Numer. Math. Theor. Meth. Appl. Vol. **10**, No. 1, pp. 84-97
doi: 10.4208/nmtma.2017.y14034 February 2017

The Disc Theorem for the Schur Complement of Two Class Submatrices with γ-Diagonally Dominant Properties

Guangqi Li[1], Jianzhou Liu[1,*] and Juan Zhang[1,2]

[1] *Department of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China*
[2] *College of Science, National University of Defense Technology, Changsha, Hunan 410073, China*

Received 16 December 2014; Accepted 20 July 2016

Abstract. The distribution for eigenvalues of Schur complement of matrices plays an important role in many mathematical problems. In this paper, we firstly present some criteria for H-matrix. Then as application, for two class matrices whose submatrices are γ-diagonally dominant and product γ-diagonally dominant, we show that the eigenvalues of the Schur complement are located in the Geršgorin discs and the Ostrowski discs of the original matrices under certain conditions.

AMS subject classifications: 15A24, 15A48

Key words: Schur complement, Geršgorin Theorem, Ostrowski Theorem, H-matrix.

1. Introduction and notations

In many fields such as control theory and computational mathematics, the theory of Schur complement plays an important role. A lot of work have been done on it. Based on the Geršgorin discs and Gassini ovals, Liu and Zhang firstly presented the notations of disc separations and considered the disc separations for diagonally dominant matrix and their Schur complement ([1]). Further, Liu obtained some estimates for dominant degree of the Schur complement and some bounds for the eigenvalues of the Schur complement by the entries the original matrix ([1–5]). For another, as the eigenvalue distribution problem on the Schur complement has important applications (see e.g., [2–4]), thus there are many researchers pay attention to it. Liu and Zhang considered the relation between the eigenvalues of the Schur complement and the submatrix for diagonally dominant matrix A with real diagonal elements in the paper ([1]). Cvetković and Nedović [6] generalized this result to the S-strictly diagonally dominant matrix.

*Corresponding author. *Email addresses:* `liujz@xtu.edu.cn` (J.-Z. Liu), `zhangjuan@xtu.edu.cn` (J. Zhang)

In [7], Liu and Huang obtained the number of eigenvalues with positive real part and with negative real part for the Schur complement of H-matrix with real diagonal elements. Later, Zhang *et al.* [8] generalized this result to the H-matrix with complex diagonal elements. Liu *et al.* presented some bounds for the eigenvalues of the Schur complement by the entries of original matrix ([2–5]). As stated in these papers above, if the eigenvalues of the Schur complement can be estimated by the elements of the original matrix, it easy to know whether a linear system could be transformed into two smaller one which can be solved by iteration. This kind of iteration, which has many advantages, is called the Schur-based iteration, as it converts the original system into two smaller ones by the Schur complement. Hence, investigating the distribution for eigenvalues of Schur complement is of great significance.

In the following, we recall some notations and definitions. Let $C^{n\times n}$ denote the set of all $n\times n$ complex matrices, $N=\{1,\dots,n\}$ and $A=(a_{ij})\in C^{n\times n}$, where $n\geq 2$ and let $N_1\cup N_2=N, N_1\cap N_2=\emptyset$. Denote

$$\alpha_i(A)=\sum_{j\in N_1,j\neq i}|a_{ij}|,\quad \beta_i(A)=\sum_{j\in N_2,j\neq i}|a_{ij}|,\quad P_i(A)=\alpha_i(A)+\beta_i(A);$$
$$\alpha'_i(A)=\sum_{j\in N_1,j\neq i}|a_{ji}|,\quad \beta'_i(A)=\sum_{j\in N_2,j\neq i}|a_{ji}|,\quad S_i(A)=\alpha'_i(A)+\beta'_i(A).$$

Take

$$N_r(A)=\Big\{i:i\in N,|a_{ii}|>P_i(A)\Big\};\qquad N_c(A)=\Big\{j:j\in N,|a_{jj}|>S_j(A)\Big\}.$$

The comparison matrix of A, which is denoted by $\mu(A)=(t_{ij})$, is defined as

$$t_{ij}=\begin{cases}|a_{ij}|, & \text{if } i=j,\\ -|a_{ij}|, & \text{if } i\neq j.\end{cases}$$

It is known that A is a (row) diagonally dominant matrix (D_n) if for all $i=1,\dots,n$,

$$|a_{ii}|\geq P_i(A). \tag{1.1}$$

A is a γ-diagonally dominant matrix (D_n^γ) if there exists $\gamma\in[0,1]$ such that

$$|a_{ii}|\geq\gamma P_i(A)+(1-\gamma)S_i(A),\quad \forall i\in N. \tag{1.2}$$

And A is called a product γ-diagonally dominant matrix (PD_n^γ) if there exists $\gamma\in[0,1]$ such that

$$|a_{ii}|\geq[P_i(A)]^\gamma[S_i(A)]^{1-\gamma},\quad \forall i\in N. \tag{1.3}$$

If all inequalities in (1.1)-(1.3) hold, A is said to be strictly (row) diagonally dominant (SD_n), strictly γ-diagonally dominant (SD_n^γ), and strictly product γ-diagonally dominant (SPD_n^γ), respectively. If there exists a diagonal matrix D, with positive diagonal elements, such that AD is strictly diagonally dominant, strictly γ-diagonally dominant and strictly product γ-diagonally dominant respectively, we call A generalized diagonally dominant (GSD_n), generalized γ-diagonally dominant (GSD_n^γ) and generalized product γ-diagonally dominant $(GSPD_n^\gamma)$, respectively.

