Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2017.m1426

Vol. **10**, No. 1, pp. 145-166
February 2017

Nonconforming Finite Element Methods for Wave Propagation in Metamaterials

Changhui Yao [1,*] and Lixiu Wang [2]

[1] *School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, 450001, P. R. China.*
[2] *Beijing Computational Science Research Center, Beijing, 100193, P. R. China.*

Received 5 November 2014; Accepted 19 October 2016

Abstract. In this paper, nonconforming mixed finite element method is proposed to simulate the wave propagation in metamaterials. The error estimate of the semi-discrete scheme is given by convergence order $O(h^2)$, which is less than 40 percent of the computational costs comparing with the same effect by using Nédélec-Raviart element. A Crank-Nicolson full discrete scheme is also presented with $O(\tau^2+h^2)$ by traditional discrete formula without using penalty method. Numerical examples of 2D TE, TM cases and a famous re-focusing phenomena are shown to verify our theories.

AMS subject classifications: 65N30 , 65N15 , 35J25

Key words: Wave Propagation, Metamaterials, Nonconforming Mixed Finite Element, Error Estimates.

1. Introduction

The investigations of wave propagation in Metamaterials have attracted researchers from many areas such as construction of perfect lens, sub-wavelength imaging and cloaking devices. Many numerical simulations have been done on some interesting exotic properties such as negative refractive index and amplification of evanescent waves in Metamaterials which structured electromagnetic composite materials [1, 2].

Generally, numerical simulations in electromagnetic system employ edge finite element method [3–6, 17]. The main advantage is that the spurious solutions can be avoided simultaneously because of the property of curl conforming. In [9], the authors considered three popular dispersives medium models (the isotropic cold plasma medium, one-pole Debye medium and two-pole Lorentz medium) of time-dependent Maxwell's equations in a bounded three-dimensional domain by Nédélec's element, and obtained the optimal order error estimates. In [12], the authors derived the global

*Corresponding author. *Email addresses:* `chyao@lsec.cc.ac.cn` (C.-H. Yao), `lxwang@csrc.ac.cn` (L.-X. Wang)

superconvergence results for semi-discrete scheme. In [13–15], they developed a leap-frog mixed finite element scheme for solving Maxwell's equations. The more merit discrete schemes in time direction can be found in [20, 21]. In [16], the interior penalty discontinuous Galerkin (DG) methods for the time-dependent Maxwell's equations in cold plasma were set up. The above studies are only concentrated on the family of Nédélec's element. However, the Nédélec's element broke down for large-scale computations due to the fact that they could not represent purely TE fields [8]. The others of finite element methods such as C^0-conforming vector nodal finite element methods [7] and nonconforming finite element method [23–25] were also explored by penalty techniques.

The first constructive theoretical and numerical analysis for Maxwell's equations by nonconforming finite element methods can be found in S. C. Brenner's works [23–25], where the Crouzeix-Raviart type triangular nonconforming finite element approximating to two dimensional $curl - curl$ system was studied. And numerical experiments indicated that the traditional weak formula could not lead to a convergence scheme even if the mesh is refined. Therefore, the discrete formula was modified by adding penalty terms, which involved the tangent and normal jumps. The crucial difference is that the piecewise broken $H(curl)\bigcap H(div)$ semi-norm, unlike the piecewise broken H^1 semi-norm for Poisson problem, is too weak to control the jumps. Hence the two terms involving the jumps have to be included in the discrete formula so as to control the consistency error.

Based on the above discussion, it is necessary to reestablish a framework of nonconforming mixed finite element methods approximate the electromagnetic system by traditional discrete scheme without penalty techniques. In [2], the authors summarized a list of ten interesting topics to be explored which concluded the investigations of nonconforming finite element methods. In [19,22], the authors provided a family of rectangular nonconforming mixed finite element to approximate electromagnetic system, whose theoretical and numerical analysis demonstrated the modeling problems worked successfully. How do these nonconforming mixed finite elements perform in applications?

In this paper, we consider wave propagation in metamaterials by nonconforming mixed finite element method. Re-focusing property of metamaterials can be found clearly. The main advantages conclude the three facts: the first one is the curl conforming in the means of integration for the approximation space of $H(curl,\Omega)$; the second is the transformational relation between the differential operator and the interpolation operator, which will be shown in the following section; the third is the lower computational cost than the corresponding Nédélec's element, which can be reflected by the degrees of freedoms. Another wonderful merit is the superconvergence of consistency term, which leads to overcome the weakness of the discrete norm shown in [23–25]. In the meaning time, we provide the error estimates of the semi-discrete scheme and Crank-Nicolson full discrete scheme for wave propagation model in metamaterials.

The rest of this paper is organized as follows. In Section 2, model of wave propagation presented and a variational formula is provided based on Helmholtz decomposition. In Section 3, new nonconforming mixed finite element space is constructed and some important properties are established. Especially, we use Lin-method to proof the consistency error. Error estimate of the semi-discrete scheme is set up. In Section 4, Crank-Nicolson full discrete scheme for wave propagation model in metamaterials is given and convergence order is obtained. In the last section, TE, TM case of wave propagation model in metamaterials and the re-focusing property are shown numerically.

2. Equations of wave propagation in meta-materials

The wave propagation in meta-materials can be employed the governing equations [1,2]

$$\epsilon_0\mathbf{E}_t - curl\mathbf{H} = -\mathbf{J} + \mathbf{f}, \qquad (0,T]\times\Omega, \tag{2.1}$$
$$\mu_0\mathbf{H}_t + curl\mathbf{E} = -\mathbf{K} + \mathbf{g}, \qquad (0,T]\times\Omega, \tag{2.2}$$
$$\frac{1}{\epsilon_0\omega_{pe}^2}\mathbf{J}_t + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\mathbf{J} = \mathbf{E}, \qquad (0,T]\times\Omega, \tag{2.3}$$
$$\frac{1}{\mu_0\omega_{pm}^2}\mathbf{K}_t + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}\mathbf{K} = \mathbf{H}, \qquad (0,T]\times\Omega, \tag{2.4}$$

where ϵ_0 denotes the permittivity of free space and μ_0 denotes the permeability of free space, ω_{pe}, ω_{pm} are the electric and magnetic plasma frequencies, respectively. Γ_e, Γ_m are the electric and magnetic damping frequencies. $\mathbf{E}(\mathbf{x},t), \mathbf{H}(\mathbf{x},t)$ are the electric and magnetic fields,respectively, and $\mathbf{J}(\mathbf{x},t), \mathbf{K}(\mathbf{x},t)$ are the induced electric and magnetic currents,respectively. $\mathbf{f}, \mathbf{g}$ are added source terms. We also assume that the boundary of Ω is perfect conducting

$$\mathbf{n}\times\mathbf{E} = 0, \quad \text{on} \quad \partial\Omega, \tag{2.5}$$

where $\mathbf{n}$ is the unit outward norm to $\partial\Omega$. Furthermore, the initial conditions are

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}), \quad \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}), \tag{2.6}$$
$$\mathbf{J}(\mathbf{x},0) = \mathbf{J}_0(\mathbf{x}), \quad \mathbf{K}(\mathbf{x},0) = \mathbf{K}_0(\mathbf{x}), \tag{2.7}$$

where $\mathbf{E}_0(\mathbf{x}), \mathbf{H}_0(\mathbf{x}, \mathbf{J}_0(\mathbf{x}), \mathbf{K}_0(\mathbf{x})$ are some given functions.

The existence,uniqueness and stability of equations (2.2)-(2.7) can be found in [1].

Define the following notations:

$$\begin{aligned}
&H(curl) = \{\mathbf{v} = (v_1,v_2,v_3)\in[L^2(\Omega)]^3 : curl\ \mathbf{v}\in[L^2(\Omega)]^3\},\\
&H_0(curl) = \{\mathbf{v}\in H(curl) : \mathbf{n}\times\mathbf{v} = 0, \text{on} \ \ \partial\Omega\},\\
&H(div^0) = \{\mathbf{v} = (v_1,v_2,v_3)\in[L^2(\Omega)]^3 : div\mathbf{v} = 0\},\\
&H^s(\Omega) = \{\phi\in L^2(\Omega)|\partial^\alpha\phi\in L^2(\Omega), \forall\ |\alpha|\le s\}.
\end{aligned}$$

Since Gauss laws must be successful, we have to employ the Helmholtz decomposition. A function $\hat{\mathbf{u}}\in H_0(curl,\Omega)$ can be written uniquely as

$$\hat{\mathbf{u}} = \mathbf{u} + \nabla w,$$

where $\mathbf{u} \in H_0(curl,\Omega) \cap H(div^0,\Omega)$ and $w \in H_0^1(\Omega)$. With $H_0(curl,\Omega) \cap H(div^0,\Omega) \hookrightarrow [H^s(\Omega)]^3, s \geq \frac{1}{2}$, the variational problem of (2.2)-(2.7) is: find $\mathbf{E} \in C^1(0,T;H_0(curl,\Omega) \cap H(div^0,\Omega))$, $\mathbf{J} \in C^1(0,T;H(curl,\Omega) \cap H(div^0,\Omega))$ and $\mathbf{H},\mathbf{K} \in C^1(0,T;[L^2(\Omega)]^3)$ such that $\forall \mathbf{\Phi} \in H_0(curl,\Omega) \cap H(div^0,\Omega)$, $\bar{\mathbf{\Phi}} \in H(curl,\Omega) \cap H(div^0,\Omega)$, $\mathbf{\Psi},\bar{\mathbf{\Psi}} \in [L^2(\Omega)]^3$

$$\epsilon_0(\mathbf{E}_t,\mathbf{\Phi}) - (\mathbf{H}, curl\mathbf{\Phi}) = -(\mathbf{J},\mathbf{\Phi}) + (\mathbf{f},\mathbf{\Phi}), \tag{2.8}$$

$$\mu_0(\mathbf{H}_t,\mathbf{\Psi}) + (curl\mathbf{E},\mathbf{\Psi}) = -(\mathbf{K},\mathbf{\Psi}) + (\mathbf{g},\mathbf{\Psi}), \tag{2.9}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}(\mathbf{J}_t,\bar{\mathbf{\Phi}}) + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}(\mathbf{J},\bar{\mathbf{\Phi}}) = (\mathbf{E},\bar{\mathbf{\Phi}}), \tag{2.10}$$

$$\frac{1}{\mu_0\omega_{pm}^2}(\mathbf{K}_t,\bar{\mathbf{\Psi}}) + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}(\mathbf{K},\bar{\mathbf{\Psi}}) = (\mathbf{H},\bar{\mathbf{\Psi}}). \tag{2.11}$$

3. New nonconforming finite element methods

Assume $\Omega = [0,1]^3$ and $\mathcal{J}_h$ be the uniform partition. Considering the reference element $\hat{e} = [-1,1]^3$, we can define $\widehat{V} = \widehat{V}_x \times \widehat{V}_y \times \widehat{V}_z$ be on $\widehat{e}$ by

$$\widehat{V}_{\hat{x}} = span\{1,\hat{y},\hat{z},\hat{y}^2-\hat{z}^2\},$$
$$\widehat{V}_{\hat{y}} = span\{1,\hat{z},\hat{x},\hat{z}^2-\hat{x}^2\},$$
$$\widehat{V}_{\hat{z}} = span\{1,\hat{x},\hat{y},\hat{x}^2-\hat{y}^2\}.$$

Define the interpolation by $\widehat{\pi} : [H^2(\widehat{e})]^3 \to \widehat{V}(\widehat{e})$ by

$$\frac{1}{|\hat{F}_i|}\int_{\hat{F}_i}(\hat{\pi}\hat{\varphi} - \hat{\varphi})d\hat{s} = 0, \quad 1 \leq i \leq 6.$$

Let $\widehat{W} = \widehat{W}_x \times \widehat{W}_y \times \widehat{W}_z$, where

$$\widehat{W}_{\hat{x}} = span\{1,\hat{y},\hat{z}\}, \quad \widehat{W}_{\hat{y}} = span\{1,\hat{z},\hat{x}\}, \quad \widehat{W}_{\hat{z}} = span\{1,\hat{x},\hat{y}\}.$$

Define the interpolation operator $\widehat{P} : [L^2(\widehat{e})]^3 \to \widehat{W}(\widehat{e})$ by: for $\overrightarrow{\hat{\psi}} = (\hat{\psi}_{\hat{x}},\hat{\psi}_{\hat{y}},\hat{\psi}_{\hat{z}})$,

$$\int_{\widehat{e}}(\widehat{P}\hat{\psi}_{\hat{l}} - \hat{\psi}_{\hat{l}})\mathbf{q}d\hat{x}d\hat{y}d\hat{z} = 0, \quad \forall \mathbf{q} \in P_1,$$

where P_1 is the $1th$ polynomial space.

