efficient and accurate numerical approximation of the wave equations involved in modeling acoustic, elastic or electromagnetic wave propagation is of fundamental importance in many real-life problems. In geophysics, it helps for instance in the interpretation of field data and to predict the damage patterns due to earthquakes. Finite difference methods have been widely used for the simulation of time dependent waves because of their simplicity and their efficiency on structured Cartesian meshes [1, 9, 21, 33]. However, in the presence of heterogeneous media and complex geometry or small geometric futures that require locally refined meshes, their usefulness is somewhat limited.

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Alternatively, finite element methods (FEMs) can easily handle complex geometry and heterogeneous media, and it is easy to incorporate free surface boundary conditions and nonmatching grids. They also have the advantage of local discretization techniques using error indicators. Their extension to high order is straightforward, even in the presence of curved boundaries or material interfaces. Attempts at wave simulation using finite elements have used continuous Galerkin methods [3, 5, 8, 14, 15,24,30,31], discontinuous Galerkin methods [17,18,20,28], and mixed finite element methods [11,13,16,19].

In this paper, we are interested in the finite element approximation of the acoustic wave equation

$$u_{tt} - \nabla \cdot (c^2 \nabla u) = f, \quad \text{in} \ \ \Omega \times J, \tag{1.1}$$

with boundary and initial conditions given by

$$\begin{aligned} u &= 0, && \text{on} \ \ \partial\Omega \times J, && (1.2a) \\ u|_{t=0} &= u^0, && \text{in} \ \ \Omega, && (1.2b) \\ u_t|_{t=0} &= v^0, && \text{in} \ \ \Omega, && (1.2c) \end{aligned}$$

where $J=(0,T)$ is a finite time interval, $T>0$, and Ω is a bounded, convex polygonal domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, with boundary $\partial\Omega$. The (known) source term $f(x,t)$ lies in $L^2(J;L^2(\Omega))$, while $u^0(x)\in H_0^1(\Omega)$ and $v^0(x)\in L^2(\Omega)$ are prescribed initial conditions. We assume that the speed of propagation, $c(x)$, is piecewise smooth and satisfies the bounds

$$0 < c_{\min} \le c(x) \le c_{\max} < \infty, \quad x \in \bar{\Omega}.$$

The standard weak formulation of problem (1.1)-(1.2c) is stated as follows: find $u\in L^2(J;H_0^1(\Omega))$, satisfying (1.2b) and (1.2c), with $u_t\in L^2(J;L^2(\Omega))$ and $u_{tt}\in L^2(J;H^{-1}(\Omega))$, such that

$$\langle u_{tt}, v\rangle + a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \quad \text{a.e. in} \ \ J. \tag{1.3}$$

Here, the time derivatives are understood in the sense of distributions, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $L^2(\Omega)$, $(\cdot,\cdot)$ is the usual inner product in $L^2(\Omega)$, and $a(\cdot,\cdot)$ is the elliptic bilinear form given by

$$a(u,v) = (c\nabla u, c\nabla v). \tag{1.4}$$

Existence and uniqueness of a solution to the variational problem is proved, for instance, in [23]. It is shown that the weak solution u is continuous in time; that is

$$u \in C^0\big(\bar{J}; H_0^1(\Omega)\big), \qquad u_t \in C^0\big(\bar{J}; L^2(\Omega)\big).$$

This result implies in particular that the initial conditions (1.2b) and (1.2c) are well defined; see Chapter 3 in [23] and Chapter 8 in [27] for more details. Additional regularity assumptions will be made throughout the paper to carry out the convergence analysis. We finally recall that because the bilinear form $a(\cdot,\cdot)$ is symmetric and coercive, in the absence of forcing ($f\equiv 0$), the (continuous) energy

$$E(t)=\frac{1}{2}(u_t,u_t)+\frac{1}{2}a(u,u),$$

is conserved for all time.

A priori error estimates for continuous Galerkin approximations for problem (1.1)-(1.2c) have been first derived by Dupont [14] using a standard energy argument. These estimates were improved by Baker [3], who used a technique that can be interpreted as a nonstandard energy arguments. Baker showed that optimal estimates for the error can be obtained using L^2-projections of the initial data as starting values, with minimal smoothness requirements on the exact solution. In [15], Gekeler analyzed general multistep methods for the time discretization of second-order hyperbolic equations, with a Galerkin procedure in space. High-order accurate two-step approximations for second-order hyperbolic equations were derived in [4]. In [32], high-order Taylor-Galerkin schemes combined with an adaptive h-p procedure were presented for second-order hyperbolic problems.

Mixed finite element approximations to the acoustic wave equation have been considered in [11,13,16,19]. In [16], Geveci derived L^∞-in-time, L^2-in-space error bounds for the continuous-in-time mixed finite element approximations of velocity and stress. In [11,13], a priori error estimates were obtained for the mixed finite element approximation of displacement requiring less regularity than was needed in [16]. Stability for a family of discrete-in-time schemes was also demontratred in [11,13]. In [19], an alternative mixed finite element displacement formulation, that requires less regularity on the displacement solution than the approach in [11,13], was proposed.

Standard continuous (conforming) Galerkin methods generally impose significant restrictions on the underlying mesh and discretization; in particular, they do not easily accomodate hanging nodes for local mesh refinement. In contrast, discontinuous Galerkin (DG) methods offer greater flexibility in handling elements of various types and shapes, irregular nonmatching grids, and even locally varying polynomial order. The origin of the DG methods can be traced back to the 1970s, where they were proposed for the numerical solution of hyperbolic neutron transport equations. Based on discontinuous finite element spaces, DG methods weakly enforce continuity by adding suitable bilinear forms, so-called numerical fluxes, to standard variational formulations; see [6] for a review of the development of DG methods.

When applied to second-order hyperbolic problems, most DG methods reformulate the problem as a first-order hyperbolic system, for which various DG methods are available [7,25]. Recently, Rivière and Wheeler [28,29] proposed a DG method for the acoustic wave equation on its original second-order formulation. The method is based on a nonsymmetric interior penalty formulation. In [18], a symmetric interior penalty discontinuous Galerkin method was presented for the time dependent wave equation. Optimal error bounds in the energy norm and in the L^2-norm were derived for the semidiscrete formulation. Recently, this error analysis was extended in [17] to the fully discrete numerical scheme, where a second-order scheme is used for the time discretization. The aim of this paper is to extend the results of Grote et al. [17] to the case of implicit time integration. Precisely, we investigate the stability and convergence of a family of implicit finite difference schemes in time and DG methods in space for the wave problem. Stability results and detailed error estimates are obtained.

The paper is organized as follows. In Section 2, we briefly review the discontinuous Galerkin finite element approach. We introduce the time-stepping procedure and derive a family of fully discrete schemes. Stability analysis of the fully discrete schemes is presented in Section 3. In Section 4, we state a priori error estimates which are optimal in both time and space. Concluding remarks are presented in the last section.

2 Fully discrete Galerkin schemes for the wave equation

We shall first discretize (1.1)-(1.2c) in space by using a discontinuous Galerkin discretization, while leaving time continuous. We will consider the symmetric interior penalty method presented in [18] as an example of a discontinuous Galerkin method. Standard notations from the finite element theory will be used. We consider regular and quasi-uniform meshes $T_h=\{K\}$ that partition the domain Ω into disjoint elements $\{K\}$, such that

$$\bar{\Omega} = \cup_{K\in T_h}\bar{K}.$$

The elements are assumed to be triangles in two space dimensions and tetrahedra in three space dimensions. The diameter of element K is denoted by h_k and the mesh size, h, is given by

$$h = \max_{K\in T_h} h_K.$$

The analysis carries out for quadrilaterals in two space dimensions and hexaedra in three space dimensions as well.