A matrix A is an M-matrix, if it can be written in the form $A = mI - P$, where P is a nonnegative matrix and $m > \rho(P)$, the spectral radius of P. A matrix A is an H-matrix, if $\mu(A)$ is an M-matrix. we denote by H_n and M_n the sets of $n \times n$ H-matrices and M-matrices, respectively.

For nonempty index sets $\alpha, \beta \subseteq N$ whose elements are both conventionally arranged in increasing order, we denote by $|\alpha|$ the cardinality of α and $\alpha^c = N - \alpha$ the complement of α in N. We write $A(\alpha, \beta)$ to mean the submatrix of $A \in C^{n\times n}$ lying in the rows indexed by α and the columns indexed by β. $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Assuming that $A(\alpha)$ is nonsingular, the Schur complement of A with respect to $A(\alpha)$, which is denoted by $A/A(\alpha)$ or simply A/α, is defined to be

$$A(\alpha^c) - A(\alpha^c, \alpha)[A(\alpha)]^{-1} A(\alpha, \alpha^c). \tag{1.4}$$

In this paper, we first present some criteria for H-matrix. Then as application, for two class matrices whose submatrices are γ-diagonally dominant and product γ-diagonally dominant, we show that the eigenvalues of the Schur complement are located in the Geršgorin discs and the Ostrowski discs of the original matrices under certain conditions.

2. The criterion for H-matrix

In order to obtain the eigenvalue distribution of the Schur complement by the entries of the original matrix, in this section, we present two new criteria for the H-matrix.

Lemma 2.1. ([9, p.137]) *A matrix $A \in C^{n\times n}$ is an H-matrix if and only if A is generalized diagonally dominant.*

Lemma 2.2. ([7, Theorem 2]) *Let $A \in C^{n\times n}$. Then the following conditions are equivalent:*
(i) $A \in H_n$.
(ii) There exists $\gamma \in [0, 1]$ such that $A \in GSD_n^\gamma$.
(iii) There exists $\gamma \in [0, 1]$ such that $A \in GSPD_n^\gamma$.

Lemma 2.3. ([10, p.114-115]) *A matrix $A \in C^{n\times n}$ is an H-matrix if and only if there exists positive diagonal matrices D_1 and D_2 such that D_1AD_2 is an H-matrix.*

Theorem 2.1. *Let $A \in C^{n\times n}$. If there exists proper subsets N_1, N_2 of N with $N_1 \cup N_2 = N, N_1 \cap N_2 = \emptyset$ and $\gamma \in [0, 1]$ such that $A(N_2)$ is strictly γ-diagonally dominant and*

$$\frac{|a_{ii}| - \gamma\alpha_i(A) - (1-\gamma)\alpha_i'(A)}{\gamma\beta_i(A) + (1-\gamma)\beta_i'(A)} > \frac{\gamma\alpha_j(A) + (1-\gamma)\alpha_j'(A)}{|a_{jj}| - \gamma\beta_j(A) - (1-\gamma)\beta_j'(A)}, \tag{2.1}$$

for all $i \in N_1, j \in N_2$, *then* A *is an* H*-matrix. Note when* $\gamma\beta_i(A) + (1-\gamma)\beta_i'(A) = 0$, *we let*

$$\frac{|a_{ii}| - \gamma\alpha_i(A) - (1-\gamma)\beta_i(A)}{\gamma\beta_i(A) + (1-\gamma)\beta_i'(A)} = +\infty.$$

Proof. By (2.1), we can choose d such that

$$\min_{i\in N_1} \frac{|a_{ii}| - \gamma\alpha_i(A) - (1-\gamma)\alpha_i'(A)}{\gamma\beta_i(A) + (1-\gamma)\beta_i'(A)} > d > \max_{j\in N_2} \frac{\gamma\alpha_j(A) + (1-\gamma)\alpha_j'(A)}{|a_{jj}| - \gamma\beta_j(A) - (1-\gamma)\beta_j'(A)}.$$

Since $A(N_2)$ is strictly γ-diagonally dominant, we have so $d > 0$ and we can construct a positive diagonal matrix $D = diag(d_i | d_i = 1, i \in N_1; d_i = d, i \in N_2)$. Denote $B = DAD = (b_{ij})$. Then

$$b_{ij} = \begin{cases} a_{ij}, & \text{if} \quad i \in N_1, j \in N_1; \\ da_{ij}, & \text{if} \quad i \in N_1, j \in N_2; \\ da_{ij}, & \text{if} \quad i \in N_2, j \in N_1; \\ d^2 a_{ij}, & \text{if} \quad i \in N_2, j \in N_2. \end{cases}$$

Hence for all $i \in N_1$,

$$\begin{aligned} & |b_{ii}| - \gamma P_i(B) - (1-\gamma)S_i(B) \\ = & |a_{ii}| - \gamma\alpha_i(A) - \gamma\beta_i(A)d - (1-\gamma)\alpha_i'(A) - (1-\gamma)\beta_i'(A)d \\ = & |a_{ii}| - \gamma\alpha_i(A) - (1-\gamma)\alpha_i'(A) - [\gamma\beta_i(A) + (1-\gamma)\beta_i'(A)]d \\ > & |a_{ii}| - \gamma\alpha_i(A) - (1-\gamma)\alpha_i'(A) - [\gamma\beta_i(A) + (1-\gamma)\beta_i'(A)] \\ & \quad \frac{|a_{ii}| - \gamma\alpha_i(A) - (1-\gamma)\alpha_i'(A)}{\gamma\beta_i(A) + (1-\gamma)\beta_i'(A)} \\ = & 0, \end{aligned}$$

