AN ERROR ANALYSIS METHOD SPP–BEAM AND A CONSTRUCTION GUIDELINE OF NONCONFORMING FINITE ELEMENTS FOR FOURTH ORDER ELLIPTIC PROBLEMS

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Abstract

Under two hypotheses of nonconforming finite elements of fourth order elliptic problems, we present a side–patchwise projection based error analysis method (SPP–BEAM for short). Such a method is able to avoid both the regularity condition of exact solutions in the classical error analysis method and the complicated bubble function technique in the recent medius error analysis method. In addition, it is universal enough to admit generalizations. Then, we propose a sufficient condition for these hypotheses by imposing a set of in some sense necessary degrees of freedom of the shape function spaces. As an application, we use the theory to design a $P_3$ second order triangular $H^2$ non-conforming element by enriching two $P_4$ bubble functions and, another $P_3$ second order triangular $H^2$ nonconforming finite element, and a $P_3$ second order tetrahedral $H^2$ non-conforming element by enriching eight $P_4$ bubble functions, adding some more degrees of freedom.

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Key words: Nonconforming finite element, A priori error analysis, Biharmonic equation.

1. Introduction

We solve a biharmonic equation:

\[ \Delta^2 u = f, \quad \text{in } \Omega, \]
\[ u = u_n := \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial \Omega, \]

(1.1)

where $\Omega$ is a bounded 2D polygonal domain or 3D polyhedral domain, and $n$ is the unit outer normal to $\partial \Omega$. Doing integration by parts twice, the weak formulation of (1.1) is: Find $u \in H^2_0(\Omega)$ such that

\[ a(u, v) = (f, v) \quad \forall v \in H^2_0(\Omega). \]

(1.2)

Here $H^2_0(\Omega) := \{ v \in H^2(\Omega) \mid v = v_n = 0 \text{ on } \partial \Omega \}$ and $H^2(\Omega)$ is the standard Sobolev space $[3]$. The bilinear forms are

\[ a(u, v) := \int_{\Omega} D^2 u : D^2 v dx, \]
\[ (f, v) := \int_{\Omega} f v dx, \]
where

\[
D^2 u := \begin{pmatrix}
\partial_{x_1 x_1} u & \partial_{x_1 x_2} u \\
\partial_{x_2 x_1} u & \partial_{x_2 x_2} u
\end{pmatrix}
\quad \text{for } 2D,
\]

\[
D^2 u := \begin{pmatrix}
\partial_{x_1 x_1} u & \partial_{x_1 x_2} u & \partial_{x_1 x_3} u \\
\partial_{x_2 x_1} u & \partial_{x_2 x_2} u & \partial_{x_2 x_3} u \\
\partial_{x_3 x_1} u & \partial_{x_3 x_2} u & \partial_{x_3 x_3} u
\end{pmatrix}
\quad \text{for } 3D,
\]

with \(\partial_{x_1 x_2} u := \frac{\partial^2 u}{\partial x_1 \partial x_2}\).

In traditional finite element methods of problem (1.2), degree 2 is the minimum polynomial degree. We need polynomials of degree 2 or above so that the second derivatives are not identically zero in the weak variational form. On a macro-triangle grid, the Powell-Sabin element [14] is a \(P_2 H^2\)-conforming finite element in 2D. That is, the finite element space is \(C_1\), a subspace of the \(H^2\) Sobolev space. The Hsieh-Clough-Tocher \(P_3\) element is an \(H^2\)-conforming finite element on the 1-to-3 splitting macro-triangle grids, [6]. When the polynomial degree is 5 or above, single-triangle \(H^2\)-conforming elements can be constructed on general triangular grids [2]; when the polynomial degree is 9 or above, single-tetrahedron \(H^2\)-conforming elements can be constructed on general tetrahedral grids [21,22]. On both non-macro-triangle grids and non-macro-tetrahedra grids, Morley element is a \(P_2\) non-conforming finite element, i.e., the finite element space is not a subspace of the \(H^2\) space. The remaining gap is the \(P_3\) and \(P_4\) non-conforming finite elements for the 4-th order differential equations in 2D and the \(P_5\), \(P_6\), \(P_7\) non-conforming finite elements for the 4-th order differential equations in 3D.

In this paper, we first present two hypotheses of nonconforming finite elements. Then, under them, we generalize the idea of [10,12] to develop a side–patchwise projection based error analysis method (SPP–BEAM for short). Such a method only assumes the basic \(H^2\) regularity for the exact solution. Compared with the classical a priori error analysis of nonconforming finite elements [3,6,16], the analysis herein applies integration by parts to discrete functions in nonconforming finite element spaces rather than the exact solution of the problem under consideration. This in particular allows to remove the indispensable regularity condition of the exact solution in the classical analysis. Compared with the recent medius analysis of [8,10,13], the analysis herein does not involve the bubble function technique which was first introduced to analyze efficiency of a posteriori error estimators [17]. Note that the bubble function technique will be very complicated for high dimensional cases and high order problems [8].

As an application of the theory, we propose a sufficient condition for these two hypotheses. More precisely, we give a set of in some sense necessary degrees of freedom of the possible shape functions space. Based on these degrees of freedom, we construct a \(P_3\) second order triangular \(H^2\) non-conforming finite element by enriching two \(P_4\) bubble functions and imposing some additional degrees of freedom. As a result, the shape functions space is of 12 dimensions, and the corresponding degrees of freedom are the function value at three vertices and the average of function, the average of the normal derivative, and the first moment of the normal derivative, on three edges. Then, we construct another \(P_4\), but still of 2nd order, \(H^2\) non-conforming element on triangular grids. After that we design a \(P_3\) second order tetrahedral \(H^2\) non-conforming finite element by enriching eight \(P_4\) bubble functions and adding some necessary degrees of freedom. The shape functions space of this three dimensional element is of 28 dimensions, and the corresponding degrees of freedom are the average and the first moment of function on six edges, and the average of function, the average of the normal derivative, and the first moment of the normal derivative, on four faces. We show that the three elements are well-defined
and converge at the second order. Moreover, we give explicitly the basis functions for them. Numerical tests confirm the theory.

Previously, for 2D, Wang, Zu and Zhang enriched the \( P_3 \) polynomials by six \( P_3 \) bubble functions in [20], to obtain a 2nd-order triangular \( H^2 \) non-conforming element. Independently, but similarly, Chen and Chen constructed a 2nd-order triangular \( H^2 \) non-conforming element in [4], by enriching the \( P_3 \) polynomials by three \( P_3 \) bubble functions and three \( P_6 \) bubble functions, cf. [4]. Both elements have 16 degrees of freedom on each element. In [7], Gao, Zhang and Wang did construct another 12 dof 2nd order triangular \( H^2 \) non-conforming element. In their construction, they enriched the \( P_3 \) space by two special \( P_6 \) bubbles, cf. [7], such that the resulting finite element space is \( C^0 \). This construction seems not natural, but intelligent. We numerically compare the two new triangular elements with three existing finite elements of 2nd order, showing that the new elements are more stable and more accurate. For 3D, see [5] for numerically compare the two new triangular elements with three existing finite elements of 2nd order, showing that the new elements are more stable and more accurate. For 3D, see [5] for a second order \( P_3 \) plus 12 \( P_6 \) element bubbles \( H^2 \) nonconforming tetrahedron element and [20] for a second order \( P_3 \) plus 12 \( P_7 \) element bubbles \( H^2 \) nonconforming tetrahedron element.

2. Hypothesis and SPP–BEAM

This section presents hypotheses and consequently the SPP–BEAM of nonconforming finite element methods satisfying the hypotheses. This leads to abstract a priori error estimates. To this end, we introduce some more notation. Let \( \mathcal{T}_h = \{ K \} \) be a regular triangulation on \( \Omega \), cf. [3]. Let \( \mathcal{E}_h \) be the set of \( n - 1 \) dimensional sides of \( \mathcal{T}_h \), and \( \mathcal{E}_h(\Omega) \) be the set of internal sides. For each internal \( n - 1 \) dimensional side \( e \), let \( \omega_e \) be the union of two elements sharing the side \( e \); for each boundary side \( e \), let \( \omega_e \) be the unique element which takes \( e \) as one of its \( n - 1 \) dimensional sides. Given integer \( \ell \geq 0 \), let \( \Pi_{\ell,\omega_e} u \) be the \( L^2 \) projection of \( u \) to the degree \( \ell \) polynomial space on \( \omega_e \), and \( \Pi_{\ell-1,K} u \) be the \( L^2 \) projection of \( u \) onto the degree \( \ell - 1 \) polynomial space over element \( K \); for the case \( \ell = 0 \), let \( \Pi_{\ell-1,K} u = 0 \).

Let \( V^n_{h,\ell} \) be a nonconforming finite element space of \( H^2(\Omega) \) defined over the mesh \( \mathcal{T}_h \) with \( n = 2, 3 \). The condition of the wellposedness of the discrete problem and a priori error estimate of the finite solution is presented under the following two hypotheses.

**Hypothesis 2.1.** For any internal \( n - 1 \) dimensional side \( e \), it holds that

\[
\int_e [\nabla_h v] \cdot q ds = 0 \quad \forall q \in (P_\ell(e))^n, \forall v \in V^n_{h,\ell},
\]

here and throughout this paper, \([\cdot]\) denotes the average and \([\cdot]\) the jump across a side \( e \), \( \nabla_h \) and \( D_h^2 \) are the discrete counterpart of \( \nabla \) and \( D^2 \), respectively, defined piecewise; for any boundary \( n - 1 \) dimensional side \( e \), it holds that

\[
\int_e \nabla v \cdot q ds = 0 \quad \forall q \in (P_\ell(e))^n, \quad \forall v \in V^n_{h,\ell}.
\]

**Hypothesis 2.2.** For any internal \( n - 1 \) dimensional side \( e \), it holds that

\[
\int_e [v] q ds = 0 \quad \forall q \in P_{\ell-1}(e), \quad \forall v \in V^n_{h,\ell};
\]

for any boundary \( n - 1 \) dimensional side \( e \), it holds that

\[
\int_e v q ds = 0 \quad \forall q \in P_{\ell-1}(e), \quad \forall v \in V^n_{h,\ell}.
\]
For two dimensions, a sufficient condition for both Hypothesis 2.1 and 2.2 is up to additional possible degrees of freedom for the uni–solvency to take the following degrees of freedom on each element $K$:

1. the moments $\int \partial^{(\cdot)} q ds$ for all $q \in P_\ell(e)$ on each edge $e$;
2. the moments $\int q ds$ for all $q \in P_{\ell-1}(e)$ on each edge $e$;
3. the value of $(\cdot)$ at each vertex.

For three dimensions, a sufficient condition for both Hypothesis 2.1 and 2.2 is up to additional possible degrees of freedom for the uni–solvency to take the following degrees of freedom on each element $K$:

1. the moments $\int \partial^{(\cdot)} q ds$ for all $q \in P_\ell(e)$ on each face $e$;
2. the moments $\int q ds$ for all $q \in P_{\ell-1}(e)$ on each face $e$;
3. the moments $\int q ds$ for all $q \in P_{\ell}(s)$ on each edge $s$.

The finite element problem, discretizing the biharmonic equation (1.1), is: Find $u_h \in V_{h,\ell}$ such that

$$(D^2 u_h, D^2 v_h)_h = (f, v_h) \quad \forall v_h \in V_{h,\ell}^n,$$

where the discrete inner product is defined element-wise, $(\cdot, \cdot)_h = \sum_{K \in \mathcal{T}_h} (\cdot, \cdot)_K$, and the semi-norm $|v_h|_{2,h} = (\sum_{K \in \mathcal{T}_h} (D^2 v_h, D^2 v_h)_K)^{1/2}$.

For the analysis, we introduce the projection averaging interpolation operator defined in [16]. To this end, let $V_h^c$ be the $P_5$ Argyris element space for 2D [2] and the $P_9 H^2$ conforming element space for 3D [21,22], respectively. For the $P_5$ Argyris element, its degrees of freedom on triangle $K$ are as follows

1. the values of $D^{\ell}(\cdot)$, $\ell = 0, 1, 2$, at each vertex;
2. the first order normal derivative $\partial^{(\cdot)}$ at the mid-point of each edge.

For the $P_9 H^2$ conforming element in 3D, its degrees of freedom on tetrahedron $K$ are as follows

1. the values of $D^{\ell}(\cdot)$, $\ell = 0, 1, 2, 3, 4$, at each vertex;
2. the first order normal derivatives $\partial^{(\cdot)}$ and $\partial^{(\cdot)}$ at the mid-point of each edge;
3. the second order normal derivatives $\partial^{2(\cdot)}$, $\partial^{2(\cdot)}$, $\partial^{2(\cdot)}$ at 2 equally-distributed internal points of each edge;
4. the value $(\cdot)$ at the mid-point of each face;
5. the first order normal derivatives $\partial^{(\cdot)}$ at three standardly-distributed internal points of each face which are able to uniquely determine a 2D $P_1$ polynomial;
6. the values $(\cdot)$ at four standardly-distributed internal points of element which are able to uniquely determine a 3D $P_1$ polynomial.
Lemma 2.1. Assume that Hypothesis 2.1 and 2.2 hold with \( \ell \geq 0 \). Then, for \( 0 \leq \ell \leq 2 \), it holds that
\[
\|D_h^\ell (v_h - \tilde{I}_h v_h)\|_0^2 \leq C \sum_{K \in T_h} h_K^{(2-\ell)2} \|D^2 v_h\|_{0,K}^2,
\]
for any \( v_h \in V_{h,\ell}^n \). Here and throughout the paper \( D_h = \nabla_h \).