2.1 Discontinuous Galerkin formulation

We briefly review the symmetric interior penalty DG formulation from [2] and [18]. For a given partition T_h of Ω and an approximation of order $p\geq 1$, we define the discontinuous Galerkin finite element space

$$V_h = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_p(K), \quad \forall K \in T_h\}, \tag{2.1}$$

where $\mathcal{P}_p(K)$ is the set of polynomials of total degree at most p on K. Then, we consider the following semidiscrete discontinuous Galerkin approximation of (1.1)-(1.2c): find $u_h : \bar{J} \times V_h \to \mathbb{R}$, such that

$$(u_{htt}, v) + a_h(u_h, v) = (f, v), \quad \forall v \in V_h, \;\; t \in J, \tag{2.2a}$$

$$u_h|_{t=0} = P_h u^0, \tag{2.2b}$$

$$u_{ht}|_{t=0} = P_h v^0, \tag{2.2c}$$

where P_h denotes the L^2-projection onto V_h. The discrete bilinear form a_h is the standard symmetric interior penalty form for the Laplacian given by

$$\begin{aligned} a_h(u,v) = & \sum_{K\in T_h} \int_K c^2 \nabla u \cdot \nabla v dx - \sum_{F\in \mathcal{F}_h} \int_F [[u]] \cdot \{\{c^2 \nabla v\}\} ds \\ & - \sum_{F\in \mathcal{F}_h} \int_F [[v]] \cdot \{\{c^2 \nabla u\}\} ds + \sum_{F\in \mathcal{F}_h} \gamma h_F^{-1} \int_F c^2 [[v]] \cdot [[u]] ds. \end{aligned} \tag{2.3}$$

The last three terms in (2.3) correspond to jump and flux terms at element boundaries, with h_F denoting the diameter of the edge or the face F; see [18] for further details. The parameter $\gamma>0$ is the interior penalty stabilization parameter that has to be chosen sufficiently large, independent of the mesh size. The bilinear form a_h is clearly symmetric:

$$a_h(u,v) = a_h(v,u).$$

The key properties of the bilinear form a_h are given in [2]. If we consider the broken norm

$$|||u|||^2 = \sum_{K\in T_h} ||\nabla u||_{0,K}^2 + \sum_{K\in T_h} h_K^2 ||D^2 u||_{0,K}^2 + \sum_{F\in \mathcal{F}_h} h_K^{-1} ||[[u]]||_{0,F}^2,$$

where $D^2 u$ denotes the matrix of the second derivatives of u, then there exists a threshold value $\gamma_0>0$, independent of the mesh size, such that for $\gamma \geq \gamma_0$ there holds

$$a_h(u,u) \geq C_C |||u|||^2, \quad \forall u \in V_h, \tag{2.4}$$

with a coercivity constant $C_C>0$ independent of the mesh size. Moreover, we have

$$|a_h(u,v)| \leq C_A c_{\max}^2 \max\{1,\gamma\} |||u|||\ |||v|||, \quad \forall u,v \in H^2(\Omega) + V_h,$$

with a continuity constant $C_A>0$ independent of the mesh size, c^2 and γ. Finally, the following spectral estimate will play a crucial role in our analysis: for quasi-uniform meshes T_h, there holds

$$a_h(u,u) \leq M_\gamma h^{-2} ||u||_0^2, \quad \forall u \in V_h, \tag{2.5}$$

where $||\cdot||_0$ is the standard norm on $L^2(\Omega)$ and

$$M_\gamma = C_S c_{\max}^2 \max\{1,\gamma\},$$

with $C_S>0$ a constant independent of the mesh size, c^2 and γ. Inequality (2.5) results from the fact that, for quasi-uniform meshes, the following inequality

$$|||u|||^2 \leq C C_A c_{\max}^2 \max\{1,\gamma\} h^{-2} ||u||_0^2,$$

holds, with a constant $C>0$ independent of the mesh size, c^2 and γ; see [17].

Our analysis will be based on the four key assumptions on the bilinear form a_h: symmetry, continuity, coercivity, and adjoint-consistency in the sense of [2]. Hence, it immediately extends to other spatial DG finite element methods as long as these four assumptions on a_h hold.

2.2 Time discretization

We now describe the time discretization procedure applied to the semidiscrete problem (2.2a)-(2.2c) and formulate the fully discrete Galerkin finite element scheme. Stability and convergence will be analyzed in the next sections. We first introduce the following notations. Let $\Delta t{=}T/N$ be the time step, where N is a positive integer, and define the discrete times $t^n{=}n\Delta t$, for $n{=}0,\cdots,N$. For any function v of time, let v^n denote $v(t^n)$. We shall use this notation for functions defined for all times as well as those defined only at discrete times.

For the time discretization, the simplest scheme consists of using the classical leapfrog scheme with three time levels. The fully discrete numerical solution to the wave equation (1.1)-(1.2c) is then defined by finding the sequence $\{U^n\}_{n=0}^N$ in V_h such that

$$(\bar{\partial}_{tt}U^n,v)+a_h(U^n,v)=(f^n,v),\quad \forall v\in V_h,\;\; n=1,\cdots,N-1, \tag{2.6}$$

where

$$\bar{\partial}_{tt}U^n=\frac{U^{n+1}-2U^n+U^{n-1}}{\Delta t^2}.$$

This scheme yields a second-order accuracy with respect to Δt which is generally not sufficient for a higher-order finite element method. As a stability constraint, the scheme requires to choose

$$\Delta t=\mathcal{O}(h).$$

To overcome this stability restriction, one can use a local time-stepping scheme allowing small time steps precisely where small elements in the mesh are located; see for instance [12] and the references therein. However, to preserve the accuracy provided by the space discretization, one must use higher-order schemes with respect to time. A convenient scheme can be obtained by using the so-called modified equation approach [9, 10, 33]. Such a scheme can be seen as an appropriate modification of the leapfrog scheme (2.6) constructed by looking at the truncation error associated with the leapfrog scheme.

In this paper, we define the discrete time Galerkin approximation to be a sequence $\{U^n\}_{n=0}^N$ in V_h, such that

$$(\bar{\partial}_{tt}U^n,v)+a_h(U^{n;\theta},v)=(f^{n;\theta},v),\quad \forall v\in V_h,\;\; n=1,\cdots,N-1, \tag{2.7}$$

where

$$U^{n;\theta}=\theta U^{n+1}+(1-2\theta)U^n+\theta U^{n-1},$$

$f^{n;\theta}$ is analogously defined, and θ is a parameter to be chosen in $[0,0.5]$ so that the weights used in the expression of $U^{n;\theta}$ are nonnegative. The same one-parameter scheme has been considered in [21, 22] in the context of finite difference methods in space. We remark that (2.7) is implicit if $\theta{\neq}0$. The case $\theta{=}0$ corresponds to the explicit leapfrog scheme (2.6). The fourth-order accurate scheme in time derived by the modified equation approach turns out to be a special case of (2.7) obtained with $\theta{=}1/12$;

see [21]. The time-stepping algorithm (2.7) and its influence on temporal dispersion and dissipation have been analyzed in the literature (see, e.g., [8, 9, 21]). It is found in [21] that the implicit procedure results in less dispersive solutions than the explicit one, which is advantageous for the numerical solution in very oscillatory media. Similar observations have been made, for instance, in [8, 9].

