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Lacunary Interpolation by Fractal Splines with Variable Scaling Parameters

P. Viswanathan[1,*], A.K.B. Chand [2], K.R. Tyada [2]

[1] *Department of Mathematics, Indian Institute of Technology Delhi, New Delhi, India.*
[2] *Department of Mathematics, Indian Institute of Technology Madras, Chennai, India.*

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Abstract. For a prescribed set of lacunary data $\{(x_\nu, f_\nu, f''_\nu) : \nu = 0, 1, \ldots, N\}$ with equally spaced knot sequence in the unit interval, we show the existence of a family of fractal splines $S_b^\alpha \in \mathcal{C}^3[0,1]$ satisfying $S_b^\alpha(x_\nu) = f_\nu$, $(S_b^\alpha)^{(2)}(x_\nu) = f''_\nu$ for $\nu = 0, 1, \ldots, N$ and suitable boundary conditions. To this end, the unique quintic spline introduced by A. Meir and A. Sharma [SIAM J. Numer. Anal. 10(3) 1973, pp. 433-442] is generalized by using fractal functions with variable scaling parameters. The presence of scaling parameters that add extra "degrees of freedom", self-referentiality of the interpolant, and "fractality" of the third derivative of the interpolant are additional features in the fractal version, which may be advantageous in applications. If the lacunary data is generated from a function Φ satisfying certain smoothness condition, then for suitable choices of scaling factors, the corresponding fractal spline S_b^α satisfies $\|\Phi^r - (S_b^\alpha)^{(r)}\|_\infty \to 0$ for $0 \le r \le 3$, as the number of partition points increases.

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1. Introduction

Through his land mark papers [1,3], Barnsley commenced the study of fractal interpolation by using the framework of Iterated Function System (IFS). Since then, many researchers have explored the technique and earnestly attempted to generalize the notion of Fractal Interpolation Function (FIF) in many different ways. As a new type of interpolant, FIF enjoys more advantages than the classical interpolation methods, which are based on polynomials, trigonometric functions, rational functions, and splines. To put in a nutshell, the main advantages of FIFs over traditional nonrecursive interpolants

*Corresponding author. *Email addresses:* amritaviswa@gmail.com (P. Viswanathan), chand@iitm.ac.in (A.K.B. Chand), kurmaths86@gmail.com (K.R. Tyada)

are: (i) they provide a method to render non-smooth approximants (ii) by suitable selection of parameters of the underlying IFS, FIFs can be made smooth and these smooth FIFs include traditional interpolants as special cases (iii) interpolation scheme produced by fractal functions can have local or global dependence on data points, depending on the choice of scaling factors (iv) interpolants possess self-referentiality (v) the interpolant or a certain derivative of it has a non-integer box-counting dimension, which can be controlled by scaling factors.

If the IFS is chosen appropriately in terms of a prescribed continuous function f, then the notion of fractal interpolation can be used to produce a family of fractal functions $\{f^\alpha\}$, which includes f as a very special case. This was first observed by Barnsley and later popularized by Navascués through a series of papers (see, for instance, [9, 10]). The free parameter α, which is a suitable vector in the Euclidean space, enables us to preserve or modify properties of the original function f. In particular, each element of this class can be made to preserve smoothness of the original. The methodology is so versatile and the corresponding notion of α-fractal function acts as a medium by which the theory of fractal interpolation overlaps and interacts quite fruitfully with many other fields of mathematics. In the perspective of numerical analysis, the notion of α-fractal function is used to generalize some well-known traditional interpolation techniques such as Hermite interpolation and splines [5, 11], but not explored in the area of lacunary interpolation. Furthermore, in much of the researches in fractal functions, the free parameters termed scaling factors, which have decisive influence on the properties of the "perturbed function", are restricted to be constants. Deriving principal influence from these facts, the present article targets to invite fractal functions with variable scalings to the field of lacunary interpolation.

To achieve the intended goal, a family of fractal splines is constructed as fractal perturbation (having function scaling parameters) of a quintic spline with $\mathcal{C}^3$-continuity introduced in [8]. This perturbation process allows one to replace the unicity of the traditional quintic spline that solves the lacunary interpolation problem with unicity up to a particular choice of scaling vector. This has practical advantage: the lack of unicity opens up the possibility of choosing an interpolant that fit a certain application best, for instance, in a problem that involves both approximation and optimization. Further, in contrast to the traditional quintic spline $S \in \mathcal{C}^3(I)$, the perturbed function $S_b^\alpha \in \mathcal{C}^3(I)$ has the property that its third derivative $(S_b^\alpha)^{(3)}$ may reveal, in general, non-smooth or fractal characteristic which can be quantified in terms of Minkowski dimension [6]. The fractal characteristic of the interpolant may be explored in various nonlinear and nonequilibrium phenomena. On the other hand, for suitable choice of scaling functions, the fractal spline introduced herein has same approximation properties as that of its classical counterpart. Thus, the current article may be considered as a humble attempt to (i) re-investigate [8] using fractal interpolation, a methodology which is not yet very familiar to the "traditional" numerical analysts, (ii) reiterate the ubiquity of fractal function by taking lacunary interpolation - a field where fractal splines are not yet explored - as a medium, and (iii) pronounce that approximation by fractal functions can provide more flexibility, which may be exploited in various practical applications.

2. Background and Preliminaries

In this section, we review some basic definitions and results that are needed in the sequel. For details, the reader is referred to [1,9].

2.1. Fractal functions

We begin with the definition of an Iterated Function System (IFS), which forms a standard framework to define fractal functions.

Definition 2.1. *Let (X, d) be a complete metric space. For a positive integer $N > 1$, let $W_i : X \to X$, $i = 1, 2, \ldots, N$ be continuous maps. Then the collection $\{X; W_1, W_2, \ldots, W_N\}$ is called an Iterated Function System. Further, if each of the maps W_i is a contraction, i.e.,*

$$d\big(W_i(x), W_i(y)\big) \leq \lambda_i \, d(x, y) \quad \textit{for all } x, y \in X, \quad \textit{and for some } 0 < \lambda_i < 1,$$

then the IFS is referred to as a contractive or hyperbolic IFS.

With a given IFS $\mathcal{F} = \{X; W_1, W_2, \ldots, W_N\}$ one can associate a set-valued map, which is termed a collage map, as follows. Let $\mathcal{H}(X)$ denote the collection of all non-empty compact subsets of X endowed with the Hausdorff metric

$$h(A, B) = \max\Big\{\max_{a\in A}\min_{b\in B} d(a, b), \max_{b\in B}\min_{a\in A} d(b, a)\Big\} \quad \forall \quad A, B \in \mathcal{H}(X).$$

Define $W : \mathcal{H}(X) \to \mathcal{H}(X)$ by

$$W(B) = \cup_{i=1}^{N} W_i(B),$$

where $W_i(B) = \{W_i(b) : b \in B\}$.

Definition 2.2. *A nonempty compact subset A of X is called an attractor of an IFS $\mathcal{F} = \{X; W_1, W_2, \ldots, W_N\}$ if*

1. *A is a fixed point of W, that is $W(A) = A$*
2. *there exists an open set $U \subseteq X$ such that $A \subset U$ and $\lim_{k\to\infty} h\big(W^k(B), A\big) = 0$ for all $B \in \mathcal{H}(U)$, where W^k is the k-fold autocomposition of W.*

The largest open set U for which (2) holds is called the basin of attraction for the attractor A of the IFS $\mathcal{F}$. The attractor A is also referred to as a fractal or self-referential set owing to the fact that A is a union of transformed copies of itself.