After obtaining the basis function on the reference hexahedron $\hat{e}$, we can derive the basis function on a general element e by mapping F_e. To make the degrees of freedom invariant, we need the following special transformation

$$\mathbf{u} \circ F_e = B_e^{-1}\hat{\mathbf{u}}, \tag{3.1}$$

where F_e is the affine mapping. For technical reasons, we assume that B_e is a transform matrix.

The unit outward normal vector $\mathbf{n}$ to e is obtained by the transformation

$$\mathbf{n}\circ F_e=\frac{B_e^{-1}\hat{\mathbf{n}}}{|B_e^{-1}\hat{\mathbf{n}}|}, \tag{3.2}$$

$$curl\mathbf{u}=\frac{1}{|det(B_e)|}B_e\widehat{curl}\hat{\mathbf{u}}. \tag{3.3}$$

Using the scaling argument, we denote the operator π_h and P_h translating from $\hat{\pi}$ and $\hat{P}$, respectively.

Define the nonconforming finite element space

$$V_h=\Big\{\varphi:\hat{\varphi}|_{\hat{e}}=B_e\varphi\circ F_e\in\hat{V}\Big\},\quad W_h=\Big\{\psi:\hat{\psi}|_{\hat{e}}=B_e\psi\circ F_e\in\hat{W}\Big\},$$
$$V_{0h}=\Big\{\varphi\in V_h,\frac{1}{|F|}\int_F\mathbf{n}\times\varphi=0,F\subset(\partial e\cap\partial\Omega)\Big\}.$$

Semi-discrete variational weak form can be formulated: find $\mathbf{E}_t^h\in V_{0h},\mathbf{J}\in V_h,\mathbf{H},\mathbf{K}\in W_h$ such that

$$\epsilon_0(\mathbf{E}_t^h,\mathbf{\Phi}^h)-(\mathbf{H}^h,curl\mathbf{\Phi}^h)=-(\mathbf{J}^h,\mathbf{\Phi}^h)+(\mathbf{f},\mathbf{\Phi}^h),\qquad\forall\mathbf{\Phi}^h\in V_{0h}, \tag{3.4}$$
$$\mu_0(\mathbf{H}_t^h,\mathbf{\Psi}^h)+(curl\mathbf{E}^h,\mathbf{\Psi}^h)=-(\mathbf{K}^h,\mathbf{\Psi}^h)+(\mathbf{g},\mathbf{\Psi}^h),\qquad\forall\mathbf{\Psi}^h\in W_h, \tag{3.5}$$
$$\frac{1}{\epsilon_0\omega_{pe}^2}(\mathbf{J}_t^h,\bar{\mathbf{\Phi}}^h)+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}(\mathbf{J}^h,\bar{\mathbf{\Phi}}^h)=(\mathbf{E}^h,\bar{\mathbf{\Phi}}^h),\qquad\forall\bar{\mathbf{\Phi}}^h\in V_h, \tag{3.6}$$
$$\frac{1}{\mu_0\omega_{pm}^2}(\mathbf{K}_t^h,\bar{\mathbf{\Psi}}^h)+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}(\mathbf{K}^h,\bar{\mathbf{\Psi}}^h)=(\mathbf{H}^h,\bar{\mathbf{\Psi}}^h),\qquad\forall\bar{\mathbf{\Psi}}^h\in W_h. \tag{3.7}$$

By [19,22], we have the following important properties.

Lemma 3.1. *The interpolation operators π_h and P_h can be uniquely determined.*

Lemma 3.2. *Let e_1 and e_2 be the two adjoin non-overlapping elements with a common interface such that $e_1\cap e_2=F$. Assume that $\mathbf{u}\in V_h$ defined by*

$$\mathbf{u}=\mathbf{u}_1,\ on\ e_1,\qquad\mathbf{u}=\mathbf{u}_2,\ on\ e_2. \tag{3.8}$$

Then $\int_F\mathbf{u}_1\times\mathbf{n_1}ds=\int_F\mathbf{u}_2\times\mathbf{n_2}ds$ on F, where $\mathbf{n_1}$ (resp. $\mathbf{n_2}$) is the unit normal vector of F pointing towards outside of e_1 (resp. e_2).

Proof. The proof can be carried out in exactly the same way by using the following identity: for any function $\phi\in[C_0^\infty(e_1\cup e_2\cup F)]^3$,

$$\int_{e_1\cup e_2\cup F}\mathbf{u}\cdot curl\phi d\mathbf{x}$$
$$=\int_{e_1}curl\mathbf{u}_1\cdot\phi d\mathbf{x}+\int_{e_2}curl\mathbf{u}_2\cdot\phi d\mathbf{x}+\int_F(\mathbf{u}_1\times\mathbf{n}_1+\mathbf{u}_2\times\mathbf{n}_2)\cdot\phi ds.$$

By the property of degrees of freedom $\int_F[\mathbf{u}]\times\mathbf{n}ds=0$, the proof is completed. □

Lemma 3.3. *For all $\phi \in V_h$, $div_h\phi = 0$.*

Proof. By calculating, we have

$$div_h\phi = \sum_e (div\phi|_e) = \sum_e \left(\frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial y} + \frac{\partial\phi_3}{\partial z}\right) = 0. \qquad \square$$

Lemma 3.4. *For the space pairs V_h and W_h,there holds*

$$curl_h V_h \subseteq W_h. \tag{3.9}$$

And if ϕ is a function such that both the interpolants $\pi_h\phi$ and $P_h(curl\phi)$, then $curl_h(\pi_h\phi) = P_h(curl\phi)$.

Proof. Based on the definitions of the operator π_h and P_h, for $\phi \in H(curl,\Omega) \cap [H^2(\Omega)]^3$, we have

$$\begin{aligned}
&\int_e (curl\pi_e\phi - P_e(curl\phi))dxdydz = \int_e (curl\pi_e\phi - (curl\phi))dxdydz \\
=&\int_{\partial e} \mathbf{n}\times\pi_e\phi ds - \int_e curl\phi dxdydz = \int_{\partial e}\mathbf{n}\times\phi ds - \int_e curl\phi dxdydz \\
=&\int_e curl\phi dxdydz - \int_e curl\phi dxdydz = 0.
\end{aligned}$$

Therefore, $curl\pi_h\phi = P_h(curl\phi)$. $\square$

Lemma 3.5. *Assume that $\mathbf{E},\mathbf{H},\mathbf{J},\mathbf{K} \in [H^2(\Omega)]^3$, the following interpolation error estimate holds*

$$\|\mathbf{E}-\pi_h\mathbf{E}\|_0 + \|\mathbf{H}-P_h\mathbf{H}\|_0 + \|\mathbf{J}-\pi_h\mathbf{J}\|_0 + \|\mathbf{K}-P_h\mathbf{K}\|_0 \le Ch^2. \tag{3.10}$$

Lemma 3.6. *Assume that $\mathbf{H} \in [H^2(\Omega)]^3$ and $\mathbf{\Phi} \in V^h$, the following consistency error estimate holds* [22]

$$\Big|\sum_{e\in\mathcal{J}^h}\int_{\partial e}\mathbf{H}\mathbf{n}\times\mathbf{\Phi}ds| \le Ch|\mathbf{H}|_2\|\mathbf{\Phi}\|_0. \tag{3.11}$$

And if $\mathbf{H} \in [H^3(\Omega)]^3$, the following superconvergence estimate of consistency error holds

$$\Big|\sum_{e\in\mathcal{J}^h}\int_{\partial e}\mathbf{H}\mathbf{n}\times\mathbf{\Phi}ds| \le Ch^2|\mathbf{H}|_3\|\mathbf{\Phi}\|_0. \tag{3.12}$$

Proof. Define six faces $g_i, i = 1,2,\cdots,6$ on every element e which unit outward normal vector

$$\begin{aligned}
&n = (1,0,0), \quad \text{on } g_1, \qquad && n = (-1,0,0), \quad \text{on } g_2, \\
&n = (0,1,0), \quad \text{on } g_3, \qquad && n = (0,-1,0), \quad \text{on } g_4,
\end{aligned}$$

$$n=(0,0,1),\quad \text{on } g_5,\qquad n=(0,0,-1),\quad \text{on } g_5.$$

Then, we have

$$\begin{aligned}&< \mathbf{H}, n\times\phi >\\ &=\sum_e\int_{\partial e} H_1\cdot(n_2\phi_3-n_3\phi_2)-H_2\cdot(n_2\phi_3-n_3\phi_1)+H_3\cdot(n_1\phi_2-n_2\phi_1)\\ &=\int_{g_1-g_2}(H_3\phi_2-H_2\phi_3)ds+\int_{g_3-g_4}(H_1\phi_3-H_3\phi_1)ds+\int_{g_5-g_6}(H_2\phi_1-H_1\phi_2)ds.\end{aligned}$$

Define

$$\overline{\phi_i}|_{g_j}=\frac{1}{|g_j|}\int_{g_j}\phi_i dS,\quad i\le i\le 3,\quad 1\le j\le 6.$$

By $\phi\times n=0$, on $\partial\Omega$ and $\int_F[\phi]ds=0, F\subset e\cap e', e,e'\in J_h$, we have

$$\sum_e\int_{g_1-g_2}\Big(H_3\overline{\phi_2}-H_2\overline{\phi_3}\Big)ds+\int_{g_3-g_4}\Big(H_1\overline{\phi_3}-H_3\overline{\phi_1}\Big)ds+\int_{g_5-g_6}\Big(H_2\overline{\phi_1}-H_1\overline{\phi_2}\Big)ds=0.$$

Then the consistency error estimate is:

$$\begin{aligned}< \mathbf{H}, n\times\phi =\sum_e\Big(&\int_{g_1-g_2}H_3(\phi_2-\overline{\phi_2})-H_2(\phi_3-\overline{\phi_3})ds+\int_{g_3-g_4}H_1(\phi_3-\overline{\phi_3})\\ &-H_3(\phi_1-\overline{\phi_1})ds+\int_{g_5-g_6}H_2(\phi_1-\overline{\phi_1})-H_1(\phi_2-\overline{\phi_2})ds\Big).\end{aligned}\tag{3.13}$$