The three-level scheme (2.7) requires appropriate initial conditions $U^0 \in V_h$ and $U^1 \in V_h$. We select the initial condition U^0 to be the L^2-projection of u^0 onto V_h,

$$U^0 = P_h u^0,$$

and we define the fictitious value U^{-1} satisfying

$$\frac{U^1 - U^{-1}}{2\Delta t} = P_h v^0.$$

By considering (2.7) with $n=0$, we obtain

$$\begin{aligned}&\big(2\Delta t^{-2}(U^1 - U^0), v\big) + 2\theta a_h(U^1 - U^0, v)\\ =&2\Delta t^{-1}(P_h v^0, v) + 2\Delta t\theta a_h(P_h v^0, v) - a_h(U^0, v) + \big(f^{0;\theta}, v\big).\end{aligned} \tag{2.8}$$

We remark that

$$f^{0;\theta} = f^0 + \theta(f^1 - 2f^0 + f^{-1}).$$

So, formally we can write

$$f^{0;\theta} = f^0 + \theta\Delta t^2 f^0_{tt} + \mathcal{O}(\Delta t^4).$$

Substituting the expression for $f^{0;\theta}$ into (2.8), multiplying by $\Delta t^2/2$, and dropping high-order terms in Δt, we obtain

$$\big(U^1 - U^0, v\big) + \Delta t^2\theta a_h(U^1 - U^0, v) = \Delta t(v^0, v) + \frac{\Delta t^2}{2}(\tilde{U}^0, v), \tag{2.9}$$

where $\tilde{U}^0 \in V_h$ is defined by

$$(\tilde{U}^0, v) = (f^0, v) - a_h(u^0, v), \quad \forall v \in V_h. \tag{2.10}$$

From the consistency of the method, it is clear that $\tilde{U}^0$ is the L^2-projection of u^0_{tt} onto V_h.

For the special case $\theta=1/12$, a higher-order approximation to the fictitious value U^{-1} has to be considered. Assuming the exact solution has enough regularity, we define U^{-1} as follows

$$\frac{U^1 - U^{-1}}{2\Delta t} = P_h v^0 + \frac{\Delta t^2}{6} P_h u^0_{ttt}.$$

Considering (2.7) with n=0, performing calculations similar to the previous case, and dropping high-order terms in Δt, we then define $U^1 \in V_h$ by requiring that

$$\begin{aligned}&(U^1-U^0,v)+\frac{\Delta t^2}{12}a_h(U^1-U^0,v)\\ =&\Delta t(v^0,v)+\frac{\Delta t^2}{2}(\tilde{U}^0,v)-\frac{\Delta t^3}{12}a_h(v^0,v)+\frac{\Delta t^3}{6}(f_t^0,v)+\frac{\Delta t^4}{24}(f_{tt}^0,v),\end{aligned} \tag{2.11}$$

for all $v\in V_h$, where $\tilde{U}^0$ was previously defined. One has to notice that in the derivation of (2.11) we substituted the term $a_h(P_h v^0,v)$ by $a_h(v^0,v)$.

The fully discrete problem is now defined by (2.7) and (2.9) if $\theta \neq 1/12$ and by (2.7) and (2.11) for $\theta=1/12$. In each case, the solution of a Galerkin elliptic problem is required to start the time-stepping procedure using (2.7).

Existence and uniqueness of the numerical approximations are given in the following proposition.

Proposition 1. *The fully discrete approximations $\{U^n\}_{n=0}^N$ are uniquely defined in V_h by* (2.7) *and* (2.9) *if $\theta\neq 1/12$ and by* (2.7) *and* (2.11) *for $\theta=1/12$.*

Proof. Clearly U^0 and U^1 are uniquely defined in V_h for any θ. By noticing that

$$U^{n;\theta}=U^n+\theta\Delta t^2\bar{\partial}_{tt}U^n, \tag{2.12}$$

and setting

$$Q^n=\bar{\partial}_{tt}U^n,$$

we deduce that Q^n satisfies

$$c_h(Q^n,v)=l^n(v),\quad \forall v\in V_h,\ \ n\geq 2,$$

where c_h is the bilinear form given by

$$c_h(u,v)=(u,v)+\theta\Delta t^2 a_h(u,v),\quad \forall u,v\in V_h,$$

and l^n is the linear operator given by

$$l^n(v)=(f^{n;\theta},v)-a_h(U^n,v),\quad \forall v\in V_h.$$

Clearly, the operator l^n is continuous and the bilinear form c_h is coercive on V_h. Hence, Q^n exists uniquely in V_h for each $n=1,\cdots,N-1$, which implies that U^{n+1} is uniquely defined in V_h for $n=1,\cdots,N-1$. □

A general class of time discretization methods well-known in the engineering literature is given by the so-called Newmark scheme [26], which has been used extensively in applications. The resulting fully discrete scheme is given by

$$(\bar{\partial}_{tt}U^n,v)+a_h(U^{n,\theta,\gamma},v)=(f^{n,\theta,\gamma},v),\quad \forall v\in V_h,\ \ n=1,\cdots,N-1, \tag{2.13}$$

where

$$U^{n,\theta,\gamma}=\theta U^{n+1}+\Big(\frac{1}{2}-2\theta+\gamma\Big)U^n+\Big(\frac{1}{2}+\theta-\gamma\Big)U^{n-1},$$

and $\gamma\geq 0$ is a free parameter. The scheme reduces to (2.7) if $\gamma=1/2$ and is only first-order accurate when $\gamma\neq 1/2$.

3 Stability analysis

In this section, we introduce further notations and investigate the stability of the fully discrete scheme (2.7) in the absence of forcing. We let

$$v^{n+\frac{1}{2}} = \frac{v^{n+1} + v^n}{2},$$

and define the following terms for the discrete temporal derivatives:

$$\bar{\partial}_t v^n = \frac{v^{n+1} - v^{n-1}}{2\Delta t}, \qquad \bar{\partial}_t^- v^n = \frac{v^n - v^{n-1}}{\Delta t}, \qquad \bar{\partial}_t^+ = \frac{v^{n+1} - v^n}{\Delta t}.$$

We easily see that

$$\bar{\partial}_t v^n = \frac{\bar{\partial}_t^+ v^n + \bar{\partial}_t^- v^n}{2} = \frac{v^{n+\frac{1}{2}} - v^{n-\frac{1}{2}}}{\Delta t}, \qquad \bar{\partial}_{tt} v^n = \frac{\bar{\partial}_t^+ v^n - \bar{\partial}_t^- v^n}{\Delta t}.$$

The following stability result holds.

Theorem 1. *The fully discrete scheme* (2.7) *is stable if*

$$\Delta t^2 \Big(\frac{1}{4} - \theta\Big) \sup_{v \in V_h \backslash \{0\}} \frac{a_h(v,v)}{(v,v)} \leq 1, \tag{3.1}$$

and conserves the discrete energy

$$\begin{aligned} E_h^{n+\frac{1}{2}} =& \frac{1}{2}\Big[(\bar{\partial}_t^+ U^n, \bar{\partial}_t^+ U^n) + \Delta t^2 \Big(\theta - \frac{1}{4}\Big) a_h(\bar{\partial}_t^+ U^n, \bar{\partial}_t^+ U^n) \\ &+ a_h\big(U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\big)\Big]. \end{aligned} \tag{3.2}$$

The scheme is unconditionally stable when $\theta \geq 1/4$.