\( \square \)

Proof. The proof can be found in [16, Lemma 5.6.4].

Theorem 2.1. Let \( u_h \) be the solution of (2.1), and \( u \) be the solution of (1.1). Assume that Hypothesis 2.1 and 2.2 hold with \( \ell \geq 0 \), and that the seminorm \( \|D_h^2 \cdot \|_0 \) defines a norm over the nonconforming finite element space \( V_{h,\ell}^n \). Then,
\[
\|D_h^2 (u - u_h)\|_0 \leq C \inf_{s_h \in V_{h,\ell}^n} \|D_h^2 (u - s_h)\|_0 + C \left( \sum_{e \in E_h} \| (I - \Pi_{\ell,e}) D^2 u \|_{0,e}^2 \right)^{1/2} \sum_{e \in E_h} \| (I - \Pi_{\ell-1,e}) f \|_{e}^2 \left( \sum_{K \in T_h} h_K^{1/2} \| \Pi_{\ell-1,K} f \|_{0,K}^2 \right)^{1/2}.
\]

Remark 2.1. The SPP–BEAM is closely related to the medius error analysis method in [8,10,13]. The major new idea herein is to introduce the side patch projection operator \( \Pi_{\ell,e} \). This enables us to avoid the bubble function technique which is the main tool of [8,10,13]. Note that the bubble function technique will become very complicated for high dimensional cases and high order problems.

Proof. Since the seminorm \( \|D_h^2 \cdot \|_0 \) defines a norm over the nonconforming finite element space \( V_{h,\ell}^n \), the discrete problem (2.1) is well-defined. For the nonconforming finite element solution, it holds that
\[
(D^2(s_h - u_h), D^2(s_h - u_h))_h = (D^2 u, D^2(s_h - u_h))_h - (f, s_h - u_h) + (D^2(s_h - u), D^2(s_h - u_h))_h,
\]
for any \( s_h \in V_{h,\ell}^n \). By the Schwarz inequality and the triangle inequality,
\[
|s_h - u_h|_{2,h} \leq \sup_{0 \neq v_h \in V_{h,\ell}^n} \frac{(D^2 u, D^2 v_h)_h - (f, v_h)}{|v_h|_{2,h}} + |u - s_h|_{2,h}, \tag{2.2}
\]
\[
|u - u_h|_{2,h} \leq \sup_{0 \neq v_h \in V_{h,\ell}^n} \frac{(D^2 u, D^2 v_h)_h - (f, v_h)}{|v_h|_{2,h}} + 2|u - s_h|_{2,h}. \tag{2.3}
\]
Let \( \tilde{v}_h = \tilde{I}_hv_h \) be the averaging interpolation of \( v_h \) defined above. Next we shall follow \cite{8,10,13} to use such an averaging interpolation to analyze the first term on the right-hand side of (2.3). This shall allow for an application of the usual Green formulation to a discrete function in the finite element space but not to the exact solution.

\[
\frac{(D^2u, D^2v_h)_h - (f, v_h)}{|v_h|_{2,h}} = \frac{(D^2u, D^2(v_h - \tilde{v}_h))_h - (f, (v_h - \tilde{v}_h))}{|v_h|_{2,h}} = \frac{(D^2(u - s_h), D^2(v_h - \tilde{v}_h))_h + (D^2s_h, D^2(v_h - \tilde{v}_h))_h - (f, v_h - \tilde{v}_h)}{|v_h|_{2,h}} \\
\leq \frac{(D^2s_h, D^2(v_h - \tilde{v}_h))_h - (f, v_h - \tilde{v}_h)}{|v_h|_{2,h}} + C|u - s_h|_{2,h},
\]

where we use the stability of the averaging interpolation operator \( \tilde{I}_h \) that
\[
(D^2(u - s_h), D^2(v_h - \tilde{v}_h))_h \\
\leq |u - s_h|_{2,h}|v_h - \tilde{v}_h|_{2,h} \leq (1 + C)|u - s_h|_{2,h}|v_h|_{2,h}.
\]

We start to estimate the consistency error. An integration by parts leads to
\[
(D^2s_h, D^2(v_h - \tilde{v}_h))_h - (f, v_h - \tilde{v}_h) \\
= \sum_{e \in \mathcal{E}_h} \left( \int_e [D_h^2 s_h] \mathbf{n} \cdot \{ \nabla_h (v_h - \tilde{v}_h) \} ds - \int_e [\text{div}_h D_h^2 s_h] \cdot \mathbf{n} \{ v_h - \tilde{v}_h \} ds \right) \\
+ \sum_{e \in \mathcal{E}_h} \left( \int_e \{ D_h^2 s_h \} \mathbf{n} \cdot \{ \nabla_h (v_h - \tilde{v}_h) \} ds - \int_e [\text{div}_h D_h^2 s_h] \cdot \mathbf{n} \{ v_h - \tilde{v}_h \} ds \right) \\
+ \sum_{K \in \mathcal{T}_h} \int_K \Delta^2 (s_h - u) (v_h - \tilde{v}_h) dx.
\]

Here, \( \mathbf{n} \) is a fixed unit normal vector to the side \( e \). We estimate the five error terms one by one.

One observation is that \( [\Pi_{\ell, \omega} D^2 u]_e = 0 \) for any internal \( e \). The main idea here is to use it to analyze the first term on the right-hand side of (2.4) which avoids the bubble function technique used in the medius error analysis from \cite{8,10,13}. This leads to
\[
\int_e [D_h^2 s_h] \mathbf{n} \cdot \{ \nabla_h (v_h - \tilde{v}_h) \} ds \\
= \int_e [D_h^2 s_h - \Pi_{\ell, \omega} D^2 u] \mathbf{n} \cdot \{ \nabla_h (v_h - \tilde{v}_h) \} ds \\
\leq \| D_h^2 s_h - \Pi_{\ell, \omega} D^2 u \|_{0,e} \| \{ \nabla_h (v_h - \tilde{v}_h) \} \|_{0,e} \\
\leq \frac{1}{2} \| D^2 s_h - \Pi_{\ell, \omega} D^2 u \|_{0,e+\text{We}} \| \nabla (v_h - \tilde{v}_h) \|_{0,e+\text{We}}.
\]

Here and throughout the paper, \( e^+ \) and \( e^- \) denote the restrictions from both side of \( e \). Using the inverse inequality, we get
\[
\| D_h^2 s_h - \Pi_{\ell, \omega} D^2 u \|_{0,e} \|_{0,e+\text{We}} \\
\leq C h^{-1/2} \| D_h^2 s_h - \Pi_{\ell, \omega} D^2 u \|_{0,\omega} + h^{1/2} | D_h^2 s_h - \Pi_{\ell, \omega} D^2 u |_{1,\omega}\| \\
\leq C h^{-1/2}(\| D_h^2 (s_h - u) \|_{0,\omega} + \| (I - \Pi_{\ell, \omega}) D^2 u \|_{0,\omega}).
\]
For the other term, we apply the stability of the averaging interpolation operator to get
\[
\|\nabla_h (v_h - \bar{v}_h)\|_{0,e + \cup_e} \leq C h^{-1/2} \|\nabla_h (v_h - \bar{v}_h)\|_{0,\omega_e} + C h^{1/2} \|\nabla_h (v_h - \bar{v}_h)\|_{1,\omega_e}
\]
\[
\leq C h^{1/2} \|\nabla_h (v_h - \bar{v}_h)\|_{1,\omega_e}
\]
\[
\leq C h^{1/2} (D_h^2 v_h\|_{0,\omega_e} + \|D_h^2 v_h\|_{0,\bar{\omega}_e}) \leq C h^{1/2} \|D_h^2 v_h\|_{0,\bar{\omega}_e},
\]
where \(\bar{\omega}_e\) is the union of all the elements which shares at least one vertex with the two elements of \(\omega_e\). Combining the two estimates, we estimate the first term in (2.4) to get
\[
\sum_{e \in \mathcal{E}_h(\Omega)} \int_e [D_h^2 s_h] \mathbf{n} \cdot \{\nabla_h (v_h - \bar{v}_h)\} ds \leq C \left( \sum_{e \in \mathcal{E}_h(\Omega)} \| (I - \Pi_{\ell,\omega_e}) D_h^2 u \|_{0,\omega_e}^2 \right)^{1/2} \|v_h\|_{2,h} + C \|D_h^2 (u - s_h)\|_0 \|v_h\|_{2,h}.
\]

For the third error term in (2.4), we need to use the \(P_\ell\) continuity of \(\nabla_h v_h\) in the sense that \(\int_e \nabla \nabla_h v_h \cdot p_\ell ds = 0\) for any \(P_\ell\) polynomial \(p_\ell\).
\[
\int_e [D_h^2 s_h] \mathbf{n} \cdot \{\nabla_h (v_h - \bar{v}_h)\} ds = \int_e [D_h^2 s_h - \Pi_{\ell,\omega_e} D_h^2 u] \mathbf{n} \cdot \{\nabla_h (v_h - \bar{v}_h)\} ds \leq \| [D_h^2 s_h - \Pi_{\ell,\omega_e} D_h^2 u] \|_{0,e} \|\nabla_h (v_h - \bar{v}_h)\|_{0,e} \leq C h^{1/2} \| [D_h^2 s_h - \Pi_{\ell,\omega_e} D_h^2 u] \|_{0,\omega_e} \|\nabla_h (v_h - \bar{v}_h)\|_{0,\omega_e} \leq C (\|D_h^2 (s_h - u)\|_{0,\omega_e} + \|\nabla_h (v_h - \bar{v}_h)\|_{0,\omega_e} + \| (I - \Pi_{\ell,\omega_e}) D_h^2 u \|_{0,\omega_e}) \|D_h^2 v_h\|_{0,\bar{\omega}_e}.
\]
The analysis in the last two steps is the same as that in the last two steps of (2.5). As in (2.6), a summation over all sides gives
\[
\sum_{e \in \mathcal{E}_h} \int_e \| (I - \Pi_{\ell,\omega_e}) D_h^2 u \|_{0,\omega_e} \left( \|v_h\|_{2,h} + C \|D_h^2 (u - s_h)\|_0 \|v_h\|_{2,h} \right)^{1/2} \|v_h\|_{2,h}.
\]
The second error term in (2.4) is estimated similarly to the first term:
\[
\int_e \| \text{div}_h D_h^2 s_h \| \mathbf{n} (v_h - \bar{v}_h) ds = \int_e \| \text{div}_h D_h^2 s_h - \text{div} \Pi_{\ell,\omega_e} D_h^2 u \| \mathbf{n} (v_h - \bar{v}_h) ds \leq C h^{1/2} \| \text{div}_h D_h^2 s_h - \text{div} \Pi_{\ell,\omega_e} D_h^2 u \|_{0,\omega_e} \|v_h - \bar{v}_h\|_{0,\omega_e} \leq C (\|D_h^2 (s_h - u)\|_{0,\omega_e} + \|\nabla_h (v_h - \bar{v}_h)\|_{0,\omega_e} + \|\nabla_h (v_h - \bar{v}_h)\|_{0,\omega_e}) \|v_h\|_{2,\bar{\omega}_e}.
\]
The fourth error term in (2.4) is analyzed similarly to that of the second term. Using \( \int_E [v_h]_{p-1} ds = 0 \) for any \( P_{p-1} \) polynomial \( p_r \), we get
\[
\int_E \{ \text{div}_h D_h^2 s_h \} \cdot n [v_h - \tilde{v}_h] ds
= \int_E \{ \text{div}_h D_h^2 s_h - \text{div}_h \Pi_{\ell, \omega_r} D^2 u \} \cdot n [v_h - \tilde{v}_h] ds
\leq C(\|D_h^2(s_h - u)\|_{0, \omega_r} + \| (I - \Pi_{\ell, \omega_r}) D^2 u \|_{0, \omega_r}) |v_h|_{2, \omega_r}.
\]

