

The Relaxation Limits of the Two-Fluid Compressible Euler-Maxwell Equations

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Abstract. In this paper we consider the relaxation limits of the two-fluid Euler-Maxwell systems with initial layer. We construct an asymptotic expansion with initial layer functions and prove the convergence between the exact solutions and the approximate solutions.

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1 Introduction

In this paper, we consider the three-dimensional two-fluid (including electrons and ions) Euler-Maxwell equations in a torus $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^3$:

$$\partial_t n_\alpha + \operatorname{div}(n_\alpha u_\alpha) = 0, \quad (1.1)$$

$$m_\alpha [\partial_t (n_\alpha u_\alpha) + \operatorname{div}(n_\alpha u_\alpha \otimes u_\alpha)] + \nabla p(n_\alpha) = q_\alpha n_\alpha (E + u_\alpha \times B) - \frac{m_\alpha n_\alpha u_\alpha}{\tau_\alpha}, \quad (1.2)$$

$$\varepsilon \partial_t E - \frac{1}{\mu} \nabla \times B = n_e u_e - n_i u_i, \quad (1.3)$$

$$\partial_t B + \nabla \times E = 0, \quad (1.4)$$

$$\varepsilon \operatorname{div} E = n_i - n_e, \quad \operatorname{div} B = 0, \quad (1.5)$$

where $\alpha = e, i$, $q_i = 1$, $q_e = -1$; n_e and n_i stand for the density of the electrons and ions; u_e and u_i stand for the velocity of the electrons and ions; E and B are respectively the electric

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field and magnetic field; $p = p(n_\alpha)$ is the pressure function which is sufficiently smooth and strictly increasing for $n_\alpha > 0$. These variables are functions of a three-dimensional position vector $x \in \mathbb{T}$ and of the time $t > 0$. In the above systems the physical parameters are the electron mass m_e and the ion mass m_i , the momentum relaxation times τ_e and τ_i , and the permittivity ε and the permeability μ .

For simplicity, we denote $m_\alpha = 1$, $\varepsilon, \mu = 1$ and $\tau_e = \tau_i = \tau$, then we obtain the following systems:

$$\partial_t n_\alpha + \operatorname{div}(n_\alpha u_\alpha) = 0, \quad (1.6)$$

$$\partial_t(n_\alpha u_\alpha) + \operatorname{div}(n_\alpha u_\alpha \otimes u_\alpha) + \nabla p(n_\alpha) = q_\alpha n_\alpha (E + u_\alpha \times B) - \frac{n_\alpha u_\alpha}{\tau_\alpha}, \quad (1.7)$$

$$\partial_t E - \nabla \times B = n_e u_e - n_i u_i, \quad \operatorname{div} E = n_i - n_e, \quad (1.8)$$

$$\partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0. \quad (1.9)$$

Furthermore, we make the time scaling by replacing t by $\frac{t}{\tau}$ and define the enthalpy function $h(n_\alpha)$ by

$$h(n_\alpha) = \int_1^{n_\alpha} \frac{p'(s)}{s} ds. \quad (1.10)$$

So the system we considered is rewritten the following reduced two-fluid Euler-Maxwell systems:

$$\partial_t n_\alpha + \frac{1}{\tau} \operatorname{div}(n_\alpha u_\alpha) = 0, \quad (1.11)$$

$$\partial_t u_\alpha + \frac{1}{\tau} (u_\alpha \cdot \nabla) u_\alpha + \frac{1}{\tau} \nabla h(n_\alpha) = \frac{q_\alpha (E + u_\alpha \times B)}{\tau} - \frac{u_\alpha}{\tau^2}, \quad (1.12)$$

$$\partial_t E - \frac{1}{\tau} \nabla \times B = \frac{n_e u_e - n_i u_i}{\tau}, \quad \operatorname{div} E = n_i - n_e, \quad (1.13)$$

$$\partial_t B + \frac{1}{\tau} \nabla \times E = 0, \quad \operatorname{div} B = 0, \quad (1.14)$$

with initial data:

$$(n_\alpha, u_\alpha, E, B)|_{t=0} = (n_{\alpha,0}^\tau, u_{\alpha,0}^\tau, E_0^\tau, B_0^\tau). \quad (1.15)$$

The study of compressible Euler-Maxwell equations began in 2000, Chen, Jerome and Wang [1] prove the existence of global weak solutions of the simplified Euler-Maxwell equations by using the method of step by step Godunov scheme combined with compensated compactness; in 2007 and 2008, Peng and Wang [2,3] study the non relativistic limit convergence problem for compressible Euler-Maxwell equations to compressible Euler-Poisson equations and the composite limits of the quasi neutral limit and the non relativistic limit for compressible Euler-Maxwell equations; Peng, Wang and Gu [4] discuss the relaxation limit of compressible Euler-Maxwell equations and the existence of global smooth solution in 2011; in the same year, Wang, Yang and Zhao [5] research the relaxation limit of the plasma two-fluid Euler-Maxwell equations with the help of Maxwell

iteration and energy method for the well-prepared initial data; Mohamed-Lasmer Hajjej and Peng [6] study the relaxation limits of the one-fluid Euler-Maxwell equation with initial layer by the method of asymptotic expansion.