Proof. By using (2.12), we first rewrite (2.7) in the following form

$$(\bar{\partial}_{tt} U^n, v) + \Delta t^2 \theta a_h(\bar{\partial}_{tt} U^n, v) + a_h(U^n, v) = 0, \quad \forall v \in V_h. \tag{3.3}$$

By noticing that

$$U^n + \frac{\Delta t^2}{4} \bar{\partial}_{tt} U^n = \frac{1}{2}(U^{n+\frac{1}{2}} + U^{n-\frac{1}{2}}),$$

we rearrange (3.3) as

$$(\bar{\partial}_{tt} U^n, v) + \Delta t^2 \Big(\theta - \frac{1}{4}\Big) a_h(\bar{\partial}_{tt} U^n, v) + \frac{1}{2} a_h\big(U^{n+\frac{1}{2}} + U^{n-\frac{1}{2}}, v\big) = 0.$$

Now we choose $\bar{\partial}_t U^n$ as a test function for the previous equation to have

$$\begin{aligned} (\bar{\partial}_{tt} U^n, \bar{\partial}_t U^n) &+ \Delta t^2 \Big(\theta - \frac{1}{4}\Big) a_h(\bar{\partial}_{tt} U^n, \bar{\partial}_t U^n) \\ &+ \frac{1}{2} a_h(U^{n+\frac{1}{2}} + U^{n-\frac{1}{2}}, \bar{\partial}_t U^n) = 0. \end{aligned} \tag{3.4}$$

We next examine the three main terms in (3.4). We have

$$\begin{aligned}(\bar{\partial}_{tt}U^n,\bar{\partial}_tU^n) =&\frac{1}{2\Delta t}(\bar{\partial}_t^+U^n-\bar{\partial}_t^-U^n,\bar{\partial}_t^+U^n+\bar{\partial}_t^-U^n)\\ =&\frac{1}{2\Delta t}\big[(\bar{\partial}_t^+U^n,\bar{\partial}_t^+U^n)-(\bar{\partial}_t^-U^n,\bar{\partial}_t^-U^n)\big],\end{aligned}$$

and similarly

$$a_h(\bar{\partial}_{tt}U^n,\bar{\partial}_tU^n)=\frac{1}{2\Delta t}\big[a_h(\bar{\partial}_t^+U^n,\bar{\partial}_t^+U^n)-a_h(\bar{\partial}_t^-U^n,\bar{\partial}_t^-U^n)\big].$$

The last term is

$$\begin{aligned}a_h\big(U^{n+\frac{1}{2}}+U^{n-\frac{1}{2}},\bar{\partial}_tU^n\big) =&\frac{1}{\Delta t}a_h\big(U^{n+\frac{1}{2}}+U^{n-\frac{1}{2}},U^{n+\frac{1}{2}}-U^{n-\frac{1}{2}}\big)\\ =&\frac{1}{\Delta t}\big[a_h\big(U^{n+\frac{1}{2}},U^{n+\frac{1}{2}}\big)-a_h\big(U^{n-\frac{1}{2}},U^{n-\frac{1}{2}}\big)\big].\end{aligned}$$

If $E_h^{n+\frac{1}{2}}$ is the discrete energy defined by (3.2), then (3.4) is equivalent to

$$\frac{1}{\Delta t}\big(E_h^{n+\frac{1}{2}}-E_h^{n-\frac{1}{2}}\big)=0.$$

That is, the scheme conserves the discrete energy $E_h^{n+\frac{1}{2}}$, which guarantees stability if and only if $E_h^{n+\frac{1}{2}}$ is positive semidefinite. A sufficient condition for $E_h^{n+\frac{1}{2}}$ to be positive semidefinite is that the bilinear form

$$c(u,v):=(u,v)+\Delta t^2\Big(\theta-\frac{1}{4}\Big)a_h(u,v),$$

is positive semidefinite on V_h, that is

$$(v,v)+\Delta t^2\Big(\theta-\frac{1}{4}\Big)a_h(v,v)\geq 0,\quad \forall v\in V_h. \tag{3.5}$$

It is easy to verify that (3.5) is equivalent to condition (3.1) in the theorem, which completes the proof. $\square$

Remark 3.1. Let $\tilde{A}_h$ denote the bounded operator on V_h associated with the bilinear form $a_h(\cdot,\cdot)$ and the inner product $(\cdot,\cdot)$:

$$(\tilde{A}_hu,v)=a_h(u,v),\quad \forall u,v\in V_h.$$

The norm of the operator $\tilde{A}_h$ is defined by

$$||\tilde{A}_h||=\sup_{v\in V_h\setminus\{0\}}\frac{a_h(v,v)}{(v,v)}.$$

Hence, (3.1) can be written as

$$\Delta t^2\Big(\frac{1}{4}-\theta\Big)||\tilde{A}_h|| \le 1. \tag{3.6}$$

For θ=0, we then recover the well-known CFL condition of the leapfrog scheme (see [10])

$$\Delta t^2 \frac{||\tilde{A}_h||}{4} \le 1.$$

For $\theta=1/12$, the stability condition reads

$$\Delta t^2 \frac{||\tilde{A}_h||}{6} \le 1.$$

It is clear from (3.6) that, among all θ in the interval $[0,1/4)$, the leapfrog scheme has the most restrictive stability condition. For $\theta\ge1/4$ the scheme becomes unconditionally stable. The time truncation is minimized over this class by taking $\theta=1/4$. This latter case is particularly interesting because the form of the discrete energy it yields is similar to that of the continuous problem.

4 Convergence analysis

In this section, we state our main results: optimal a priori error estimates for the fully discrete finite element schemes (2.7)-(2.9) and (2.7)-(2.11). Some of the techniques used in the proofs can be found in previous works [3,13] and in the recent work [17] where the special case θ=0 has been considered. For $u\in H^2(\Omega)$, the elliptic projection $\Pi_h u$ of u onto V_h is defined by requiring that

$$a_h(\Pi_h u,v)=a_h(u,v), \quad \forall v\in V_h. \tag{4.1}$$

We begin by recalling the main estimate that we shall use in our analysis: If $u\in H^{p+1}(\Omega)$, $p\ge1$ is the polynomial degree, then we have

$$||u-\Pi_h u||_0 \le Ch^{p+1}||u||_{p+1}, \tag{4.2}$$

where $C>0$ is a constant independent of the mesh size [17] and $||\cdot||_s$ denotes the standard Sobolev norm on $H^s(\Omega)$. Moreover, since the operators ∂_t^i and Π_h commute, there follows from (4.2) that

$$||\partial_t^i(u-\Pi_h u)||_0 \le Ch^{p+1}||\partial_t^i u||_{p+1}, \tag{4.3}$$

if $\partial_t^i u\in H^{p+1}(\Omega)$.

In order to estimate the errors in the finite element approximations, we define the auxiliary functions

$$\omega^n=\Pi_h u^n, \qquad \phi^n=\omega^n-U^n, \qquad \eta^n=u^n-\omega^n,$$

so that the error e^n at time t^n,

$$e^n = u^n - U^n,$$

can be written as

$$e^n = \phi^n + \eta^n. \tag{4.4}$$

We also define $r^n \in V_h$ by

$$(r^n, v) = (\bar{\partial}_{tt}\omega^n - u_{tt}^{n;\theta}, v), \quad \forall v \in V_h, \; n \geq 1,$$

where

$$u_{tt}^{n;\theta} = \theta u_{tt}^{n-1} + (1-2\theta) u_{tt}^n + \theta u_{tt}^{n+1},$$

and

$$(r^0, v) = \Delta t^{-2}(\phi^1 - \phi^0, v) + \theta a_h(\phi^1 - \phi^0, v), \quad \forall v \in V_h, \; n = 0.$$

We finally set

$$R^n = \Delta t \sum_{m=0}^{n} r^m.$$

We suppose that the mesh size h and the time step Δt satisfy the CFL condition

$$\frac{\Delta t^2}{h^2}\Big(\frac{1}{4} - \theta\Big) < \frac{1}{M_\gamma}, \tag{4.5}$$

where M_γ is given by (2.5). This CFL condition will naturally arise in the proofs of the next results. We also assume that the solution u to the wave problem (1.1)-(1.2c) satisfies the regularity properties

$$u \in C^2(\bar{J}; H^{p+1}(\Omega)), \qquad \partial_t^3 u \in C(\bar{J}; L^2(\Omega)), \qquad \partial_t^4 u \in L^1(J; L^2(\Omega)). \tag{4.6}$$

For the error analysis with $\theta = 1/12$, we further require that

$$u \in C^2(\bar{J}; H^{p+1}(\Omega)), \qquad \partial_t^5 u \in C(\bar{J}; L^2(\Omega)), \qquad \partial_t^6 u \in L^1(J; L^2(\Omega)). \tag{4.7}$$

We notice that under the regularity assumptions (4.6), the exact solution u to the wave problem satisfies

$$(u_{tt}, v) + a_h(u, v) = (f, v), \quad \forall v \in V_h, \; t \in J. \tag{4.8}$$

This follows from the consistency of the bilinear form a_h (cf [2], Sec. 3.3). Now, we have the following error bound.