For convenience, we consider the term $\int_{g_3-g_4}H_3(\phi_1-\overline{\phi_1})dxdz$. By computation:

$$(\phi_1-\overline{\phi_1})|_{g_3,g_4}=(z-z_e)\phi_{1z}-\Big((z-z_e)^2+\frac{h_z^2}{3}\Big)\frac{\phi_{1zz}}{2},$$

we have

$$\begin{aligned}&\int_{g_3-g_4}H_3(\phi_1-\overline{\phi_1})dxdz\\ =&\int_{g_3-g_4}H_3\Big[(z-z_e)\phi_{1z}-\Big((z-z_e)^2+\frac{h_z^2}{3}\Big)\frac{\phi_{1zz}}{2}\Big]dxdz\\ =&\int_e H_{3y}\Big[(z-z_e)\phi_{1z}-\Big((z-z_e)^2+\frac{h_z^2}{3}\Big)\frac{\phi_{1zz}}{2}\Big]dxdydz\\ =&\frac{1}{h_y}\int_{\widehat{e}}\widehat{H}_{3\widehat{y}}\Big[\widehat{z}\widehat{\phi}_{1\widehat{z}}-\Big(\widehat{z}^2+\frac{1}{3}\Big)\frac{\widehat{\phi}_{1\widehat{z}\widehat{z}}}{2}\Big]h_xh_yh_zd\widehat{x}d\widehat{y}d\widehat{z}.\end{aligned}\tag{3.14}$$

Define the bilinear functional

$$B(\widehat{H}_3,\widehat{\phi}_1)=\int_{\widehat{e}}\widehat{H}_{3\widehat{y}}\Big[\widehat{z}\widehat{\phi}_{1\widehat{z}}-\Big(\widehat{z}^2+\frac{1}{3}\Big)\frac{\widehat{\phi}_{1\widehat{z}\widehat{z}}}{2}\Big]d\widehat{x}d\widehat{y}d\widehat{z}.$$

Table 1: Basic functions of P_4 space.

$\widehat{H}_3$	1	$\hat{x}$	$\hat{y}$	$\hat{z}$	$\hat{x}^2$	$\hat{y}^2$	$\hat{z}^2$	$\hat{x}\hat{y}$	$\hat{x}\hat{z}$	$\hat{y}\hat{z}$
$\widehat{H}_{3\hat{y}}$	0	0	1	0	0	$2\hat{y}$	0	$\hat{x}$	0	$\hat{z}$
$\widehat{H}_3$	$\hat{x}^3$	$\hat{y}^3$	$\hat{z}^3$	$\hat{x}^2\hat{y}$	$\hat{x}^2\hat{z}$	$\hat{y}^2\hat{x}$	$\hat{y}^2\hat{z}$	$\hat{z}^2\hat{x}$	$\hat{z}^2\hat{y}$	$\hat{x}\hat{y}\hat{z}$
$\widehat{H}_{3\hat{y}}$	0	$3\hat{y}^2$	0	$\hat{x}^2$	0	$2\hat{x}\hat{y}$	$2\hat{y}\hat{z}$	0	$\hat{z}^2$	$\hat{x}\hat{z}$
$\widehat{H}_3$	$\hat{x}^4$	$\hat{y}^4$	$\hat{z}^4$	$\hat{x}^3\hat{y}$	$\hat{x}^3\hat{z}$	$\hat{y}^3\hat{x}$	$\hat{y}^3\hat{z}$	$\hat{z}^3\hat{x}$		
$\widehat{H}_{3\hat{y}}$	0	$4\hat{y}^3$	0	$\hat{x}^3$	0	$3\hat{x}\hat{y}^2$	$3\hat{y}^2\hat{z}$	0		
$\widehat{H}_3$	$\hat{z}^3\hat{y}$	$\hat{x}^2\hat{y}\hat{z}$	$\hat{x}\hat{y}^2\hat{z}$	$\hat{x}\hat{y}\hat{z}^2$	$\hat{x}^2\hat{y}^2$	$\hat{x}^2\hat{z}^2$	$\hat{y}^2\hat{z}^2$			
$\widehat{H}_{3\hat{y}}$	$\hat{z}^3$	$\hat{x}^2\hat{z}$	$2\hat{x}\hat{y}\hat{z}$	$\hat{x}\hat{z}^2$	$2\hat{x}^2\hat{y}$	0	$2\hat{y}\hat{z}^2$			

Denote

$$\widehat{\phi}_1=(1,\widehat{y},\widehat{z},\widehat{y}^2-\widehat{z}^2),\quad \widehat{\phi}_{1\widehat{z}}=(0,0,1,-2\widehat{z}),\quad \widehat{\phi}_{1\widehat{z}\widehat{z}}=(0,0,0,-2).$$

We have

$$|B(\widehat{H}_3,\widehat{\phi}_1)|\leq C\|\widehat{H}_3\|_2\|\widehat{\phi}_1\|_0.$$

Obviously, $B(P_1,\widehat{\phi}_1)=0$. Therefore,

$$|B(\widehat{H}_3,\widehat{\phi}_1)|\leq C|\widehat{H}_3|_2\|\widehat{\phi}_1\|_0.$$

When $\widehat{H}_3=\widehat{y}\widehat{z}$, we have

$$\begin{aligned} B(\widehat{H}_3,\widehat{\phi}_1)&=\int_{\widehat{e}}\Big[(0,0,\widehat{z}^2,-2\widehat{z}^3)-\Big(\widehat{z}^3+\frac{\widehat{z}}{3}\Big)(0,0,0,-1)\Big]d\widehat{x}d\widehat{y}d\widehat{z}\\ &=\int_{\widehat{e}}(0,0,\widehat{z}^2,0)=\frac{8}{3}(0,0,1,0)=\frac{1}{3}\int_{\widehat{e}}\widehat{H}_{3\widehat{y}\widehat{z}}\widehat{\phi}_{1\widehat{z}}d\widehat{x}d\widehat{y}d\widehat{z}. \end{aligned} \tag{3.15}$$

Let

$$G(\widehat{H}_3,\widehat{\phi}_1)=B(\widehat{H}_3,\widehat{\phi}_1)-\frac{1}{3}\int_{\widehat{e}}\widehat{H}_{3\widehat{y}\widehat{z}}\widehat{\phi}_{1\widehat{z}}d\widehat{x}d\widehat{y}d\widehat{z}.$$

Based on $|G(\widehat{H}_3,\widehat{\phi}_1)|\leq C\|\widehat{H}_3\|_3\|\widehat{\phi}_1\|_0$, by $G(P_2,\widehat{\phi}_1)=0$, we have

$$|G(\widehat{H}_3,\widehat{\phi}_1)|\leq C|\widehat{H}_3|_3\|\widehat{\phi}_1\|_0.$$

Therefore,

$$B(\widehat{H}_3,\widehat{\phi}_1)=\frac{1}{3}\int_{\widehat{e}}\widehat{H}_{3\widehat{y}\widehat{z}}\widehat{\phi}_{1\widehat{z}}d\widehat{x}d\widehat{y}d\widehat{z}+O(1)|\widehat{H}_3|_3\|\widehat{\phi}_1\|_0.$$

Similarly, when $\widehat{H}_3=\widehat{y}\widehat{z}^2$

$$G(\widehat{H}_3,\widehat{\phi}_1)=\Big(0,0,0,\frac{128}{45}\Big)=-\frac{4}{45}\int_{\widehat{e}}\widehat{H}_{3\widehat{y}\widehat{z}\widehat{z}}\widehat{\phi}_{1\widehat{z}\widehat{z}}d\widehat{x}d\widehat{y}d\widehat{z}.$$

Let

$$L(\widehat{H}_3,\widehat{\phi}_1)=B(\widehat{H}_3,\widehat{\phi}_1)-\frac{1}{3}\int_{\widehat{e}}\widehat{H}_{3\widehat{y}\widehat{z}}\widehat{\phi}_{1\widehat{z}}d\widehat{x}d\widehat{y}d\widehat{z}+\frac{4}{45}\int_{\widehat{e}}\widehat{H}_{3\widehat{y}\widehat{z}\widehat{z}}\widehat{\phi}_{1\widehat{z}\widehat{z}}d\widehat{x}d\widehat{y}d\widehat{z}.$$

Based on $|L(\widehat{H}_3,\widehat{\phi}_1)|\le C\|\widehat{H}_3\|_4\|\widehat{\phi}_1\|_0$, by $L(P_3,\widehat{\phi}_1)=0$, we have

$$|L(P_3,\widehat{\phi}_1)|\le C|\widehat{H}_3|_4\|\widehat{\phi}_1\|_0.$$

Therefore,

$$B(\widehat{H}_3,\widehat{\phi}_1)=\frac{1}{3}\int_{\widehat{e}}\widehat{H}_{3\widehat{y}\widehat{z}}\widehat{\phi}_{1\widehat{z}}d\widehat{x}d\widehat{y}d\widehat{z}-\frac{4}{45}\int_{\widehat{e}}\widehat{H}_{3\widehat{y}\widehat{z}\widehat{z}}\widehat{\phi}_{1\widehat{z}\widehat{z}}d\widehat{x}d\widehat{y}d\widehat{z}+O(1)|\widehat{H}_3|_4|\widehat{\phi}_1|_0.$$

From integrating by parts and the fact $\mathbf{n}\times\phi=0$ on $\partial\Omega$, we have

$$\begin{aligned}&\sum_e\int_{g_3-g_4}H_3(\phi_1-\overline{\phi_1})dxdz\\&=\sum_e\Big[\frac{h_z^2}{3}\int_e H_{3yz}\phi_{1z}-\frac{4h_z^4}{45}\int_e H_{3yzz}\phi_{1zz}+o(h^2)|H_3|_4|\phi_1|_0\Big]\\&=\sum_e\Big[-\frac{h_z^2}{3}\int_e H_{3yzz}\phi_1-\frac{4h_z^4}{45}\int_e H_{3yzzzz}\phi_1+o(h^2)|H_3|_4|\phi_1|_0\Big]\\&\le Ch^2|\widehat{H}_3|_2|\widehat{\phi}_1|_0.\end{aligned}\tag{3.16}$$

The same way can be done on the other five term in $<\mathbf{H},\mathbf{n}\times\phi>$. Then the proof is finished. □

Theorem 3.1. *Let* $(\mathbf{E}(t),H(t),\mathbf{J}(t),K(t))$ *and* $(\mathbf{E}^h(t),H^h(t),\mathbf{J}^h(t),K^h(t))$ *be the solutions of* (2.8)-(2.11) *and* (3.4)-(3.7)*, respectively. Then there exists a constant* $C=C(\epsilon_0,\mu_0,\Gamma_e,\Gamma_m,\omega_{pe},\omega_{pm})$ *independent of mesh size* h*, such that*

$$\begin{aligned}&\epsilon_0\|\mathbf{E}-\mathbf{E}^h\|_0+\mu_0\|\mathbf{H}-H^h\|_0+\frac{1}{\epsilon_0\omega_{pe}^2}\|\mathbf{J}-\mathbf{J}^h\|_0+\frac{1}{\mu_0\omega_{pm}^2}\|\mathbf{K}-K^h\|_0\\&\le Ch^2\int_0^t\Big(\|\mathbf{E}_t\|_2+\|\mathbf{E}\|_2+\|\mathbf{H}\|_3+\|\mathbf{J}_t\|_2+\|\mathbf{J}\|_2\Big)dt.\end{aligned}\tag{3.17}$$