For later use in this paper, we recall some inequalities in Sobolev spaces [7, 8] and the local existence of smooth solutions for symmetrizable hyperbolic equation. For any $s > 0$, we denote by $\|\cdot\|_s$ the norm of the usual Sobolev space $H^s(\mathbb{T})$, and by $\|\cdot\|$ and $\|\cdot\|_\infty$ the norms of $L^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$, respectively.

Lemma 1.1. (See [9, 10]) *Let $s \geq 3$ be an integer and $(n_{\alpha,0}^\tau, u_{\alpha,0}^\tau, E_0^\tau, B_0^\tau) \in H^s(\mathbb{T})$ with $n_{\alpha,0}^\tau \geq k$ for some given constant $k > 0$, independent of τ . Then there exist $T_1^\tau > 0$ and a unique smooth solution $(n_\alpha^\tau, u_\alpha^\tau, E^\tau, B^\tau)$ to the periodic problem (1.6)-(1.9) defined in the time interval $[0, T_1^\tau]$, with $(n_\alpha^\tau, u_\alpha^\tau, E^\tau, B^\tau) \in C^1([0, T_1^\tau]; H^{s-1}(\mathbb{T})) \cap C([0, T_1^\tau]; H^s(\mathbb{T}))$.*

The main result is as follows:

Theorem 1.1. *For any integer $s \geq 3$, under the Proposition 1.1 and the following conditions:*

$$\operatorname{div} E_\tau = n_{i,\tau} - n_{e,\tau}, \quad \operatorname{div} B_\tau = 0, \quad (1.16)$$

$$\sup_{0 \leq t \leq T_1} \|(n_{\alpha,\tau}, E_\tau, B_\tau)(t, \cdot)\|_s \leq C, \quad \sup_{0 \leq t \leq T_1} \|(u_{\alpha,\tau}(t, \cdot))\|_s \leq C, \quad (1.17)$$

$$\|(n_{\alpha,0}^\tau - n_{\alpha,\tau}(0, \cdot), u_{\alpha,0}^\tau - u_{\alpha,\tau}(0, \cdot), E_0^\tau - E_\tau(0, \cdot), B_0^\tau - B_\tau(0, \cdot))\|_s \leq C\tau^2, \quad (1.18)$$

here $(n_{\alpha,\tau}, u_{\alpha,\tau}, E_\tau, B_\tau)(t, x)$ is the approximate solutions of (1.11)-(1.14), such that as $\tau \rightarrow 0$ we have $T_1^\tau \geq T_1$ and the solution $(n_\alpha^\tau, u_\alpha^\tau, E^\tau, B^\tau)$ satisfies:

$$\|(n_\alpha^\tau, u_\alpha^\tau, E^\tau, B^\tau) - (n_{\alpha,\tau}, u_{\alpha,\tau}, E_\tau, B_\tau)\|_s \leq C\tau^2, \quad (1.19)$$

$$\|u_\alpha^\tau - u_{\alpha,\tau}\|_{L^2(0, T_\tau; H^s)} \leq C\tau^3. \quad (1.20)$$

for any $t \in [0, T^\tau]$, here $T^\tau \leq T_1$.

Our main purpose in this paper is to study the relaxation limits of the two-fluid Euler-Maxwell systems (1.11)-(1.14) with initial data (1.15). We consider the problem with initial layer. In order to establish our result, we make an asymptotic expansion including initial layer. In Section 2, we will take the approximate expression into systems (1.11)-(1.14) such that we get the error estimates of the remainders. In Section 3, we prove the main result about the convergence.

2 Asymptotic expansion

In this section, we make an asymptotic expansion with initial layer functions and take it into the systems, then we can get the expression of the first-order initial layer functions and second-order initial layer functions. Furthermore, we obtain the estimates of the remainders which produced by approximate solutions and extra solutions.

We know that

$$(n_{\alpha,\tau}, u_{\alpha,\tau}, E_\tau, B_\tau) = (\tilde{n}_{\alpha,\tau}, \tilde{u}_{\alpha,\tau}, \tilde{E}_\tau, \tilde{B}_\tau) + (n_{\alpha,I}, u_{\alpha,I}, E_I, B_I),$$

where $(n_{\alpha,\tau}, u_{\alpha,\tau}, E_\tau, B_\tau)$ is the approximate solution, $(\tilde{n}_{\alpha,\tau}, \tilde{u}_{\alpha,\tau}, \tilde{E}_\tau, \tilde{B}_\tau)$ the inner function, and $(n_{\alpha,I}, u_{\alpha,I}, E_I, B_I)$ is the initial layer function.