Proposition 2. *Assume that the CFL condition* (4.5) *holds. Then we have*

$$\max_{n=0}^{N} ||e^n||_0 \leq C\Big(||e^0||_0 + \max_{n=0}^{N} ||\eta^n||_0 + \Delta t \sum_{n=0}^{N-1} ||R^n||_0\Big),$$

with a constant $C > 0$ independent of h, Δt and T.

Proof. By the triangle inequality, we have that

$$\max_{n=0}^{N}||e^n||_0 \le \max_{n=0}^{N}||\phi^n||_0 + \max_{n=0}^{N}||\eta^n||_0, \tag{4.9}$$

and so we need to further bound $\max_{n=0}^{N}||\phi^n||_0$. First, we notice that u satisfies

$$(u_{tt}^{n;\theta}, v) + a_h(u^{n;\theta}, v) = (f^{n;\theta}, v), \quad \forall v \in V_h, \;\; n = 1, \cdots, N-1. \tag{4.10}$$

To see this, average (4.8) at time t^{n+1}, t^n and t^{n-1} with weights θ, $1-2\theta$ and θ, respectively. We next subtract (2.7) from (4.10) and conclude that

$$(u_{tt}^{n;\theta} - \bar{\partial}_{tt}\omega^n + \bar{\partial}_{tt}\omega^n - \bar{\partial}_{tt}U^n, v) + a_h(u^{n;\theta} - \omega^{n;\theta} + \omega^{n;\theta} - U^{n;\theta}, v) = 0, \tag{4.11}$$

for all $v \in V_h$ and $n=1, \cdots, N-1$. Since

$$a_h(u^{n;\theta} - \omega^{n;\theta}, v) = 0,$$

by the definition of the elliptic projection, we have that

$$(\bar{\partial}_{tt}\phi^n, v) + a_h(\phi^{n;\theta}, v) = (r^n, v),$$

which can be rearranged in the form

$$(\bar{\partial}_{tt}\phi^n, v) + \Delta t^2\Big(\theta - \frac{1}{4}\Big)a_h(\bar{\partial}_{tt}\phi^n, v) + \frac{1}{2}a_h(\phi^{n+\frac{1}{2}} + \phi^{n-\frac{1}{2}}, v) = (r^n, v), \tag{4.12}$$

for all $v \in V_h$ and $n = 1, \cdots, N-1$.

Summing over time levels from $n=1$ to $n=m$, multiplying by Δt and taking into account cancelation, we readily obtain

$$\begin{aligned}
&\Big(\frac{\phi^{m+1} - \phi^m}{\Delta t}, v\Big) - \Big(\frac{\phi^1 - \phi^0}{\Delta t}, v\Big) + \Delta t^2\Big(\theta - \frac{1}{4}\Big)a_h\Big(\frac{\phi^{m+1} - \phi^m}{\Delta t}, v\Big) \\
&- \Delta t^2\Big(\theta - \frac{1}{4}\Big)a_h\Big(\frac{\phi^1 - \phi^0}{\Delta t}, v\Big) + \frac{\Delta t}{2}\sum_{n=1}^{m} a_h\big(\phi^{n+\frac{1}{2}} + \phi^{n-\frac{1}{2}}, v\big) \\
&= \Delta t \sum_{n=1}^{m} (r^n, v).
\end{aligned}$$

Upon defining

$$\Phi^0 = -\frac{1}{2}\phi^0, \qquad \Phi^m = -\frac{1}{2}\phi^0 + \sum_{n=0}^{m-1}\phi^{n+\frac{1}{2}}, \tag{4.13}$$

we verify that

$$\sum_{n=1}^{m}\phi^{n+\frac{1}{2}} + \sum_{n=1}^{m}\phi^{n-\frac{1}{2}} + \frac{1}{2}(\phi^1 - \phi^0) = \Phi^{m+1} + \Phi^m,$$

and hence we have after rearrangements

$$\Big(\frac{\phi^{m+1}-\phi^m}{\Delta t},v\Big)+\Delta t^2\Big(\theta-\frac{1}{4}\Big)a_h\Big(\frac{\phi^{m+1}-\phi^m}{\Delta t},v\Big)+\frac{\Delta t}{2}a_h(\Phi^{m+1}+\Phi^m,v)=(R^m,v),$$

for all $v\in V_h$ and $0\leq m\leq N-1$. We now choose

$$v=\phi^{m+1}+\phi^m=2\big(\Phi^{m+1}-\Phi^m\big),$$

as a test function for the previous equation and multiply the resulting expression by Δt. This results in

$$\begin{aligned}&||\phi^{m+1}||_0^2-||\phi^m||_0^2+\Delta t^2\Big(\theta-\frac{1}{4}\Big)a_h(\phi^{m+1},\phi^{m+1})-\Delta t^2\Big(\theta-\frac{1}{4}\Big)a_h(\phi^m,\phi^m)\\&+\Delta t^2a_h(\Phi^{m+1},\Phi^{m+1})-\Delta t^2a_h(\Phi^m,\Phi^m)=\Delta t(R^m,\phi^{m+1}+\phi^m),\end{aligned}$$

for $0\leq m\leq N-1$. Summation from $m=0$ to $m=n-1$, for $1\leq n\leq N$, yields

$$\begin{aligned}&||\phi^n||_0^2-||\phi^0||_0^2+\Delta t^2\Big(\theta-\frac{1}{4}\Big)a_h(\phi^n,\phi^n)-\Delta t^2\Big(\theta-\frac{1}{4}\Big)a_h(\phi^0,\phi^0)\\&+\Delta t^2a_h(\Phi^n,\Phi^n)-\Delta t^2a_h(\Phi^0,\Phi^0)=\Delta t\sum_{m=0}^{n-1}(R^m,\phi^{m+1}+\phi^m).\end{aligned}$$

Taking into account (4.13), we deduce that, for $1\leq n\leq N$,

$$\begin{aligned}&||\phi^n||_0^2+\Delta t^2\Big(\theta-\frac{1}{4}\Big)a_h(\phi^n,\phi^n)\\&\leq||\phi^0||_0^2+\Delta t^2\theta a_h(\phi^0,\phi^0)+\Delta t\sum_{m=0}^{n-1}(R^m,\phi^{m+1}+\phi^m).\end{aligned}$$

If the CFL condition (4.5) holds, then we have

$$\tilde{C}||\phi^n||_0^2\leq(1+\theta\Delta t^2h^{-2}M_\gamma)||\phi^0||_0^2+\Delta t\sum_{m=0}^{n-1}(R^m,\phi^{m+1}+\phi^m),\tag{4.14}$$

where

$$\tilde{C}=1-\frac{\Delta t^2}{h^2}\Big(\frac{1}{4}-\theta\Big)M_\gamma>0.$$

From (4.14), we can now derive the following bound

$$\max_{n=0}^{N}||\phi^n||_0\leq\Big[\frac{2(1+\theta\Delta t^2h^{-2}M_\gamma)}{\tilde{C}}\Big]^{\frac{1}{2}}||\phi^0||_0+\frac{2\Delta t}{\tilde{C}}\sum_{n=0}^{N-1}||R^n||_0,\tag{4.15}$$

based on arguments similar to those presented in [3] and [17]. The desired estimate in the proposition follows now from (4.9), (4.15) and the fact that

$$||\phi^0||_0\leq||e^0||_0+||\eta^0||_0,$$

which completes the proof. $\square$

Now, we wish to bound the terms $||R^n||_0$ on the right-hand side of the inequality in Proposition 2. To that end, we first estimate the L^2-norms of the functions r^n. We distinguish the cases n=0 and n≥1. The corresponding results are given in Lemma 1 and Lemma 2, respectively.