and for all $i \in N_2$,

$$\begin{aligned} & |b_{ii}| - \gamma P_i(B) - (1-\gamma)S_i(B) \\ = & |a_{ii}|d^2 - \gamma\alpha_i(A)d - \gamma\beta_i(A)d^2 - (1-\gamma)\alpha_i'(A)d - (1-\gamma)\beta_i'(A)d^2 \\ = & d\left\{[|a_{ii}| - \gamma\beta_i(A) - (1-\gamma)\beta_i'(A)]d - \gamma\alpha_i(A) - (1-\gamma)\alpha_i'(A)\right\} \\ > & d\{[|a_{ii}| - \gamma\beta_i(A) - (1-\gamma)\beta_i'(A)] \frac{\gamma\alpha_i(A) + (1-\gamma)\alpha_i'(A)}{|a_{ii}| - \gamma\beta_i(A) - (1-\gamma)\beta_i'(A)} \\ & \quad -\gamma\alpha_i(A) - (1-\gamma)\alpha_i'(A)\} \\ = & 0. \end{aligned}$$

Hence B is strictly γ-diagonally dominant and by Lemma 2.2 we know B is an H-matrix. By Lemma 2.3 it is easy to know A is an H-matrix. □

Theorem 2.2. *Let* $A \in C^{n\times n}$. *If there exists proper subsets* N_1, N_2 *of* N *with* $N_1 \cup N_2 = N, N_1 \cap N_2 = \emptyset$ *and* $\gamma \in (0,1]$ *such that* $|a_{jj}| > \beta_j(A)^\gamma S_j(A)^{1-\gamma}$ *and*

$$\frac{|a_{ii}|^{\frac{1}{\gamma}} - \alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}}{\beta_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}} > \frac{\alpha_j(A)S_j(A)^{\frac{1-\gamma}{\gamma}}}{|a_{jj}|^{\frac{1}{\gamma}} - \beta_j(A)S_j(A)^{\frac{1-\gamma}{\gamma}}} \tag{2.2}$$

for all $i \in N_1, j \in N_2$*, then* A *is an* H*-matrix. Note when* $\beta_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}} = 0,$ *we let*

$$\frac{|a_{ii}|^{\frac{1}{\gamma}} - \alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}}{\beta_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}} = +\infty.$$

Proof. By (2.2) we can choose d such that

$$\min_{i\in N_1} \frac{|a_{ii}|^{\frac{1}{\gamma}} - \alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}}{\beta_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}} > d > \max_{j\in N_2} \frac{\alpha_j(A)S_j(A)^{\frac{1-\gamma}{\gamma}}}{|a_{jj}|^{\frac{1}{\gamma}} - \beta_j(A)S_j(A)^{\frac{1-\gamma}{\gamma}}}.$$

Since $|a_{jj}| > \beta_j(A)^\gamma S_j(A)^{1-\gamma}$ for all $j \in N_2$, we know $d > 0$. Hence we can construct a positive diagonal matrix $D = diag(d_i | d_i = 1, i \in N_1; d_i = d, i \in N_2)$ and denote $B = AD = (b_{ij})$, then

$$b_{ij} = \begin{cases} a_{ij}, & \text{if } \ j \in N_1; \\ da_{ij}, & \text{if } \ j \in N_2. \end{cases}$$

By direct calculation, we know for all $i \in N_1$,

$$\begin{aligned} &|b_{ii}|^{\frac{1}{\gamma}} - P_i(B)S_i(B)^{\frac{1-\gamma}{\gamma}} \\ &= |a_{ii}|^{\frac{1}{\gamma}} - \alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}} - d\beta_i(A)S_i(A)^{\frac{1}{\gamma}} \\ &> |a_{ii}|^{\frac{1}{\gamma}} - \alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}} - \frac{|a_{ii}|^{\frac{1}{\gamma}} - \alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}}{\beta_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}} \beta_i(A)S_i(A)^{\frac{1}{\gamma}} \\ &= 0; \end{aligned}$$

and for all $i \in N_2$,

$$\begin{aligned} &|b_{ii}|^{\frac{1}{\gamma}} - P_i(B)S_i(B)^{\frac{1-\gamma}{\gamma}} \\ &= (d|a_{ii}|)^{\frac{1}{\gamma}} - \alpha_i(A)(dS_i(A))^{\frac{1-\gamma}{\gamma}} - d\beta_i(A)(dS_i(A))^{\frac{1-\gamma}{\gamma}} \\ &= d^{\frac{1-\gamma}{\gamma}} \left[|a_{ii}|^{\frac{1}{\gamma}} d - \alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}} - d\beta_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}} \right] \\ &= d^{\frac{1-\gamma}{\gamma}} \left\{ \left[a_{ii}|^{\frac{1}{\gamma}} - \beta_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}} \right] d - \alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}} \right\} \\ &> d^{\frac{1-\gamma}{\gamma}} \left\{ \left[|a_{ii}|^{\frac{1}{\gamma}} - \beta_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}} \right] \frac{\alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}}{|a_{ii}|^{\frac{1}{\gamma}} - \beta_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}}} - \alpha_i(A)S_i(A)^{\frac{1-\gamma}{\gamma}} \right\} \\ &= 0. \end{aligned}$$

For all $i \in N, |b_{ii}|^{\frac{1}{\gamma}} > P_i(B)S_i(B)^{\frac{1-\gamma}{\gamma}}$, i.e., $|b_{ii}| > P_i(B)^\gamma S_i(B)^{1-\gamma}$, so B is a strictly product γ-diagonally dominant matrix and A is a generalized product γ-diagonally dominant matrix. By Lemma 2.2 it is easy to know A is an H-matrix. □

If we choose $\gamma = 1$ in the Theorems 2.1 or 2.2, we obtain the following corollary.