Proof. Multiplying (2.2)-(2.4) by $\mathbf{\Phi}^h$, $\mathbf{\Psi}^h$, $\bar{\mathbf{\Phi}}^h$, $\bar{\Psi}^h$ respectively and integrating in Ω, we have the error equations

$$\epsilon_0(\mathbf{E}_t,\mathbf{\Phi}^h)-(H,curl\mathbf{\Phi}^h)=-(\mathbf{J},\mathbf{\Phi}^h)-\sum_e\int_{\partial e}H\mathbf{n}\times\mathbf{\Phi}^h+(\mathbf{f},\mathbf{\Phi}^h),\tag{3.18}$$

$$\mu_0(\mathbf{H}_t,\mathbf{\Psi}^h)+(curl\mathbf{E},\mathbf{\Psi}^h)=-(\mathbf{K},\mathbf{\Psi}^h)+(\mathbf{g},\mathbf{\Psi}^h),\tag{3.19}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}(\mathbf{J}_t,\bar{\mathbf{\Phi}}^h)+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}(\mathbf{J},\bar{\mathbf{\Phi}}^h)=(\mathbf{E},\bar{\mathbf{\Phi}}^h),\tag{3.20}$$

$$\frac{1}{\mu_0\omega_{pm}^2}(\mathbf{K}_t,\bar{\Psi}^h)+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}(\mathbf{K},\bar{\Psi}^h)=(\mathbf{H},\bar{\Psi}^h).\tag{3.21}$$

Let $\xi=(\pi_h\mathbf{E}-\mathbf{E}^h), \theta=(P_h\mathbf{H}-\mathbf{H}^h), \bar{\xi}=(\pi_h\mathbf{J}-\mathbf{J}^h)$ and $\bar{\theta}=(P_h\mathbf{K}-\mathbf{K}^h)$. We then have

$$\begin{aligned}
&\epsilon_0(\xi_t,\xi)-(\theta,curl\xi)\\
&\quad=\epsilon_0((\pi_h\mathbf{E}-\mathbf{E})_t,\xi)-(P_h\mathbf{H}-\mathbf{H},curl\xi)-(\mathbf{J}-\pi_h\mathbf{J}+\bar{\xi},\xi)-\sum_e\int_{\partial e}\mathbf{H}\cdot\mathbf{n}\times\xi,\\
&\mu_0(\theta_t,\theta)+(curl\xi,\theta)\\
&\quad=\mu_0((P_h\mathbf{H}-\mathbf{H})_t,\theta)+(curl(\pi_h\mathbf{E}-\mathbf{E}),\theta)-(\mathbf{K}-P_h\mathbf{K}+\bar{\theta},\theta),\\
&\frac{1}{\epsilon_0\omega_{pe}^2}(\bar{\xi}_t,\bar{\xi})+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}(\bar{\xi},\bar{\xi})\\
&\quad=(\pi_h\mathbf{E}-\mathbf{E}+\xi,\bar{\xi})+\frac{1}{\epsilon_0\omega_{pe}^2}((\pi_h\mathbf{J}-\mathbf{J})_t,\bar{\xi})+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}(\pi_h\mathbf{J}-\mathbf{J},\bar{\xi}),\\
&\frac{1}{\mu_0\omega_{pm}^2}(\bar{\theta}_t,\bar{\theta})+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}(\bar{\theta},\bar{\theta})\\
&\quad=(\mathbf{H}-P_h\mathbf{H}+\theta,\bar{\theta})+\frac{1}{\mu_0\omega_{pm}^2}((P_h\mathbf{K}-\mathbf{K})_t,\bar{\theta})+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}(P_h\mathbf{K}-\mathbf{K},\bar{\theta}).
\end{aligned}$$

Adding the above four equations together, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\epsilon_0\|\xi\|_0^2+\mu_0\|\theta\|_0^2+\frac{1}{\epsilon_0\omega_{pe}^2}\|\bar{\xi}\|_0^2+\frac{1}{\mu_0\omega_{pm}^2}\|\bar{\theta}\|_0^2\right)\\
&\qquad+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\|\bar{\xi}\|_0^2+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\|\bar{\theta}\|_0^2=\sum_{i=1}^{13}Err_i.
\end{aligned}\tag{3.22}$$

We can estimate them by Lemmas 3.4-3.6 and Young's inequality,

$$\begin{aligned}
Err_1&=\epsilon_0((\pi_h\mathbf{E}-\mathbf{E})_t,\xi)\le C_1\epsilon_0\|(\pi_h\mathbf{E}-\mathbf{E})_t\|_0\|\xi\|_0\\
&\le C_1\epsilon_0h^2\|\mathbf{E}_t\|_2\|\xi\|_0\le\frac{C_1\epsilon_0^2h^4}{4\delta_1}\|\mathbf{E}_t\|_2^2+\delta_1\|\xi\|_0^2, &&(3.23a)\\
|Err_2|&=(P_h\mathbf{H}-\mathbf{H},curl\xi)=0, &&(3.23b)\\
Err_3&=-(\mathbf{J}-\pi_h\mathbf{J}+\bar{\xi},\xi)=-(\mathbf{J}-\pi_h\mathbf{J}+\bar{\xi},\xi)-(\bar{\xi},\xi), &&(3.23c)\\
&-(\mathbf{J}-\pi_h\mathbf{J},\xi)\le\frac{C_2h^4}{4\delta_2}\|\mathbf{J}\|_2^2+\delta_2\|\xi|_0^2, &&(3.23d)\\
Err_4&=\sum_e\int_{\partial e}\mathbf{H}\mathbf{n}\times\xi ds\le\frac{C_3h^4}{4\delta_3}\|\mathbf{H}\|_3^2+\delta_3\|\xi\|_0^2, &&(3.23e)\\
Err_5&=\mu_0((P_h\mathbf{H}-\mathbf{H})_t,\theta)=0, &&(3.23f)\\
Err_6&=(curl(\pi_h\mathbf{E}-\mathbf{E}),\theta)=(P_hcurl\mathbf{E}-curl\mathbf{E},\theta)=0, &&(3.23g)\\
Err_7&=-(\mathbf{K}-P_h\mathbf{K}+\bar{\theta},\theta)=-(\bar{\theta},\theta), &&(3.23h)
\end{aligned}$$

$$Err_8 = (\mathbf{E} - \pi_h\mathbf{E} + \xi, \bar{\xi}) \leq \frac{C_4h^4}{4\delta_4}\|\mathbf{E}\|_2^2 + \delta_4\|\bar{\xi}|_0^2 + (\bar{\xi}, \xi), \tag{3.23i}$$

$$Err_9 = \frac{1}{\epsilon_0\omega_{pe}^2}((\pi_h\mathbf{J} - \mathbf{J})_t, \bar{\xi}) \leq \frac{C_5h^4}{4\epsilon_0^2\omega_{pe}^4\delta_5}\|\mathbf{J}_t\|_2^2 + \delta_5\|\bar{\xi}|_0^2, \tag{3.23j}$$

$$Err_{10} = \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}(\pi_h\mathbf{J} - \mathbf{J}, \bar{\xi}) \leq \frac{C_6\Gamma_e^2h^4}{4\epsilon_0^2\omega_{pe}^4\delta_6}\|\mathbf{J}\|_2^2 + \delta_6\|\bar{\xi}|_0^2, \tag{3.23k}$$

$$Err_{11} = (\mathbf{H} - P_h\mathbf{H} + \theta, \bar{\theta}) = (\theta, \bar{\theta}), \tag{3.23l}$$

$$Err_{12} = \frac{1}{\mu_0\omega_{pm}^2}((P_h\mathbf{K} - \mathbf{K})_t, \bar{\theta}) = 0, \tag{3.23m}$$

$$Err_{13} = \frac{\Gamma_m}{\mu_0\omega_{pm}^2}(P_h\mathbf{K} - \mathbf{K}, \bar{\theta}) = 0. \tag{3.23n}$$

Therefor,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\epsilon_0\|\xi\|_0^2 + \mu_0\|\theta\|_0^2 + \frac{1}{\epsilon_0\omega_{pe}^2}\|\bar{\xi}\|_0^2 + \frac{1}{\mu_0\omega_{pm}^2}\|\bar{\theta}\|_0^2\right) + \left(\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\right)\|\bar{\xi}\|_0^2 + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}\|\bar{\theta}\|_0^2 \\
\leq& \frac{C_1\epsilon_0^2h^4}{4\delta_1}\|\mathbf{E}_t\|_2^2 + \frac{C_2h^4}{4\delta_2}\|\mathbf{J}\|_2^2 + \frac{C_3h^4}{4\delta_3}\|\mathbf{H}\|_3^2 + \frac{C_4h^4}{4\delta_4}\|\mathbf{E}\|_2^2 + \frac{C_5h^4}{4\epsilon_0^2\omega_{pe}^4\delta_5}\|\mathbf{J}_t\|_2^2 \\
&+ \frac{C_6\Gamma_e^2h^4}{4\epsilon_0^2\omega_{pe}^4\delta_6}\|\mathbf{J}\|_2^2 + (\delta_1 + \delta_2 + \delta_3)\|\xi\|_0^2 + (\delta_4 + \delta_5 + \delta_6)\|\bar{\xi}|_0^2.
\end{aligned} \tag{3.24}$$

Integrating both side of (3.24) with respect to t, noticing the facts $\xi(0) = \theta(0) = \bar{\xi}(0) = \bar{\theta}(0) = 0$, from the Gronwall's inequality, we have

$$\begin{aligned}
&\left(\epsilon_0\|\xi\|_0^2 + \mu_0\|\theta\|_0^2 + \frac{1}{\epsilon_0\omega_{pe}^2}\|\bar{\xi}\|_0^2 + \frac{1}{\mu_0\omega_{pm}^2}\|\bar{\theta}\|_0^2\right) \\
\leq& Ch^4\int_0^t \left(\|\mathbf{E}_t\|_2^2 + \|\mathbf{E}\|_2^2 + \|\mathbf{H}\|_3^2 + \|\mathbf{J}_t\|_2^2 + \|\mathbf{J}\|_2^2\right)dt.
\end{aligned} \tag{3.25}$$

With the help of Lemma 3.5 and triangular inequality, we can finish the proof. □

4. Full-discrete error estimates

To define a fully discrete scheme,we divide the time interval $(0, T]$ into uniform subintervals by points $0 = t_0 < t_1 < \ < t_N = T$, where $t_k = k\tau$,and $\tau = T/N$. we use the central difference and average operators at time lever $k + \frac{1}{2}$:

$$\delta_\tau w^n = \frac{w^n - w^{n-1}}{\tau}, \quad \bar{w}^n = \frac{w^n + w^{n-1}}{2},$$

where $w^n = w(n\tau)$.