Lemma 1. *There holds*

$$||r^0||_0 \leq C\Big(\Delta t^{-1}h^{p+1}||u_t||_{C(\bar{J};H^{p+1}(\Omega))} + \Delta t||u_{ttt}||_{C(\bar{J};L^2(\Omega))} + \Delta t||f_t||_{C(\bar{J};L^2(\Omega))}\Big),$$

if $\theta \neq 1/12$, *and*

$$||r^0||_0 \leq C\Big(\Delta t^{-1}h^{p+1}||u_t||_{C(\bar{J};H^{p+1}(\Omega))} + \Delta t^3||\partial_t^5 u||_{C(\bar{J};L^2(\Omega))} + \Delta t^3||f_{ttt}||_{C(\bar{J};L^2(\Omega))}\Big),$$

if θ=$1/12$, *with a constant* C>0 *independent of* h, Δt *and* T *in each case.*

Proof. We recall that r^0 is defined by

$$(r^0, v) = \Delta t^{-2}(\phi^1 - \phi^0, v) + \theta a_h(\phi^1 - \phi^0, v), \quad \forall v \in V_h. \tag{4.16}$$

We have

$$\begin{aligned}
&(\phi^1 - \phi^0, v)\\
=&(\omega^1 - U^1, v) - (\omega^0 - U^0, v)\\
=&(\omega^1 - u^1, v) + (u^1 - U^1, v) - (\omega^0 - u^0, v) - (u^0 - U^0, v)\\
=&((\Pi_h - I)(u^1 - u^0), v) + (u^1 - u^0, v) - (U^1 - U^0, v), \qquad (4.17)\\
&a_h(\phi^1 - \phi^0, v)\\
=&a_h(\omega^1 - U^1, v) - a_h(\omega^0 - U^0, v)\\
=&a_h(\omega^1 - u^1, v) + a_h(u^1 - U^1, v) - a_h(\omega^0 - u^0, v) - a_h(u^0 - U^0, v)\\
=&a_h(u^1 - u^0, v) - a_h(U^1 - U^0, v), \qquad (4.18)
\end{aligned}$$

where we have used that

$$a_h(\omega^1 - u^1, v) = a_h(\Pi_h u^1 - u^1, v) = 0, \quad a_h(\omega^0 - u^0, v) = a_h(\Pi_h u^0 - u^0, v) = 0.$$

From Taylor's formula with integral remainder, we have

$$u^1 = u^0 + \Delta t v^0 + \frac{\Delta t^2}{2}u_{tt}^0 + \frac{1}{2}\int_0^{t^1}(\Delta t - s)^2 u_{ttt}(\cdot, s)ds. \tag{4.19}$$

If we consider (4.8) with n=0 and n=1, and subtract the resulting equations, we get

$$(u_{tt}^1 - u_{tt}^0, v) + a_h(u^1 - u^0, v) = (f^1 - f^0, v), \quad \forall v \in V_h. \tag{4.20}$$

Using (4.19) and (4.20), we readily obtain

$$\begin{aligned}&(u^1-u^0,v)+\theta\Delta t^2 a_h(u^1-u^0,v)\\ =&\Delta t(v^0,v)+\frac{\Delta t^2}{2}(u^0_{tt},v)+\theta\Delta t^2(f^1-f^0,v)-\theta\Delta t^2(u^1_{tt}-u^0_{tt},v)\\ &+\frac{1}{2}\int_0^{t^1}(\Delta t-s)^2(u_{ttt}(\cdot,s),v)ds.\end{aligned}\tag{4.21}$$

Recall that U^1 is the solution of the Galerkin elliptic problem

$$(U^1-U^0,v)+\Delta t^2\theta a_h(U^1-U^0,v)=\Delta t(v^0,v)+\frac{\Delta t^2}{2}(\tilde{U}^0,v),\quad \forall v\in V_h.\tag{4.22}$$

Subtracting (4.22) from (4.21) and taking into account (2.10) yields

$$\begin{aligned}\Delta t^2(r^0,v)=&\big((\Pi_h-I)(u^1-u^0),v\big)-\theta\Delta t^2(u^1_{tt}-u^0_{tt},v)\\ &+\theta\Delta t^2(f^1-f^0,v)+\frac{1}{2}\int_0^{t^1}(\Delta t-s)^2(u_{ttt}(\cdot,s),v)ds.\end{aligned}\tag{4.23}$$

We now estimate the terms on the right-hand side of (4.23). To bound the first term $((\Pi_h-I)(u^1-u^0),v)$, we use standard arguments; see for instance [17]. We have

$$|((\Pi_h-I)(u^1-u^0),v)|\le\int_0^{t^1}|(\partial_t(\Pi_h-I)u,v)|dt=\int_0^{t^1}|((\Pi_h-I)u_t,v)|dt,$$

and thanks to (4.3), we derive the bound

$$|((\Pi_h-I)(u^1-u^0),v)|\le C\Delta t h^{p+1}||u_t||_{C(\bar{J};H^{p+1}(\Omega))}||v||_0.\tag{4.24}$$

Next, we have the following bounds which are easy to verify,

$$\begin{aligned}|(f^1-f_0,v)|&\le\int_0^{t^1}|(f_t(\cdot,s),v)|ds\le\Delta t||f_t||_{C(\bar{J};L^2(\Omega))}||v||_0,\\ |(u^1_{tt}-u^0_{tt},v)|&\le\int_0^{t^1}|(u_{ttt}(\cdot,s),v)|ds\le\Delta t||u_{ttt}||_{C(\bar{J};L^2(\Omega))}||v||_0,\end{aligned}$$

and

$$\begin{aligned}\Big|\int_0^{t^1}(\Delta t-s)^2(u_{ttt}(\cdot,s),v)ds\Big|\le&\Delta t^2\int_0^{t^1}|(u_{ttt}(\cdot,s),v)|\,ds\\ \le&\Delta t^3||u_{ttt}||_{C(\bar{J};L^2(\Omega))}||v||_0.\end{aligned}$$

Since $r^0\in V_h$, referring to (4.23), (4.24) and the three previous bounds, we conclude that

$$\begin{aligned}\Delta t^2||r^0||_0\le&C\Delta t h^{p+1}||u_t||_{C(\bar{J};H^{p+1}(\Omega))}+\Delta t^3\Big(\theta+\frac{1}{2}\Big)||u_{ttt}||_{C(\bar{J};L^2(\Omega))}\\ &+\theta\Delta t^3||f_t||_{C(\bar{J};L^2(\Omega))}.\end{aligned}\tag{4.25}$$

Dividing (4.25) by Δt^2 yields the desired estimate for $||r^0||_0$.