Corollary 2.1. *Let $A \in C^{n\times n}$. If there exists proper subsets N_1, N_2 of N with $N_1 \cup N_2 = N, N_1 \cap N_2 = \emptyset$ such that $A(N_2)$ is strictly diagonally dominant and*

$$\frac{|a_{ii}| - \alpha_i(A)}{\beta_i(A)} > \frac{\alpha_j(A)}{|a_{jj}| - \beta_j(A)}, \tag{2.3}$$

for all $i \in N_1, j \in N_2$, then A is an H-matrix.

Remark 2.1. Theorems 1 and 2 of the paper [11] are exactly Corollary 2.1 of this paper. In [12], R. S. Varga called this matrix which satisfy the condition of Corollary 2.1 S-strictly diagonally dominant matrix and by using it obtained an eigenvalue inclusion set in the complex plane.

3. Distribution for eigenvalues of Schur complement

In this section, for two class matrices whose submatrices are γ-diagonally dominant and product γ-diagonally dominant, we show that the eigenvalues of the Schur complement are located in the Geršgorin discs and Ostrowski discs of the original matrices under certain conditions.

Lemma 3.1. ([13, p.19]) *Let $A \in C^{n\times n}$ and $\alpha \subset N$. If $A(\alpha)$ is nonsingular, then*

$$\det A = \det(A/\alpha) \det A(\alpha). \tag{3.1}$$

Lemma 3.2. ([14, p.349]) *Let $A = (a_{ij}) \in C^{n\times n}$ be strictly diagonally dominant. Then A is invertible.*

Theorem 3.1. *Let $A \in C^{n\times n}$, $\alpha = \{i_1, i_2, \ldots, i_k\} = N_1$, $\alpha^c = \{j_1, j_2, \ldots, j_l\} = N_2$, with $N_1 \cup N_2 = N, N_1 \cap N_2 = \emptyset$. If there exists $\gamma \in [0,1]$ such that $A(\alpha)$ is strictly γ-diagonally dominant, then*

$$\begin{aligned}\sigma(A/\alpha) \subseteq & \bigcup_{t=1}^{l} \left\{ z \in C : |z - a_{j_t j_t}| \le \gamma P_{j_t}(A) + (1-\gamma) S_{j_t}(A) - w_{j_t}^{(\gamma)} \right\} \\ & \equiv G_1^{(\gamma)}(A),\end{aligned} \tag{3.2}$$

where

$$w_{j_t}^{(\gamma)} = \min_{1\le w\le k} \frac{|a_{i_w i_w}| - \gamma P_{i_w}(A) - (1-\gamma) S_{i_w}(A)}{|a_{i_w i_w}| - \gamma \alpha_{i_w}(A) - (1-\gamma)\alpha'_{i_w}(A)} \left[\gamma \alpha_{j_t}(A) + (1-\gamma)\alpha'_{j_t}(A)\right].$$

Proof. Suppose that (3.2) is not valid, then there exists an eigenvalue λ of A/α such that $\lambda \notin G_1^{(\gamma)}(A)$ and hence $\lambda \in [G_1^{(\gamma)}(A)]^c$. By the definition and property of complement set, we know

$$\begin{aligned}[G_1^{(\gamma)}(A)]^c &= \left\{\bigcup_{t=1}^{l}\left\{z\in C: |z-a_{j_tj_t}|\le \gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A)-w_{j_t}^{(\gamma)}\right\}\right\}^c\\ &=\bigcap_{t=1}^{l}\left\{z\in C: |z-a_{j_tj_t}|\le \gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A)-w_{j_t}^{(\gamma)}\right\}^c\\ &=\bigcap_{t=1}^{l}\left\{z\in C: |z-a_{j_tj_t}|> \gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A)-w_{j_t}^{(\gamma)}\right\},\end{aligned}$$

hence, for all $1\le t\le l$,

$$|\lambda-a_{j_tj_t}|>\gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A)-w_{j_t}^{(\gamma)}.$$

By direct calculation,

$$\begin{aligned}&|\lambda-a_{j_tj_t}|>\gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A)-w_{j_t}^{(\gamma)}\\ =&\gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A)-\min_{1\le w\le k}\frac{|a_{i_wi_w}|-\gamma P_{i_w}(A)-(1-\gamma)S_{i_w}(A)}{|a_{i_wi_w}|-\gamma\alpha_{i_w}(A)-(1-\gamma)\alpha'_{i_w}(A)}\\ &\left[\gamma\alpha_{j_t}(A)+(1-\gamma)\alpha'_{j_t}(A)\right]\\ \ge&\gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A)-\frac{|a_{i_wi_w}|-\gamma P_{i_w}(A)-(1-\gamma)S_{i_w}(A)}{|a_{i_wi_w}|-\gamma\alpha_{i_w}(A)-(1-\gamma)\alpha'_{i_w}(A)}\\ &\left[\gamma\alpha_{j_t}(A)+(1-\gamma)\alpha'_{j_t}(A)\right]\\ =&\gamma\alpha_{j_t}(A)+\gamma\beta_{j_t}(A)+(1-\gamma)\alpha'_{j_t}(A)+(1-\gamma)\beta'_{j_t}(A)\\ &-\frac{|a_{i_wi_w}|-\gamma\alpha_{i_w}(A)-\gamma\beta_{i_w}(A)-(1-\gamma)\alpha'_{i_w}(A)-(1-\gamma)\beta'_{i_w}(A)}{|a_{i_wi_w}|-\gamma\alpha_{i_w}(A)-(1-\gamma)\alpha'_{i_w}(A)}\\ &\left[\gamma\alpha_{j_t}(A)+(1-\gamma)\alpha'_{j_t}(A)\right]\\ =&\gamma\beta_{j_t}(A)+(1-\gamma)\beta'_{j_t}(A)+\frac{\gamma\beta_{i_w}(A)+(1-\gamma)\beta'_{i_w}(A)}{|a_{i_wi_w}|-\gamma\alpha_{i_w}(A)-(1-\gamma)\alpha'_{i_w}(A)}\\ &\left[\gamma\alpha_{j_t}(A)+(1-\gamma)\alpha'_{j_t}(A)\right].\end{aligned}$$