The full discrete scheme is provided by:for $k = 1, 2, \cdots N$, find $\mathbf{E}_h^k \in V_{0h}, \mathbf{J}_h^k \in V_h, \mathbf{H}_h^k, \mathbf{K}_h^k \in W_h$, such that

$$(\epsilon_0 \delta_\tau \mathbf{E}_h^k, \mathbf{\Phi}_h) - (\bar{\mathbf{H}}_h^k, curl\mathbf{\Phi}_h) + (\bar{\mathbf{J}}_h^k, \mathbf{\Phi}_h) = (\mathbf{f}^{k-\frac{1}{2}}, \mathbf{\Phi}), \qquad \forall \mathbf{\Phi} \in V_{0h}, \tag{4.1}$$

$$(\mu_0 \delta_\tau \mathbf{H}_h^k, \mathbf{\Psi}_h) + (curl\bar{\mathbf{E}}_h^k, \mathbf{\Psi}_h) + (\bar{K}_h^k, \mathbf{\Psi}_h) = (\mathbf{g}^{k-\frac{1}{2}}, \mathbf{\Psi}), \qquad \forall \Psi_h \in W_h, \tag{4.2}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2}(\delta_\tau \mathbf{J}_h^k, \tilde{\phi}_h) + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2}(\bar{\mathbf{J}}_h^k, \tilde{\phi}_h) = (\bar{\mathbf{E}}_h^k, \tilde{\mathbf{\Phi}}_h), \qquad \forall \tilde{\phi}_h \in V_h, \tag{4.3}$$

$$\frac{1}{\epsilon_0 \omega_{pm}^2}(\delta_\tau \mathbf{K}_h^k, \tilde{\psi}) + \frac{\Gamma_m}{\epsilon_0 \omega_{pm}^2}(\bar{\mathbf{K}}_h^k, \tilde{\psi}) = (\bar{\mathbf{H}}_h^k, \tilde{\psi}), \qquad \forall \tilde{\psi} \in W_h, \tag{4.4}$$

subject to the initial approximation:

$$\mathbf{E}_h^0(\mathbf{x}) = \pi_h \mathbf{E}_0(\mathbf{x}), \mathbf{J}_h^0(\mathbf{x}) = \pi_h \mathbf{J}_0(\mathbf{x}), \mathbf{H}_h^0(\mathbf{x}) = P_h \mathbf{H}_0(\mathbf{x}), \mathbf{K}_h^0(\mathbf{x}) = P_h \mathbf{K}_0(\mathbf{x}).$$

In fact, we first solve (4.3) and (4.4) for $\mathbf{J}_h^k$ and K_h^k by

$$\mathbf{J}_h^{k+1} = \frac{2 - \tau\Gamma_e}{2 + \tau\Gamma_e}\mathbf{J}_h^k + \frac{\tau\omega_{pe}^2}{2 + \tau\Gamma_e}(\mathbf{E}_h^{k+1} + \mathbf{E}_h^k), \tag{4.5}$$

$$\mathbf{K}_h^{k+1} = \frac{2 - \tau\Gamma_m}{2 + \tau\Gamma_m}\mathbf{K}_h^k + \frac{\tau\omega_{pm}^2}{2 + \tau\Gamma_m}(\mathbf{H}_h^{k+1} + \mathbf{H}_h^k). \tag{4.6}$$

Then,substituting (4.5) and (4.6) into (4.1) and (4.2), respectively, we obtain

$$\epsilon_0\Big(1 + \frac{\tau^2\omega_{pe}^2}{2(2 + \tau\Gamma_e)}\Big)(\mathbf{E}_h^{k+1}, \mathbf{\Phi}_h) - \frac{\tau}{2}(\mathbf{H}_h^{k+1}, curl\mathbf{\Phi}_h) \tag{4.7}$$
$$= \epsilon_0\Big(1 - \frac{\tau^2\omega_{pe}^2}{2(2 + \tau\Gamma_e)}\Big)(\mathbf{E}_h^k, \mathbf{\Phi}_h) + \frac{\tau}{2}(\mathbf{H}_h^k, curl\mathbf{\Phi}_h) - \frac{2\tau}{2 + \tau\Gamma_e}(\mathbf{J}_h^k, \mathbf{\Phi}_h) + \tau(\mathbf{f}^{k+\frac{1}{2}}, \mathbf{\Phi}_h),$$

$$\mu_0\Big(1 + \frac{\tau^2\omega_{pm}^2}{2(2 + \tau\Gamma_m)}\Big)(\mathbf{H}_h^{k+1}, \mathbf{\Psi}_h) + \frac{\tau}{2}(curl\mathbf{E}_h^{k+1}, \mathbf{\Psi}_h) \tag{4.8}$$
$$= \mu_0\Big(1 - \frac{\tau^2\omega_{pm}^2}{2(2 + \tau\Gamma_m)}\Big)(\mathbf{H}_h^k, \mathbf{\Psi}_h) - \frac{\tau}{2}(curl\mathbf{E}_h^k, \mathbf{\Psi}_h) - \frac{2\tau}{2 + \tau\Gamma_m}(\mathbf{K}_h^k, \mathbf{\Psi}_h) + \tau(\mathbf{g}^{k+\frac{1}{2}}, \mathbf{\Psi}_h),$$

where $\mathbf{f}, \mathbf{g}$ are the added source terms.

Lemma 4.1. *At each time step, the system* (4.7) *and* (4.8) *is uniquely solvable.*

Proof. The system (4.7) and (4.8) can be rewritten by the algebra form

$$\begin{pmatrix} A & -B \\ B' & C \end{pmatrix} \begin{pmatrix} \mathbf{E_h} \\ H_h \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}} \\ \tilde{g} \end{pmatrix},$$

where matrix $B = \frac{\tau}{2}(\Psi_h, curl\mathbf{\Phi}_h)$,

$$A = \epsilon_0\Big(1 + \frac{\tau^2\omega_{pe}^2}{2(2 + \tau\Gamma_e)}\Big)(\mathbf{\Phi}_h, \mathbf{\Phi}_h), \quad C = \mu_0\Big(1 + \frac{\tau^2\omega_{pm}^2}{2(2 + \tau\Gamma_m)}\Big)(\Psi_h, \Psi_h),$$

and $\mathbf{\Phi}_h \in V_{0h}, \Psi_h \in W_h$.

It is easy to check that the coefficient matrix determinant equals $det(A)det(D + B'A^{-1}B)$, which is obviously non-zero. Hence, the coefficient matrix is non-singular, which concludes the proof. □

Theorem 4.1. *Let* $(\mathbf{E}^k, \mathbf{H}^k, \mathbf{J}^k, \mathbf{K}^k)$ *and* $(\mathbf{E}_h^k, \mathbf{H}_h^k, \mathbf{J}_h^k, \mathbf{K}_h^k)$ *be the solution of* (3.4)-(3.7) *and* (4.1)-(4.4) *at time* $t = t^k$*, respectively. Assume that , Then there exists*

$$\max_{1\le n\le N}\Big(\|\mathbf{E}^n - \mathbf{E}_h^n\|_0 + \|\mathbf{H}^n - \mathbf{H}_h^n\|_0 + \|\mathbf{J}^n - \mathbf{J}_h^n\|_0 + \|\mathbf{K}^n - \mathbf{K}_h^n\|_0\Big) \le C(\tau^2 + h^2).$$

Proof. Multiplying (2.2)-(2.4) by $\frac{1}{\tau}\mathbf{\Phi}_h \in V_{0h}, \frac{1}{\tau}\Psi_h \in W_h, \frac{1}{\tau}\bar{\phi}_h \in V_h, \frac{1}{\tau}\bar{\psi}_h \in W_h$ repectively and integrating in time over $I^k = [x_{k-1}, x_k]$ and in space Ω, then using the Green's formula,

$$(curl\mathbf{H}, \mathbf{\Phi}_h) = (\mathbf{H}, curl\mathbf{\Phi}_h) - < \mathbf{H}, \mathbf{n}\times\mathbf{\Phi}_h >,$$

where $< \mathbf{H}, \mathbf{n}\times\mathbf{\Phi}_h >= \sum_{e\in\mathcal{J}^h} \int_{\partial e} H\mathbf{n}\times\mathbf{\Phi}_h ds$, we have

$$\begin{aligned}
&\epsilon_0(\delta_\tau \mathbf{E}^k, \mathbf{\Phi}_h) - \left(\frac{1}{\tau}\int_{I^k}\mathbf{H}dt, curl\mathbf{\Phi}_h\right) + \left(\frac{1}{\tau}\int_{I^k}\mathbf{J}dt, \mathbf{\Phi}_h\right)\\
&\qquad = - < \frac{1}{\tau}\int_{I^k}\mathbf{H}dt, \mathbf{n}\times\mathbf{\Phi} >, \qquad (4.9)\\
&\mu_0(\delta_\tau\mathbf{H}^k, \mathbf{\Psi}) + \left(\frac{1}{\tau}\int_{I^k} curl\mathbf{E}dt, \mathbf{\Psi}_h\right) + \left(\frac{1}{\tau}\int_{I^k}\mathbf{K}dt, \mathbf{\Psi}_h\right) = 0, \qquad (4.10)\\
&\frac{1}{\epsilon_0\omega_{pe}^2}(\delta_\tau\mathbf{J}^k, \tilde{\phi}_h) + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}(\frac{1}{\tau}\int_{I^k}\mathbf{J}dt, \tilde{\phi}_h) = \left(\frac{1}{\tau}\int_{I^k}\mathbf{E}dt, \tilde{\phi}_h\right), \qquad (4.11)\\
&\frac{1}{\mu_0\omega_{pm}^2}(\delta_\tau\mathbf{K}^k, \tilde{\psi}_h) + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}\left(\frac{1}{\tau}\int_{I^k}\mathbf{K}dt, \tilde{\psi}_h\right) = \left(\frac{1}{\tau}\int_{I^k}\mathbf{H}dt, \tilde{\psi}_h\right). \qquad (4.12)
\end{aligned}$$