For the case with $\theta=1/12$, we make use of the higher-order Taylor's formula

$$u^1 = u^0 + \Delta t v^0 + \frac{\Delta t^2}{2} u^0_{tt} + 2\theta \Delta t^3 u^0_{ttt} + \frac{\theta}{2}\Delta t^4 u^0_{tttt} + \frac{1}{24}\int_0^{t^1} (\Delta t - s)^4 \partial_t^5 u(\cdot,s) ds,$$

which combined with (4.20) gives

$$\begin{aligned}
&(u^1 - u^0, v) + \theta \Delta t^2 a_h(u^1 - u^0, v) \\
=& \Delta t(v^0, v) + \frac{\Delta t^2}{2}(u^0_{tt}, v) + 2\theta \Delta t^3 (u^0_{ttt}, v) \\
&+ \theta \Delta t^2 (f_1 - f^0, v) - \theta \Delta t^2 a_h(u^1_{tt} - u^0_{tt}, v) \\
&+ \frac{\theta}{2}\Delta t^4 (u^0_{tttt}, v) + \frac{\theta}{2}\int_0^{t^1} (\Delta t - s)^4 (\partial_t^5 u(\cdot,s), v) ds.
\end{aligned}$$

Using the fact that

$$u^1_{tt} = u^0_{tt} + \Delta t u^0_{ttt} + \frac{\Delta t^2}{2} u^0_{tttt} + \frac{1}{2}\int_0^{t^1} (\Delta t - s)^2 \partial_t^5 u(\cdot,s) ds,$$

we have

$$\begin{aligned}
&(u^1 - u^0, v) + \theta \Delta t^2 a_h(u^1 - u^0, v) \\
=& \Delta t(v^0, v) + \frac{\Delta t^2}{2}(u^0_{tt}, v) + \theta \Delta t^3 (u^0_{ttt}, v) + \frac{\theta}{2}\int_0^{t^1} (\Delta t - s)^4 (\partial_t^5 u(\cdot,s), v) ds \\
&- \frac{\theta}{2}\Delta t^2 \int_0^{t^1} (\Delta t - s)^2 (\partial_t^5 u(\cdot,s), v) ds + \theta \Delta t^2 (f_1 - f^0, v).
\end{aligned} \tag{4.26}$$

If we differentiate (4.8) with respect to t and consider the resulting equation with $n=0$, we obtain

$$a_h(u^0_t, v) = (f^0_t, v) - (u^0_{ttt}, v). \tag{4.27}$$

Subtracting (2.11) from (4.26) and taking into account (4.27) shows that

$$\begin{aligned}
\Delta t^2 (r^0, v) =& ((\Pi_h - I)(u^1 - u^0), v) + \theta \Delta t^2 \Big(f^1 - f_0 - \Delta t f^0_t - \frac{\Delta t^2}{2} f^0_{tt}, v \Big) \\
&+ \frac{\theta}{2}\int_0^{t^1} (\Delta t - s)^4 (\partial_t^5 u(\cdot,s), v) ds - \frac{\theta}{2}\Delta t^2 \int_0^{t^1} (\Delta t - s)^2 (\partial_t^5 u(\cdot,s), v) ds.
\end{aligned} \tag{4.28}$$

By noticing that

$$\begin{aligned}
\Big| \Big(f^1 - f_0 - \Delta t f^0_t - \frac{\Delta t^2}{2} f^0_{tt}, v \Big) \Big| \le& \frac{\Delta t^2}{2}\int_0^{t^1} |(f_{ttt}(\cdot,s), v)| ds \\
\le& \frac{\Delta t^3}{2} ||f_{ttt}||_{C(\bar{J};L^2(\Omega))} ||v||_0,
\end{aligned}$$

and bounding the two integrals in (4.28) as done in the previous case, we conclude that

$$\begin{aligned}\Delta t^2||r^0||_0 \le& C\Delta t h^{p+1}||u_t||_{C(\bar{J};H^{p+1}(\Omega))} + \theta\Delta t^5||\partial_t^5 u||_{C(\bar{J};L^2(\Omega))}\\ &+ \theta\Delta t^5||f_{ttt}||_{C(\bar{J};L^2(\Omega))}.\end{aligned} \tag{4.29}$$

Dividing (4.29) by Δt^2 yields the second estimate for $||r^0||_0$. □

Lemma 2. *For* $1\le n\le N-1$, *there holds*

$$||r^n||_0 \le C\Big(\Delta t^{-1}h^{p+1}\int_{t^{n-1}}^{t^{n+1}}||u_{tt}(\cdot,s)||_{p+1}ds + \Delta t\int_{t^{n-1}}^{t^{n+1}}||\partial_t^4(\cdot,s)||_0 ds\Big),$$

if $\theta\neq 1/12$, *and*

$$||r^n||_0 \le C\Big(\Delta t^{-1}h^{p+1}\int_{t^{n-1}}^{t^{n+1}}||u_{tt}(\cdot,s)||_{p+1}ds + \Delta t^3\int_{t^{n-1}}^{t^{n+1}}||\partial_t^6(\cdot,s)||_0 ds\Big),$$

if $\theta=1/12$, *with a constant* $C>0$ *independent of* h, Δt *and* T.

Proof. By the triangle inequality, we have

$$||r^n||_0 = ||\bar{\partial}_{tt}\omega^n - u_{tt}^{n;\theta}||_0 \le ||\bar{\partial}_{tt}(\Pi_h - I)u^n||_0 + ||\bar{\partial}_{tt}u^n - u_{tt}^{n;\theta}||_0. \tag{4.30}$$

From Taylor's formulas with integral remainders, we find that

$$\bar{\partial}_{tt}u^n = \frac{1}{\Delta t^2}\int_{-\Delta t}^{\Delta t}(\Delta t - |s|)\partial_t^2 u(\cdot,t^n+s)ds.$$

By using (4.3), we have

$$\begin{aligned}||\bar{\partial}_{tt}(\Pi_h - I)u^n||_0 \le& \frac{1}{\Delta t^2}\int_{-\Delta t}^{\Delta t}(\Delta t - |s|)||\partial_t^2(\Pi_h - I)u||_0 ds\\ \le& C\frac{h^{p+1}}{\Delta t}\int_{-\Delta t}^{\Delta t}||u_{tt}(\cdot,t^n+s)||_{p+1}ds,\end{aligned} \tag{4.31}$$

where we used the fact that

$$(\Delta t - |s|) \le \Delta t, \quad \text{when} \quad s\in[-\Delta t,\Delta t].$$

To estimate the second term on the right-hand side of (4.30), we make use of the identity

$$\bar{\partial}_{tt}u^n = u_{tt}^n + \frac{1}{6\Delta t^2}\int_{-\Delta t}^{\Delta t}(\Delta t - |s|)^3\partial_t^4 u(\cdot,t^n+s)ds. \tag{4.32}$$

From the Taylor's expansions of u_{tt}^{n+1} and u_{tt}^{n-1} about u_{tt}^n, we have

$$\begin{aligned}u_{tt}^{n+1} &= u_{tt}^n + \Delta t u_{ttt}^n + \int_0^{\Delta t}(\Delta t - |s|)\partial_t^4 u(\cdot,t^n+s)ds,\\ u_{tt}^{n-1} &= u_{tt}^n - \Delta t u_{ttt}^n + \int_{-\Delta t}^0(\Delta t - |s|)\partial_t^4 u(\cdot,t^n+s)ds,\end{aligned}$$

and we deduce that

$$u_{tt}^{n;\theta} = u_{tt}^n + \theta \int_{-\Delta t}^{\Delta t} (\Delta t - |s|)\partial_t^4 u(\cdot, t^n + s) ds. \tag{4.33}$$

Subtracting (4.33) from (4.32) yields

$$\begin{aligned}\bar{\partial}_{tt} u^n - u_{tt}^{n;\theta} =& \frac{1}{6\Delta t^2} \int_{-\Delta t}^{\Delta t} (\Delta t - |s|)^3 \partial_t^4 u(\cdot, t^n + s) ds \\ &- \theta \int_{-\Delta t}^{\Delta t} (\Delta t - |s|)\partial_t^4 u(\cdot, t^n + s) ds.\end{aligned}$$

This leads to the bound

$$||\bar{\partial}_{tt} u^n - u_{tt}^{n;\theta}||_0 \le \Big(\frac{1}{6} + \theta\Big)\Delta t \int_{-\Delta t}^{\Delta t} ||\partial_t^4 u(\cdot, t^n + s)||_0 \, ds. \tag{4.34}$$

Referring to (4.30), (4.31) and (4.34) proves the first estimate in the lemma.