Hence for all $1\le t\le l$ and $1\le w\le k$

$$\frac{|\lambda-a_{j_tj_t}|-\gamma\beta_{j_t}(A)-(1-\gamma)\beta'_{j_t}(A)}{\gamma\alpha_{j_t}(A)+(1-\gamma)\alpha'_{j_t}(A)}>\frac{\gamma\beta_{i_w}(A)+(1-\gamma)\beta'_{i_w}(A)}{|a_{i_wi_w}|-\gamma\alpha_{i_w}(A)-(1-\gamma)\alpha'_{i_w}(A)}.$$

Because $A(\alpha)$ is strictly γ-diagonally dominant, by Theorem 2.4 we know $B(\lambda)$ is an H-matrix, where

$$B(\lambda)=\begin{pmatrix}A(\alpha) & A(\alpha,\alpha^c)\\ A(\alpha^c,\alpha) & A(\alpha^c)-\lambda I\end{pmatrix}.$$

Notice that $B(\lambda)/A(\alpha) = A(\alpha^c) - \lambda I - A(\alpha^c,\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha^c) = A/\alpha - \lambda I$. By Lemma 3.2 and Lemma 2.1 we know H-matrices and diagonally dominant matrices are nonsingular, as $B(\lambda)$ is an H-matrix and $A(\alpha)$ is a strictly γ-diagonally dominant matrix, $B(\lambda)$ and $A(\alpha)$ are nonsingular, and by Lemma 3.1 we know that

$$\det(A/\alpha - \lambda I) = \det[B(\lambda)/A(\alpha)] = \frac{\det B(\lambda)}{\det A(\alpha)} \neq 0,$$

which is in contradiction with the assumption that λ is an eigenvalue of A/α. Thus we have completed the proof. $\square$

Theorem 3.2. *Let $A \in C^{n\times n}$, $\alpha = \{i_1, i_2, \ldots, i_k\} = N_1$ and $\alpha^c = \{j_1, j_2, \ldots, j_l\} = N_2$ with $N_1 \cup N_2 = N, N_1 \cap N_2 = \emptyset$. If there exists $\gamma \in (0,1]$ such that $|a_{i_w i_w}| > \alpha_{i_w}(A)^\gamma S_{i_w}(A)^{1-\gamma}$ for all $1 \leq w \leq k$, then*

$$\sigma(A/\alpha) \subseteq \bigcup_{t=1}^{l} \left\{ z \in C : |z - a_{j_t j_t}|^{\frac{1}{\gamma}} \leq P_{j_t}(A) S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} - r_{j_t}^{(\gamma)} \right\} \equiv G_2^{(\gamma)}(A), \tag{3.3}$$

where

$$r_{j_t}^{(\gamma)} = \min_{1\leq w\leq k} \frac{|a_{i_w i_w}|^{\frac{1}{\gamma}} - P_{i_w}(A) S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}}{|a_{i_w i_w}|^{\frac{1}{\gamma}} - \alpha_{i_w}(A) S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}} \left[\alpha_{j_t}(A) S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} \right].$$

Proof. In order to prove (3.3), we suppose that (3.3) is not valid, then there exists an eigenvalue λ of A/α such that $\lambda \notin G_2^{(\gamma)}(A)$ and hence $\lambda \in \{G_2^{(\gamma)}(A)\}^c$, by the definition and property of complement set,

$$\begin{aligned}
\left\{G_2^{(\gamma)}(A)\right\}^c &= \left\{ \bigcup_{t=1}^{l} \left\{ z \in C : |z - a_{j_t j_t}|^{\frac{1}{\gamma}} \leq P_{j_t}(A) S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} - r_{j_t}^{(\gamma)} \right\} \right\}^c \\
&= \bigcap_{t=1}^{l} \left\{ z \in C : |z - a_{j_t j_t}|^{\frac{1}{\gamma}} \leq P_{j_t}(A) S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} - r_{j_t}^{(\gamma)} \right\}^c \\
&= \bigcap_{t=1}^{l} \left\{ z \in C : |z - a_{j_t j_t}|^{\frac{1}{\gamma}} > P_{j_t}(A) S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} - r_{j_t}^{(\gamma)} \right\}.
\end{aligned}$$

Therefore we obtain that for all $1 \leq t \leq l$

$$|\lambda - a_{j_t j_t}|^{\frac{1}{\gamma}} > P_{j_t}(A) S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} - r_{j_t}^{(\gamma)}.$$