If the IFS $\mathcal{F}$ is contractive, then the existence of a unique attractor is ensured by the Banach fixed point theorem, and in this case, the basin of attraction is X.

In what follows, the question of how to obtain continuous functions whose graphs are fractals (in the above sense) is readdressed.

Let $\{(x_\nu, y_\nu) \in \mathbb{R}^2 : \nu = 0, 1, \ldots, N\}$ denote the given interpolation points. Let I denote the closed bounded interval $[x_0, x_N]$ and $I_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, N$. Suppose $L_i : I \to I_i$ are contraction homeomorphisms such that

$$L_i(x_0) = x_{i-1},\ L_i(x_N) = x_i, \quad |L_i(x) - L_i(x^*)| \le\ l_i|x - x^*| \\ \forall\, x, x^* \in I,\ \text{ for some } l_i \in (0,1).$$

Further, assume that $F_i : I \times \mathbb{R} \to \mathbb{R}$ are continuous maps satisfying

$$F_i(x_0, y_0) = y_{i-1}, \quad F_i(x_N, y_N) = y_i, \\ \big|F_i(x,y) - F_i(x, y^*)\big| \le s_i|y - y^*| \quad \forall\ y, y^* \in \mathbb{R},$$

and for some $s_i \in (0,1)$. Define functions $W_i : I \times \mathbb{R} \to I_i \times \mathbb{R}$ by

$$W_i(x,y) = \big(L_i(x), F_i(x,y)\big).$$

The following is a fundamental theorem in the subject of fractal functions.

Theorem 2.1. ([1]) *Let $\mathcal{C}(I)$, the space of all real-valued continuous functions on a compact interval I, be endowed with the Chebyshev norm $\|h\|_\infty := \max\{|h(x)| : x \in I\}$ and consider the closed metric subspace*

$$\mathcal{C}_{y_0,y_N}(I) := \{h \in \mathcal{C}(I) :\ \ h(x_0) = y_0,\ \ h(x_N) = y_N\}.$$

The following hold.

1. *The IFS $\{I \times \mathbb{R}; W_i, i = 1, 2, \ldots, N\}$ has a unique attractor $G(g)$ which is the graph of a continuous function $g : I \to \mathbb{R}$ satisfying $g(x_\nu) = y_\nu$ for $\nu = 0, 1, \ldots, N$.*

2. *The function g is the fixed point of the Read-Bajraktarević (RB) operator $T : \mathcal{C}_{y_0,y_N}(I) \to \mathcal{C}_{y_0,y_N}(I)$ defined via*

$$(Th)(x) = F_i\big(L_i^{-1}(x), h \circ L_i^{-1}(x)\big), \quad x \in I_i, \quad i \in \{1, \ldots, N\}.$$

Definition 2.3. *The function f appearing in the foregoing theorem is termed Fractal Interpolation Function (FIF) corresponding to the data $\{(x_\nu, y_\nu) : \nu = 0, 1, \ldots, N\}$.*

The most widely studied FIFs so far are defined by the system of maps

$$L_i(x) = a_i x + b_i, \quad F_i(x,y) = \alpha_i y + q_i(x), \tag{2.1}$$

where $|\alpha_i| < 1$ and $q_i : I \to \mathbb{R}$ are continuous functions satisfying

$$q_i(x_0) = y_{i-1} - \alpha_i y_0, \quad q_i(x_N) = y_i - \alpha_i y_N.$$

The parameter α_i is referred to as vertical scaling factor of the transformation F_i and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in (-1,1)^N$ is called scale vector. Note that here and throughout the article, we denote by A^m the Cartesian product $A \times A \times \cdots \times A$ (m times) of a set A.

Let $f \in \mathcal{C}(I)$. Choose a partition $\{x_0, x_1, \ldots, x_N\}$ of I and consider

$$q_i(x) = f\big(L_i(x)\big) - \alpha_i b(x),$$

where $b : I \to \mathbb{R}$ is a continuous function satisfying

$$b \neq f, \quad b(x_0) = f(x_0), \quad \text{and} \quad b(x_N) = f(x_N).$$

Then the IFS in (2.1) provides a fractal function denoted by $f^{\alpha}_{\Delta,b}$, which interpolates the data $\big\{\big(x_\nu, f(x_\nu)\big) : \nu = 0, 1, \ldots, N\big\}$. The function $f^{\alpha}_{\Delta,b}$ is termed as fractal perturbation of f or α-fractal function associated to f with respect to the partition Δ, base function b, and scale vector α. For a fixed partition Δ of I and scaling vector α, the operator which associates f to $f^{\alpha}_{\Delta,b}$ is a linear map, provided b depends on f linearly, say $b = Lf$, where $L : \mathcal{C}(I) \to \mathcal{C}(I)$ is a linear operator. That is,

$$\mathcal{F}^{\alpha}_{\Delta,b} : \mathcal{C}(I) \to \mathcal{C}(I), \quad \mathcal{F}^{\alpha}_{\Delta,b}(f) = f^{\alpha}_{\Delta,b}$$

is a linear operator.

In this paper, following [17], we replace constant scaling α_i with continuous functions $\alpha_i : I \to \mathbb{R}$ satisfying

$$\|\alpha_i\|_\infty := \max\big\{|\alpha_i(x)| : x \in I\big\} < 1.$$

2.2. Lacunary interpolation

Given n points $\{x_\nu : \nu = 1, 2, \ldots, n\}$ and corresponding to each x_ν a set of nonnegative integers $\{m_{1,\nu}, m_{2,\nu}, \ldots, m_{\alpha_\nu,\nu}\}$ and arbitrary numbers $\omega_{1,\nu}, \omega_{2,\nu}, \ldots, \omega_{\alpha_\nu,\nu}$, the central problem of lacunary interpolation is to find a polynomial P of degree less than or equal to $\alpha_1 + \alpha_2 + \cdots + \alpha_n - 1$ satisfying $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ conditions

$$P^{(m_{k,\nu})}(x_\nu) = \omega_{k,\nu} \quad \text{for} \ \ k = 1, \ldots, \alpha_\nu \quad \text{and} \quad \nu = 1, \ldots, n.$$

The term lacunary is chosen to suggest that there may be lacunae (gaps) in the sequence $m_{1,\nu}, m_{2,\nu}, \ldots, m_{\alpha_\nu,\nu}$. It reduces to the Hermite interpolation problem when $m_{k,\nu} = k - 1$ for $k = 1, \ldots, \alpha_\nu$. In contrast to the Hermite interpolation, for the problem of lacunary interpolation or Hermite-Birkhoff interpolation the existence and uniqueness of solution cannot be guaranteed in general. Schoenberg [12] used splines in lacunary interpolation. Various special cases of lacunary interpolation have received attention in the literature. For a historical background of lacunary interpolation, we refer the reader to the nice survey article [13].

3. Fractal Splines in Lacunary Interpolation

In this section, we initiate $(0,2)$-lacunary interpolation with fractal splines. We shall construct a class of fractal splines such that each member of the class will satisfy the desired lacunary interpolation condition. For a special choice of the parameter α, we reobtain the Meir-Sharma quintic spline interpolant [8]. For the sake of completeness of the exposition, a brief review of the Meir-Sharma quintic spline interpolant is provided in the Appendix. Note that the word spline in Meir-Sharma interpolant is used in a more general sense to mean a piecewise defined polynomial interpolant, a more accurate term could be a deficient spline. In the sequel, the space of all three times continuously differentiable real-valued functions on I will be denoted by $\mathcal{C}^3(I)$.