For the fifth error term in (2.4), we have
\[
\int_K (\Delta^2 s_h - f)(v_h - \tilde{v}_h) dx \leq C h_K^2 \| \Delta^2 s_h - f \|_{0, K} |v_h|_{2, K},
\] (2.8)
where \( \tilde{K} \) is the union of elements sharing at least one vertex with \( K \). To estimate
\[
\| \Delta^2 s_h - f \|_{0, K} \leq \| \Delta^2 s_h - \Pi_{\ell-1, K} f \|_{0, K} + \| (I - \Pi_{\ell-1, K}) f \|_{0, K},
\] (2.9)
we follow the idea of [17] to introduce a \( P_{n+1} \) bubble function on element \( K \):
\[
b = \lambda_1 \cdots \lambda_{n+1},
\]
where \( \lambda_i \) is a barycentric coordinate on \( K \), i.e., the linear function \( \lambda_i \) assumes value 1 at vertex \( x_i \), and 0 on the opposite side. The two norms, the weighted norm and the standard norm, are equivalent.
\[
\| \Delta^2 s_h - \Pi_{\ell-1, K} f \|_{0, K}^2 \leq C b(\Delta^2 s_h - \Pi_{\ell-1, K} f) \|_{0, K}^2
= C(\Delta^2 s_h - \Pi_{\ell-1, K} f, w_K)_K,
\] (2.10)
where we let
\[
w_K = b^2(\Delta^2 s_h - \Pi_{\ell-1, K} f) \in H^2_0(K).
\]
In order to use the inverse estimate, we need the following decomposition:
\[
(\Delta^2 s_h - \Pi_{\ell-1, K} f, w_K)_K = (\Delta^2 s_h - f, w_K)_K + ((I - \Pi_{\ell-1, K}) f, w_K)_K.
\]
We apply the integration by parts twice on one element to get
\[
(\Delta^2 s_h - f, w_K) = (D^2 s_h, D^2 w_K) - (D^2 u, D^2 w_K)
\leq |u - s_h|_{2, K} |w_K|_{2, K}.
\]
Combining the above two inequalities, we obtain
\[
(\Delta^2 s_h - \Pi_{\ell-1, K} f, w_K) \leq (C h_K^2 |u - s_h|_{2, K} + \| (I - \Pi_{\ell-1, K}) f \|_{0, K}) |w_K|_{0, K}.
\] (2.11)
By (2.9)–(2.11), we have
\[
\| \Delta^2 s_h - f \|_{0, K} \leq C h_K^{-2} |u - s_h|_{2, K} + C \| (I - \Pi_{\ell-1, K}) f \|_{0, K}.
\]
By (2.8), the fifth term of (2.4) is bounded by
\[
\sum_{K \in T_h} \int_K \Delta^2 (s_h - u)(v_h - \tilde{v}_h) dx
\leq C(\|D_h^2(u - s_h)\|_0 + \left( \sum_{K \in T_h} h_K^4 \| (I - \Pi_{\ell-1, K}) f \|_{0, K}^2 \right)^{1/2}) |v_h|_{2, h}.
\]
Putting together the above five bounds into (2.4), we obtain the estimate of the finite element solution in the theorem.
3. A $P_3 + 2B_4$ Element in 2D

On each triangle, the finite element shape functions are formed by the following polynomials

$$P_3^+(K) := \text{span}\{P_3(x_1,x_2), x_1P_3,0(K), x_2P_3,0(K)\},$$

where $P_3,0(K)$ denotes the space of $P_3$ polynomials vanishing on $\partial K$. Note that $P_3,0(K) := \text{span}\{\lambda_1 \lambda_2 \lambda_3\}$ where $\lambda_i$ is the barycentric coordinate for vertex $x_i$, a linear polynomial vanishing on one edge. The dual basis of the finite element space, i.e., the degrees of freedom of the finite element, is

$$v(x_i) \quad \text{at three vertices of } K, \quad \text{(3.1a)}$$

$$\int_{e_j} vds, \quad \int_{e_j} \partial_n vds, \quad \int_{e_j} s \partial_n vds \quad \text{on three edges of } K. \quad \text{(3.1b)}$$

The global finite element space is defined by

$$V_{h,HZ} := \left\{ v \in L^2(\Omega) \mid v|_K \in P_3^+(K), \quad v \text{ is continuous at internal vertices of } T_h, \quad \int_e vds, \quad \int_e \partial_n vds \text{ and } \int_e s \partial_n vds \text{ are continuous across internal edges of } T_h, \quad v(x_i) = \int_e vds = \int_e \partial_n vds = \int_e s \partial_n vds = 0 \right\} \quad \text{at boundary vertices and edges of } T_h. \quad \text{(3.2)}$$

**Theorem 3.1.** The 12 degrees of freedom in (3.1a) uniquely determine a polynomial of $P_3^+(K)$, defined in (3.2).

**Proof.** Let $u_h \in P_3^+(K)$ such that the 12 degrees of freedom of $u_h$ in (3.1a) and (3.1b) having zero value. We need to show $u_h = 0$. On a general triangle $K = \triangle x_1 x_2 x_3$, we recall that $\lambda_i$ is the barycentric coordinate, that is a linear function having values

$$\lambda_i(x_j) = \delta_{ij}, \quad i,j = 1, 2, 3.$$

Then we can choose such a homogeneous basis for the space of the $P_3 + 2B_4$ element that

$$u_h = \sum_{i=1}^{12} u_i b_i(x), \quad \text{(3.3)}$$

where

$$b_1 = \lambda_1^3, \quad b_2 = \lambda_2^3, \quad b_3 = \lambda_3^3,$$

$$b_4 = \lambda_1^2 \lambda_2, \quad b_5 = \lambda_1 \lambda_2^2, \quad b_6 = \lambda_2^2 \lambda_3,$$

$$b_7 = \lambda_2 \lambda_3^2, \quad b_8 = \lambda_3^2 \lambda_1, \quad b_9 = \lambda_3 \lambda_1^2,$$

$$b_{10} = \lambda_1^2 \lambda_2 \lambda_3, \quad b_{11} = \lambda_1 \lambda_2^2 \lambda_3, \quad b_{12} = \lambda_1 \lambda_2 \lambda_3^2.$$

At $x_1$, all $b_i(x_1) = 0$ except $i = 1$. So the three vertex values would force $u_1 = u_2 = u_3 = 0$ in (3.3). Next, when restricted on an edge $e_1$, only two basis functions $b_6$ and $b_7$ are nonzero.
By computing the zeroth moment on $e_1$, we get
\[
\int_{e_1} b_7 ds = \int_{e_1} b_6 ds = \frac{1}{12} |e_1|.
\]
A combination of two basis leads to $\int_{e_1} (b_6 - b_7) ds = 0$. So we change a basis for the expansion of $u_h$ in (3.3) to
\[
u_h = \tilde{c}_4 \hat{b}_4 + \tilde{c}_6 \hat{b}_6 + \tilde{c}_8 \hat{b}_8 + \tilde{c}_{10} \hat{b}_{10} + \tilde{c}_{11} \hat{b}_{11} + \tilde{c}_{12} \hat{b}_{12},
\] where
\[
\hat{b}_4 = (\lambda_1^2 \lambda_2 - \lambda_1 \lambda_2^2), \quad \hat{b}_6 = (\lambda_2^2 \lambda_3 - \lambda_2 \lambda_3^2), \quad \hat{b}_8 = (\lambda_3^2 \lambda_1 - \lambda_3 \lambda_1^2),
\] \[
\hat{b}_{10} = \lambda_1^2 \lambda_2 \lambda_3, \quad \hat{b}_{11} = \lambda_1 \lambda_2^2 \lambda_3, \quad \hat{b}_{12} = \lambda_1 \lambda_2 \lambda_3^2.
\]
Next, consider the 0th and 1st moments of $\partial_1 u_h = \nabla u_h \cdot n_1$ on an edge $e_1$. For the linear functions $\lambda_1$ and $\lambda_3$, cf. Fig. 3.1,
\[
\partial_1 \lambda_1 = \nabla \lambda_1 \cdot n_1 = \frac{1}{r_1},
\]
\[
\partial_1 \lambda_3 = -\frac{r_{31}}{r_1 |e_1|}, \quad \partial_1 \lambda_2 = -\frac{|e_1| - r_{31}}{r_1 |e_1|}.
\]
Consequently, we have
\[
\int_{e_1} \partial_1 \hat{b}_6 s ds = \int_{e_1} \left[ -\frac{|e_1| - r_{31}}{r_1 |e_1|} (2\lambda_2 \lambda_3 - \lambda_2^2) - \frac{r_{31}}{r_1 |e_1|} (\lambda_2^2 - 2\lambda_2 \lambda_3) \right] s ds
\]
\[
= \frac{|e_1|^2}{r_1} \int_0^1 \left[ (1 - \tilde{r}_{31})(2(1 - t)t - t^2) - \tilde{r}_{31}((1 - t)^2 - 2(1 - t)t) \right] t dt
\]
\[
= \frac{|e_1|^2}{r_1} \frac{1}{12},
\]
where $\tilde{r}_{31} := r_{31}/|e_1| := |x_2 o_1|/|e_1|$. Note that the answer is independent of $r_{31}$. Similarly, we get
\[
\int_{e_1} \partial_1 \begin{pmatrix} \hat{b}_6 \\ \hat{b}_8 \\ \hat{b}_4 \\ \hat{b}_{11} \\ \hat{b}_{12} \end{pmatrix} (s \ s) ds = \frac{|e_1|^2}{60 r_1} \begin{pmatrix} 5 & -5 \\ 15 & 5 \\ -5 & -15 \\ 2 & 3 \\ 3 & 2 \end{pmatrix},
\] (3.5)
where \( \hat{s} = |e_1| - s \). The two columns of the matrix make two equations for the unknown \( \tilde{c}_i \). Rotating the above calculation, on \( e_2 \) and \( e_3 \), together we would get 6 equations that

\[
\begin{pmatrix}
-5 & 5 & 15 & | & 2 & 3 \\
-15 & -5 & 5 & | & 3 & 2 \\
15 & -5 & 5 & | & 3 & 2 \\
5 & -15 & -5 & | & 2 & 3 \\
5 & 15 & -5 & | & 2 & 3 \\
5 & 5 & -15 & | & 3 & 2
\end{pmatrix}
\begin{pmatrix}
\tilde{c}_4 \\
\tilde{c}_6 \\
\tilde{c}_8 \\
\tilde{c}_{10} \\
\tilde{c}_{11} \\
\tilde{c}_{12}
\end{pmatrix} = 0.
\]

The determinant of the coefficient matrix is \(-135000\). Thus we have a unique solution \( \tilde{c}_i = 0 \) and \( u_h = 0 \).

Theorem 3.2. With \( V^2_{h,\ell} = V_{h,HZ12} \) in (2.1), the finite element equations have a unique solution. Moreover,

\[
||D^2_h(u - u_h)||_0 \leq C h^2 |u|_4,
\]

provided that \( u \in H^4(\Omega) \cap H^2_0(\Omega) \).

Proof. For the finite square system of equations, the uniqueness implies the existence of a solution. We show a unique solution of (2.1) when \( f = 0 \). Letting \( v_h = u_h \) in (2.1), we get

\[
\sum_{K \in \mathcal{T}_h} |D^2 u_h|^2_{0,K} = 0.
\]

As all second derivatives of \( u_h \) are zero, \( u_h \) is a linear function on each triangle \( K \). Since \( u_h \) is continuous at the vertices of \( \mathcal{T}_h \), \( u_h \) is a function in the space \( H^1_0(\Omega) \). As \( \nabla u_h \) is continuous at the midpoints of the internal edges and vanishes at the midpoints of the boundary edges, \( \nabla u_h = 0 \) and consequently \( u_h = 0 \). The error estimate follows from the abstract theory in Theorem 2.1. \( \square \)

4. The \( P_4 \) Finite Element in 2D

On a triangular grid, the dual basis of the \( P_4 \) finite element space, i.e., the degrees of freedom of the finite element, is

\[
v(x_i) \quad \text{at three vertices of } K, \\
\int_{e_j} vds, \int_{e_j} \partial_n vds, \int_{s_j} s\partial_n vds \quad \text{on three edges of } K, \\
\int_K vdx, \int_K x_1 vdx, \int_K x_2 vdx \quad \text{on the triangle } K.
\]

The global finite element space is defined by

\[
V_{h,HZ15} := \left\{ v \in L^2(\Omega) \mid v \bigg|_K \in P_4(K), \quad v \text{ is continuous at internal vertices of } \mathcal{T}_h, \\
\int_e vds, \int_e \partial_n vds \quad \text{and} \quad \int_e s\partial_n vds \quad \text{are continuous across internal edges of } \mathcal{T}_h, \\
v(x_i) = \int_e vds = \int_e \partial_n vds = \int_e s\partial_n vds = 0 \quad \text{at boundary vertices and edges of } \mathcal{T}_h \right\}.
\]
Theorem 4.1. The 15 degrees of freedom in (4.1) uniquely determine a \( P_3 \) polynomial in the space \( V_{h,Hz15} \) of (4.2).