By direct calculation,

$$\begin{aligned}
&|\lambda - a_{j_tj_t}|^{\frac{1}{\gamma}} \\
&> P_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} - r_{j_t}^{(\gamma)} \\
&= P_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} - \min_{1\le w\le k} \frac{|a_{i_wi_w}|^{\frac{1}{\gamma}} - P_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}}{|a_{i_wi_w}|^{\frac{1}{\gamma}} - \alpha_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}} \Big[\alpha_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}}\Big] \\
&\ge P_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} - \frac{|a_{i_wi_w}|^{\frac{1}{\gamma}} - P_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}}{|a_{i_wi_w}|^{\frac{1}{\gamma}} - \alpha_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}} \Big[\alpha_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}}\Big] \\
&= \alpha_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} + \beta_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} \\
&\qquad - \frac{|a_{i_wi_w}|^{\frac{1}{\gamma}} - \alpha_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}} - \beta_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}}{|a_{i_wi_w}|^{\frac{1}{\gamma}} - \alpha_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}} \Big[\alpha_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}}\Big] \\
&= \beta_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} + \frac{\beta_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}}{|a_{i_wi_w}|^{\frac{1}{\gamma}} - \alpha_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}} \Big[\alpha_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}}\Big].
\end{aligned}$$

Hence for all $j_t \in \alpha^c$ and $i_w \in \alpha$

$$\frac{|\lambda - a_{j_tj_t}|^{\frac{1}{\gamma}} - \beta_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}}}{\alpha_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}}} > \frac{\beta_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}}{|a_{i_wi_w}|^{\frac{1}{\gamma}} - \alpha_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}}}.$$

Since $|a_{i_wi_w}| > \alpha_{i_w}(A)^\gamma S_{i_w}(A)^{1-\gamma}$ for all $1 \le w \le k$, by Theorem 2.5, we know that matrix $B(\lambda)$ is an H-matrix, where

$$B(\lambda) = \begin{pmatrix} A(\alpha) & A(\alpha,\alpha^c) \\ A(\alpha^c,\alpha) & A(\alpha^c) - \lambda I \end{pmatrix}.$$

Observe that $B(\lambda)/A(\alpha) = A(\alpha^c) - \lambda I - A(\alpha^c,\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha^c) = A/\alpha - \lambda I$ and H-matrices are nonsingular, therefore by Lemma 3.1, we know

$$\det(A/\alpha - \lambda I) = \det(B(\lambda)/A(\alpha)) = \frac{\det(B(\lambda))}{\det(A(\alpha))} \ne 0.$$

It implies λ is not an eigenvalue of A/α. Thus we have completed the proof. □

Remark 3.1. If $\alpha \subseteq \{i \in N : |a_{ii}| > P_{j_t}(A)^\gamma S_{j_t}(A)^{1-\gamma}\}$, it easy to see for all $1 \le w \le k$,

$$|a_{i_wi_w}| > P_{i_w}(A)^\gamma S_{i_w}(A)^{1-\gamma},$$

and therefore

$$|a_{i_wi_w}|^{\frac{1}{\gamma}} > P_{i_w}(A)S_{i_w}(A)^{\frac{1-\gamma}{\gamma}},$$

and $r_{j_t}^{(\gamma)} > 0$. By (3.3), we obtain

$$\begin{aligned}\sigma(A/\alpha) &\subseteq \bigcup_{t=1}^{l}\left\{z\in C: |z-a_{j_tj_t}|^{\frac{1}{\gamma}} \le P_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}} - r_{j_t}^{(\gamma)}\right\}\\ &\subseteq \bigcup_{t=1}^{l}\left\{z\in C: |z-a_{j_tj_t}|^{\frac{1}{\gamma}} \le P_{j_t}(A)S_{j_t}(A)^{\frac{1-\gamma}{\gamma}}\right\}\\ &= \bigcup_{t=1}^{l}\left\{z\in C: |z-a_{j_tj_t}| \le P_{j_t}(A)^{\gamma}S_{j_t}(A)^{1-\gamma}\right\}.\end{aligned}$$

Hence if $\alpha \subseteq \{i\in N: |a_{ii}| > P_i(A)^{\gamma}S_i(A)^{1-\gamma}\}$, the eigenvalues of the Schur complement are located in the Ostrowski discs of the original matrices.

Remark 3.2. Suppose $A\in C^{n\times n}, \alpha=\{i_1,i_2,\dots,i_k\}, \alpha^c = N-\alpha = \{j_1,j_2,\dots,j_l\}$ and denote

$$w_{j_t} = \min_{1\le v\le k}\frac{|a_{i_vi_v}|-P_{i_v}(A)}{|a_{i_vi_v}|}\sum_{u=1}^{k}|a_{j_ti_u}|;\quad w_{j_t}^{T} = \min_{1\le v\le k}\frac{|a_{i_vi_v}|-S_{i_v}(A)}{|a_{i_vi_v}|}\sum_{u=1}^{k}|a_{i_uj_t}|.$$

Under the conditions $\alpha\subseteq N_r(A)\cap N_c(A)\neq\emptyset$, J. Z. Liu and Z. J. Huang [2] obtained the following eigenvalue distributions for the Schur complement A/α :

$$\sigma(A/\alpha) \subseteq \bigcup_{t=1}^{l}\left\{z\in C: |z-a_{j_tj_t}| \le \gamma P_{j_t}(A) + (1-\gamma)S_{j_t} - \gamma w_{j_t} - (1-\gamma)w_{j_t}^{T}\right\} \quad (3.4)$$

$$\sigma(A/\alpha) \subseteq \bigcup_{t=1}^{l}\left\{z\in C: |z-a_{j_tj_t}| \le (P_{j_t}(A)-w_{j_t})^{\gamma}(S_{j_t}-w_{j_t}^{T})^{1-\gamma}\right\}. \quad (3.5)$$