Denoting $\xi_h^k = \pi_h\mathbf{E}^k - \mathbf{E}_h^k, \theta_h^k = P_h\mathbf{H}^k - \mathbf{H}_h^k, \tilde{\xi}_h^k = \pi_h\mathbf{J}^k - \mathbf{J}_h^k, \tilde{\theta}_h^k = P_h\mathbf{K}^k - \mathbf{K}_h^k$ in (4.1)-(4.4) and choosing $\mathbf{\Phi}^k = (\xi^k + \xi^{k-1}), \mathbf{\Psi}^k = (\theta^k + \theta^{k-1}), \phi^k = (\tilde{\xi}^k + \tilde{\xi}^{k-1}), \psi^k = (\tilde{\theta}^k + \tilde{\theta}^{k-1})$ in (4.9)-(4.12), we can get the discrete error equations

$$\begin{aligned}
&\epsilon_0(\delta_\tau\xi_h^k, \overline{\xi}_h^k) - (\overline{\theta}^k, curl\overline{\xi}_h^k)\\
&= \epsilon_0(\delta_\tau(\pi_h\mathbf{E}^k - \mathbf{E}^k), \overline{\xi}_h^k) - \left(P_h\overline{\mathbf{H}}^k - \frac{1}{\tau}\int_{I^k}\mathbf{H}dt, curl\overline{\xi}_h^k\right)\\
&\qquad - \left(\frac{1}{\tau}\int_{I^k}\mathbf{J}dt - \overline{\mathbf{J}}_h^k, \overline{\xi}_h^k\right) - < \frac{1}{\tau}\int_{I^k}\mathbf{H}dt, \mathbf{n}\times\overline{\xi}_h^k >, \qquad (4.13)\\
&\mu_0(\delta_\tau\theta_h^k, \overline{\theta}_h^k) + (curl\overline{\xi}_h^k, \overline{\theta}_h^k) = \mu_0(\delta_\tau(P_h\mathbf{H}^k - \mathbf{H}^k), \overline{\theta}_h^k)
\end{aligned}$$

$$+\Big(curl(\pi_h\overline{\mathbf{E}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{E}dt),\overline{\theta}_h^k\Big)-\Big(\frac{1}{\tau}\int_{I^k}\mathbf{K}dt-\overline{\mathbf{K}}_h^k,\overline{\theta}_h^k\Big), \tag{4.14}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}(\delta_\tau\tilde{\xi}_h^k,\bar{\tilde{\xi}}_h^k)+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}(\bar{\tilde{\xi}}_h^k,\bar{\tilde{\xi}}_h^k)=\frac{1}{\epsilon_0\omega_{pe}^2}(\delta_\tau(\pi_h\mathbf{J}^k-\mathbf{J}^k),\bar{\tilde{\xi}}_h^k)$$

$$+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\Big(\pi_h\overline{\mathbf{J}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{J}dt,\bar{\tilde{\xi}}_h^k\Big)+\Big(\frac{1}{\tau}\int_{I^k}\mathbf{E}dt-\overline{\mathbf{E}}_h^k,\bar{\tilde{\xi}}_h^k\Big), \tag{4.15}$$

$$\frac{1}{\mu_0\omega_{pm}^2}(\delta_\tau\tilde{\theta}_h^k,\bar{\tilde{\theta}}_h^k)+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}(\bar{\tilde{\theta}}_h^k,\bar{\tilde{\theta}}_h^k)=\frac{1}{\mu_0\omega_{pm}^2}(\delta_\tau(P_h\mathbf{K}^k-\mathbf{K}^k),\bar{\tilde{\theta}}_h^k)$$

$$+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\Big(P_h\overline{\mathbf{K}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{K}dt,\bar{\tilde{\theta}}_h^k\Big)+\Big(\frac{1}{\tau}\int_{I^k}\mathbf{H}dt-\overline{\mathbf{H}}_h^k,\bar{\tilde{\theta}}_h^k\Big). \tag{4.16}$$

Adding (4.13)-(4.16), multiplying the time step τ and employing the inequality $a(a-b)\geq\frac{1}{2}(a^2-b^2)$, we obtain

$$\begin{aligned}
&\frac{\epsilon_0}{2}\Big(\|\xi_{\mathbf{h}}{}^k\|_0^2-\|\xi_h^{k-1}\|_0^2\Big)+\frac{\mu_0}{2}\Big(\|\theta_h^k\|_0^2-\|\theta_h^{k-1}\|_0^2\Big)+\frac{1}{2\epsilon_0\omega_{pe}^2}\Big(\|\tilde{\xi}_h^k\|_0^2-\|\tilde{\xi}_h^{k-1}\|_0^2\Big)\\
&\quad+\frac{1}{2\mu_0\omega_{pm}^2}\Big(\|\tilde{\theta}_h^k\|_0^2-\|\tilde{\theta}_h^{k-1}\|_0^2\Big)+\frac{\Gamma_e}{2\epsilon_0\omega_{pe}^2}\|\bar{\tilde{\xi}}_h^{k+1}\|_0^2+\frac{\Gamma_m}{2\mu_0\omega_{pm}^2}\|\bar{\tilde{\theta}}_h^{k+1}\|_0^2\\
&=\tau\epsilon_0(\delta_\tau(\pi_h\mathbf{E}^k-\mathbf{E}^k),\overline{\xi}_h^k)-\tau\Big(\overline{\mathbf{H}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{H}dt,curl\overline{\xi}_h^k\Big)\\
&\quad-\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{J}dt-\overline{\mathbf{J}}_h^k,\overline{\xi}_h^k\Big)-\tau<\frac{1}{\tau}\int_{I^k}\mathbf{H}dt,\mathbf{n}\times\overline{\xi}_h^k>+\tau\mu_0(\delta_\tau(P_h\mathbf{H}^k-\mathbf{H}^k),\overline{\theta}_h^k)\\
&\quad+\tau\Big(curl(\pi_h\overline{\mathbf{E}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{E}dt),\overline{\theta}_h^k\Big)-\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{K}dt-\overline{\mathbf{K}}_h^k,\overline{\theta}_h^k\Big)\\
&\quad+\frac{\tau}{\epsilon_0\omega_{pe}^2}(\delta_\tau(\pi_h\mathbf{J}^k-\mathbf{J}^k),\bar{\tilde{\xi}}_h^k)+\frac{\tau\Gamma_e}{\epsilon_0\omega_{pe}^2}\Big(\pi_h\overline{\mathbf{J}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{J}dt,\bar{\tilde{\xi}}_h^k\Big)\\
&\quad+\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{E}dt-\overline{\mathbf{E}}_h^k,\bar{\tilde{\xi}}_h^k\Big)+\frac{\tau}{\mu_0\omega_{pm}^2}(\delta_\tau(P_h\mathbf{H}^k-\mathbf{H}^k),\bar{\tilde{\theta}}_h^k)\\
&\quad+\frac{\tau\Gamma_m}{\mu_0\omega_{pm}^2}\Big(\overline{\mathbf{K}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{K}dt,\bar{\tilde{\theta}}_h^k\Big)+\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{H}dt-\overline{\mathbf{H}}_h^k,\bar{\tilde{\theta}}_h^k\Big)=\sum_{i=1}^{13}\mathcal{T}.
\end{aligned}$$

In order to estimate $\mathcal{T}, 1\leq i\leq 13$, we will need the following two inequalities

$$\|\delta_\tau w^k\|_0^2\leq\frac{1}{\tau}\int_{I^k}\|w_t(t)\|_0^2dt,\ \ \forall w\in C^1(0,T;H^1(\Omega)), \tag{4.17}$$

$$\|w^k-\frac{1}{\tau}\int_{I^k}w(t)dt\|_0^2\leq\tau\int_{I^k}\|w_t(t)\|_0^2dt,\ \ \forall w\in C^1(0,T;H^1(\Omega)), \tag{4.18}$$

$$\|\overline{w}^k-\frac{1}{\tau}\int_{I^k}w(t)dt\|_0^2\leq\frac{\tau^3}{4}\int_{I^k}\|w_{tt}(t)\|_0^2dt,\ \ \forall w\in C^2(0,T;L^2(\Omega)). \tag{4.19}$$

With the help of Lemmas 3.4-3.6 , Yong's inequality and (4.17)-(4.19), we have

$$\mathcal{T}_1=\tau\epsilon_0(\delta_\tau(\pi_h\mathbf{E}^k-\mathbf{E}^k),\overline{\xi}_h^k)\leq\delta_7\tau\epsilon_0\Big(\|\xi_h^k\|_0^2+\|\xi_h^{k-1}\|_0^2\Big)+\frac{C\epsilon_0h^4}{4\delta_7}\int_{I^k}\|\mathbf{E}_t(t)\|_2^2dt,$$

$$\begin{aligned}\mathcal{T}_2&=-\tau\Big(P_h\mathbf{H}^k-\frac{1}{\tau}\int_{I^k}\mathbf{H}dt,curl\xi^k\Big)=-\tau\Big(rot(\overline{\mathbf{H}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{H}dt),\xi^k\Big)\\&\leq\tau\delta_8(\|\xi_h^k\|_0^2+\|\xi_h^{k-1}\|_0^2)+\frac{\tau^4}{4\delta_8}\int_{I^k}\|rotH_{tt}(t)\|_0^2dt,\end{aligned}$$

$$\begin{aligned}\mathcal{T}_3&=-\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{J}dt-\overline{\mathbf{J}}_h^k,\overline{\xi}_h^k\Big)=-\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{J}d-\overline{\mathbf{J}}^k+\overline{\mathbf{J}}^k-\pi_h\overline{\mathbf{J}}^k+\overline{\tilde{\xi}}_h^k,\overline{\xi}_h^k\Big)\\&\leq-\tau(\overline{\tilde{\xi}}_h^k,\overline{\xi}_h^k)+\tau\delta_9(\|\xi_h^k\|_0^2+\|\xi_h^{k-1}\|_0^2)+\frac{\tau^4}{4\delta_9}\int_{I^k}\mathbf{J}_{tt}(t)\|_0^2dt+C\tau h^4\|\mathbf{J}\|_2^2,\end{aligned}$$

$$\mathcal{T}_4=-\tau<\frac{1}{\tau}\int_{I^k}\mathbf{H}dt,\mathbf{n}\times\overline{\xi}_h^k>\leq\delta_{10}(\|\xi_h^k\|_0^2+\|\xi_h^{k-1}\|_0^2)+\frac{Ch^4}{4\delta_{10}}\|\mathbf{H}\|_3^2,$$

$$\mathcal{T}_5=\tau\mu_0(\delta_\tau(P_h\mathbf{H}^k-\mathbf{H}^k),\overline{\theta}_h^k)=0,$$

$$\begin{aligned}\mathcal{T}_6&=\tau\Big(curl(\pi_h\overline{\mathbf{E}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{E}dt),\overline{\theta}_h^k\Big)=\tau\Big(curl(\pi_h\overline{\mathbf{E}}^k-\overline{\mathbf{E}}^k+\overline{\mathbf{E}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{E}dt),\overline{\theta}_h^k\Big)\\&=\tau(P_hcurl\overline{\mathbf{E}}^k-curl\overline{\mathbf{E}}^k,\overline{\theta}_h^k)+\tau\Big(curl\overline{\mathbf{E}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{E}dt),\overline{\theta}_h^k\Big)\\&\leq\tau\delta_{11}\Big(\|\theta_h^k\|_0^2+\|\theta_h^{k-1}\|_0^2\Big)+\frac{C\tau^4}{4\delta_{11}}\int_{I^k}\|curl\mathbf{E}_{tt}(t)\|_0^2dt,\end{aligned}$$

$$\begin{aligned}\mathcal{T}_7&=-\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{K}dt-\overline{\mathbf{K}}_h^k,\overline{\theta}_h^k\Big)=-\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{K}dt-\overline{\mathbf{K}}^k+\overline{\mathbf{K}}^k-P_h\overline{\mathbf{K}}^k+\overline{\tilde{\theta}}_h^k,\overline{\theta}_h^k\Big)\\&\leq-\tau(\overline{\tilde{\theta}}_h^k,\overline{\theta}_h^k)+\tau\delta_{12}\Big(\|\theta_h^k\|_0^2+\|\theta_h^{k-1}\|_0^2\Big)+\frac{C\tau^4}{4\delta_{12}}\int_{I^k}\|\mathbf{K}_{tt}(t)\|_0^2dt,\end{aligned}$$

$$\mathcal{T}_8=\frac{\tau}{\epsilon_0\omega_{pe}^2}(\delta_\tau(\pi_h\mathbf{J}^k-\mathbf{J}^k),\overline{\tilde{\xi}}_h^k)\leq\tau\delta_{13}\Big(\|\tilde{\xi}_h^k\|_0^2+\|\tilde{\xi}_h^{k-1}\|_0^2\Big)+\frac{h^4}{4\delta_{13}}\int_{I^k}\|\mathbf{J}_t(t)\|_2^2dt,$$