Proof. Let \( u_h \in P_4 \) such that the 15 degrees of freedom of \( u_h \) in (4.1) having zero value. We need to show \( u_h = 0 \). On a general triangle \( K = \Delta x_1 x_2 x_3 \), we can choose a homogeneous basis for the space of \( P_4 \) such that

\[
u_h = \sum_{i=1}^{15} u_i b_i(x), \tag{4.3}\]

where

\[
b_1 = \lambda_1^4, \quad b_2 = \lambda_2^4, \quad b_3 = \lambda_3^4,
b_4 = \lambda_1^3 \lambda_2, \quad b_5 = \lambda_1^2 \lambda_2^2, \quad b_6 = \lambda_1 \lambda_2^3,
b_7 = \lambda_2^3 \lambda_3, \quad b_8 = \lambda_2^2 \lambda_3^2, \quad b_9 = \lambda_2 \lambda_3^3,
b_{10} = \lambda_3^3 \lambda_1, \quad b_{11} = \lambda_3^2 \lambda_1^2, \quad b_{12} = \lambda_3 \lambda_1^3,
b_{13} = \lambda_1^2 \lambda_2 \lambda_3, \quad b_{14} = \lambda_1 \lambda_2^2 \lambda_3, \quad b_{15} = \lambda_1 \lambda_2 \lambda_3^2.
\]

At \( x_1 \), all \( b_i(x_1) = 0 \) except \( i = 1 \). So the three vertex values would force \( u_1 = u_2 = u_3 = 0 \) in (4.3). Next, when restricted on an edge \( e_1 \), only three basis functions \( b_7, b_8, \) and \( b_9 \) nonzero. By computing the zeroth moment on \( e_1 \), we get

\[
\int_{e_1} b_7 ds = \int_{e_1} b_8 ds = \frac{|e_1|}{20}, \quad \int_{e_1} b_9 ds = \frac{|e_1|}{30}.
\]

A combination of two basis yields \( \int_{e_1} (2b_7 - 3b_9) ds = 0 \). The constraint of zero 0th moment on \( e_1 \) would force the coefficient of the only non-zero term \( b_8 \) to be zero. So we change a basis for the expansion of \( u_h \) in (4.3) to

\[
u_h = \tilde{c}_4 \tilde{b}_4 + \tilde{c}_9 \tilde{b}_9 + \tilde{c}_7 \tilde{b}_7 + \tilde{c}_6 \tilde{b}_6 + \tilde{c}_{10} \tilde{b}_{10} + \tilde{c}_{12} \tilde{b}_{12} + \tilde{c}_{13} \tilde{b}_{13} + \tilde{c}_{14} \tilde{b}_{14} + \tilde{c}_{15} \tilde{b}_{15}, \tag{4.4}\]

where

\[
\tilde{b}_4 = (2\lambda_1^4 \lambda_2 - 3\lambda_1^2 \lambda_2^2), \quad \tilde{b}_6 = (2\lambda_1 \lambda_2^3 - 3\lambda_1^2 \lambda_2^2),
\tilde{b}_7 = (2\lambda_1^3 \lambda_2 - 3\lambda_1^2 \lambda_2^2), \quad \tilde{b}_9 = (2\lambda_1 \lambda_2^3 - 3\lambda_1^2 \lambda_2^2),
\tilde{b}_{10} = (2\lambda_1^3 \lambda_3 - 3\lambda_1^2 \lambda_2 \lambda_3), \quad \tilde{b}_{12} = (2\lambda_1 \lambda_2^3 - 3\lambda_1^2 \lambda_2^2),
\tilde{b}_{13} = \lambda_1^2 \lambda_2 \lambda_3, \quad \tilde{b}_{14} = \lambda_1 \lambda_2^2 \lambda_3, \quad \tilde{b}_{15} = \lambda_1 \lambda_2 \lambda_3^2.
\]

Next, consider the 0th and 1st moments of \( \partial_1 u_h = \nabla u_h \cdot n_1 \) on an edge \( e_1 \). For the linear functions \( \lambda_1 \) and \( \lambda_3 \), cf. Figure 3.1,

\[
\partial_1 \lambda_1 = \nabla \lambda_1 \cdot n_1 = \frac{1}{r_1}, \quad \partial_1 \lambda_3 = \frac{r_{31}}{r_1 |e_1|}, \quad \partial_1 \lambda_2 = \frac{|e_1| - r_{31}}{r_1 |e_1|}.
\]
Consequently, we have
\[
\int_{e_1} \partial_1 \hat{b}_7 s ds = \int_{e_1} \left[ -\frac{|e_1|}{r_1} - r_{31} \left( 6(1 - t)^2 t - 6(1 - t)^2 \right) \right] ds
\]
\[
= \frac{|e_1|^2}{r_1} \int_0^1 \left[ -(1 - \tilde{r}_{31})(6(1 - t)^2 t - 6(1 - t)^2) \right] t dt
\]
\[
= \frac{|e_1|^2}{r_1} \frac{1}{10},
\]
where \( \tilde{r}_{31} = r_{31}/|e_1| \). Note that the answer is independent of \( r_{31} \). Similarly, we get
\[
\int_{e_1} \partial_1 \left( \begin{array}{c} \hat{b}_7 \\ \hat{b}_9 \\ \hat{b}_{10} \\ \hat{b}_6 \\ \hat{b}_{15} \\ \hat{b}_{14} \end{array} \right) (s \ s) ds = \frac{|e_1|^2}{r_1} \left( \begin{array}{ccccccc} 1/10 & -1/10 \\ 1/10 & 1/10 \\ 1/30 & 1/10 \\ 1/10 & 1/30 \\ 1/10 & 1/30 \\ 1/30 & 1/30 \end{array} \right),
\]
where \( \hat{s} = |e_1| - s \). The two columns of the matrix make two equations for the unknown \( \tilde{c}_i \).

Rotating the above calculation, on \( e_2 \) and \( e_3 \), we would get 6 equations that
\[
\left( \begin{array}{cccccc} 6 & 6 & -6 & 24 & 2 & 3 \\ 24 & -6 & 6 & 6 & 3 & 2 \\ 6 & 24 & -6 & 6 & 2 & 3 \\ 6 & -6 & 24 & 6 & 2 & 3 \\ -6 & 6 & 6 & 24 & 3 & 2 \end{array} \right) \left( \begin{array}{c} \tilde{c}_4 \\ \tilde{c}_6 \\ \tilde{c}_7 \\ \tilde{c}_9 \\ \tilde{c}_{10} \\ \tilde{c}_{12} \\ \tilde{c}_{13} \\ \tilde{c}_{14} \\ \tilde{c}_{15} \end{array} \right) = 0.
\]

The reduced row-echelon form is
\[
\left( \begin{array}{cccc} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right) \left( \begin{array}{c} \tilde{c}_4 \\ \tilde{c}_6 \\ \tilde{c}_7 \\ \tilde{c}_9 \\ \tilde{c}_{10} \\ \tilde{c}_{12} \\ \tilde{c}_{13} \\ \tilde{c}_{14} \\ \tilde{c}_{15} \end{array} \right) = 0.
\]

Thus, if we choose basis functions \( \hat{b}_i = \hat{b}_i, i \leq 12, \) and
\[
\hat{b}_{13} = \tilde{b}_{13} - \frac{1}{6} \hat{b}_4 - \frac{1}{6} \hat{b}_{12}, \quad \hat{b}_{14} = \tilde{b}_{14} - \frac{1}{6} \hat{b}_6 - \frac{1}{6} \hat{b}_7,
\]
\[
\hat{b}_{15} = \tilde{b}_{15} - \frac{1}{6} \hat{b}_8 - \frac{1}{6} \hat{b}_{10},
\]
then the expansion of $u_h$ is forced to a simple form, by above 12 conditions, that

$$u_h = \hat{c}_{13} \hat{b}_{13} + \hat{c}_{14} \hat{b}_{14} + \hat{c}_{15} \hat{b}_{15}.$$ 

Finally, evaluating $u_h$ at the three internal points, $(2x_i + x_j + x_k)/4$, we get

$$\begin{bmatrix} 16 & 21 & 21 \\ 21 & 16 & 21 \\ 21 & 21 & 16 \end{bmatrix} \begin{bmatrix} \hat{c}_{13} \\ \hat{c}_{14} \\ \hat{c}_{15} \end{bmatrix} = 0.$$ 

Adding 5 to the diagonal entries of the above matrix, the eigenvalues are $0, 0, 3 \times 21$. So the eigenvalues of the above matrix are $-5, -5, 58$, non zero. Thus, $\hat{c}_i = 0$ and $u_h = 0$. 

\[ \square \]

**Theorem 4.2.** With $V_{h,k}^2 = V_{h,HZ15}$ in (2.1), the finite element equations have a unique solution. Moreover,

$$\|D_h^2(u - u_h)\|_0 \leq Ch^2|u|_4,$$

provided that $u \in H^4(\Omega) \cap H_0^2(\Omega)$.

**Proof.** The proof is the same as that for Theorem 3.2. \[ \square \]

### 5. The Basis Functions of the Two New Elements in 2D

In this section, we give explicitly the basis functions on each element $K$.

#### 5.1. The basis functions for the $P_3 + 2B_4$ element

The first six nodal basis functions are associated to the 0th and 1st moments of normal derivative on three edges of element $K$.

$$\varphi_{1,1}(x) = r_1 \left(4(\lambda_2^2 \lambda_3 - \lambda_2 \lambda_3^2) - 2(\lambda_2^2 \lambda_1 - \lambda_3 \lambda_1^2) - 2\lambda_2^2 \lambda_3 - 2\lambda_1 \lambda_2^2 \lambda_3 + 18\lambda_1 \lambda_2 \lambda_3^2\right),$$

$$\varphi_{1,2}(x) = r_1 \left(2(\lambda_2^2 \lambda_2 - \lambda_1 \lambda_2^2) - 4(\lambda_2^2 \lambda_1 - \lambda_2 \lambda_1^2) - 2\lambda_2^2 \lambda_2 + 18\lambda_1 \lambda_2 \lambda_1 - 22\lambda_1 \lambda_2 \lambda_2^2\right),$$

$$\varphi_{2,1}(x) = r_2 \left(2(\lambda_2^2 \lambda_3 - \lambda_2 \lambda_3^2) - 4(\lambda_2^2 \lambda_1 - \lambda_3 \lambda_1^2) - 2\lambda_2^2 \lambda_3 - 2\lambda_1 \lambda_2^2 \lambda_3 + 18\lambda_1 \lambda_2 \lambda_3^2\right),$$

$$\varphi_{2,2}(x) = r_2 \left(-2(\lambda_2^2 \lambda_2 - \lambda_1 \lambda_2^2) + 4(\lambda_2^2 \lambda_1 - \lambda_2 \lambda_1^2) + 18\lambda_1^2 \lambda_2 \lambda_3 - 2\lambda_1 \lambda_2 \lambda_3^2 - 22\lambda_1 \lambda_2 \lambda_2^2\right),$$

$$\varphi_{3,1}(x) = r_3 \left(4(\lambda_2^2 \lambda_2 - \lambda_2 \lambda_2^2) - 2(\lambda_2^2 \lambda_1 - \lambda_2 \lambda_1^2) - 2\lambda_2^2 \lambda_2 + 18\lambda_1 \lambda_2 \lambda_2^2 - 2\lambda_1 \lambda_2 \lambda_3^2\right),$$

$$\varphi_{3,2}(x) = r_3 \left(-4(\lambda_2^2 \lambda_2 - \lambda_2 \lambda_2^2) + 2(\lambda_2^2 \lambda_1 - \lambda_2 \lambda_1^2) + 18\lambda_1^2 \lambda_2 \lambda_3 - 2\lambda_1 \lambda_2 \lambda_2^2 - 22\lambda_1 \lambda_2 \lambda_3^2 - 2\lambda_1 \lambda_2 \lambda_2^2\right).$$

These six nodal basis functions satisfy the usual orthogonal conditions in the sense that

$$\int_{e_k} \frac{\partial \varphi_{li}(x)}{\partial n_k} \lambda_{ij}^{(l)} ds = \delta_{l,l} \delta_{i,j}, \quad l, k = 1, 2, 3, \quad i, j = 1, 2.$$ 

Here and in the following, $\lambda_{ij}^{(l)} = \lambda_j$, $j = 1, 2$, are the edge barycenter coordinates on $e_k$, for instance, $\lambda_{i1}^{(1)} = \lambda_2$, $\lambda_{i2}^{(1)} = \lambda_3$, on $e_1$. Note that the six nodal basis functions vanish for the remaining degrees of freedom in (3.1a), namely,

$$\varphi_{l,i}(x_k) = 0, \quad \int_{e_k} \varphi_{l,j} ds = 0, \quad l, k = 1, 2, 3, \quad i = 1, 2.$$
Using the edge barycenter coordinates, these six basis functions can be expressed in a unifying way:

\[
\varphi_{l,i}(x) = r_l \left( 4 \left( (\lambda_{e_l}^{(i)})^2 \lambda_{e_l}^{(i+1)} - (\lambda_{e_l}^{(i+1)})^2 \lambda_{e_l}^{(i)} \right) - \lambda_{e_l} \lambda_{e_l} \lambda_{e_l} \left( 22 \lambda_{e_l}^{(i)} - 18 \lambda_{e_l}^{(i+1)} + 2 \lambda_{e_l} \right) \right), \quad l = 1, 2, 3, \quad i = 1, 2.
\]

The second three nodal basis functions are associated to the 0th moment of function on three edges of element \( K \).