However, according to Theorem 3.1, we just need the condition $A(\alpha)$ is strictly γ-diagonally dominant, then we can obtain the eigenvalue distributions (3.2), and by Theorem 3.4, to get the eigenvalue distributions (3.3) we just need the condition $|a_{i_wi_w}| > \alpha_{i_w}(A)^{\gamma}S_{i_w}(A)^{1-\gamma}$ for all $1\le w\le k$. Furthermore if $\alpha\subseteq N_r(A)\cap N_c(A)$, we can see that under certain conditions the eigenvalue distributions (3.2) is better than (3.5) for some matrices from the later numerical examples.

We obtain the following results if we choose $\gamma = 1$ in Theorem 3.1.

Corollary 3.1. *Let $A\in C^{n\times n}$, $\alpha=\{i_1,i_2,\dots,i_k\}\subset N$ and $\alpha^c=\{j_1,j_2,\dots,j_l\}$. If $A(\alpha)$ is strictly diagonally dominant, then*

$$\sigma(A/\alpha) \subseteq \bigcup_{t=1}^{l}\left\{z\in C: |z-a_{j_tj_t}| \le P_{j_t}(A) - w_{j_t}^{(1)}\right\}, \quad (3.6)$$

where

$$w_{j_t}^{(1)} = \min_{1\le w\le k}\frac{|a_{i_wi_w}|-P_{i_w}(A)}{|a_{i_wi_w}|-\alpha_{i_w}(A)}\alpha_{j_t}(A).$$

Remark 3.3. Under the condition $\alpha \subseteq N_r(A)$, J. Z. Liu and Z. J. Huang [2] obtain the following eigenvalue distribution for the Schur complement:

$$\sigma(A/\alpha) \subseteq \bigcup_{t=1}^{l} \Big\{ z \in C : |z - a_{j_t j_t}| \leq P_{j_t}(A) - w_{j_t} \Big\}. \tag{3.7}$$

But in Corollary 3.1 we know we just need the condition $A(\alpha)$ is strictly diagonally dominant, then we can obtain (3.6), furthermore if $\alpha \subseteq N_r(A)$ it is easy to see

$$w_{j_t}^{(1)} = \min_{1\leq w\leq k} \frac{|a_{i_w i_w}| - P_{i_w}(A)}{|a_{i_w i_w}| - \alpha_{i_w}(A)} \alpha_{j_t}(A) \geq \min_{1\leq w\leq k} \frac{|a_{i_w i_w}| - P_{i_w}(A)}{|a_{i_w i_w}|} \alpha_{j_t}(A) = w_{jt},$$

hence our result is better than (3.7).

4. Numerical examples

In this section, we demonstrate the effectiveness of our results by the following examples.

Example 4.1. Let

$$A = \begin{pmatrix} 4 & 1 & 2 & 3 & 1 \\ 1 & 4 & 3 & 1 & 1 \\ 1 & 0 & 10 & 4 & 6 \\ 1 & 1 & 8 & 15 & 4 \\ 1 & 0 & 5 & 2 & 20 \end{pmatrix}, \qquad \alpha = \{1, 2\}.$$

Then

$$\begin{aligned} &P_1(A) = 7, \quad P_2(A) = 6, \quad P_3(A) = 11, \quad P_4(A) = 14, \quad P_5(A) = 8; \\ &S_1(A) = 4, \quad S_2(A) = 2, \quad S_3(A) = 18, \quad S_4(A) = 10, \quad S_5(A) = 12; \\ &r_3^{(\frac{1}{2})} = -18, \qquad r_4^{(\frac{1}{2})} = -20, \qquad r_5^{(\frac{1}{2})} = -12. \end{aligned}$$

Obviously $\alpha \nsubseteq N_r(A)$. Hence we could not directly use the results in [2]− [3] to obtain the eigenvalue distribution for the Schur complement A/α. However, since $|a_{11}|^2 > |a_{12}|S_1(A)$ and $|a_{22}|^2 > |a_{21}|S_2(A)$, by Theorem 3.2 the eigenvalue λ satisfies

$$\begin{aligned} \lambda \in &\{z : |z - 10| \leq 14.70\} \cup \{z : |z - 15| \leq 16.49\} \cup \{z : |z - 20| \leq 10.39\} \\ &\equiv G_2. \end{aligned} \tag{4.1}$$

Further, we use Figure 4.1 to illustrate (4.1).

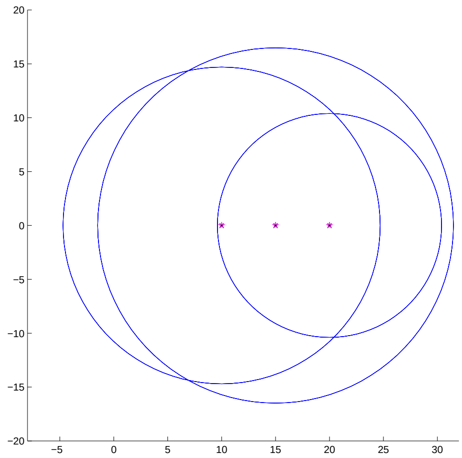

Figure 1: The dotted line denotes the corresponding discs of (4.1).