Firstly, we make the following ansatz for inner function

$$(\tilde{n}_{\alpha,\tau}, \tilde{u}_{\alpha,\tau}, \tilde{E}_\tau, \tilde{B}_\tau)(t, x) = \sum_{j>0} \tau^{2j} (n_{\alpha,I}^j, \tau u_{\alpha,I}^j, E^j, \tau B^j), \quad (2.1)$$

and take it into systems (1.11) – (1.14), we obtain

$$\partial_t n_\alpha^0 + \operatorname{div}(n_\alpha^0 u_\alpha^0) = 0, \quad (2.2)$$

$$\nabla h(n_\alpha^0) = q_\alpha E^0 - u_\alpha^0, \quad (2.3)$$

$$\partial_t E^0 - \nabla \times B^0 = \frac{n_e^0 u_e^0 - n_i^0 u_i^0}{\tau}, \quad \operatorname{div} E^0 = n_i^0 - n_e^0, \quad (2.4)$$

$$\nabla \times E^0 = 0, \quad \operatorname{div} B^0 = 0, \quad (2.5)$$

there $\nabla \times E^0 = 0$ implies the existence of a potential ϕ^0 such that $E^0 = -\nabla \phi^0$. Then n_α^0 solves a classical system drift-diffusion equations:

$$\begin{cases} \partial_t n_\alpha^0 - \operatorname{div}(n_\alpha^0 (\nabla h(n_\alpha^0) + q_\alpha \nabla \phi^0)) = 0, \\ -\Delta \phi^0 = n_i^0 - n_e^0, \end{cases}$$

with initial condition:

$$n_\alpha^0(0, x) = n_{\alpha,0}.$$

Then we can get the first order compatibility condition:

$$u_{\alpha,0} = -\nabla h(n_{\alpha,0}) - q_\alpha \nabla \phi_0, \quad E_0 = -\nabla \phi_0, \quad B_0 = B^0,$$

where ϕ_0 is determined by $-\Delta \phi_0 = n_{i,0} - n_{e,0}$. It is similar to [6].

Secondly, let the initial data of an approximate solution $(n_{\alpha,\tau}, u_{\alpha,\tau}, E_\tau, B_\tau)(t, x)$ have an asymptotic expansion of the form:

$$(n_{\alpha,\tau}, u_{\alpha,\tau}, E_\tau, B_\tau)(t, x) |_{t=0} = (n_{\alpha,0}, \tau u_{\alpha,0}, E_0, \tau B_0) + O(\tau^2), \quad (2.6)$$

where $(n_{\alpha,0}, \tau u_{\alpha,0}, E_0, \tau B_0)$ are the given smooth functions solutions; moreover the asymptotic expansion including initial layer correction is

$$\begin{aligned} (n_{\alpha,\tau}, u_{\alpha,\tau}, E_\tau, B_\tau)(t, x) = & (n_\alpha^0, \tau u_\alpha^0, E^0 + \tau^2 E_c^1, \tau B^0)(t, x) + ((n_{\alpha,I}^0, \tau u_{\alpha,I}^0, E_I^0, \tau B_I^0)(z, x) \\ & + \tau^2 (n_{\alpha,I}^1, \tau u_{\alpha,I}^1, E_I^1, \tau B_I^1))(z, x) + O(\tau^2), \end{aligned} \quad (2.7)$$

where $z = t/\tau^2$ is the fast variable, the subscript I stands for the initial layer variable and E_c^1 is the correction term defined by:

$$\nabla \times E_c^{m+1} = -\partial_t B^m, \quad \operatorname{div} E_c^{m+1} = 0, \quad (2.8)$$

where $m = 0, 1, 2, \dots$

Obviously, (n^0, u^0, E^0, B^0) satisfies the systems (2.2)-(2.5). Putting expression (2.7) into (1.11) and (1.13), we obtain $\partial_z n_{\alpha, I}^0 = 0, \partial_z E_I^0 = 0$ which imply

$$n_{\alpha, I}^0 = E_I^0 = 0. \quad (2.9)$$

Putting expression (2.7) into (1.14) and using (2.5) and (2.9), we have

$$\partial_z B_I^0 = 0, \quad (2.10)$$

which imply

$$B_I^0 = 0. \quad (2.11)$$

Putting expression (2.7) into (1.12) and using (2.2), we get

$$\partial_z u_{\alpha, I}^0 + u_{\alpha, I}^0 = 0. \quad (2.12)$$

From (2.6)-(2.7), we have

$$u_{\alpha}^0(0, x) + u_{\alpha, I}^0(0, x) = u_{\alpha, 0}(x), \quad (2.13)$$

together with (2.12), we get the solution about first order initial layers for variable u

$$u_{\alpha, I}^0(z, x) = u_{\alpha, I}^0(0, x)e^{-z} = (u_{\alpha, 0}(x) - u_{\alpha}^0(0, x))e^{-z}. \quad (2.14)$$

Using the similar way, we can obtain the second order initial layers

$$u_{\alpha, I}^1(z, x) = 0, \quad (2.15)$$

$$\partial_z n_{\alpha, I}^1(z, x) + \operatorname{div}(n_{\alpha}^0(0, x)u_{\alpha, I}^0(z, x)) = 0, \quad (2.16)$$

$$\partial_z E_I^1(z, x) = n_e^0(0, x)u_{e, I}^0(z, x) - n_i^0(0, x)u_{i, I}^0(z, x), \quad (2.17)$$

$$\partial_z B_I^1(z, x) + \nabla \times E_I^1(z, x) = 0. \quad (2.18)$$

Suppose that $(n_{\alpha, 1}, E_1, B_1)$ is smooth function, and let

$$(n_{\alpha, I}^1, E_I^1, B_I^1)(0, x) = (n_{\alpha, 1}, E_1, B_1)(x). \quad (2.19)$$