\[
\begin{align*}
\phi_1(x) &= 12 \lambda_{e_1} \lambda_{e_1}^2 + (n_1)^T \left( \frac{n_2}{r_2} + \frac{2 n_3}{r_3} \right) \varphi_{1,1}(x) + (3 \frac{n_2}{r_2} + 2 \frac{n_3}{r_3}) \varphi_{1,2}(x) + \frac{\varphi_{2,1}(x) + 3 \varphi_{2,2}(x)}{r_2}, \\
\phi_2(x) &= 12 \lambda_{e_2} \lambda_{e_2}^2 + (n_2)^T \left( \frac{2 n_1}{r_1} + 3 \frac{n_3}{r_3} \right) \varphi_{2,1}(x) + (2 \frac{n_1}{r_1} + \frac{n_3}{r_3}) \varphi_{2,2}(x) + \frac{3 \varphi_{3,1}(x) + \varphi_{3,2}(x)}{r_3}, \\
\phi_3(x) &= 12 \lambda_{e_3} \lambda_{e_3}^2 + (n_3)^T \left( \frac{n_1}{r_1} + 2 \frac{n_2}{r_2} \right) \varphi_{3,1}(x) + (3 \frac{n_1}{r_1} + 2 \frac{n_2}{r_2}) \varphi_{3,2}(x) + \frac{3 \varphi_{3,1}(x) + \varphi_{1,2}(x)}{r_1}.
\end{align*}
\]

These three nodal basis functions satisfy the usual orthogonal conditions in the sense that

\[
\frac{1}{|e_k|} \int_{e_k} \phi_l(x) dx = \delta_{l,k}, \quad l, k = 1, 2, 3,
\]

and vanish for the remaining degrees of freedom in (3.1a), namely,

\[
\frac{1}{|e_k|} \int_{e_k} \frac{\partial \phi_l(x)}{\partial n_k} \lambda_{e_k}^{(j)} dx = 0, \quad \phi_l(x_k) = 0, \quad l, k = 1, 2, 3, \quad j = 1, 2.
\]

The last three nodal basis functions are associated to the value of function at three vertices of element \( K \).

\[
\begin{align*}
\theta_1(x) &= \lambda_{e_1}^3 + (n_1)^T n_2 \left( \frac{3}{4} \varphi_{2,1}(x) + \frac{1}{4} \varphi_{2,2}(x) \right)/r_1 \\
&\quad + (n_1)^T n_3 \left( \frac{3}{4} \varphi_{3,1}(x) + \frac{1}{4} \varphi_{3,2}(x) \right)/r_1 - \frac{1}{4} (\varphi_2(x) + \varphi_3(x)), \\
\theta_2(x) &= \lambda_{e_2}^3 + (n_2)^T n_1 \left( \frac{3}{4} \varphi_{1,1}(x) + \frac{1}{4} \varphi_{1,2}(x) \right)/r_2 \\
&\quad + (n_2)^T n_3 \left( \frac{1}{4} \varphi_{3,1}(x) + \frac{3}{4} \varphi_{3,2}(x) \right)/r_2 - \frac{1}{4} (\varphi_1(x) + \varphi_3(x)), \\
\theta_3(x) &= \lambda_{e_3}^3 + (n_3)^T n_1 \left( \frac{1}{4} \varphi_{1,1}(x) + \frac{3}{4} \varphi_{1,2}(x) \right)/r_3 \\
&\quad + (n_3)^T n_2 \left( \frac{1}{4} \varphi_{2,1}(x) + \frac{3}{4} \varphi_{2,2}(x) \right)/r_3 - \frac{1}{4} (\varphi_1(x) + \varphi_2(x)).
\end{align*}
\]

These three nodal basis functions satisfy the usual orthogonal conditions in the sense that

\[
\theta_l(x_k) = \delta_{l,k}, \quad l, k = 1, 2, 3,
\]

and vanish for the remaining degrees of freedom in (3.1a), namely,

\[
\frac{1}{|e_k|} \int_{e_k} \frac{\partial \theta_l(x)}{\partial n_k} \lambda_{e_k}^{(j)} dx = 0, \quad \frac{1}{|e_k|} \int_{e_k} \theta_l(x) dx = 0, \quad l, k = 1, 2, 3, \quad j = 1, 2.
\]
5.2. The basis functions for the $P_1$ element

The first six nodal basis functions are associated to the normal derivatives.

$$\varphi_{1,1}(x) = r_1 \left( -\frac{3}{2} (2\lambda_1^3 \lambda_2 - 3\lambda_2^3 \lambda_2^2) - (2\lambda_1 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) + (2\lambda_2^3 \lambda_2 - 3\lambda_2^3 \lambda_2^2) \right)$$

$$-\frac{1}{2} (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) + \frac{1}{2} (2\lambda_2^3 \lambda_3 - 3\lambda_3^3 \lambda_3^2) - \frac{1}{2} (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) + 9\lambda_2 \lambda_3^2 \lambda_3 - 3\lambda_1 \lambda_2 \lambda_3^2 \right),$$

$$\varphi_{1,2}(x) = r_1 \left( -\frac{1}{2} (2\lambda_1^3 \lambda_2 - 3\lambda_2^3 \lambda_2^2) + \frac{1}{2} (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) - \frac{1}{2} (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) + 9\lambda_1 \lambda_2 \lambda_3 - 3\lambda_1 \lambda_2 \lambda_3^2 \right).$$

$$\varphi_{2,1}(x) = r_2 \left( -\frac{1}{2} (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) - \frac{1}{2} (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) - \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) - \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) + 9\lambda_2 \lambda_3 \lambda_1 - 3\lambda_2 \lambda_3 \lambda_1^2 \right),$$

$$\varphi_{2,2}(x) = r_2 \left( \frac{1}{2} (2\lambda_1 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) - \frac{1}{2} (2\lambda_1 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) - \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) - \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) + 9\lambda_2 \lambda_3 \lambda_1 - 3\lambda_2 \lambda_3 \lambda_1^2 \right).$$

$$\varphi_{3,1}(x) = r_3 \left( (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) - \frac{1}{2} (2\lambda_1 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) + \frac{1}{2} (2\lambda_1 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) + \frac{1}{2} (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) - \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) - \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) + 9\lambda_2 \lambda_3 \lambda_1 - 3\lambda_2 \lambda_3 \lambda_1^2 \right),$$

$$\varphi_{3,2}(x) = r_3 \left( -\frac{1}{2} (2\lambda_1 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) + (2\lambda_1 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) - (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) - (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) + \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) + \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) + 9\lambda_2 \lambda_3 \lambda_1 - 3\lambda_2 \lambda_3 \lambda_1^2 \right).$$

Using the edge barycenter coordinates, the above six basis functions can be expressed in a unifying way:

$$\varphi_{l,i}(x) = r_l \left( -\frac{3}{2} (2\lambda_2^3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) - (2\lambda_1 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) + (2\lambda_2 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) \right)$$

$$-\frac{1}{2} (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) + \frac{1}{2} (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) - \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) + 9\lambda_2 \lambda_3 \lambda_1 - 3\lambda_2 \lambda_3 \lambda_1^2 \right),$$

$$\varphi_{l,i}(x) = r_l \left( -\frac{1}{2} (2\lambda_1 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) + (2\lambda_1 \lambda_2^3 - 3\lambda_2^3 \lambda_2^2) - (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) - (2\lambda_2 \lambda_3^3 - 3\lambda_3^3 \lambda_3^2) + \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) + \frac{1}{2} (2\lambda_3 \lambda_1^3 - 3\lambda_1^3 \lambda_1^2) + 9\lambda_2 \lambda_3 \lambda_1 - 3\lambda_2 \lambda_3 \lambda_1^2 \right),$$

$$l = 1, 2, 3, \quad i = 1, 2.$$
The second three nodal basis functions are associated to the 0th and 1st moments of function on element $K$.

$$\psi_1(x) = \left(120(2\lambda_1^3\lambda_2 - 3\lambda_1^2\lambda_2^2) - 90(2\lambda_1^3\lambda_3 - 3\lambda_1^2\lambda_3^2) - 90(2\lambda_2^3\lambda_1 - 3\lambda_2^2\lambda_1^2) - 90(2\lambda_3^3\lambda_1 - 3\lambda_3^2\lambda_1^2) - 720\lambda_1^3\lambda_2\lambda_3 + 540\lambda_1\lambda_2^2\lambda_3 + 540\lambda_1\lambda_2\lambda_3^2\right),$$

$$\psi_2(x) = \left(-90(2\lambda_1^3\lambda_2 - 3\lambda_1^2\lambda_2^2) - 120(2\lambda_1^3\lambda_3 - 3\lambda_1^2\lambda_3^2) - 120(2\lambda_2^3\lambda_1 - 3\lambda_2^2\lambda_1^2) - 90(2\lambda_2^3\lambda_3 - 3\lambda_2^2\lambda_3^2) + 540\lambda_1^3\lambda_2\lambda_3 - 720\lambda_1\lambda_2^2\lambda_3 + 540\lambda_1\lambda_2\lambda_3^2\right),$$

$$\psi_3(x) = \left(-90(2\lambda_1^3\lambda_2 - 3\lambda_1^2\lambda_2^2) - 90(2\lambda_1^3\lambda_3 - 3\lambda_1^2\lambda_3^2) - 90(2\lambda_2^3\lambda_1 - 3\lambda_2^2\lambda_1^2) + 120(2\lambda_2^3\lambda_3 - 3\lambda_2^2\lambda_3^2) + 120(2\lambda_3^3\lambda_1 - 3\lambda_3^2\lambda_1^2) + 90(2\lambda_3^3\lambda_3 - 3\lambda_3^2\lambda_3^2) + 90(2\lambda_2^3\lambda_3 - 3\lambda_2^2\lambda_3^2) - 720\lambda_1\lambda_2\lambda_3^2 + 540\lambda_1\lambda_2^2\lambda_3 - 540\lambda_1\lambda_2\lambda_3^2\right),$$

The above three basis functions can be expressed in a unifying way:

$$\psi_1(x) = \left\{ \begin{array}{l}
120(2\lambda_1^3\lambda_2 - 3\lambda_1^2\lambda_2^2) + 120(2\lambda_1^3\lambda_3 - 3\lambda_1^2\lambda_3^2) - 90(2\lambda_2^3\lambda_1 - 3\lambda_2^2\lambda_1^2) - 90(2\lambda_3^3\lambda_1 - 3\lambda_3^2\lambda_1^2) - 720\lambda_1^3\lambda_2\lambda_3 + 540\lambda_1\lambda_2^2\lambda_3 + 540\lambda_1\lambda_2\lambda_3^2,
\end{array} \right.$$

where $(i, j, k)$ is a permutation of $(1, 2, 3)$. The third three nodal basis functions are associated to the value of function at three vertices.

$$\phi_1(x) = 30\lambda_2^2\lambda_3^2 + (n_1)^T \left(\frac{2n_2}{r_2} + 3\frac{n_3}{r_3}\right)\varphi_{1,1}(x) + (3\frac{n_2}{r_2} + 2\frac{n_3}{r_3})\varphi_{1,2}(x)$$

$$- \frac{1}{42}\psi_1(x) - \frac{1}{14}\psi_2(x) - \frac{1}{14}\psi_3(x),$$

$$\phi_2(x) = 30\lambda_2^2\lambda_3^2 + (n_2)^T \left(2\frac{n_1}{r_1} + 3\frac{n_3}{r_3}\right)\varphi_{2,1}(x) + (3\frac{n_1}{r_1} + 2\frac{n_3}{r_3})\varphi_{2,2}(x)$$

$$- \frac{1}{42}\psi_1(x) - \frac{1}{14}\psi_2(x) - \frac{1}{14}\psi_3(x),$$

$$\phi_3(x) = 30\lambda_2^2\lambda_3^2 + (n_3)^T \left(2\frac{n_1}{r_1} + 3\frac{n_2}{r_2}\right)\varphi_{3,1}(x) + (3\frac{n_1}{r_1} + 2\frac{n_2}{r_2})\varphi_{3,2}(x)$$

$$- \frac{1}{42}\psi_1(x) - \frac{1}{14}\psi_2(x) - \frac{1}{14}\psi_3(x).$$

The last three nodal basis functions are associated to the value of function at three vertices.
of element $K$.

\[
\begin{align*}
\theta_1(x) &= \lambda_1^4 + \frac{n_1^T n_2}{5r_1}(\varphi_{1,1}(x) + \varphi_{2,1}(x)) + \frac{n_1^T n_3}{5r_1}(\varphi_{1,1}(x) + \varphi_{3,1}(x)) \\
&\quad - \frac{1}{5}(\phi_2(x) + \phi_3(x)) - \frac{1}{42}\psi_1(x) - \frac{1}{210}\psi_2(x) - \frac{1}{210}\psi_3(x), \\
\theta_2(x) &= \lambda_2^4 + \frac{n_2^T n_1}{5r_2}(\varphi_{1,1}(x) + \varphi_{1,2}(x)) + \frac{n_2^T n_3}{5r_2}(\varphi_{1,1}(x) + 4\varphi_{3,2}(x)) \\
&\quad - \frac{1}{5}(\phi_1(x) + \phi_2(x)) - \frac{1}{210}\psi_1(x) - \frac{1}{42}\psi_2(x) - \frac{1}{210}\psi_3(x), \\
\theta_3(x) &= \lambda_3^4 + \frac{n_3^T n_1}{5r_3}(\varphi_{1,1}(x) + 4\varphi_{1,2}(x)) + \frac{n_3^T n_3}{5r_3}(\varphi_{2,1}(x) + 4\varphi_{2,2}(x)) \\
&\quad - \frac{1}{5}(\phi_1(x) + \phi_2(x)) - \frac{1}{210}\psi_1(x) - \frac{1}{42}\psi_2(x) - \frac{1}{210}\psi_3(x).
\end{align*}
\]

For these thirty nodal basis functions, the twelve basis functions $\varphi_{1,1}(x)$, $\phi_1(x)$ and $\theta_1(x)$ are similarly defined as those for the enriched $P_3$ element in the previous subsection. In addition, they vanish for the following three conditions

\[
\int_K (\cdot)\lambda_l dx = 0, \quad l = 1, 2, 3. \tag{5.1}
\]

Compared to the enriched $P_3$ element, there are three more basis functions $\psi_l(x)$ which are associated to the three moments on element $K$ in (5.1) and satisfy the following orthogonal conditions

\[
\frac{1}{|K|} \int_K \psi_l(x)\lambda_k dx = \delta_{l,k}, \quad l, k = 1, 2, 3.
\]

Moreover $\psi_l(x)$ vanish for the remaining degrees of freedom in (4.1).