Theorem 3.1. *Let N be an odd integer and let a set of $2N+4$ real numbers*

$$\{f_0, f_1, \ldots, f_N; f_0'', f_1'', \ldots, f_N''; f_0''', f_N'''\}$$

be given. There exist smooth functions α_i, S, and b so that the α-fractal function S_b^α obtained through the IFS with maps $L_i : [0,1] \to [\frac{i-1}{N}, \frac{i}{N}]$ and $F_i : [0,1] \times \mathbb{R} \to \mathbb{R}$

$$L_i(x) = \frac{x + (i-1)}{N}, \quad F_i(x,y) = \alpha_i(x)y + S(L_i(x)) - \alpha_i(x)b(x), \quad i = 1, \ldots, N,$$

has $\mathcal{C}^3$-continuity and satisfies the $(0,2)$-interpolation conditions

$$(S_b^\alpha)\left(\frac{\nu}{N}\right) = f_\nu, \quad (S_b^\alpha)^{(2)}\left(\frac{\nu}{N}\right) = f_\nu'' \quad \textit{for} \quad \nu = 0, 1, \ldots, N,$$

together with the boundary conditions

$$(S_b^\alpha)^{(3)}(0) = f_0''' \quad \textit{and} \quad (S_b^\alpha)^{(3)}(1) = f_N'''.$$

Proof. Let $S \in \mathcal{C}^3[0,1]$ be a function which is a quintic polynomial in each of the subintervals determined by an equally spaced partition points in $[0,1]$ and which satisfies the lacunary interpolatory conditions

$$\begin{aligned} &S\left(\frac{\nu}{N}\right) = f_\nu, \quad S^{(2)}\left(\frac{\nu}{N}\right) = f_\nu'', \quad \nu = 0, 1, \ldots, N; \\ &S^{(3)}(0) = f_0''', \quad S^{(3)}(1) = f_N'''. \end{aligned}$$

The existence and uniqueness of such a quintic spline is established in [8] (see Appendix). We call this quintic spline S solving $(0,2)$-lacunary interpolation problem as the Meir-Sharma interpolant. For $i = 1, 2, \ldots, N$, choose $\alpha_i \in \mathcal{C}^3[0,1]$ such that

$$\|\alpha_i\|_{\mathcal{C}^3} := \max\left\{\|\alpha_i^{(j)}\|_\infty : j = 0, 1, 2, 3\right\} < \frac{1}{(2N)^3}, \quad i = 1, \ldots, N.$$

Further, choose $b \in \mathcal{C}^3[0,1]$ such that

$$b^{(j)}(0) = S^{(j)}(0), \quad b^{(j)}(1) = S^{(j)}(1), \quad j = 0, 1, 2, 3. \tag{3.1}$$

We shall obtain S_b^α as the fixed point of a suitable contraction map acting on a complete metric space. To this end, consider the Banach space $\mathcal{C}^3[0,1]$ endowed with the norm

$$\|g\|_{\mathcal{C}^3} := \max\left\{\|g^{(j)}\|_\infty : j = 0,1,2,3\right\}$$

and its (metric) subspace

$$\mathcal{C}_S^3[0,1] := \Big\{g \in \mathcal{C}^3[0,1] : g^{(j)}(0) = S^{(j)}(0), \quad g^{(j)}(1) = S^{(j)}(1), \text{ for } j = 0,1,2,3\Big\}.$$

It follows at once that $\mathcal{C}_S^3[0,1]$ is a closed subspace of $\mathcal{C}^3[0,1]$ and hence a complete metric space. Define an operator $T : \mathcal{C}_S^3[0,1] \to \mathcal{C}_S^3[0,1]$ such that for $x \in I_i = [\frac{i-1}{N}, \frac{i}{N}],\ i = 1,\dots,N$

$$\begin{aligned}(Tg)(x) &= F_i\big(L_i^{-1}(x), g(L_i^{-1}(x))\big)\\ &= S(x) + \alpha_i\big(L_i^{-1}(x)\big)\Big(g\big(L_i^{-1}(x)\big) - b\big(L_i^{-1}(x)\big)\Big).\end{aligned}$$

By the selection of functions S, b and α_i, $i = 1,\dots,N$, it is apparent that Tg and its first three derivatives are continuous on $I_i,\ i = 1,2,\dots,N$. Using the fact that $L_i : [0,1] \to I_i$ are affine maps such that $L_i(1) = L_{i+1}(0) = \frac{i}{N}$ for $i = 1,\dots,N-1$ and applying the Leibnitz rule of differentiation we obtain that for $k = 0,1,2,3$:

$$\begin{aligned}(Tg)^{(k)}\left(\frac{i}{N}^-\right) &= S^{(k)}\left(\frac{i}{N}\right) + N^k\sum_{j=0}^{k}\binom{k}{j}(g-b)^{(j)}(1)\ \alpha_i^{(k-j)}(1),\\ (Tg)^{(k)}\left(\frac{i}{N}^+\right) &= S^{(k)}\left(\frac{i}{N}\right) + N^k\sum_{j=0}^{k}\binom{k}{j}(g-b)^{(j)}(0)\ \alpha_{i+1}^{(k-j)}(0).\end{aligned}$$

Since $g^{(j)}(0) = b^{(j)}(0) = S^{(j)}(0)$ and $g^{(j)}(1) = b^{(j)}(1) = S^{(j)}(1)$ for each $g \in \mathcal{C}_S^3[0,1]$ and $j = 0,1,2,3$, it follows from the previous equations that for $k = 0,1,2,3$:

$$(Tg)^{(k)}\left(\frac{i}{N}^-\right) = (Tg)^{(k)}\left(\frac{i}{N}^+\right) = S^{(k)}\left(\frac{i}{N}\right) \quad \text{for } i = 1,\dots,N-1. \tag{3.2}$$

Consequently, $Tg \in \mathcal{C}^3[0,1]$. Using $L_1(0) = 0$ and $L_N(1) = 1$ we may deduce that for $k = 0,1,2,3$:

$$(Tg)^{(k)}(0) = S^{(k)}(0) + N^k\sum_{j=0}^{k}\binom{k}{j}(g-b)^{(j)}(0)\ \alpha_1^{(k-j)}(0) = S^{(k)}(0), \tag{3.3a}$$

$$(Tg)^{(k)}(1) = S^{(k)}(1) + N^k\sum_{j=0}^{k}\binom{k}{j}(g-b)^{(j)}(1)\ \alpha_N^{(k-j)}(1) = S^{(k)}(1), \tag{3.3b}$$

and from which it follows that $Tg \in \mathcal{C}^3_S[0,1]$. Suppose $g, h \in \mathcal{C}^3_S[0,1]$ and $x \in [\frac{i-1}{N}, \frac{i}{N}]$. We see that

$$\begin{aligned}
&\big|(Tg)^{(k)}(x) - (Th)^{(k)}(x)\big| \\
&= N^k \Big| \sum_{j=0}^{k} \binom{k}{j} \alpha_i^{(k-j)}\big(L_i^{-1}(x)\big)(g-h)^{(j)}\big(L_i^{-1}(x)\big) \Big| \\
&\leq N^k \, \|\alpha_i\|_{\mathcal{C}^k} \, \|g-h\|_{\mathcal{C}^k} \sum_{j=0}^{k} \binom{k}{j}.
\end{aligned} \tag{3.4}$$