Together with (2.13), (2.14) and (2.16), we have

$$n_{\alpha, I}^1(z, x) = n_{\alpha, 1}(x) - \operatorname{div}[n_{\alpha}^0(0, x)(u_{\alpha, 0}(x) - u_{\alpha}^0(0, x))](1 - e^{-z}). \quad (2.20)$$

Similarly, together with (2.14), (2.17) and (2.19), we obtain

$$E_I^1(z, x) = E_1 + [n_e^0(0, x)(u_{e, 0}(x) - u_e^0(0, x)) - n_i^0(0, x)(u_{i, 0}(x) - u_i^0(0, x))](1 - e^{-z}). \quad (2.21)$$

There we take

$$E_1(x) = -n_e^0(0,x)(u_{e,0}(x) - u_e^0(0,x)) - n_i^0(0,x)(u_{i,0}(x) - u_i^0(0,x)), \quad (2.22)$$

$$\operatorname{div} E_1(x) = n_{i,1}(x) - n_{e,1}(x), \quad \operatorname{div} B_1(x) = 0. \quad (2.23)$$

Then we have

$$\operatorname{div} E_I^1 = n_{e,I}^1 - n_{i,I}^1. \quad (2.24)$$

From a series of calculations, we get

$$E_I^1(z,x) = -[n_e^0(0,x)(u_{e,0}(x) - u_e^0(0,x)) - n_i^0(0,x)(u_{i,0}(x) - u_i^0(0,x))]e^{-z}, \quad (2.25)$$

$$B_I^1(z,x) = B_1(x) + \nabla \times [n_\alpha^0(0,x)(u_{\alpha,0}(x) - u_\alpha^0(0,x))](1 - e^{-z}). \quad (2.26)$$

Then we have

$$\operatorname{div} B_I^1 = 0. \quad (2.27)$$

According to the asymptotic expansions above, set

$$n_{\alpha,\tau,I}(t,x) = n_\alpha^0(t,x) + \tau^2 n_{\alpha,I}^1(z,x), \quad (2.28)$$

$$u_{\alpha,\tau,I}(t,x) = \tau(u_\alpha^0(t,x) + u_{\alpha,I}^0(z,x)), \quad (2.29)$$

$$E_{\tau,I}(t,x) = E^0(t,x) + \tau^2(E_c^1(t,x) + E_I^1(z,x)), \quad (2.30)$$

$$B_{\tau,I}(t,x) = \tau B^0(t,x) + \tau^3 B_I^1(z,x). \quad (2.31)$$

Then we have

$$(n_{\alpha,\tau,I}, u_{\alpha,\tau,I}, E_{\tau,I}, B_{\tau,I})|_{t=0} = (n_{\alpha,0}, \tau u_{\alpha,0}, E_0, \tau B_0) + \tau^2(n_{\alpha,1}, 0, E_1 + E_c^1(0,x), \tau B_1). \quad (2.32)$$

Moreover, equations (2.4), (2.8) and (2.24) imply that

$$\operatorname{div} E_{\tau,I} = n_{i,\tau,I} - n_{e,\tau,I}, \quad (2.33)$$

and equations (2.5) and (2.27) imply that

$$\operatorname{div} B_{\tau,I} = 0. \quad (2.34)$$

Define the remainders $R_{n_\alpha}^{\tau,I}$, $R_{u_\alpha}^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ by

$$\partial_t n_{\alpha,\tau,I} + \frac{1}{\tau} \operatorname{div}(n_{\alpha,\tau,I} u_{\alpha,\tau,I}) = R_{n_\alpha}^{\tau,I}, \quad (2.35)$$

$$\partial_t u_{\alpha,\tau,I} + \frac{1}{\tau} (u_{\alpha,\tau,I} \cdot \nabla) u_{\alpha,\tau,I} + \frac{1}{\tau} \nabla h(n_{\alpha,\tau,I}) = \frac{q_\alpha(E_{\tau,I} + u_{\alpha,\tau,I} \times B_{\tau,I})}{\tau} - \frac{u_{\alpha,\tau,I}}{\tau^2} + R_{u_\alpha}^{\tau,I}, \quad (2.36)$$

$$\partial_t E_{\tau,I} - \frac{1}{\tau} \nabla \times B_{\tau,I} = \frac{n_{e,\tau,I} u_{e,\tau,I} - n_{i,\tau,I} u_{i,\tau,I}}{\tau} + R_E^{\tau,I}, \quad (2.37)$$

$$\partial_t B_{\tau,I} + \frac{1}{\tau} \nabla \times E_{\tau,I} = R_B^{\tau,I}. \quad (2.38)$$

Because that there is $\eta \in [0, t] \subset [0, T_1]$, such that

$$n_\alpha^0(t, x) - n_\alpha^0(0, x) = t \partial_t n_\alpha^0(\eta, x) = \tau^2 z \partial_t n_\alpha^0(\eta, x), \quad (2.39)$$

then we have

$$(n_\alpha^0(t, x) - n_\alpha^0(0, x)) u_{\alpha, I}^0 = O(\tau^2). \quad (2.40)$$