6. A $P_3 + 8B_4$ Element in 3D

On a tetrahedral grid $\mathcal{T}_h = \{K\}$ of $\Omega$, the finite element shape functions space can be chosen as

\[
P_3^+(K) := P_3(K) + \text{span}\{\lambda_1^2\lambda_2\lambda_4, \lambda_2^2\lambda_3\lambda_4, \lambda_3^2\lambda_2\lambda_4, \lambda_1\lambda_2\lambda_3\lambda_4, \lambda_1\lambda_2\lambda_2^2\lambda_4, \lambda_1\lambda_2\lambda_4^2\lambda_3, \lambda_1\lambda_3\lambda_2^2\lambda_4, \lambda_1\lambda_3\lambda_4^2\lambda_2\}
\]

where $P_3(K)$ denotes the space of $P_3$ polynomials, $\lambda_i$ is the barycentric coordinate for vertex $x_i$, a linear polynomial vanishing on the face triangle opposite to the vertex.

6.1. The degrees of freedom

The dual basis of the finite element space, i.e., the degrees of freedom of the finite element, is

\[
\begin{align*}
E_{j,k}(\cdot) := \frac{1}{|e_j|} \int_{e_j} (\cdot)\lambda_{e_j}^{(k)} ds & \quad \text{on edge } e_j, j = 1, \cdots, 6, k = 1, 2, \\
F_{j}(\cdot) := \frac{1}{|F_j|} \int_{F_j} (\cdot) df & \quad \text{on face } F_j, j = 1, \cdots, 4, \\
N_{j,k}(\cdot) := \frac{1}{|F_j|} \int_{F_j} \partial_n (\cdot)\lambda_{F_j}^{(k)} df & \quad \text{on face } F_j, j = 1, \cdots, 4, k = 1, 2, 3.
\end{align*}
\]
Here \( \lambda^{(k)}_{v_j} \) are two 1D barycentric coordinates on the edge \( v_j \), \( \lambda^{(k)}_{F_j} \) are three 2D barycenter coordinates on the face triangle \( F_j \), in the sense that \( \lambda^{(1)}_{F_j} = \lambda_2, \lambda^{(2)}_{F_j} = \lambda_3 \) and \( \lambda^{(3)}_{F_j} = \lambda_4 \). In addition, let \( r_j \) be the distance from \( x_j \) to face \( F_j \). The global finite element space is defined by

\[
V_{h,IZ28} := \left\{ v \in L^2(\Omega) \mid v|_K \in P_3^+(K), \int_F vdf \text{ is continuous around internal edges of } T_h, \right. \\
\left. \int_F vdf \text{ and } \int_F \partial_n vdf \text{ are continuous across internal face triangles of } T_h \right. \\
\int_F vdf = \int_F \partial_n vdf = 0 \\
\text{at boundary edges and face triangles of } T_h \right\},
\]

where \( p_1 \) is a linear function on an edge or a face triangle.

**Theorem 6.1.** The 28 degrees of freedom in (6.1) uniquely determine a polynomial of \( P_3^+(K) \).

**Proof.** Let \( u \in P_3^+(K) \) such that the 28 degrees of freedom of \( u \) in (6.1) have zero value. Since we use only 2 face bubbles instead of 3 face bubbles, the nodal basis on a general element depends on the numbering of vertex. First we list twelve functions which have zero edge moments and face value moments. In fact, these functions vanish on all edges and faces, except one face where only the average vanishes.

\[
b_1 = \lambda_1 \lambda_2 \lambda_4 (3\lambda_1 - 1), \quad b_2 = \lambda_1 \lambda_3 \lambda_4 (3\lambda_1 - 1), \\
b_3 = \lambda_2 \lambda_1 \lambda_3 (3\lambda_2 - 1), \quad b_4 = \lambda_2 \lambda_3 \lambda_4 (3\lambda_2 - 1), \\
b_5 = \lambda_3 \lambda_1 \lambda_2 (3\lambda_3 - 1), \quad b_6 = \lambda_3 \lambda_2 \lambda_4 (3\lambda_3 - 1), \\
b_7 = \lambda_1 \lambda_1 \lambda_2 (3\lambda_4 - 1), \quad b_8 = \lambda_4 \lambda_1 \lambda_3 (3\lambda_4 - 1).
\]

Before we find the rest 4 basis functions which have 16 vanishing edge and face moments, we find 16 dual-basis functions (they do not have vanishing normal derivative moments and they are not unique).

\[
b_{13} = 12\lambda_1 \lambda_2 (5\lambda_1 - 2), \quad b_{14} = 12\lambda_1 \lambda_2 (5\lambda_2 - 2), \quad b_{15} = 12\lambda_1 \lambda_3 (5\lambda_1 - 2), \\
b_{16} = 12\lambda_1 \lambda_3 (5\lambda_3 - 2), \quad b_{17} = 12\lambda_1 \lambda_4 (5\lambda_1 - 2), \quad b_{18} = 12\lambda_1 \lambda_4 (5\lambda_4 - 2), \\
b_{19} = 12\lambda_2 \lambda_3 (5\lambda_2 - 2), \quad b_{20} = 12\lambda_2 \lambda_3 (5\lambda_3 - 2), \quad b_{21} = 12\lambda_2 \lambda_4 (5\lambda_2 - 2), \\
b_{22} = 12\lambda_2 \lambda_4 (5\lambda_4 - 2), \quad b_{23} = 12\lambda_3 \lambda_4 (5\lambda_3 - 2), \quad b_{24} = 12\lambda_3 \lambda_4 (5\lambda_4 - 2), \\
b_{25} = 60\lambda_1 \lambda_2 \lambda_3, \quad b_{26} = 60\lambda_1 \lambda_2 \lambda_4, \quad b_{27} = 60\lambda_1 \lambda_3 \lambda_4, \quad b_{28} = 60\lambda_2 \lambda_3 \lambda_4.
\]

For the first 16 dual basis functionals, we have

\[
E_{j,k}(b_{i+12}) = \delta_{i,j-k-1}, \quad E_{m}(b_{i+12}) = 0, \quad 1 \leq j \leq 6, \quad k = 1, 2, \quad 1 \leq m \leq 4, \quad 1 \leq i \leq 12, \\
F_{m}(b_{i+24}) = \delta_{i,m}, \quad E_{j,k}(b_{i+24}) = 0, \quad 1 \leq j \leq 6, \quad k = 1, 2, \quad 1 \leq m \leq 4, \quad 1 \leq i \leq 4.
\]
Now we correct the four $\lambda^3_i$ functions to get four new basis functions with vanishing edge and face moments.

$$b_9 = -20\lambda^3_1 + 24\lambda^2_1 - 6\lambda_1, \quad b_{10} = -20\lambda^3_2 + 24\lambda^2_2 - 6\lambda_2,$$

$$b_{11} = -20\lambda^3_3 + 24\lambda^2_3 - 6\lambda_3, \quad b_{12} = -20\lambda^3_4 + 24\lambda^2_4 - 6\lambda_4.$$

Under the basis, $u = \sum_{i=1}^{28} c_i b_i$. As $E_{j,k}(u) = F_i(u) = 0$, we get $c_{i+12} = 0, \quad 1 \leq i \leq 16$.

Hence $u = \sum_{i=1}^{12} c_i b_i$. Next we show that the remaining 12 coefficients are also zero. An application of the functionals $r_{jN_{j,k}}$, $j = 1, \ldots, 4$, $k = 1, 2, 3$, to $u$, yields the following matrix $A = \ldots$. Its determinant is nonzero. Thus, we have $c_i = 0$ for $i \leq 12$. □

It follows from Theorem 2.1 that

**Theorem 6.2.** With $V^3_{h,k} = V_{h,HZ28}$ in (2.1), the finite element solution $u_h$ of (2.1) approximates the biharmonic solution $u$ of (1.1) at the second order.

$$|u - u_h|_{2,h} \leq C h^2 |u|_4$$

provided that $u \in H^4(\Omega) \cap H^3_0(\Omega)$.

### 6.2. The basis functions

We present a basis for $P^+_3(K)$ with respect to degrees of freedom defined in (6.1). Let $b_i, i = 1, \ldots, 8$ and $i = 9, \ldots, 12$ be defined in (6.3), (6.4a) and (6.4b), respectively. We first
give the 12 basis functions for the functionals \( N_{j,k} \) as follows

\[
\begin{align*}
\phi_{1,1}(x) &= 20r_1(b_1 - b_3 + b_5 + 2b_7) + \frac{r_1}{2}(b_{10} - b_{11} - b_{12}), \\
\phi_{1,2}(x) &= 20r_1(b_2 + b_3 - b_5 + 2b_8) + \frac{r_1}{2}(-b_{10} + b_{11} - b_{12}), \\
\phi_{1,3}(x) &= -20r_1(b_1 + b_2 + 2b_7 + 2b_8) + \frac{r_1}{2}(b_9 + b_{12}), \\
\phi_{2,1}(x) &= 20r_2(-b_1 + b_3 + 2b_5 + b_7) + \frac{r_2}{2}(b_9 - b_{11} - b_{12}), \\
\phi_{2,2}(x) &= -20r_2(b_3 + b_4 + 2b_5 + 2b_6) + \frac{r_2}{2}(b_{10} + 2b_{11}), \\
\phi_{2,3}(x) &= 20r_2(b_1 + b_4 + 2b_6 - b_7) + \frac{r_2}{2}(-b_9 - b_{11} + b_{12}), \\
\phi_{3,1}(x) &= 20r_3(-b_2 + 2b_3 + b_5 + b_8) + \frac{r_3}{2}(b_9 - b_{10} - b_{12}), \\
\phi_{3,2}(x) &= -20r_3(2b_3 + 2b_4 + b_5 + b_6) + \frac{r_3}{2}(2b_{10} + b_{11}), \\
\phi_{3,3}(x) &= 20r_3(b_2 + 2b_4 + b_6 - b_8) + \frac{r_3}{2}(-b_9 - b_{10} + b_{12}), \\
\phi_{4,1}(x) &= -20r_4(2b_1 + 2b_2 + b_7 + b_8) + \frac{r_4}{2}(2b_9 + b_{12}), \\
\phi_{4,2}(x) &= 20r_4(b_1 - b_4 + b_6 + b_7) + \frac{r_4}{2}(-b_9 + b_{10} - b_{11}), \\
\phi_{4,3}(x) &= 20r_4(b_2 + b_4 - b_6 + b_8) + \frac{r_4}{2}(-b_9 + b_{10} + b_{11}).
\end{align*}
\]

These 12 basis functions satisfy

\[
N_{i,l}(\phi_{j,k}) = \delta_{i,j} \delta_{l,k}, \quad i, j = 1, \cdots, 4, \quad l, k = 1, 2, 3
\]

and vanish for the remaining functionals in (6.1).