Suppose that for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \big(\mathcal{C}^3[0,1]\big)^N$, the product norm is defined by

$$\|\alpha\|_k = \max\big\{\|\alpha_i\|_{\mathcal{C}^k} : i = 1, \ldots, N\big\}.$$

From (3.4) we infer that for $k = 0, 1, 2, 3$,

$$\|(Tg)^{(k)} - (Th)^{(k)}\|_\infty \leq (2N)^k \, \|\alpha\|_k \, \|g-h\|_{\mathcal{C}^k},$$

and hence that

$$\|Tg - Th\|_{\mathcal{C}^3} \leq (2N)^3 \, \|\alpha\|_3 \, \|g-h\|_{\mathcal{C}^3}.$$

Since $\|\alpha_i\|_{\mathcal{C}^3} < \frac{1}{(2N)^3}$ for all $i = 1, \ldots, N$, the previous inequality implies that T is a contraction on $\mathcal{C}^3_S[0,1]$. Therefore, by the Banach fixed point theorem T has a unique fixed point S^α_b satisfying the self-referential equation: for $i = 1, 2, \ldots, N$

$$S^\alpha_b(x) = S(x) + \alpha_i\big(L_i^{-1}(x)\big)\big[S^\alpha_b\big(L_i^{-1}(x)\big) - b\big(L_i^{-1}(x)\big)\big], \quad x \in I_i. \tag{3.5}$$

From (3.2)-(3.3), it follows that

$$\begin{aligned}
&(S^\alpha_b)\left(\frac{\nu}{N}\right) = S\left(\frac{\nu}{N}\right) = f_\nu, \quad (S^\alpha_b)^{(2)}\left(\frac{\nu}{N}\right) = S^{(2)}\left(\frac{\nu}{N}\right) = f''_\nu, \quad \nu = 0, 1, \ldots, N, \\
&(S^\alpha_b)^{(3)}(0) = S^{(3)}(0) = f'''_0, \quad (S^\alpha_b)^{(3)}(1) = S^{(3)}(1) = f'''_N.
\end{aligned}$$

This completes the proof. □

Several straightforward but noteworthy comments are in order.

Remark 3.1. It is easy to see that in the definition of a fractal function, for the recursion to take place, we need at least three knot points. Hence, we may assume in the previous theorem and throughout the paper that N is an odd integer greater than or equal to 3. For $N = 1$, additional knots may be inserted for the construction of fractal function.

Remark 3.2. From the functional equation of S^α_b (cf. (3.5)) it follows at once that if each scaling function is assumed to be zero function, then S^α_b reduces to the Meir-Sharma interpolant.

Remark 3.3. A natural choice for the function b satisfying (3.1) is the Hermite polynomial of degree 7 with contact of order 3 corresponding to the function S and knot points $x_0 = 0$, $x_N = 1$.

Remark 3.4. If we choose constant scaling factors, i.e., $\alpha_i : I \to \mathbb{R}$ as constant functions for $i = 1, 2, \ldots, N$, then a look back at (3.4) reveals that only one term in the summation, which corresponds to $j = k$, is nonzero. Consequently, the condition

$$|\alpha|_\infty := \max\{|\alpha_i| : i = 1, \ldots, N\} < \frac{1}{N^3}$$

assures that T is a contraction. In Reference [11], by using the Barnsley-Harrington (BH) theorem [3], a similar but in a slightly general setting, namely, the condition $|\alpha|_\infty < \frac{1}{N^p}$ is derived for the p-smoothness of a α-fractal function. Note that our treatment does not rely on BH-theorem but effectively uses RB operator, and allows the scaling factors to be functions rather than constants.

Remark 3.5. If N is even, then the existence of quintic spline S solving the aforementioned $(0,2)$-lacunary interpolation problem cannot be ensured (see [8]), and so is the existence of S_b^α. However, if we replace the boundary conditions in Theorem 3.1 with $(S_b^\alpha)^{(1)}(0) = f_0'$ and $(S_b^\alpha)^{(3)}(0) = f_0'''$, then for any integer $N > 2$, a fractal spline S_b^α solving this modified lacunary interpolation problem can be constructed similar to that in Theorem 3.1. Here we start with the traditional nonrecursive quintic spline S solving the lacunary interpolation problem with modified boundary conditions, the existence and uniqueness of which is established in [8].

Remark 3.6. Following our procedure, one can establish the fractal analogue of lacunary quartic spline with uniform knot sequence in $[0,1]$ constructed in [7]. Since the idea is already inherent in the proof of Theorem 3.1, we avoid the computational details.

4. Approximation Properties

This section is intended to establish that if the lacunary data set is generated by a function Φ satisfying certain smoothness condition, then the fractal spline $S_b^\alpha \in \mathcal{C}^3[0,1]$ converges to Φ with respect to the $\mathcal{C}^3$-norm, as number of partition points $N \to \infty$. To this end, we find upper bounds for the uniform norm $\|(S_b^\alpha)^{(k)} - \Phi^{(k)}\|_\infty$ for $k = 0, 1, 2,$ and 3. The bounds are not claimed to be optimal, but serve to establish the desired convergence. Let us record the following theorem as a prelude.

Theorem 4.1. ([8]) *Let $\Phi \in \mathcal{C}^4[0,1]$ and N be an odd integer. Then for the unique Meir-Sharma quintic spline S satisfying*

$$S\left(\frac{\nu}{N}\right) = \Phi\left(\frac{\nu}{N}\right), \quad S^{(2)}\left(\frac{\nu}{N}\right) = \Phi^{(2)}\left(\frac{\nu}{N}\right), \ \nu = 0, 1, \ldots, N;$$
$$S^{(3)}(0) = \Phi^{(3)}(0), \quad S^{(3)}(1) = \Phi^{(3)}(1),$$

we have

$$\|S^{(k)} - \Phi^{(k)}\|_\infty \le 75N^{k-3}\,\omega_4\left(\frac{1}{N}\right) + 8N^{k-4}\|\Phi^{(4)}\|_\infty, \quad k = 0,1,2,3,$$

where $\omega_4(.)$ *denotes the modulus of continuity of* $\Phi^{(4)}$. *If* $\Phi^{(4)} \in Lip_\beta[0,1]$, *then*

$$\|S^{(k)} - \Phi^{(k)}\|_\infty \le CN^{k-\beta-3}, \quad k = 0,1,2,3,$$

with a constant C *independent of* N. *If* Φ *is such that* $\Phi^{(4)}$ *is a Riemann integrable function on* $[0,1]$, *then we have*

$$\|S^{(k)} - \Phi^{(k)}\|_\infty \le o(N^{k-3}) \quad \text{as} \quad N \to \infty, \quad k = 0,1,2,3.$$