After a simple calculation, we get

$$R_{n_\alpha}^{\tau, I} = O(\tau^2), \quad R_{u_\alpha}^{\tau, I} = O(\tau), \quad R_E^{\tau, I} = O(\tau^2), \quad R_B^{\tau, I} = 0. \quad (2.41)$$

Lemma 2.1. *Let $s \geq 3$ be an integer. For given smooth data, the remainders $R_{n_\alpha}^{\tau, I}$, $R_{u_\alpha}^{\tau, I}$, $R_E^{\tau, I}$ and $R_B^{\tau, I}$ satisfy*

$$\sup_{0 \leq t \leq T_1} \|(R_{n_\alpha}^{\tau, I}, R_E^{\tau, I})(t, \cdot)\|_s \leq C\tau^2, \quad \sup_{0 \leq t \leq T_1} \|R_{u_\alpha}^{\tau, I}(t, \cdot)\|_s \leq C\tau, \quad R_B^{\tau, I} = 0, \quad (2.42)$$

where $C > 0$ is a constant independent of τ .

3 Proof of the convergence result

In this section we prove the main convergence result from approximate periodic solution to exact solution to two-fluid Euler-Maxwell equations. Let $(n_\alpha^\tau, u_\alpha^\tau, E^\tau, B^\tau)$ be the exact solution to (1.11)-(1.14) with initial data $(n_{\alpha, 0}^\tau, u_{\alpha, 0}^\tau, E_0^\tau, B_0^\tau)$ and $(n_{\alpha, \tau}, u_{\alpha, \tau}, E_\tau, B_\tau)$ be an approximate periodic solution defined on $[0, T_1]$, with

$$(n_{\alpha, \tau}, u_{\alpha, \tau}, E_\tau, B_\tau) \in C([0, T_1], H^{s+1}(\mathbb{T})) \cap C^1([0, T_1], H^s(T)).$$

By Lemma 1.1, the exact solution $(n_\alpha^\tau, u_\alpha^\tau, E^\tau, B^\tau)$ is defined in a time interval $[0, T_1^\tau]$ with $T_1^\tau > 0$. Since $n_\alpha^\tau \in C([0, T_1^\tau], H^s(\mathbb{T}))$ and the embedding from $H^s(\mathbb{T})$ to $C(\mathbb{T})$ is continuous, we have $n_\alpha^\tau \in C([0, T_1^\tau] \times \mathbb{T})$. From (1.17) and (1.18) and assumption $n_{\alpha, 0}^\tau \geq k > 0$, we deduce that there exist $T_2^\tau \in (0, T_1^\tau)$ and a constant $C_0 > 0$, independent of τ , such that

$$\frac{k}{2} \leq n_\alpha^\tau \leq C_0, \quad \forall (t, x) \in [0, T_2^\tau] \times \mathbb{T}.$$

Similarly, the function $t \mapsto \|(n_\alpha^\tau(t, \cdot), u_\alpha^\tau(t, \cdot), E^\tau(t, \cdot), B^\tau(t, \cdot))\|_s$ is continuous in $C([0, T_2^\tau])$. From (1.17), the sequence $(\|(n_\alpha^\tau(0, \cdot), u_\alpha^\tau(0, \cdot), E^\tau(0, \cdot), B^\tau(0, \cdot))\|_s)_{\tau > 0}$ is bounded. Then there exist $T_3^\tau \in (0, T_2^\tau]$ and a constant, still denoted by C_0 , such that

$$\|(n_\alpha^\tau(t, \cdot), u_\alpha^\tau(t, \cdot), E^\tau(t, \cdot), B^\tau(t, \cdot))\|_s \leq C_0, \quad \forall t \in (0, T_3^\tau].$$

Then we define $T^\tau = \min(T_1, T_3^\tau) > 0$, such that the exact solution and the approximate are both defined on $[0, T^\tau]$, and the exact solution satisfy:

$$\frac{k}{2} \leq n_\alpha^\tau \leq C, \quad \|(n_\alpha^\tau, u_\alpha^\tau, E^\tau, B^\tau)\|_s \leq C, \quad (3.1)$$

where $C > 0$ is a constant independent of τ . Obviously it is valid from the paper [6].

Let

$$(N_\alpha^\tau, U_\alpha^\tau, F^\tau, G^\tau) = (n_\alpha^\tau - n_{\alpha,\tau}, u_\alpha^\tau - u_{\alpha,\tau}, E^\tau - E_\tau, B^\tau - B_\tau), \tag{3.2}$$

obviously the error function $(N_\alpha^\tau, U_\alpha^\tau, F^\tau, G^\tau)$ satisfies:

$$\begin{aligned} & \partial_t N_\alpha^\tau + \frac{1}{\tau} ((U_\alpha^\tau + u_{\alpha,\tau}) \cdot \nabla) N_\alpha^\tau + \frac{1}{\tau} (N_\alpha^\tau + n_{\alpha,\tau}) \operatorname{div} U_\alpha^\tau \\ &= -\frac{1}{\tau} (N_\alpha^\tau \operatorname{div} u_{\alpha,\tau} + (U_\alpha^\tau \cdot \nabla) u_{\alpha,\tau}) - R_{n_\alpha}^\tau, \end{aligned} \tag{3.3}$$