Example 4.2. Let

$$A = \begin{pmatrix} 10 & 5 & 2 & 1 & 0 \\ 5 & 10 & 0 & 1 & 2 \\ 1 & 1 & 4 & 2 & 0 \\ 2 & 0 & 0 & 6 & 2 \\ 0 & 2 & 2 & 0 & 8 \end{pmatrix}, \qquad \alpha = \{1,2\}.$$

Then

$$\begin{aligned} &P_1(A) = 8,\ P_2(A) = 8,\ P_3(A) = 4,\ P_4(A) = 4,\ P_5(A) = 4; \\ &S_1(A) = 8,\ S_2(A) = 8,\ S_3(A) = 4,\ S_4(A) = 4,\ S_5(A) = 4; \\ &w_3 = w_4 = w_5 = w_3^T = w_4^T = w_5^T = 0.4; \\ &w_3^{(\frac{1}{2})} = w_4^{(\frac{1}{2})} = w_4^{(\frac{1}{2})} = 0.8. \end{aligned}$$

Since $\alpha \subseteq N_r(A) \cap N_c(A)$, if we take $\gamma = \frac{1}{2}$, by Theorem 4 of [2], we know the eigenvalue λ of A/α satisfies

$$\lambda \in \Big\{z : |z-4| \le 3.6\} \cup \{z : |z-6| \le 3.6\} \cup \{z : |z-8| \le 3.6\Big\} \equiv G_2. \tag{4.2}$$

On the other hand, by Theorem 3.1, we know the eigenvalue λ of A/α satisfies

$$\lambda \in \Big\{z : |z-4| \le 3.2\} \cup \{z : |z-6| \le 3.2\} \cup \{z : |z-8| \le 3.2\Big\} \equiv G_3. \tag{4.3}$$

Further, we use Figure 4.2 to illustrate (4.2) and (4.3).
It is clear that $G_3 \subset G_2$, from both (4.2), (4.3) and Figure 4.2.

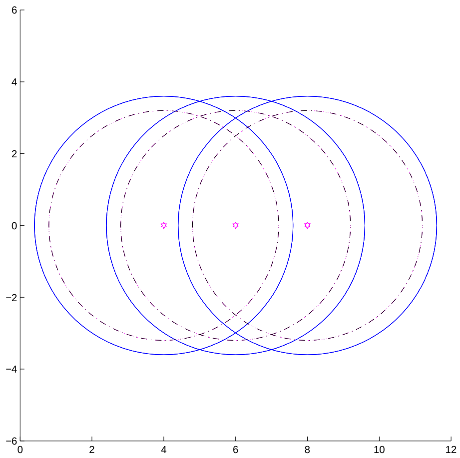

Figure 2: The dotted line and dashed line denote the corresponding discs of (4.2) and (4.3), respectively.

Acknowledgments The work was supported by National Natural Science Foundation of China (11571292, 11471279), National Natural Science Foundation for Youths of China (11401505), the Key Project of National Natural Science Foundation of China (91430213), the first class General Financial Grant from the China Postdoctoral Science Foundation (2015M582819), the General Project of Hunan Provincial Natural Science Foundation of China (2015JJ2134).

References

[1] J. Z. Liu and F. Z. Zhang, *Disc separation of the Schur complements of diagonally dominant matrices and determinantal bounds*, SIAM J. Matrix Anal. Appl., vol. 27, no. 3 (2005), pp. 665–674.

[2] J. Z. Liu and Z. J. Huang, *The Schur complements of γ-diagonally and product γ-diagonally dominant matrix and their disc separation*, Linear Algebra Appl., vol. 432 (2010), pp. 1090–1104.

[3] J. Z. Liu, Z. J. Huang, and J. Zhang, *The dominant degree and disc theorem for the Schur complement of matrix*, Appl. Math. Comput., vol. 215 (2010), pp. 4055–4066.

[4] J. Z. Liu, Z. H. Huang, L. Zhu, and Z. J. Huang, *Theorems on Schur complement of block diagonally dominant matrices and their application in reducing the order for the solution of large scale linear systems*, Linear Algebra Appl., vol. 435 (2011), pp. 3085–3100.

[5] J. Z. Liu, J. Zhang, and Y. Liu, *The Schur complement of strictly doubly diagonally dominant matrices and its application*, Linear Algebra Appl., vol. 437 (2012), pp. 168–183.

[6] L. Cvetković and M. Nedović, *Eigenvalue localization refinements for the Schur complement*, Appl. Math. Comput., vol. 218 (2012), pp. 8341–8346.

[7] J. Z. Liu and Y. Q. Huang, *Some properties on Schur complements of H-matrices and diagonally dominant matrices*, Linear Algebra Appl., vol. 389 (2004), pp. 365–380.

[8] C. Y. Zhang, C. X. Xu, and Y. T. Li, *The eigenvalue distribution on Schur complements of H-matrices*, Linear Algebra Appl., vol. 422 (2007), pp. 250–264.

[9] A. BERMAN AND R. J. PLEMMONS, *Nonnegative Matrices in the Mathematical Science*, Academic Press, New York, 1979.

[10] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge university press, New York, 1990.

[11] Y. M. GAO AND X. H. WANG, *Criteria for generalized diagonally dominant matrices and M-matrices*, Linear Algebra Appl., vol. 169 (1992), pp. 257–268.

[12] R. S. VARGA, *Geršgorin and His Circles*, Springer, Berlin, 2004.

[13] F.-Z. ZHANG, *The Schur complement and its Applications*, Springer-Verlag, New York, 2005.

[14] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Cambridge university press, New York, 1990.