Proposition 4.1. *Let* S_b^α *be a fractal spline solving* $(0,2)$*-lacunary interpolation problem established in Theorem* 3.1 *and* S *be the Meir-Sharma interpolant. Then the perturbation error satisfies*

$$\|S_b^\alpha - S\|_\infty \le \frac{1}{8N^3-1}\,\|S-b\|_\infty.$$

Proof. In view of (3.5), for $x \in I_i = \left[\frac{i-1}{N}, \frac{i}{N}\right]$ we have

$$\begin{aligned} &\left|S_b^\alpha(x) - S(x)\right| \\ =&\left|\alpha_i\big(L_i^{-1}(x)\big)\right|\,\left|(S_b^\alpha - b)\big(L_i^{-1}(x)\big)\right| \\ \le&\|\alpha_i\|_\infty\|S_b^\alpha - b\|_\infty \le \|\alpha\|_\infty\|S_b^\alpha - b\|_\infty, \end{aligned}$$

where $\|\alpha\|_\infty := \max\{\|\alpha_i\|_\infty : i = 1,2,\dots,N\}$. From the previous inequality it follows that

$$\begin{aligned} \|S_b^\alpha - S\|_\infty &\le \|\alpha\|_\infty\|S_b^\alpha - b\|_\infty \\ &\le \|\alpha\|_\infty\Big(\|S_b^\alpha - S\|_\infty + \|S-b\|_\infty\Big) \end{aligned}$$

and hence

$$\|S_b^\alpha - S\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|S-b\|_\infty.$$

Since $\|\alpha_i\|_\infty < \|\alpha_i\|_{\mathcal{C}^3} < \frac{1}{(2N)^3}$ for all $i = 1,\dots,N$, the multiplier in the right hand side of the above inequality is less than $\frac{1}{8N^3-1}$, and hence the proof. $\square$

Proposition 4.2. *Let* S *be the Meir-Sharma quintic spline and* S_b^α *be its fractal analogue introduced in Theorem* 3.1. *Then, we have the following estimates for the perturbation error*

$$\|S_b^\alpha - S\|_{\mathcal{C}^k} \le \frac{(2^{k+1}-1)N^k}{8N^3-N^k}\,\|S-b\|_{\mathcal{C}^k}, \quad k = 1,2,3.$$

Proof. From Theorem 3.1 we obtain the following functional equation for the derivative of S_b^α

$$(S_b^\alpha)^{(k)}(x) = S^{(k)}(x) + N^k \sum_{j=0}^{k} \binom{k}{j} \alpha_i^{(k-j)}\big(L_i^{-1}(x)\big)(S_b^\alpha - b)^{(j)}\big(L_i^{-1}(x)\big).$$

Therefore for $x \in I_i$, $i = 1, 2, \ldots, N$,

$$\begin{aligned}
&\big|(S_b^\alpha)^{(k)}(x) - S^{(k)}(x)\big| \\
&\le N^k \sum_{j=0}^{k} \binom{k}{j} \big|\alpha_i^{(k-j)}\big(L_i^{-1}(x)\big)\big| \, \big|(S_b^\alpha - b)^{(j)}\big(L_i^{-1}(x)\big)\big| \\
&\le N^k \Big(\|\alpha_i\|_{\mathcal{C}^k} \|S_b^\alpha - b\|_{\mathcal{C}^{k-1}} (2^k - 1) + \|\alpha_i\|_\infty \|(S_b^\alpha)^{(k)} - b^{(k)}\|_\infty \Big) \\
&\le N^k (2^k - 1) \|\alpha_i\|_{\mathcal{C}^k} \Big(\|S_b^\alpha - S\|_{\mathcal{C}^{k-1}} + \|S - b\|_{\mathcal{C}^{k-1}} \Big) \\
&\quad + N^k \|\alpha_i\|_\infty \Big(\|(S_b^\alpha)^{(k)} - S^{(k)}\|_\infty + \|S^{(k)} - b^{(k)}\|_\infty \Big) \\
&\le \ N^k \|\alpha_i\|_\infty \|(S_b^\alpha)^{(k)} - S^{(k)}\|_\infty + N^k \|\alpha_i\|_{\mathcal{C}^k} \Big((2^k - 1) \|S_b^\alpha - S\|_{\mathcal{C}^{k-1}} + 2^k \|S - b\|_{\mathcal{C}^k} \Big).
\end{aligned}$$

For $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathcal{C}^3[0,1])^N$, let

$$\|\alpha\|_\infty := \max_{1 \le i \le N} \{\|\alpha_i\|_\infty\}, \quad \|\alpha\|_k := \max_{1 \le i \le N} \{\|\alpha_i\|_{\mathcal{C}^k}\}.$$

Then it follows that, for $k = 1, 2, 3$,

$$\begin{aligned}
&\|(S_b^\alpha)^{(k)} - S^{(k)}\|_\infty \\
\le& N^k \|\alpha\|_\infty \|(S_b^\alpha)^{(k)} - S^{(k)}\|_\infty + N^k \|\alpha\|_k \Big((2^k - 1) \|S_b^\alpha - S\|_{\mathcal{C}^{k-1}} + 2^k \|S - b\|_{\mathcal{C}^k} \Big).
\end{aligned}$$

Transposing the first summand to the left hand side and noting that $\|\alpha\|_\infty < \frac{1}{8N^3} < \frac{1}{N^k}$ for $k = 1, 2, 3$, we see that

$$\begin{aligned}
&\|(S_b^\alpha)^{(k)} - S^{(k)}\|_\infty \\
\le& \frac{N^k \|\alpha\|_k}{1 - N^k \|\alpha\|_\infty} \Big[(2^k - 1) \|S_b^\alpha - S\|_{\mathcal{C}^{k-1}} + 2^k \|S - b\|_{\mathcal{C}^k} \Big], \quad k = 1, 2, 3.
\end{aligned}$$

The previous inequality for $k = 1$ in conjunction with Proposition 4.1 yields

$$\begin{aligned}
\|(S_b^\alpha)^{(1)} - S^{(1)}\|_\infty &\le \frac{N\|\alpha\|_1}{1 - N\|\alpha\|_\infty} \Big[\|S_b^\alpha - S\|_\infty + 2\|S - b\|_{\mathcal{C}^1} \Big] \\
\le \frac{N\|\alpha\|_1}{1 - N\|\alpha\|_\infty} &\Big[\frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|S - b\|_\infty + 2\|S - b\|_{\mathcal{C}^1} \Big] \\
\le \frac{N\|\alpha\|_1}{1 - N\|\alpha\|_\infty} &\Big[\|S - b\|_\infty + 2\|S - b\|_{\mathcal{C}^1} \Big] \le \frac{3N\|\alpha\|_1}{1 - N\|\alpha\|_\infty} \|S - b\|_{\mathcal{C}^1}.
\end{aligned}$$

Noting $\|\alpha\|_\infty \le 3N\|\alpha\|_1$ it follows that

$$\begin{aligned}&\|S_b^\alpha - S\|_{\mathcal{C}^1} = \max\left\{\|S_b^\alpha - S\|_\infty, \|(S_b^\alpha)^{(1)} - S^{(1)}\|_\infty\right\}\\ \le\ &\max\left\{\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|S-b\|_\infty,\ \frac{3N\|\alpha\|_1}{1-N\|\alpha\|_\infty}\|S-b\|_{\mathcal{C}^1}\right\}\\ =\ &\frac{3N\|\alpha\|_1}{1-N\|\alpha\|_\infty}\|S-b\|_{\mathcal{C}^1}.\end{aligned}$$

Similarly, by induction we infer that

$$\|S_b^\alpha - S\|_{\mathcal{C}^k} \le \frac{(2^{k+1}-1)N^k\|\alpha\|_k}{1-N^k\|\alpha\|_\infty}\|S-b\|_{\mathcal{C}^k}, \quad k=1,2,3.$$

Using $\|\alpha\|_k \le \|\alpha\|_3 < \frac{1}{8N^3}$ for $k=1,2,3$, and $\|\alpha\|_\infty < \frac{1}{8N^3}$ we obtain the desired assertion. □

Remark 4.1. In view of Remark 3.3, if we choose b to be a septic Hermite osculatory polynomial for S with contact of order 3 at the end points of the interval I, then it is well-known that (see, for instance, [4])

$$\|S^{(k)} - b^{(k)}\|_\infty \le \Lambda_k\|S^{(4)}\|_\infty, \quad k=0,1,2,3,$$

where Λ_k is a suitable constant depending on k and $\|S^{(4)}\|_\infty$ is the essential supremum of $S^{(4)}$ over $[0,1]$.