$$\begin{aligned} & \partial_t U_\alpha^\tau + \frac{1}{\tau} ((U_\alpha^\tau + u_{\alpha,\tau}) \cdot \nabla) U_\alpha^\tau + \frac{1}{\tau} h'(N_\alpha^\tau + n_{\alpha,\tau}) \nabla N_\alpha^\tau \\ &= -\frac{1}{\tau} [(U_\alpha^\tau \cdot \nabla) u_{\alpha,\tau} + (h'(N_\alpha^\tau + n_{\alpha,\tau}) + h'(n_{\alpha,\tau}))] \\ & \quad - \frac{1}{\tau^2} U_\alpha^\tau - \frac{1}{\tau} q_\alpha [F^\tau + (U_\alpha^\tau + u_{\alpha,\tau}) \times G^\tau + U_\alpha^\tau \times B_\tau] - R_{u_\alpha}^\tau, \end{aligned} \tag{3.4}$$

$$\begin{aligned} & \partial_t F^\tau - \frac{1}{\tau} \nabla \times G^\tau \\ &= \frac{1}{\tau} [(N_e^\tau U_e^\tau - N_i^\tau U_i^\tau) + (n_{e,\tau} U_e^\tau - n_{i,\tau} U_i^\tau) + (N_e^\tau u_{e,\tau} - N_i^\tau u_{i,\tau})] - R_E^\tau, \end{aligned} \tag{3.5}$$

$$\partial_t G^\tau + \frac{1}{\tau} \nabla \times F^\tau = 0, \quad \operatorname{div} F^\tau = N_i^\tau - N_e^\tau. \tag{3.6}$$

Set

$$\begin{aligned} W_I^\tau &= \begin{pmatrix} N_\alpha^\tau \\ U_\alpha^\tau \end{pmatrix}, \quad W_{II}^\tau = \begin{pmatrix} F^\tau \\ G^\tau \end{pmatrix}, \quad W^\tau = \begin{pmatrix} W_I^\tau \\ W_{II}^\tau \end{pmatrix} = \begin{pmatrix} N_\alpha^\tau \\ U_\alpha^\tau \\ F^\tau \\ G^\tau \end{pmatrix}, \\ H_1(W_I^\tau) &= \begin{pmatrix} -(U_\alpha^\tau \cdot \nabla) n_{\alpha,\tau} - N_\alpha^\tau \operatorname{div} u_{\alpha,\tau} \\ -(U_\alpha^\tau \cdot \nabla) u_{\alpha,\tau} - (h'_\alpha(N_\alpha^\tau + n_{\alpha,\tau}) - h'_\alpha(n_{\alpha,\tau})) \nabla n_{\alpha,\tau} \end{pmatrix}, \\ H_2(W_I^\tau) &= \begin{pmatrix} 0 \\ -U_\alpha^\tau \end{pmatrix}, \quad H_3(W^\tau) = \begin{pmatrix} 0 \\ q_\alpha [F^\tau + (U_\alpha^\tau + u_{\alpha,\tau}) \times G^\tau + U_\alpha^\tau \times B_\tau] \end{pmatrix}, \\ R^\tau &= \begin{pmatrix} R_{n_\alpha}^\tau \\ R_{u_\alpha}^\tau \end{pmatrix}, \quad A_j^I(n_\alpha^\tau, u_\alpha^\tau) = \begin{pmatrix} u_{\alpha,j}^\tau & n_\alpha^\tau e_i^t \\ h'_\alpha(n_\alpha^\tau) e_j & u_{\alpha,j}^\tau I_3 \end{pmatrix}, \quad j=1,2,3, \end{aligned}$$

where $\alpha = e, i$, (e_1, e_2, e_3) is the canonical basis of R^3 , y_j denotes the i th component of $y \in R^3$ and I_3 is the 3×3 unit matrix. Then systems (3.3) and (3.4) for unknown W_I^τ can be rewritten as

$$\partial_t W_I^\tau + \frac{1}{\tau} \sum_{i=1}^3 A_i^I(n_\alpha^\tau, u_\alpha^\tau) \partial_{x_i} W_I^\tau = \frac{1}{\tau} (H_1(W_I^\tau) + H_3(W^\tau)) + \frac{1}{\tau^2} H_2(W_I^\tau) - R^\tau. \tag{3.7}$$