The second four basis functions are for the functionals \( F_j \),

\[
\begin{align*}
\phi_1(x) &= 60\lambda_1\lambda_2\lambda_3 + (2\varphi_{1,1}(x) + 2\varphi_{1,2}(x) + \varphi_{1,3}(x))/r_1 \\
&\quad + (2\varphi_{2,1}(x) + 2\varphi_{2,2}(x) + \varphi_{2,3}(x))/r_2 + (2\varphi_{3,1}(x) + 2\varphi_{3,2}(x) + \varphi_{3,3}(x))/r_3 \\
&\quad - 2(n_4)^T\left((n_1/2r_1 + n_2/r_4)\varphi_{4,1}(x) + (n_2/2r_2 + n_4/r_4)\varphi_{4,2}(x) + (n_3/2r_3 + n_4/r_4)\varphi_{4,3}(x)\right),
\end{align*}
\]

\[
\begin{align*}
\phi_2(x) &= 60\lambda_1\lambda_2\lambda_4 + (2\varphi_{1,1}(x) + \varphi_{1,2}(x) + 2\varphi_{1,3}(x))/r_1 \\
&\quad + (2\varphi_{2,1}(x) + \varphi_{2,2}(x) + 2\varphi_{2,3}(x))/r_2 + (2\varphi_{3,1}(x) + 2\varphi_{4,2}(x) + \varphi_{4,3}(x))/r_4 \\
&\quad - 2(n_4)^T\left((n_1/2r_1 + n_2/r_3)\varphi_{3,1}(x) + (n_2/2r_2 + n_1/r_3)\varphi_{3,2}(x) + (n_4/2r_4 + n_1/r_3)\varphi_{3,3}(x)\right),
\end{align*}
\]

\[
\begin{align*}
\phi_3(x) &= 60\lambda_1\lambda_3\lambda_4 + (2\varphi_{1,1}(x) + \varphi_{1,2}(x) + 2\varphi_{1,3}(x))/r_1 \\
&\quad + (2\varphi_{2,1}(x) + \varphi_{2,2}(x) + 2\varphi_{3,3}(x))/r_3 + (2\varphi_{4,1}(x) + \varphi_{4,2}(x) + 2\varphi_{4,3}(x))/r_4 \\
&\quad - 2(n_2)^T\left((n_1/2r_1 + n_3/r_2)\varphi_{2,1}(x) + (n_3/2r_3 + n_2/r_2)\varphi_{2,2}(x) + (n_4/2r_4 + n_2/r_2)\varphi_{2,3}(x)\right),
\end{align*}
\]

\[
\begin{align*}
\phi_4(x) &= 60\lambda_2\lambda_3\lambda_4 + (\varphi_{2,1}(x) + 2\varphi_{2,2}(x) + 2\varphi_{3,3}(x))/r_2 \\
&\quad + (\varphi_{3,1}(x) + 2\varphi_{3,2}(x) + 2\varphi_{3,3}(x))/r_3 + (\varphi_{4,1}(x) + 2\varphi_{4,2}(x) + 2\varphi_{4,3}(x))/r_4 \\
&\quad - 2(n_1)^T\left((n_2/2r_2 + n_1/r_1)\varphi_{1,1}(x) + (n_3/2r_3 + n_1/r_1)\varphi_{1,2}(x) + (n_4/2r_4 + n_1/r_1)\varphi_{1,3}(x)\right).
\end{align*}
\]
These 4 basis functions satisfy

\[ F_j(\phi_k) = \delta_{j,k}, \quad j, k = 1, \ldots, 4 \]

and vanish for the remaining functionals in (6.1).

At the end we present the 12 basis functions for the functionals \( E_{j,k} \).

\[
\begin{align*}
\theta_{1,1}(x) &= 12 \lambda_1 \lambda_2 (5 \lambda_1 - 2) - 2(2 \varphi_{1,1}(x) + \varphi_{1,2}(x) + \varphi_{1,3}(x))/r_1 \\
&+ 2 \varphi_{2,1}(x)/r_2 + 2 \left( \frac{n_1}{r_1} + \frac{n_2}{r_2} \right)^T (\varphi_{3,1}(x)n_3 + \varphi_{4,1}(x)n_4), \\
\theta_{1,2}(x) &= 12 \lambda_1 \lambda_2 (5 \lambda_2 - 2) - 2(2 \varphi_{2,1}(x) + \varphi_{2,2}(x) + \varphi_{2,3}(x))/r_2 \\
&+ 2 \varphi_{1,1}(x)/r_1 + 2 \left( \frac{n_1}{r_1} + \frac{n_2}{r_2} \right)^T (\varphi_{3,2}(x)n_3 + \varphi_{4,2}(x)n_4), \\
\theta_{2,1}(x) &= 12 \lambda_1 \lambda_3 (5 \lambda_1 - 2) - 2(\varphi_{1,1}(x) + 2 \varphi_{1,2}(x) + \varphi_{1,3}(x))/r_1 \\
&+ 2 \varphi_{3,1}(x)/r_3 + 2 \left( \frac{n_1}{r_1} + \frac{n_3}{r_3} \right)^T (\varphi_{2,1}(x)n_2 + \varphi_{4,1}(x)n_4), \\
\theta_{2,2}(x) &= 12 \lambda_1 \lambda_3 (5 \lambda_3 - 2) - 2(2 \varphi_{3,1}(x) + \varphi_{3,2}(x) + \varphi_{3,3}(x))/r_3 \\
&+ 2 \varphi_{1,2}(x)/r_1 + 2 \left( \frac{n_1}{r_1} + \frac{n_3}{r_3} \right)^T (\varphi_{2,2}(x)n_2 + \varphi_{4,3}(x)n_4), \\
\theta_{3,1}(x) &= 12 \lambda_2 \lambda_4 (5 \lambda_1 - 2) - 2(\varphi_{1,1}(x) + \varphi_{1,2}(x) + 2 \varphi_{1,3}(x))/r_1 \\
&+ 2 \varphi_{4,1}(x)/r_4 + 2 \left( \frac{n_1}{r_1} + \frac{n_4}{r_4} \right)^T (\varphi_{2,1}(x)n_2 + \varphi_{3,1}(x)n_3), \\
\theta_{3,2}(x) &= 12 \lambda_2 \lambda_4 (5 \lambda_4 - 2) - 2(2 \varphi_{4,1}(x) + \varphi_{4,2}(x) + \varphi_{4,3}(x))/r_4 \\
&+ 2 \varphi_{1,3}(x)/r_1 + 2 \left( \frac{n_1}{r_1} + \frac{n_4}{r_4} \right)^T (\varphi_{2,3}(x)n_2 + \varphi_{3,3}(x)n_3), \\
\theta_{4,1}(x) &= 12 \lambda_2 \lambda_4 (5 \lambda_3 - 2) - 2(\varphi_{3,1}(x) + 2 \varphi_{3,2}(x) + \varphi_{3,3}(x))/r_3 \\
&+ 2 \varphi_{2,1}(x)/r_2 + 2 \left( \frac{n_2}{r_2} + \frac{n_3}{r_3} \right)^T (\varphi_{1,2}(x)n_1 + \varphi_{4,3}(x)n_4), \\
\theta_{4,2}(x) &= 12 \lambda_2 \lambda_4 (5 \lambda_2 - 2) - 2(\varphi_{1,1}(x) + \varphi_{2,2}(x) + 2 \varphi_{2,3}(x))/r_2 \\
&+ 2 \varphi_{4,2}(x)/r_4 + 2 \left( \frac{n_2}{r_2} + \frac{n_4}{r_4} \right)^T (\varphi_{1,3}(x)n_1 + \varphi_{3,3}(x)n_3), \\
\theta_{5,1}(x) &= 12 \lambda_3 \lambda_4 (5 \lambda_2 - 2) - 2(\varphi_{2,1}(x) + \varphi_{2,2}(x) + 2 \varphi_{2,3}(x))/r_2 \\
&+ 2 \varphi_{4,1}(x)/r_4 + 2 \left( \frac{n_2}{r_2} + \frac{n_4}{r_4} \right)^T (\varphi_{1,2}(x)n_1 + \varphi_{3,2}(x)n_2), \\
\theta_{5,2}(x) &= 12 \lambda_3 \lambda_4 (5 \lambda_4 - 2) - 2(\varphi_{4,1}(x) + \varphi_{4,2}(x) + 2 \varphi_{4,3}(x))/r_4 \\
&+ 2 \varphi_{2,3}(x)/r_2 + 2 \left( \frac{n_2}{r_2} + \frac{n_3}{r_3} \right)^T (\varphi_{1,3}(x)n_1 + \varphi_{3,3}(x)n_3), \\
\theta_{6,1}(x) &= 12 \lambda_3 \lambda_4 (5 \lambda_3 - 2) - 2(\varphi_{3,1}(x) + \varphi_{3,2}(x) + 2 \varphi_{3,3}(x))/r_3 \\
&+ 2 \varphi_{4,3}(x)/r_4 + 2 \left( \frac{n_3}{r_3} + \frac{n_4}{r_4} \right)^T (\varphi_{1,2}(x)n_1 + \varphi_{2,2}(x)n_2), \\
\theta_{6,2}(x) &= 12 \lambda_3 \lambda_4 (5 \lambda_1 - 2) - 2(\varphi_{4,1}(x) + \varphi_{4,2}(x) + 2 \varphi_{4,3}(x))/r_4 \\
&+ 2 \varphi_{3,3}(x)/r_3 + 2 \left( \frac{n_3}{r_3} + \frac{n_4}{r_4} \right)^T (\varphi_{1,3}(x)n_1 + \varphi_{2,3}(x)n_2).
\end{align*}
\]
These 12 basis functions satisfy
\[ E_{i,l}(\theta_{j,k}) = \delta_{i,j} \delta_{l,k}, \quad 1 \leq i,j \leq 6, \quad l,k = 1,2 \]
vanish for the remaining functionals in (6.1).

7. Numerical Tests

7.1. Numerical results in 2D

We first compute the 2D biharmonic solution of (1.1), where the exact solution is
\[ u = 2^6 x_1^2(1 - x_1)^2 x_2^2(1 - x_2)^2, \quad (7.1) \]
on the unit square domain \((0,1)^2\). We cut the domain by a north-east line to define the level one grid for the computation. Then each triangle is refined into four congruent triangles, to define the higher level grid.

In the first test, we use the \(P_3 + 2B_4\) element, \(V_{h,HZ12}\) in (3.2). As expected, the finite element solution does converge to the exact solution at the second order, shown in Table 7.1.

<table>
<thead>
<tr>
<th>(T_k)</th>
<th>(|e_h|_0)</th>
<th>(h^n)</th>
<th>(|e_h|_{1,h})</th>
<th>(h^n)</th>
<th>(|e_h|_{2,h})</th>
<th>(h^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0774343</td>
<td>0.0</td>
<td>0.414843</td>
<td>0.0</td>
<td>4.05172</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.0322262</td>
<td>1.3</td>
<td>0.198602</td>
<td>1.1</td>
<td>2.22543</td>
<td>0.9</td>
</tr>
<tr>
<td>3</td>
<td>0.0018858</td>
<td>4.1</td>
<td>0.020767</td>
<td>3.3</td>
<td>0.60045</td>
<td>1.9</td>
</tr>
<tr>
<td>4</td>
<td>0.0000831</td>
<td>4.5</td>
<td>0.002312</td>
<td>3.2</td>
<td>0.14839</td>
<td>2.0</td>
</tr>
<tr>
<td>5</td>
<td>0.0000039</td>
<td>4.4</td>
<td>0.000285</td>
<td>3.0</td>
<td>0.03508</td>
<td>2.0</td>
</tr>
<tr>
<td>6</td>
<td>0.0000003</td>
<td>4.0</td>
<td>0.000036</td>
<td>3.0</td>
<td>0.00882</td>
<td>2.0</td>
</tr>
<tr>
<td>7</td>
<td>0.0000000</td>
<td>3.9</td>
<td>0.000005</td>
<td>3.0</td>
<td>0.00219</td>
<td>2.0</td>
</tr>
</tbody>
</table>

In the second test, we compute the exact solution (7.1) by the \(P_4\) non-conforming finite element \(V_{h,HZ15}\) defined in (4.2). As shown by the theorem, the finite element solution converges to the exact solution at the second order, shown in Table 7.2. Comparing with the 12 dof \(P_3 + 2B_4\) element, this element is slightly more accurate though it is also of the second order.