$$\begin{aligned}\mathcal{T}_9&=\frac{\tau\Gamma_e}{\epsilon_0\omega_{pe}^2}\Big(\pi_h\overline{\mathbf{J}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{J}dt,\overline{\tilde{\xi}}_h^k\Big)=\frac{\tau\Gamma_e}{\epsilon_0\omega_{pe}^2}\Big(\pi_h\overline{\mathbf{J}}^k-\overline{\mathbf{J}}^k+\overline{\mathbf{J}}^k-\frac{1}{\tau}\int_{I^k}\mathbf{J}dt,\overline{\tilde{\xi}}_h^k\Big)\\&\leq\frac{\tau\delta_{14}\Gamma_e}{\epsilon_0\omega_{pe}^2}\Big(\|\tilde{\xi}_h^k\|_0^2+\|\tilde{\xi}_h^{k-1}\|_0^2\Big)+\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\Big[\frac{\tau^4}{4\delta_{14}}\int_{I^k}\|\mathbf{J}_{tt}(t)\|_0^2dt+Ch^4\|\mathbf{J}\|_2^2\Big],\end{aligned}$$

$$\begin{aligned}\mathcal{T}_{10}&=\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{E}dt-\overline{\mathbf{E}}_h^k,\overline{\tilde{\xi}}_h^k\Big)=\tau\Big(\frac{1}{\tau}\int_{I^k}\mathbf{E}dt-\overline{\mathbf{E}}^k+\overline{\mathbf{E}}^k-\pi_h\overline{\mathbf{E}}^k+\overline{\xi}_h^k,\overline{\tilde{\xi}}_h^k\Big)\\&\leq\tau(\overline{\xi}_h^k,\overline{\tilde{\xi}}_h^k)+\tau\delta_{15}\Big(\|\tilde{\xi}_h^k\|_0^2+\|\tilde{\xi}_h^{k-1}\|_0^2\Big)+\frac{1}{4\delta_{15}}\Big(\tau^4\int_{I^k}\|\mathbf{E}_{tt}(t)\|_0^2dt+Ch^4\|\mathbf{E}\|_2^2\Big),\end{aligned}$$

$$
\begin{aligned}
\mathcal{T}_{11} &= \frac{\tau}{\mu_0\omega_{pm}^2}(\delta_\tau(P_h\mathbf{H}^k - \mathbf{H}^k), \bar{\tilde{\theta}}_h^k) = 0,\\
\mathcal{T}_{12} &= \frac{\tau\Gamma_m}{\mu_0\omega_{pm}^2}(\overline{\mathbf{K}}^k - \left(\frac{1}{\tau}\int_{I^k}\mathbf{K}dt, \bar{\tilde{\theta}}_h^k\right)\\
&\le \frac{\Gamma_m}{\mu_0\omega_{pm}^2}\left(\tau\delta_{16}(\|\theta_h^k\|_0^2 + \|\theta_h^{k-1}\|_0^2) + \frac{\tau^4}{4\delta_{16}}\int_{I^k}\|\mathbf{K}_{tt}(t)\|_0^2 dt\right),\\
\mathcal{T}_{13} &= \tau\left(\frac{1}{\tau}\int_{I^k}\mathbf{H}dt - \overline{\mathbf{H}}_h^k, \bar{\tilde{\theta}}_h^k\right) = \tau\left(\frac{1}{\tau}\int_{I^k}\mathbf{H}dt - \overline{\mathbf{H}}^k + \overline{\mathbf{H}}^k - P_h\overline{\mathbf{H}}^k + \bar{\theta}_h^k, \bar{\tilde{\theta}}_h^k\right)\\
&\le \tau(\bar{\theta}_h^k, \bar{\tilde{\theta}}_h^k) + \tau\delta_{17}\left(\|\theta_h^k\|_0^2 + \|\theta_h^{k-1}\|_0^2\right) + \frac{\tau^4}{4\delta_{17}}\int_{I^k}\|\mathbf{H}_{tt}(t)\|_0^2 dt.
\end{aligned}
$$

Substituting the estimates of $\mathcal{T}_i, 1 \le i \le 13$, and summing up the results from $k = 1$ to n, and using the facts $N\tau \le T$, and $\xi_h^0 = \tilde{\xi}_h^0 = 0, \theta_h^0 = \tilde{\theta}_h^0 = 0$, we have

$$
\begin{aligned}
&\frac{\epsilon_0}{2}\|\xi_h^n\|_0^2 + \frac{\mu_0}{2}\|\theta_h^k\|_0^2 + \frac{1}{2\epsilon_0\omega_{pe}^2}\|\tilde{\xi}_h^n\|_0^2 + \frac{1}{2\mu_0\omega_{pm}^2}\|\tilde{\theta}_h^n\|_0^2\\
\le\; & C\tau\sum_{k=1}^{n-1}\left(\|\xi_h^n\|_0^2 + \|\theta_h^k\|_0^2 + \|\tilde{\xi}_h^n\|_0^2 + \|\tilde{\theta}_h^n\|_0^2\right) + C(\tau^4 + h^4),
\end{aligned} \tag{4.20}
$$

which, along with the discrete Gronwall inequality, the triangle inequality, the estimates Lemma 3.5, completes the proof. □

5. Numerical experiments

5.1. 2D TE model

In this section, we will give some examples to verify our theoretical analysis. To check the convergence rate, we construct the following exact solutions for the 2-D transverse electrical(TE) model, that means $\mathbf{E} = (E_x, E_y, 0), \mathbf{H} = (0, 0, H_z), \mathbf{J} = (J_x, J_y, 0), \mathbf{K} = (0, 0, K_z)$. Assume that $\Gamma_e = \Gamma_m = 1, \omega_{pe} = \omega_{pm} = \omega_e = 1$ on the domain $\Omega = [0, 1]^2$. Now let

$$
\begin{aligned}
\mathbf{E} &= (E_x, E_y) = (\sin\pi y, \sin\pi x)e^{-\Gamma_e t},\\
H_z &= \frac{1}{\pi}(\cos\pi x - \cos\pi y)e^{-\Gamma_e t}(\omega_e^2 t - \Gamma_e).
\end{aligned}
$$

The corresponding electric and magnetic currents are

$$
\begin{aligned}
\mathbf{J} &= (J_x, J_y) = (\sin\pi y, \sin\pi x)\omega_e^2 t e^{-\Gamma_e t}\\
K_z &= \frac{1}{\pi}(\cos\pi x - \cos\pi y)e^{-\Gamma_e t}\omega_e^2\left(\frac{1}{2}\omega_e^2 t^2 - \Gamma_e t\right).
\end{aligned}
$$

The corresponding source term $\mathbf{f} \equiv 0$ and

$$g = \frac{1}{\pi}(\cos \pi x - \cos \pi y)e^{-\Gamma_e t}\Big(- 2\Gamma_e \omega_e^2 t + \Gamma_e^2 + \omega_e^2 + \pi^2 \frac{1}{2}\omega_e^4 t^2\Big).$$

Define

$$\begin{aligned}
&errE = \max_{1\le n\le N} \|\mathbf{E}^n - \mathbf{E}_h^n\|_0, \quad errCurlE = \max_{1\le n\le N} \|curl\mathbf{E}^n - curl_h\mathbf{E}_h^n\|_0,\\
&errH = \max_{1\le n\le N} \|H^n - H_h^n\|_0, \; errJ = \max_{1\le n\le N} \|\mathbf{J}^n - \mathbf{J}_h^n\|_0,\\
&errK = \max_{1\le n\le N} \|K^n - K_h^n\|_0.
\end{aligned}$$

Results in Tables 2-6 are obtained on the uniform rectangular mesh after 1 time step with $\tau = 10^{-8}$. From the Tables 2-6, we can see that the numerical results are coincide

Table 2: Error Results of $\mathbf{E}$.

$N\times N$	$errE$	order
4×4	0.016501553954799	-
8×8	0.004080285965617	2.01585971806161
16×16	0.001017010619176	2.00433552333004
32×32	0.000254057311403	2.00110885473070
64×64	0.000063502054700	2.00027880519698
128×128	0.000015874745592	2.00006980160546
256×256	0.000003968638362	2.00001746213015

Table 3: Error Results of $curl\mathbf{E}$.

$N\times N$	$errCurlE$	order
4×4	0.00718076678516	-
8×8	0.00180709923725	1.99026312733970
16×16	0.00045252194078	1.99741632378998
32×32	0.00011317722648	1.99920411503889
64×64	0.00002829722867	1.99965103080372
128×128	0.00000707448981	1.99976275748336
256×256	0.00000176863387	1.99979068818369

Table 4: Error Results of H.

$N\times N$	$errH$	order
4×4	0.007253299782990	-
8×8	0.001825352764898	1.99046217355706
16×16	0.000457092869478	1.99761608539852
32×32	0.000114320430791	1.99940405544443
64×64	0.000028583059258	1.99985101590531
128×128	0.000007145949300	1.99996275375874
256×256	0.000001786498857	1.99999068725242

Table 5: Error Results of $\mathbf{J}$.

$N \times N$	$errJ$	order
4×4	0.163650926529572e-9	-
8×8	0.040759745626441e-9	2.00540481733043
16×16	0.010169119751205e-9	2.00295024442725
32×32	0.002540620600819e-9	2.00094194930721
64×64	0.000635046864314e-9	2.00024598216059
128×128	0.000158758891189e-9	2.00002567048550
256×256	0.000039691821638e-9	1.99992371055101

Table 6: Error Results of $\mathbf{K}$.

$N \times N$	$errK$	order
4×4	0.725329996432064e-10	-
8×8	0.182535281052979e-10	1.99046217355830
16×16	0.045709288090349e-10	1.99761608540270
32×32	0.011432043364721e-10	1.99940405546137
64×64	0.002858305997026e-10	1.99985101599950
128×128	0.000714594947626e-10	1.99996275412408
256×256	0.000178649889959e-10	1.99999068844379

with our theoretical analysis. Compared our nonconforming mixed finite element with Raviart-Thomas- Nédélec finite element spaces $Q_{12} \times Q_{21}$ — Q_{11}, we can see that the computational cost of our element is 40 percent less than that.

5.2. 2D TM model

We construct the following exact solutions for the 2-D transverse magnetic (TM) model, that means $\mathbf{E} = (0,0,E_z), \mathbf{H} = (H_x,H_y,0), \mathbf{J} = (0,0,J_z), \mathbf{K} = (K_x,K_y,0)$. Assume that $\Gamma_e = \Gamma_m = \pi, \omega_{pe} = \omega_{pm} = \omega_e = \pi$ on the domain $\Omega = [0,1]^2$.

$$\mathbf{H} = (H_x,H_y) = (\sin\pi x\cos\pi y, -\cos\pi x\sin\pi y)e^{-\pi t},$$
$$E_z = \sin\pi x\sin\pi y e^{-\pi t}.$$

The corresponding electric and magnetic currents are

$$J_z = \pi^2 t\sin\pi x\sin\pi y e^{-\pi t}$$
$$K = (K_x,K_y) = \pi^2 t(\sin\pi x\cos\pi y, -\cos\pi x\sin\pi y)e^{-\pi t}.$$

The corresponding source term

$$f = (-3\pi + \pi^2 t)\sin\pi x\sin\pi y e^{-\pi t}$$
$$\mathbf{g} = (g_x,g_y) = \pi^2 t(\sin\pi x\cos\pi y, -\cos\pi x\sin\pi y)e^{-\pi t}.$$

The Tables 8-9 list the convergence of $E, \mathbf{H}, J, \mathbf{K}$ after 1 time step with $\tau = 10^{-8}$.