It is symmetrizable hyperbolic with symmetrizer

$$A_0^I(n_\alpha^\tau) = \begin{pmatrix} (n_\alpha^\tau)^{-1} & 0 \\ 0 & (h'_\alpha(n_\alpha^\tau))^{-1} I_3 \end{pmatrix},$$

which is a positive definite matrix when $0 < \frac{k}{2} \leq n_\alpha^\tau = N_\alpha^\tau + n_{\alpha,\tau} \leq C$, where $\alpha = e, i$. Moreover

$$\tilde{A}_i^I(n_\alpha^\tau, u_\alpha^\tau) = A_0^I(n_\alpha^\tau) A_i^I(n_\alpha^\tau, u_\alpha^\tau) = u_{\alpha,j}^\tau A_0^I(n_\alpha^\tau) + D_j \quad (3.8)$$

is symmetric for all $1 \leq j \leq 3$, where each D_j is a constant matrix

$$D_j = \begin{pmatrix} 0 & e_j^t \\ e_j & 0 \end{pmatrix}.$$

In order to prove Theorem 1.1, it suffices to establish uniform estimates of W^τ with respect to τ . In what follows, we denote by $C > 0$ various constants independent of τ and for $\beta \in N^3$, $(W_{I,\beta}^\tau, W_{II,\beta}^\tau) = \partial_x^\beta (W_I^\tau, W_{II}^\tau)$, etc. The main estimates are contained in the following two lemmas for W_I^τ and W_{II}^τ , respectively. We first consider the estimate for W_I^τ .

Lemma 3.1. *Under the assumptions of Theorem 1.1, for all $t \in (0, T^\tau]$, as $\tau \rightarrow 0$ we have*

$$\|W_I^\tau(t)\|_s^2 + \frac{1}{\tau^2} \int_0^t \|U_\alpha^\tau(\xi)\|_s^2 d\xi \leq C \int_0^t (\|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi)\|_s^4) d\xi + C\tau^4. \quad (3.9)$$

Proof. Differentiating equation (3.7) with respect to x yields and then multiplying by $A_0^I(n_\alpha^\tau)$ and taking the inner product of the resulting equation with $W_{I,\beta}^\tau$, we obtain

$$\begin{aligned} & \frac{d}{dt} (A_0^I(n_\alpha^\tau) W_{I,\beta}^\tau, W_{I,\beta}^\tau) - \frac{2}{\tau} (A_0^I(n_\alpha^\tau) \partial_x^\beta H_2(W_I^\tau), W_{I,\beta}^\tau) \\ &= \frac{2}{\tau} \left((A_0^I(n_\alpha^\tau) [\partial_x^\beta H_1(W_I^\tau) + \partial_x^\beta H_3(W^\tau)] \cdot W_{I,\beta}^\tau, W_{I,\beta}^\tau) \right. \\ & \quad \left. + \frac{2}{\tau} (J_\beta^\tau, W_{I,\beta}^\tau) + (\operatorname{div} A_\tau^I(n_\alpha^\tau, u_\alpha^\tau) W_{I,\beta}^\tau, W_{I,\beta}^\tau) - 2(A_0^I(n_\alpha^\tau) \partial_x^\beta R^\tau, W_{I,\beta}^\tau) \right), \end{aligned} \quad (3.10)$$

where

$$J_\beta^\tau = -\sum_{j=1}^3 A_0^I(n_\alpha^\tau) \left[\partial_x^\beta (A_j^I(n_\alpha^\tau, u_\alpha^\tau) \partial_{x_j} W_I^\tau) - A_j^I(n_\alpha^\tau, u_\alpha^\tau) \partial_x^\beta \partial_{x_j} W_I^\tau \right], \quad (3.11)$$

$$\operatorname{div} A_\tau^I(n_\alpha^\tau, u_\alpha^\tau) = \partial_t A_0^I(n_\alpha^\tau) + \frac{1}{\tau} \sum_{j=1}^3 \partial_{x_j} \tilde{A}_j^I(n_\alpha^\tau, u_\alpha^\tau). \quad (3.12)$$

From estimating each term of equation (3.10), for $\forall |\beta| \leq s$, we have

$$\frac{d}{dt} (A_0^I(n_\alpha^\tau) W_{I,\beta}^\tau, W_{I,\beta}^\tau) + \frac{k}{\tau^2} \leq \frac{C_\varepsilon}{\tau^2} \|U_\alpha^\tau\|_s^2 + C_\varepsilon (\|W^\tau\|_s^2 + \|W^\tau\|_s^4) + C_\varepsilon \tau^4. \quad (3.13)$$

Integrating this equation over $(0, t)$ with $t \in (0, T^\tau)$ and summing up over all $|\beta| \leq s$, taking $\varepsilon \geq 0$ sufficiently small that the term including $\frac{C_\varepsilon}{\tau^2} \|U_\alpha^\tau\|_s^2$ can be controlled by the left-hand side, together with condition (1.18) for the initial data, we get (3.9). \square

The estimate for W_I^τ is similar to the paper [6], so we give the simple steps. Now we establish the estimate for W_{II}^τ , it will be complicated because there exist both electron and ion.