Remark 4.2. If we impose a stronger condition on the scaling functions, namely, $\|\alpha_i\|_{\mathcal{C}^3} < \frac{1}{(2N)^4}$ for all $i=1,2,\dots,N$, then from Propositions 4.1-4.2 we obtain

$$\|S_b^\alpha - S\|_{\mathcal{C}^k} \le \frac{(2^{k+1}-1)N^k}{16N^4-N^k}\|S-b\|_{\mathcal{C}^k}, \quad k=0,1,2,3.$$

Having established these results, now we can easily prove the main theorem in this section.

Theorem 4.2. *Let the function Φ generating the lacunary data $\{(x_\nu, f_\nu, f''_\nu, f'''_0, f'''_N) : \nu = 0,1,\dots,N\}$, where $\{x_\nu : \nu = 0,1,\dots,N\}$ is a uniform knot sequence on the unit interval, be such that $\Phi^{(4)}$ is Riemann integrable on $[0,1]$. Let the scaling functions satisfy $\|\alpha_i\|_{\mathcal{C}^3} < \frac{1}{(2N)^4}$. Then the lacunary fractal interpolant S_b^α satisfies*

$$\|\Phi^{(k)} - (S_b^\alpha)^{(k)}\|_\infty = o\left(\frac{1}{N^{3-k}}\right) \quad \text{as} \quad N\to\infty, \quad k=0,1,2,3.$$

In particular, S_b^α converges to Φ with respect to the $\mathcal{C}^3$-norm as $N\to\infty$.

Proof. Consider the triangle inequality

$$\|\Phi^{(k)}-(S_b^\alpha)^{(k)}\|_\infty \le \|\Phi^{(k)}-S^{(k)}\|_\infty+\|S^{(k)}-(S_b^\alpha)^{(k)}\|_\infty,\quad k=0,1,2,3.$$

The first term on the right hand side of the above inequality can be bounded from above using Theorem 4.1 and the second term on the right can be bounded from above in view of Remark 4.2. Combining these, we obtain the proof. □

Remark 4.3. With the original assumption on the scaling functions, viz., $\|\alpha_i\|_\infty < \frac{1}{(2N)^3}$ for all $i=1,2,\dots,N$, we can derive the uniform convergence of S_b^α and its first two derivatives. Hence, in this case, we obtain a weaker convergence, namely, $\|\Phi - S_b^\alpha\|_{\mathcal{C}^2}\to 0$ as $N\to\infty$.

The next theorem points to the stability of fractal spline S_b^α with respect to perturbation in data points. Proof follows from the corresponding stability result of the Meir-Sharma interpolant (see Theorem 2, [14]) adapted to the present setting and the triangle inequality

$$\|\Phi^{(k)}-(S_b^\alpha)^{(k)}\|_\infty \le \|\Phi^{(k)}-S^{(k)}\|_\infty+\|S^{(k)}-(S_b^\alpha)^{(k)}\|_\infty,\quad k=0,1,2,3.$$

and hence omitted.

Theorem 4.3. *Let $\Phi\in\mathcal{C}^4[0,1]$ and N be an odd positive integer. Let S be the unique Meir-Sharma interpolant to the data*

$$\begin{aligned}&\left(\frac{\nu}{N}\right)=\beta_{\nu,0},\quad S^{(2)}\left(\frac{\nu}{N}\right)=\beta_{\nu,2},\quad 0\le\nu\le N;\\ &S^{(3)}(0)=\beta_{0,3},\quad S^{(3)}(1)=\beta_{N,3},\end{aligned}$$

and S_b^α be the corresponding fractal spline, where the scaling functions satisfy $\|\alpha_i\|_{\mathcal{C}^3}<\frac{1}{(2N)^4}$ for all $i=1,2,\dots,N$. Suppose that there exists a function $F(\Phi,N)$ such that

$$\begin{aligned}&\max_{0\le\nu\le N}\left|\Phi\left(\frac{\nu}{N}\right)-\beta_{\nu,0}\right|\le N^{-4}F(\Phi,N),\\ &\max_{0\le\nu\le N}\left|\Phi^{(2)}\left(\frac{\nu}{N}\right)-\beta_{\nu,2}\right|\le N^{-2}F(\Phi,N),\\ &\max\left\{|\Phi^{(3)}(0)-\beta_{0,3}|,|\Phi^{(3)}(1)-\beta_{N,3}|\right\}\le N^{-1}F(\Phi,N).\end{aligned}$$

Then

$$\begin{aligned}&\|\Phi^{(k)}-(S_b^\alpha)^{(k)}\|_\infty\\ \le& KN^{k-3}\left[\omega_k\left(\frac{1}{N}\right)+F(\Phi,N)\right]+\frac{(2^{k+1}-1)N^k}{16N^4-N^k}\|S-b\|_{\mathcal{C}^k},\quad k=0,1,2,3,\end{aligned}$$

where K is a constant, independent of Φ, F, and N, and $\omega_k(.)$ is the modulus of continuity of $\Phi^{(k)}$.

Remark 4.4. The sensitivity analysis of a fractal function with respect to the scaling functions, which shows that small perturbations in the scaling functions produce only small variations in the fractal functions, can be consulted in [17]. As free parameters, the scaling functions have decisive influence on the properties of the fractal spline S_b^α. Finding an optimal scaling vector $\alpha \in (-1,1)^N$ for which the perturbation S_b^α of S is close to a given function Φ is a constrained convex optimization problem with a solution (see [15, 16]). Optimality problem in the case of scaling functions is kept open.

5. Numerical Examples

In this section, we provide graphical illustration for the proposed $(0,2)$-lacunary fractal interpolation. To this end, consider a uniform partition of the unit interval with step size $\frac{1}{5}$. Let a set of 14 real numbers

$$\begin{aligned}\mathcal{D} &= \{f_\nu; f_\nu''; f_0''', f_5''' : \nu = 0,1,\dots,5\}\\ &= \{0, 0.04, 0.12, 0.26, 0.39, 0.5; 2, 1.5646, 0.6663, -0.0636, -0.4172, -0.5; 0, 0\}\end{aligned}$$

associated with the points in this uniform partition of $[0,1]$ be prescribed. Let us recall that the truncated power function with exponent n is defined as follows:

$$x_+^n := \begin{cases} x^n & \text{if } x > 0,\\ 0 & \text{if } x \le 0.\end{cases}$$

The Meir-Sharma quintic spline corresponding to $\mathcal{D}$ is given below. Here the coefficients are four digit approximations of the solution of a linear system of equations (see Appendix).