Table 7: Error Results of E and $\mathbf{H}$ in TM case.

$N \times N$	$errE$	order	$errH$	order
4×4	0.0304	-	0.0420	-
8×8	0.0076	2.0000	0.0107	1.9728
16×16	0.0019	2.0000	0.0027	1.9866
32×32	4.7517e-004	1.9995	6.7174e-004	2.0070

Table 8: Error Results of J and $\mathbf{K}$ in TM case.

$N \times N$	$errJ$	order	$errK$	order
4×4	2.9688e-009	-	4.1418e-009	-
8×8	7.4925e-010	1.9864	1.0547e-009	1.9734
16×16	1.8755e-010	1.9982	2.6489e-010	1.9934
32×32	4.6896e-011	1.9997	6.6298e-011	1.9984

5.3. Wave propagation in a metamaterials lab

Now we want to repeat the experiment introduced in [10, 11]. In this example, a metamaterials lab is chosen to be the triangular with three vertex $[0.024, 0.002]m$, $[0.039, 0.062]m$, $[0.054, 0.062]m$,which is located inside a vacuum with dimension $[0, 0.07]m \times [0, 0.064]m$. The vacuum is surrounded by a PML with thickness $dd = 12h$, where h denotes the mesh size.The 2-D transverse magnetic PML model can be obtained from [18]

$$\mu_0 \frac{\partial H_1}{\partial t} = -\frac{\partial E}{\partial y} - K_1 + \mu_0(\sigma_x - \sigma_y)H_1, \tag{5.1}$$

$$\mu_0 \frac{\partial H_2}{\partial t} = \frac{\partial E}{\partial x} - K_2 - \mu_0(\sigma_x - \sigma_y)H_2, \tag{5.2}$$

$$\epsilon_0 \frac{\partial E}{\partial t} = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} - J - \epsilon_0(\sigma_x + \sigma_y)E, \tag{5.3}$$

$$\frac{\partial J}{\partial t} = \epsilon_0 \sigma_x \sigma_y E, \tag{5.4}$$

$$\frac{\partial K_1}{\partial t} = -\sigma_x K_1 + \mu_0(\sigma_x - \sigma_y)\sigma_x H_1, \tag{5.5}$$

$$\frac{\partial K_2}{\partial t} = -\sigma_y K_2 - \mu_0(\sigma_x - \sigma_y)\sigma_y H_2. \tag{5.6}$$

The incident source wave is imposed as E field and is excited at $x = 0.004m$ and $y \in [0.025m, 0.035m]$. The source wave varies in space as $\exp(-(x-0.03)^2/(50h)^2)$ and in time as

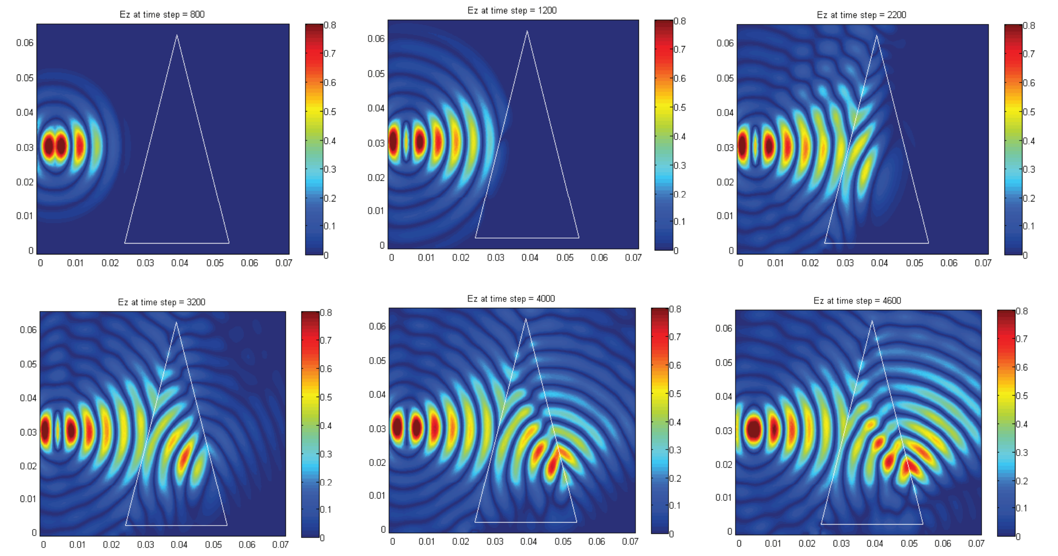

Figure 1: The re-focusing property of metamaterials for E_z at 800 1200 2200 3200 4600 time steps.

$$f(t)=\begin{cases} 0, & \text{for } t<0,\\ g_1(t)\sin(\omega_0 t), & \text{for } 0<t<mT_p,\\ \sin(\omega_0 t), & \text{for } mT_p<t<(m+k)T_p,\\ g_2(t)\sin(\omega_0 t), & \text{for } (m+k)T_p<t<(2m+k)T_p,\\ 0, & \text{for } t>(2m+k)T_p, \end{cases}$$

where $T_p=1/f_0, m=2, k=100$ and

$$g_1(t)=10x_1^3-15x_1^4+6x_1^5,\quad x_1=t/mT_p,$$

$$g_2(t)=1-(10x_2^3-15x_2^4+6x_2^5),\quad x_2=(t-(m+k)T_p)/mT_p.$$

The damping function σ_x and σ_y are choose

$$\sigma_x(x,y)=\begin{cases} \sigma_{max}(\frac{x-0.07}{dd})^4, & \text{if } x\geq 0.07,\\ \sigma_{max}(\frac{|x|}{dd})^4, & \text{if } x\leq 0.0,\\ 0, & \text{elsewhere}, \end{cases}$$

where $\sigma_{max}=-\log(err)*5*0.07*c_\nu/(2dd)$ with $err=10^{-7}$. Function σ_y has the same for but varies with respect to the y variable. We can find the re-focusing phenomena.

Acknowledgments The first author is supported by P. R. China NSFC (NO. 11471296, 11571389, 11101384). The authors would like to thank Professor Jichun Li (Department of Mathematical Science, University of Nevada Las Vegas) for helpful discussion and to thank the referees for the useful suggestions.

References

[1] J. S. Hesthaven, T. Warburton, *Nodal Discontinious Galerkin Methods:Algorithm, Analysis and Applications*, Springer-verlag, Berlin Heidelbert, 2008.

[2] J. Li, Y. Huang, *Time-Domain Finite Element Methods for Maxwell's Equations in Metamaterilas*, Springer-Verlag, Berlin Heidelbert, 2013.

[3] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford University Press, New York, 2003.

[4] P. Monk, *A mixed method for approximating Maxwell's equations*, SIAM J. Numer. Anal., 28:6(1991), pp,1610-1634.

[5] P. Monk, *Analysis of a finite element method for Maxwell's equations*, SIAM J. Numer. Anal., 29:3(1992), pp,714-729.

[6] P. Monk, *A comparison of three mixed methods for the time-dependent Maxwell's equations*, SIAM J. Sci. Statist. Comput., 13:5(1992), pp,1097-1122.

[7] H. Y. Duan, F. Jia, P. Lin and C. E. Roger Tan, *The Local L2 Projected C0 Finite Element Method for Maxwell Problem*, SIAM J. Numer. Anal., 47:2(2009), pp,1274-1303.

[8] Bo-nan Jian, Jie Wu and L. A. Povinelli, *The origin of spurious solutions in computational electromagnetics*,NASA. Technical Memorandum 106921, (1995), pp, 1-44.

[9] J.C. Li and Y. Chen, *Analysis of a time-domain finite element method for 3-D Maxwell's equations in dispersive media*,Comput. Methods Appl. Mech. Engrg., 195:33-36, (2006), pp, 4220-4229.

[10] J. Li, J.S.Hesthaven, *Analysis and application of the nodal discontinuous Galerkin method for wave propagation in metamaterials*, J. Comput. Phys., 258(2014),pp, 915-930.

[11] R.W.Ziolkowski, *Pulsed and CW Gaussian beam interactions with double negative metamaterial slabs*, Opt. Express11.(2003),pp, 662-681.

[12] Q. Lin and J.C. Li, *uperconvergence analysis for Maxwell's equations in dispersive media*,Math. Comp., 77:262, (2008), pp, 757-771.

[13] J.C. Li, *Numerical convergence and physical fidelity analysis for Maxwell's equations in metamaterials*, Comput. Methods Appl. Mech. Engrg., 198:37-40 (2009), pp,3161-3172.

[14] J.C. Li and Z.M. Zhang , *Unified analysis of time domain mixed finite element methods for maxwell's equations in dispersive media*, J. Sci. Comput., 28:5 (2010), pp,693-710.

[15] J.C. Li , *Unified analysis of leap-frog methods for solving time-domain Maxwell's equations in dispersive media*, J. Sci. Comput., 47:1 (2011), pp,1-26.

[16] Y.Q. Huang, J.C. Li and W. Yang, *Interior penalty DG methods for Maxwell's equations in dispersive media*, J. Comput. Phys., 230:12 (2011), pp,4559-4570.

[17] J.C. Li, Y. Huang, Y. Lin, *Developing Finite Element Methods for Maxwell's Equations in a Cole-Cole Dispersive Medium*, SIAM J. Sci. Comput., 33:6 (2011),pp, 3153-3174.

[18] Y. Huang, J. Li and W. Yang, *Modeling Backward wave propagation in meta- materials by the finite element time domain method*, SIAM J. Sci. Comput., 35:1(2013), pp, 248-B274.

[19] Z. Qiao, C.H. Yao and S.J. Jia, *Superconvergence and extrapolation analysis of a nonconforming mixed finite element approximation for time-harmonic Maxwell's equations*, J. Sci. Comput., 46:1 (2011),pp 1-19.

[20] Z. Qiao, *Numerical investigations of the dynamical behaviors and instabilities for the Gierer-Meinhardt system*, Comm. Comput. Phys., 3 (2008), pp, 406-426.

[21] Z. QIAO,Z. ZHANG AND T. TANG, *An adaptive time-stepping strategy for the molecular beam epitaxy models*, SIAM J. Sci. Comput., 33 (2011), pp,1395-1414.

[22] D. Y. SHI AND C. H. YAO, *Nonconforming finite element approximation of time-dependeent Maxwell's equations in Debye medium*, Numer. Methods PDEs., (2014). DOI:10.1002/num 21843.

[23] S. C. BRENNER, F. LI AND L.-Y. SUNG, *A local divergence-free interior penalty method for two-dimensional curl-curl problem*, SIAM J. Numer. Anal., 46:3 (2008), pp, 1190-1211.

[24] S. C. BRENNER, J. CUI F. LI AND L.-Y. SUNG, *A nonconforming finite element method for a two-dimensional curl-curl and grad-div problem*, Numer. Math., 109 (2008), pp, 509-533.

[25] S. C. BRENNER, F. LI AND L.-Y. SUNG, *A locally divergence-free nonconforming finite element method for the time-harmonic maxwell equations*, Math. Comp., 76:258 (2007), pp, 573-595.