Lemma 3.2. *Under the assumptions of Theorem 1.1, for all $t \in (0, T^\tau]$, as $\tau \rightarrow 0$ we have*

$$\|W_{II}^\tau(t)\|_s^2 \leq \int_0^t \left(\frac{1}{\tau^2} \|U_\alpha^\tau(t)\|_s^2 + \|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi)\|_s^4 \right) d\xi + C\tau^4. \quad (3.14)$$

Proof. For a multi-index $\beta \in N^3$ with $|\beta| \leq s$, differentiating the equations (3.5) and (3.6) with respect to x , we have

$$\begin{aligned} \partial_t F_\beta^\tau - \frac{1}{\tau} \nabla \times G_\beta^\tau = & \frac{1}{\tau} \partial_x^\beta \left[(N_e^\tau U_e^\tau - N_i^\tau U_i^\tau) + (n_{e,\tau} U_e^\tau - n_{i,\tau} U_i^\tau) \right. \\ & \left. + (N_e^\tau u_{e,\tau} - N_i^\tau u_{i,\tau}) \right] - \partial_x^\beta R_E^\tau, \end{aligned} \quad (3.15)$$

and

$$\partial_t G_\beta^\tau + \frac{1}{\tau} \nabla \times F_\beta^\tau = 0. \quad (3.16)$$

Taking the inner product of equation (3.15) with F_β^τ and taking the inner product of equation (3.16) with G_β^τ , then we have

$$\begin{aligned} \int_{\mathbb{T}} \frac{1}{2} \left[\partial_t (F_\beta^\tau)^2 + \partial_t (G_\beta^\tau)^2 \right] dx = & \frac{1}{\tau} \int_{\mathbb{T}} \partial_x^\beta \left[(N_e^\tau U_e^\tau - N_i^\tau U_i^\tau) + (n_{e,\tau} U_e^\tau - n_{i,\tau} U_i^\tau) \right. \\ & \left. + (N_e^\tau u_{e,\tau} - N_i^\tau u_{i,\tau}) \right] \cdot F_\beta^\tau - \partial_x^\beta R_E^\tau \cdot F_\beta^\tau dx, \end{aligned} \quad (3.17)$$

there because

$$\int_{\mathbb{T}} \left(\frac{1}{\tau} \nabla \times G_\beta^\tau \cdot F_\beta^\tau - \frac{1}{\tau} \nabla \times F_\beta^\tau \cdot G_\beta^\tau \right) dx = \frac{1}{\tau} \int_{\mathbb{T}} \operatorname{div} (F_\beta^\tau \times G_\beta^\tau) dx = 0. \quad (3.18)$$

Then we have

$$\begin{aligned} \frac{d}{dt} \|W_{II\beta}^\tau(t)\|^2 \leq & \int_{\mathbb{T}} \left| \partial_x^\beta \left[(N_e^\tau U_e^\tau - N_i^\tau U_i^\tau) + (n_{e,\tau} U_e^\tau - n_{i,\tau} U_i^\tau) \right. \right. \\ & \left. \left. + (N_e^\tau u_{e,\tau} - N_i^\tau u_{i,\tau}) \right] \cdot F_\beta^\tau \right| dx + \int_{\mathbb{T}} \left| \partial_x^\beta R_E^\tau \cdot F_\beta^\tau \right| dx. \end{aligned} \quad (3.19)$$

Using energy estimate, (1.21) and Lemma 2.1, we obtain

$$\frac{d}{dt} \|W_{II\beta}^\tau(t)\|^2 \leq \frac{1}{2\tau^2} \|U_\alpha^\tau(t)\|_s^2 + C \|W^\tau(\xi)\|_s^2 + C \|W^\tau(\xi)\|_s^4 + C\tau^4. \quad (3.20)$$

Integrating (3.20) over $(0, t)$, summing up over β satisfying $|\beta| \leq s$ and using (1.18), we obtain the lemma. \square

Proof of Theorem 1.1. Let $\tau \rightarrow 0$ and $\varepsilon > 0$ be sufficiently small. By Lemma 3.1 and Lemma 3.2, for $t \in (0, T^\tau]$ we have

$$\|W^\tau(t)\|_s^2 + \frac{1}{\tau^2} \int_0^t \|U_\alpha^\tau(t)\|_s^2 d\xi \leq C \int_0^t (\|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi)\|_s^4) d\xi + C\tau^4. \quad (3.21)$$

Let

$$y(t) = C \int_0^t (\|W^\tau(\xi)\|_s^2 + \|W^\tau(\xi)\|_s^4) d\xi + C\tau^4. \quad (3.22)$$

Then it follows from (3.21) that

$$\|W^\tau(t)\|_s^2 \leq y(t), \quad \frac{1}{\tau^2} \int_0^t \|U_\alpha^\tau(t)\|_s^2 d\xi \leq y(t), \quad \forall t \in (0, T^\tau], \quad (3.23)$$

$$y'(t) = C(\|W^\tau(t)\|_s^2 + \|W^\tau(t)\|_s^4) \leq C(y(t) + y^2(t)), \quad (3.24)$$

with

$$y(0) = C\tau^4. \quad (3.25)$$

From Gronwall inequality, we get

$$y(t) \leq C\tau^4 e^{Ct} \leq C\tau^4 e^{CT_1}, \quad \forall t \in [0, T^\tau]. \quad (3.26)$$

Therefore, from (3.23) we obtain

$$\|W^\tau(t)\|_s \leq \sqrt{y(t)} \leq C\tau^2, \quad \int_0^t \|U_\alpha^\tau(\xi)\|_s^2 d\xi \leq \tau^2 y(t) \leq C\tau^6. \quad (3.27)$$

By a standard argument on the time extension of smooth solutions, we obtain $T_3^\tau \geq T_1$, i.e. $T^\tau = T_1$. This finishes the proof of Theorem 1.1.

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