Now we wish to derive expressions similar to (4.32) and (4.33) by using high-order Taylor's formulas. We have

$$\bar{\partial}_{tt} u^n = u_{tt}^n + \frac{\Delta t^2}{12} u_{tttt}^n + \frac{1}{5!\Delta t^2} \int_{-\Delta t}^{\Delta t} (\Delta t - |s|)^5 \partial_t^6 u(\cdot, t^n + s) ds,$$

and it can be easily verified that

$$u_{tt}^{n;\theta} = u_{tt}^n + \theta \Delta t^2 u_{tttt}^n + \frac{\theta}{6} \int_{-\Delta t}^{\Delta t} (\Delta t - |s|)^3 \partial_t^6 u(\cdot, t^n + s) ds.$$

Combining the two previous inequalities using θ=$1/12$, we obtain

$$\begin{aligned}\bar{\partial}_{tt} u^n - u_{tt}^{n;\theta} =& \frac{1}{5!\Delta t^2} \int_{-\Delta t}^{\Delta t} (\Delta t - |s|)^5 \partial_t^6 u(\cdot, t^n + s) ds \\ &- \frac{1}{72} \int_{-\Delta t}^{\Delta t} (\Delta t - |s|)^3 \partial_t^6 u(\cdot, t^n + s) ds,\end{aligned}$$

and therefore

$$||\bar{\partial}_{tt} u^n - u_{tt}^{n;\theta}||_0 \le \Big(\frac{1}{5!} + \frac{1}{72}\Big)\Delta t^3 \int_{-\Delta t}^{\Delta t} ||\partial_t^6 u(\cdot, t^n + s)||_0 ds. \tag{4.35}$$

Referring to (4.30), (4.31) and (4.35) shows the second estimate for $||r^n||_0$, and thus completes the proof of the lemma. $\square$

The next proposition follows immediately from Lemma 1 and Lemma 2.

Proposition 3. *For* $1 \le n \le N-1$, *there holds*

$$\begin{aligned}||R^n||_0 \le& C\Delta t^2 \Big(||u_{ttt}||_{C(\bar{J};L^2(\Omega))} + ||\partial_t^4 u||_{L^1(J;L^2(\Omega))} + ||f_t||_{C(\bar{J};L^2(\Omega))} \Big) \\ &+ Ch^{p+1} \Big(||u_t||_{C(\bar{J};H^{p+1}(\Omega))} + ||u_{tt}||_{C(\bar{J};H^{p+1}(\Omega))} \Big),\end{aligned}$$

if $\theta\neq 1/12$, *and*

$$\begin{aligned}||R^n||_0 \leq & C\Delta t^4\Big(||\partial_t^5 u||_{C(\bar{J};L^2(\Omega))}+||\partial_t^6 u||_{L^1(J;L^2(\Omega))}+||f_{ttt}||_{C(\bar{J};L^2(\Omega))}\Big)\\ &+Ch^{p+1}\Big(||u_t||_{C(\bar{J};H^{p+1}(\Omega))}+||u_{tt}||_{C(\bar{J};H^{p+1}(\Omega))}\Big),\end{aligned}$$

if $\theta=1/12$, *with a constant* $C>0$ *independent in each case of* h, Δt *and* T.

Proof. Using the bounds for $||r^n||_0$ derived in Lemma 1 and Lemma 2 for $\theta=1/12$ for instance, we obtain

$$\begin{aligned}||R^n||_0 \leq & \Delta t||r^0||_0+\Delta t\sum_{m=1}^{N-1}||r^m||_0\\ \leq & C\Delta t^4\Big(||\partial_t^5 u||_{C(\bar{J};L^2(\Omega))}+||\partial_t^6 u||_{L^1(J;L^2(\Omega))}+||f_{ttt}||_{C(\bar{J};L^2(\Omega))}\Big)\\ &+Ch^{p+1}\Big(||u_t||_{C(\bar{J};H^{p+1}(\Omega))}+||u_{tt}||_{C(\bar{J};H^{p+1}(\Omega))}\Big).\end{aligned}$$

The first estimate in the proposition follows from the same argument. □

Now, we are ready to prove the main theorem.

Theorem 2. *Let* u *be the solution of the wave problem* (1.1)-(1.2c) *and let the discrete finite element approximations* $\{U^n\}_{n=0}^N$ *be defined by* (2.7) *and* (2.9) *if* $\theta\neq 1/12$ *and by* (2.7) *and* (2.11) *if* $\theta=1/12$. *Assume that the CFL condition* (4.5) *is satisfied. Then there holds the error estimate:*

$$\max_{n=0}^N ||u^n-U^n||_0\leq C(h^{p+1}+\Delta t^2).$$

Furthermore, if $\theta=1/12$ *and* u *satisfies the regularity properties* (4.7), *then the following error estimate holds:*

$$\max_{n=0}^N ||u^n-U^n||_0\leq C(h^{p+1}+\Delta t^4).$$

In each case, $C>0$ *is a constant independent of the mesh size and the time step.*

Proof. From Proposition 2, it follows that

$$\max_{n=0}^N ||e^n||_0\leq C\Big(||e^0||_0+\max_{n=0}^N||\eta^n||_0+T\max_{n=0}^{N-1}||R^n||_0\Big),$$

since

$$\Delta t\sum_{n=0}^{N-1}||R^n||_0\leq T\max_{n=0}^{N-1}||R^n||_0.$$

By the approximation properties of the L^2-projection and the elliptic projection (4.2), we respectively have

$$\begin{aligned}&||e^0||_0=||u^0-P_hu^0||_0\leq Ch^{p+1}||u_0||_{p+1}\leq Ch^{p+1}||u||_{C(\bar{J};H^{p+1}(\Omega))},\\ &\max_{n=0}^N||\eta^n||_0\leq Ch^{p+1}||u||_{C(\bar{J};H^{p+1}(\Omega))}.\end{aligned}$$

We next apply Proposition 3 to bound $\max_{n=0}^{N-1} ||R^n||_0$. This gives

$$\begin{aligned}||e^n||_0 \leq & Ch^{p+1}||u||_{C^2(\bar{J};H^{p+1}(\Omega))} + C\Delta t^2 \Big(||u_{ttt}||_{C(\bar{J};L^2(\Omega))} \\ & + ||\partial_t^4 u||_{L^1(J;L^2(\Omega))} + ||f_t||_{C(\bar{J};L^2(\Omega))} \Big),\end{aligned}$$

if $\theta \neq 1/12$, and

$$\begin{aligned}||e^n||_0 \leq & Ch^{p+1}||u||_{C^2(\bar{J};H^{p+1}(\Omega))} + C\Delta t^4 \Big(||\partial_t^5 u||_{C(\bar{J};L^2(\Omega))} \\ & + ||\partial_t^6 u||_{L^1(J;L^2(\Omega))} + ||f_{ttt}||_{C(\bar{J};L^2(\Omega))} \Big),\end{aligned}$$

if $\theta = 1/12$. The proof is now complete. □

Notice that the constant C in the theorem grows linearly with T.

5 Conclusions

We have presented and analyzed three-level implicit-in-time finite difference schemes for the acoustic wave equation, where Galerkin methods are used for the spatial approximation. The schemes cover the explicit leapfrog scheme and the fourth-order accurate scheme in time obtained by the modified equation method. Stability results covering well-known CFL conditions have been derived. The stability results are general and use only the symmetry of the underlying bilinear form, so they are not limited to finite element methods. Optimal error estimates have also been obtained. For sufficiently smooth solutions, it is demonstrated that the maximal error in the L^2-norm error over a finite time interval converges optimally as $\mathcal{O}(h^{p+1} + \Delta t^s)$, where p denotes the polynomial degree, s=2 or 4, h the mesh size, and Δt the time step. Our convergence results hold for any fully discrete DG method where the underlying DG bilinear form is symmetric, continuous, coercive, and adjoint consistent in the sense of [2]. Our study serves as a model for general second-order hyperbolic problems.

Acknowledgements

This research was supported by Sultan Qaboos University under Grant IG/SCI/DOM S/09/09.

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