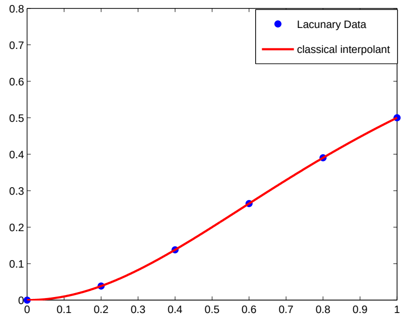
$$\begin{aligned}S(x) = {} & 0.3554B_0(x) + 2B_1(1-x) - 2.5861B_1(x) - 1.2997B_2(x)\\ & + 0.2714\,(x-0.2)_+^4 + 0.1599\,(x-0.2)_+^5 + 0.1349\,(x-0.4)_+^4\\ & - 0.4334\,(x-0.4)_+^5 - 0.0037\,(x-0.6)_+^4 - 0.2481\,(x-0.6)_+^5\\ & - 0.0406\,(x-0.8)_+^4 - 0.0074\,(x-0.8)_+^5\,,\end{aligned}$$

where

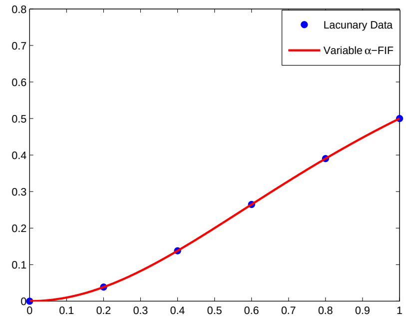
$$B_0(x) = x, \quad B_1(x) = \frac{x^4 - x^5}{10} + \frac{3(x^4 - x)}{20}, \quad B_2(x) = \frac{x^5 - x^4}{20} + \frac{x - x^4}{30}.$$

This nonrecursive quintic spline S is depicted in Fig. 1(a). Next we shall perturb the Meir-Sharma quintic spline S to obtain its self-referential analogue S_b^α, where the base function b and scaling function α satisfy conditions prescribed in Theorem 3.1. We choose b to be a septic Hermite osculatory polynomial for S with 0 and 1 as knot points (see Remark 3.3). That is,

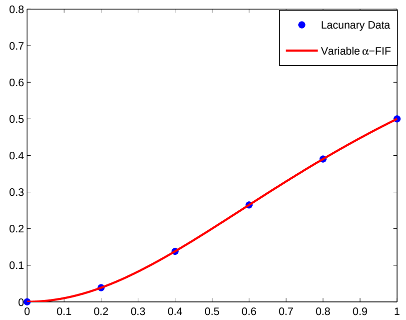
$$\begin{aligned}b(x) = & -0.000016x(1-x)^6 + 0.9999x^2(1-x)^5 + 4.9998x^3(1-x)^4\\ & + 8.7498x^4(1-x)^3 + 7.2499x^5(1-x)^2 + 2.9999x^6(1-x) + 0.5x^7.\end{aligned}$$



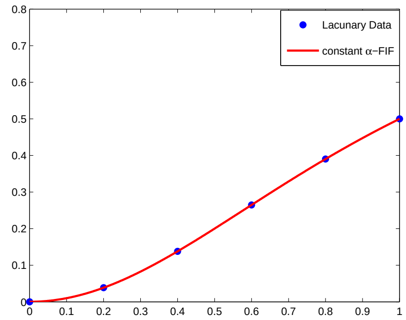
(a) Meir-Sharma quintic spline S.



(b) A fractal perturbation S^{α^1} of S with variable scaling factors.



(c) A fractal perturbation S^{α^2} of S with variable scaling factors.



(d) A fractal perturbation S^{α^3} of S with constant scaling factors.

Figure 1: Meir-Sharma quintic spline and its fractal perturbations.

We take a scaling (function) vector $\alpha^1 \in (\mathcal{C}^3[0,1])^5$ satisfying $\|\alpha_i^1\|_{\mathcal{C}^3} < 10^{-3}$ for $i = 1,\dots,5$, say

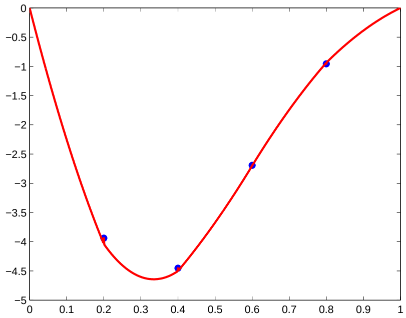
$$\begin{aligned}&\alpha_1^1(x) = 0.0008\sin x, \quad \alpha_2^1(x) = 0.00099\cos x, \quad \alpha_3^1(x) = 0.00091e^{-x},\\ &\alpha_4^1(x) = 0.000425e^{-x}\sin x, \quad \alpha_5^1(x) = 0.00042e^{-x}\cos x.\end{aligned}$$

The graph of the corresponding lacunary fractal spline $S_b^{\alpha^1}$ is depicted in Fig. 1(b). Next let us consider $\alpha^2 \in (\mathcal{C}^3[0,1])^5$ whose components are given by

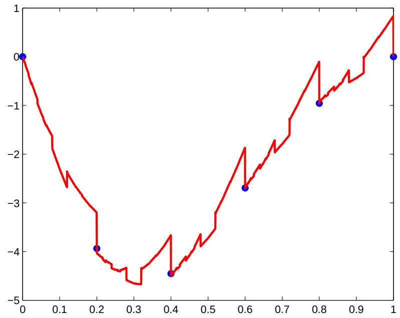
$$\begin{aligned}&\alpha_1^2(x) = 0.00016(x - x^3), && \alpha_2^2(x) = 0.000165(1 - x^2 + x^3),\\ &\alpha_3^2(x) = 0.000156(1 - x + x^2 - x^3), && \alpha_4^2(x) = 0.000025(x - x^2 + x^4 - x^5),\\ &\alpha_5^2(x) = 0.0000504(1 - x + x^3 - x^4).\end{aligned}$$

Fig. 1(c) represents the corresponding fractal spline $S_b^{\alpha^2}$. Consider a scaling vector $\alpha^3 \in (-0.008, 0.008)^5$ (see Remark 3.4), say

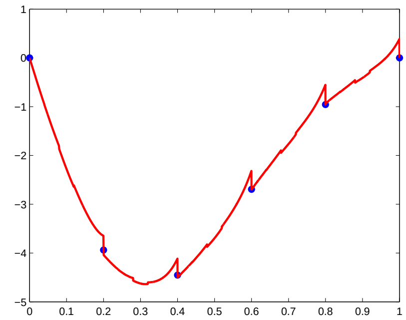
$$\alpha_1^3 = 0.0075, \quad \alpha_2^3 = 0.0065, \quad \alpha_3^3 = 0.0055, \quad \alpha_4^3 = 0.0065, \quad \alpha_5^3 = 0.0075.$$



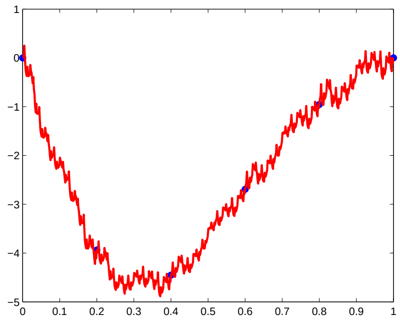
(a) The derivative S''' of Meir-Sharma quintic spline.



(b) Fractal function $(S^{\alpha^1})'''$.



(c) Fractal function $(S^{\alpha^2})'''$.



(d) Fractal function $(S^{\alpha^3})'''$.

Figure 2: Third derivatives of Meir-Sharma quintic spline and its fractal perturbations.