<table>
<thead>
<tr>
<th>(T_k)</th>
<th>(|e_h|_0)</th>
<th>(h^n)</th>
<th>(|e_h|_{1,h})</th>
<th>(h^n)</th>
<th>(|e_h|_{2,h})</th>
<th>(h^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.049596736</td>
<td>0.0</td>
<td>0.270374</td>
<td>0.0</td>
<td>2.50814</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.004671002</td>
<td>3.4</td>
<td>0.045237</td>
<td>2.6</td>
<td>0.79945</td>
<td>1.6</td>
</tr>
<tr>
<td>3</td>
<td>0.000890065</td>
<td>2.4</td>
<td>0.010920</td>
<td>2.1</td>
<td>0.30883</td>
<td>1.4</td>
</tr>
<tr>
<td>4</td>
<td>0.000076285</td>
<td>3.5</td>
<td>0.001595</td>
<td>2.8</td>
<td>0.08515</td>
<td>1.9</td>
</tr>
<tr>
<td>5</td>
<td>0.000005480</td>
<td>3.8</td>
<td>0.000214</td>
<td>2.9</td>
<td>0.02220</td>
<td>1.9</td>
</tr>
<tr>
<td>6</td>
<td>0.000000364</td>
<td>3.9</td>
<td>0.000028</td>
<td>3.0</td>
<td>0.00565</td>
<td>2.0</td>
</tr>
<tr>
<td>7</td>
<td>0.000000023</td>
<td>4.0</td>
<td>0.000004</td>
<td>3.0</td>
<td>0.00142</td>
<td>2.0</td>
</tr>
</tbody>
</table>

In the third test, we approximate the exact solution (7.1) by the 16 dof \(P_3 + 6B_6\) element constructed in [20]. On each triangle \(K\) with the barycentric-coordinates \(\lambda_i\), the finite element
Table 7.3: The error $e_h = u - u_h$ and the order of convergence, for (7.1), by the $P_3 + 6B_6$ element.

| $T_k$ | $\|e_h\|_0$ | $h^n$ | $|e_h|_{1,h}$ | $h^n$ | $|e_h|_{2,h}$ | $h^n$ |
|-------|-------------|-------|--------------|-------|--------------|-------|
| 1     | 0.0950343   | 0.0   | 0.479167     | 0.0   | 3.64488      | 0.0   |
| 2     | 0.0601461   | 0.7   | 0.308401     | 0.6   | 2.76527      | 0.4   |
| 3     | 0.0092643   | 2.7   | 0.053634     | 2.5   | 1.06496      | 1.4   |
| 4     | 0.0009260   | 3.3   | 0.066878     | 3.0   | 0.33413      | 1.7   |
| 5     | 0.0000721   | 3.7   | 0.000812     | 3.1   | 0.09269      | 1.8   |
| 6     | 0.0000049   | 3.9   | 0.000098     | 3.1   | 0.02400      | 1.9   |
| 7     | 0.0000003   | 4.0   | 0.000012     | 3.0   | 0.00605      | 2.0   |

Table 7.4: The error $e_h = u - u_h$ and the order of convergence, for (7.1), by the $P_3 + 3B_5 + 3B_6$ element.

| $T_k$ | $\|e_h\|_0$ | $h^n$ | $|e_h|_{1,h}$ | $h^n$ | $|e_h|_{2,h}$ | $h^n$ |
|-------|-------------|-------|--------------|-------|--------------|-------|
| 1     | 0.0925868   | 0.0   | 0.473937     | 0.0   | 3.56790      | 0.0   |
| 2     | 0.0590571   | 0.6   | 0.304468     | 0.6   | 2.73905      | 0.4   |
| 3     | 0.0091445   | 2.7   | 0.053343     | 2.5   | 1.06181      | 1.4   |
| 4     | 0.0009744   | 3.2   | 0.007110     | 2.9   | 0.03331      | 1.6   |
| 5     | 0.0000806   | 3.6   | 0.000851     | 3.1   | 0.09788      | 1.8   |
| 6     | 0.0000056   | 3.8   | 0.000102     | 3.1   | 0.02579      | 1.9   |
| 7     | 0.0000004   | 3.9   | 0.000012     | 3.0   | 0.00655      | 2.0   |

The dual basis of the finite element space, i.e., the degrees of freedom of the finite element, is

$$B_6(K) = \lambda_1 \lambda_2 \lambda_3 \text{span} \left\{ (1 - 3\lambda_1 - 2\lambda_2 + 3\lambda_1 \lambda_2) \lambda_2, (1 - 3\lambda_1 - 2\lambda_3 + 3\lambda_1 \lambda_3) \lambda_3, (1 - 3\lambda_2 - 2\lambda_1 + 3\lambda_2 \lambda_1) \lambda_1, (1 - 3\lambda_2 - 2\lambda_3 + 3\lambda_2 \lambda_3) \lambda_3, (1 - 3\lambda_3 - 2\lambda_1 + 3\lambda_3 \lambda_1) \lambda_1, (1 - 3\lambda_3 - 2\lambda_2 + 3\lambda_3 \lambda_2) \lambda_2 \right\}.$$  

The computational results are listed in Table 7.3. The finite element is truly of 2nd order accuracy. The local dimension of this element is 16, but the $H^2$ error is about 3 times of that of the new 12 dof $P_3 + 2B_4$ element.

In the fourth test, we approximate the exact solution (7.1) by the 16 dof $P_3 + 3B_5 + 3B_6$ element from [4]. On each triangle $K$ with barycentric-coordinates $\lambda_i$, the finite element shape functions space is

$$C_{16}(K) = P_3(K) + \lambda_1 \lambda_2 \lambda_3 \text{span} \left\{ \lambda_1 \lambda_2, \lambda_2 \lambda_3, \lambda_1 \lambda_3, \lambda_1^2 \lambda_2, \lambda_2^2 \lambda_3, \lambda_1 \lambda_1^2 \right\}.$$  

The dual basis of the finite element space, i.e., the degrees of freedom of the finite element, is
Table 7.5: The error \( e_h = u - u_h \) and the order of convergence, for (7.1), by the \( P_3 + 2B_5 \) element.

| \( T_h \) | \( ||e_h||_0 \) | \( h^n \) | \( ||e_h||_{1,h} \) | \( h^n \) | \( ||e_h||_{2,h} \) | \( h^n \) |
|------|------|------|------|------|------|------|
| 1    | 0.1017755 | 0.0 | 0.496119 | 0.0 | 3.66373 | 0.0 |
| 2    | 0.0602523 | 0.8 | 0.308833 | 0.7 | 2.77412 | 0.4 |
| 3    | 0.0096740 | 2.6 | 0.055137 | 2.5 | 1.09411 | 1.3 |
| 4    | 0.0010644 | 3.2 | 0.007374 | 2.9 | 0.35943 | 1.6 |
| 5    | 0.0000901 | 3.6 | 0.000864 | 3.1 | 0.10364 | 1.8 |
| 6    | 0.0000064 | 3.8 | 0.000100 | 3.1 | 0.02749 | 1.9 |
| 7    | 0.0000004 | 3.9 | 0.000012 | 3.0 | 0.00701 | 2.0 |

Table 7.6: The error \( e_h = u - u_h \) and the order of convergence, for (7.2), by the \( P_3 + 2B_4 \) element.

| \( T_h \) | \( ||e_h||_0 \) | \( h^n \) | \( ||e_h||_{1,h} \) | \( h^n \) | \( ||e_h||_{2,h} \) | \( h^n \) |
|------|------|------|------|------|------|------|
| 1    | 0.0257005 | 0.0 | 0.155548 | 0.0 | 1.30794 | 0.0 |
| 2    | 0.0118945 | 1.1 | 0.082354 | 0.9 | 1.21970 | 0.1 |
| 3    | 0.0007061 | 4.1 | 0.009850 | 3.1 | 0.30520 | 2.0 |
| 4    | 0.0000431 | 4.0 | 0.001374 | 2.8 | 0.08565 | 1.8 |
| 5    | 0.0000039 | 3.4 | 0.000236 | 2.5 | 0.02793 | 1.6 |
| 6    | 0.0000004 | 3.2 | 0.000048 | 2.3 | 0.01093 | 1.4 |
| 7    | 0.0000001 | 3.0 | 0.000011 | 2.2 | 0.00486 | 1.2 |

The dual basis of the finite element space, i.e., the degrees of freedom of the finite element, is

\[
v(x_i), v_x(x_i), v_y(x_i) \quad \text{at three vertices of } K,
\]

\[
v(x_c) \quad \text{at the bary-center of } K,
\]

\[
\int_{e_j} \partial_n vds, \int_{e_j} s \partial_n vds \quad \text{on three edges of } K.
\]

We compute the solution (7.1) by this \( C_{16} \) finite element. The computational results are listed in Table 7.4. The finite element is truly of 2nd order accuracy. The local dimension of this element is 16, but the \( H^2 \) error is about 3 times of that of the new 12 dof \( P_3 + 2B_5 \) element.

In the fifth test, we approximate the exact solution (7.1) by the 12 dof \( P_3 + 2B_4 \) element constructed in [7]. On each triangle \( K \) with barycentric-coordinates \( \lambda_i \), the finite element shape functions space is

\[
G_{12}(K) = P_3(K) + \lambda_1 \lambda_2 \lambda_3 \text{ span}\{(\lambda_1 - \lambda_2^2 - 2\lambda_2\lambda_3), (\lambda_2 - \lambda_3^2 - 2\lambda_1\lambda_3)\}.
\]

The dual basis of the finite element space, i.e., the degrees of freedom of the finite element, is

\[
v(x_i), v_x(x_i), v_y(x_i), \text{at three vertices of } K, \text{ and } \int_{e_j} \partial_n vds \text{ on three edges of } K.
\]

The computational results are listed in Table 7.5. The finite element is truly of 2nd order accuracy. But its \( H^2 \) error is about 3 times of that of the new \( P_3 + 2B_4 \) element.

In the next three tests, we consider a singular solution

\[
u = 2^4x^{5/2}(1-x)^2y^2(1-y)^2, \quad (x_1, x_2) \in (0, 1) \times (0, 1).
\]

This solution is not smooth enough, but in the Sobolev space \( H^{4-\epsilon}(\Omega) \). We approximate it by the \( P_3 + 2B_4 \) element, the \( P_3 + 6B_6 \) element, and the \( P_3 + 2B_5 \) element. As expected, the finite
Table 7.7: The error $e_h = u - u_h$ and the order of convergence, for (7.2), by the $P_3 + 6B_6$ element.

| $T_h$ | $\|e_h\|_0$ | $h^\alpha$ | $|e_h|_{1,h}$ | $h^\alpha$ | $|e_h|_{2,h}$ | $h^\alpha$ |
|-------|-----------|-----------|--------------|-----------|--------------|-----------|
| 1     | 0.0402600 | 0.0       | 0.267845     | 0.0       | 5.57420      | 0.0       |
| 2     | 0.0218671 | 0.9       | 0.147111     | 0.9       | 3.31137      | 0.8       |
| 3     | 0.0031348 | 2.8       | 0.023770     | 2.6       | 1.01281      | 1.7       |
| 4     | 0.0003147 | 3.3       | 0.003168     | 2.9       | 0.29331      | 1.8       |
| 5     | 0.0000253 | 3.6       | 0.000403     | 3.0       | 0.08106      | 1.9       |
| 6     | 0.0000018 | 3.8       | 0.000058     | 2.8       | 0.02393      | 1.8       |
| 7     | 0.0000001 | 3.8       | 0.000010     | 2.5       | 0.00829      | 1.5       |

Table 7.8: The error $e_h = u - u_h$ and the order of convergence, for (7.2), by the $P_3 + 2B_5$ element.

| $T_h$ | $\|e_h\|_0$ | $h^\alpha$ | $|e_h|_{1,h}$ | $h^\alpha$ | $|e_h|_{2,h}$ | $h^\alpha$ |
|-------|-----------|-----------|--------------|-----------|--------------|-----------|
| 1     | 0.0416108 | 0.0       | 0.167121     | 0.0       | 1.55539      | 0.0       |
| 2     | 0.0215182 | 1.0       | 0.133563     | 0.3       | 1.77014      | 0.0       |
| 3     | 0.0031834 | 2.8       | 0.021423     | 2.6       | 0.50960      | 1.8       |
| 4     | 0.0003534 | 3.2       | 0.002805     | 2.9       | 0.14026      | 1.9       |
| 5     | 0.0000309 | 3.5       | 0.000324     | 3.1       | 0.03780      | 1.9       |
| 6     | 0.0000023 | 3.8       | 0.000043     | 2.9       | 0.01138      | 1.7       |
| 7     | 0.0000002 | 3.8       | 0.000008     | 2.4       | 0.00424      | 1.4       |

The element solution converges at the possible highest order, in all norms, shown in Tables 7.6–7.8. For this problem, the error of the $P_3 + 2B_4$ element is about the same as that of the $P_3 + 2B_5$ element, much smaller than that of the $P_3 + 6B_6$ element.

7.2. Numerical results in 3D

We compute the 3D biharmonic solution of (1.1), where the exact solution is

$$ u = 2^{10}x_1^2(1 - x_1)^2x_2^2(1 - x_2)^2x_3^2(1 - x_3)^2, \quad (7.3) $$

on the unit square domain $(0,1)^3$.

In the computation, the level one grid is the one shown in Fig. 7.1, cutting the unit cube into 6 tetrahedra. Each grid is refined into a half-sized grid uniformly, to get a higher level grid, shown in Fig. 7.1. In Table 7.9, the error and the convergence order in various norms are listed for the true solution (7.3), by the $P_4$ enriched $P_3$ nonconforming finite element (6.2). The optimal order of convergence is achieved in Table 7.9, in all norms, confirming the theorem.

Fig. 7.1. The initial grid for (7.1), and its level 2 refinement.
Table 7.9: The error $e_h = u - u_h$ and the order of convergence, for (7.3), by the $P_3 + 8B_4$ element.

| $T_k$ | $\|e_h\|_0$ | $h^n$ | $|e_h|_{1,h}$ | $h^n$ | $|e_h|_{2,h}$ | $h^n$ |
|-------|----------|--------|---------------|--------|---------------|--------|
| 1     | 0.0463092| 0.0    | 0.3918927     | 0.0    | 3.9317928     | 0.0    |
| 2     | 0.0203641| 1.2    | 0.1970587     | 1.0    | 4.5723947     | 0.0    |
| 3     | 0.0015314| 3.7    | 0.0267813     | 2.9    | 1.3978501     | 1.7    |
| 4     | 0.0000774| 4.3    | 0.0033902     | 3.0    | 0.3961007     | 1.8    |
| 5     | 0.0000036| 4.4    | 0.0004229     | 3.0    | 0.1030880     | 1.9    |

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References