The corresponding fractal function $S_b^{\alpha^3}$ associated with S is given in Fig. 1(d). Since the "perturbations" α are small (with respect to the respective norms), the differences in the corresponding fractal functions from the germ function S are not quite apparent in Figs. 1(a)-(d) (see Remark 4.4). Values of the uniform norms $\|S_b^{\alpha^1} - S\|_\infty$, $\|S_b^{\alpha^2} - S\|_\infty$ and $\|S_b^{\alpha^3} - S\|_\infty$ estimated with their 5^5+1 values that are generated at the fourth iteration are given by 4.33155×10^{-7}, 7.708196×10^{-8} and 5.98512×10^{-6} respectively, which provide a numerical illustration for the fact that small perturbations in scaling factors provide small variations in the associated fractal function. The "fractality" of S_b^{α} is evident from its third derivative. In this regard, note that the graphs of $(S_b^{\alpha^1})'''$, $(S_b^{\alpha^2})'''$, $(S_b^{\alpha^3})'''$ depicted in Figs. 2 (b)-(d) show more irregularity than the graph of the third derivative of the traditional Meir-Sharma quintic spline (see Fig. 2 (a)). We feel that the flexibility in the choice of interpolant and fractality in the third derivative of the interpolant inherent with the proposed scheme can be exploited in some nonlinear and nonequilibrium phenomena. The fractality in the derivative may be quantified in terms of box counting dimension or Hausdorff dimension and this number can be used as an index for the complexity of the underlying phenomenon [2].

Appendix

This appendix is to recall the existence and uniqueness of a quintic spline which solves a $(0,2)$-lacunary interpolation problem given in [8]. For $N \geq 2$, let $\mathcal{S}^{(3)}_{N,5}$ denote the class of quintic spline S having the following two properties:

1. $S \in \mathcal{C}^3[0,1]$,
2. S is quintic in $[\frac{\nu}{N}, \frac{\nu+1}{N}]$, $\quad \nu = 0, 1, \ldots, N-1$.

If $S \in \mathcal{S}^{(3)}_{N,5}$, then

$$S(x) = q(x) + \sum_{\nu=1}^{N-1} \left\{ c_\nu \left(x - \frac{\nu}{N} \right)_+^4 + d_\nu \left(x - \frac{\nu}{N} \right)_+^5 \right\},$$

where $q(x)$ is a quintic polynomial and c_ν, d_ν are constants. For every odd integer N and any set of $2N+4$ real numbers $\{f_0, f_1, \ldots, f_N; f''_0, f''_1, \ldots, f''_N; f'''_0, f'''_N\}$ the existence of a unique $S \in \mathcal{S}^{(3)}_{N,5}$ satisfying

$$\begin{aligned} &S\left(\frac{\nu}{N}\right) = f_\nu, \quad S''\left(\frac{\nu}{N}\right) = f''_\nu, \quad \nu = 0, 1, \ldots, N; \\ &S'''(0) = f'''_0, \quad S'''(1) = f'''_N \end{aligned}$$

is proved in [8]. This can be seen as follows.

For a given $\mathcal{S} \in \mathcal{S}^{(3)}_{N,5}$, set $h = N^{-1}$ and

$$\begin{aligned} M_\nu &= S^{(4)}(\nu h^+), \quad \nu = 0, 1, \ldots, N-1; \\ L_\nu &= S^{(4)}(\nu h^-), \quad \nu = 1, \ldots, N. \end{aligned}$$

Since $S^{(4)}$ is linear in each interval $\big(\nu h, (\nu+1)h\big)$, it is completely determined by the $2N$ constants $\{M_\nu\}_{\nu=0}^{N-1}$ and $\{L_\nu\}_{\nu=1}^{N}$. Furthermore, a quintic polynomial P on $[0,1]$ can be expressed as

$$\begin{aligned} P(x) = &P(0)A_0(1-x) + P(1)A_0(x) + P''(0)A_1(1-x) \\ &+ P''(1)A_1(x) + P^{(4)}(0)A_2(1-x) + P^{(4)}(1)A_2(x), \end{aligned}$$

where

$$A_0(x) = x, \quad A_1(x) = \frac{1}{6}(x^3 - x), \quad \text{and} \quad A_2(x) = \frac{1}{120}(x^5 - x) - \frac{1}{36}(x^3 - x).$$

Also note that a quintic polynomial P on $[0,1]$ can be expressed in terms of its values and of its second and third derivatives at 0 and 1 as

$$\begin{aligned} P(x) = &P(0)B_0(1-x) + P(1)B_0(x) + P''(0)B_1(1-x) \\ &+ P''(1)B_1(x) - P'''(0)B_2(1-x) + P'''(1)B_2(x), \end{aligned}$$

where

$$B_0(x)=x,\quad B_1(x)=\frac{1}{10}(x^4-x^5)+\frac{1}{20}(3(x^4-x)),\quad B_2(x)=\frac{1}{20}(x^5-x^4)+\frac{1}{30}(x-x^4).$$

Therefore, using the interpolation conditions

$$S\left(\frac{\nu}{N}\right)=f_\nu,\quad S''\left(\frac{\nu}{N}\right)=f''_\nu,\qquad \nu=0,1,\dots,N$$

for $\nu h\le x\le(\nu+1)h$, $\nu=0,1,2,\dots,N-1$, it follows that S should be of the form

$$\begin{aligned}S(x)=&f_\nu A_0\left(\frac{(\nu+1)h-x}{h}\right)+f_{\nu+1}A_0\left(\frac{x-\nu h}{h}\right)\\&+h^2f''_\nu A_1\left(\frac{(\nu+1)h-x}{h}\right)+h^2f''_{\nu+1}A_1\left(\frac{x-\nu h}{h}\right)\\&+h^4M_\nu A_2\left(\frac{(\nu+1)h-x}{h}\right)+h^4L_{\nu+1}A_2\left(\frac{x-\nu h}{h}\right).\end{aligned}\tag{5.1}$$

In view of (5.1), we see that the endpoint conditions $S'''(0)=f'''_0, S'''(1)=f'''_N$ are equivalent to

$$2M_0+L_1=6h^{-2}(f''_1-f''_0-hf'''_0)\equiv 6h^{-2}\beta_0,\tag{5.2a}$$
$$M_{N-1}+2L_N=6h^{-2}(f''_{N-1}-f''_N+hf'''_N)\equiv 6h^{-2}\beta_N.\tag{5.2b}$$

By a simple calculation it can be shown that $S'(\nu h^+)=S'(\nu h^-)$, $\nu=1,\dots,N-1$, are equivalent to

$$\begin{aligned}&8(M_\nu+L_\nu)+7(M_{\nu-1}+L_{\nu+1})\\&=360h^{-4}(2f_\nu-f_{\nu-1}-f_{\nu+1}+h^2f''_\nu)+60h^{-2}(f''_{\nu+1}+f''_{\nu-1}-2f''_\nu)\\&\equiv 360h^{-4}\gamma_\nu+60h^{-2}\beta_\nu,\qquad \nu=1,\dots,N-1.\end{aligned}\tag{5.3}$$

Similarly, $S'''(\nu h^+)=S'''(\nu h^-)$, $\nu=1,\dots,N-1$, are equivalent to

$$\begin{aligned}&2(M_\nu+L_\nu)+(M_{\nu-1}+L_{\nu+1})\\&=6h^{-2}(f''_{\nu+1}+f''_{\nu-1}-2f''_\nu)\equiv 6h^{-2}\beta_\nu,\quad \nu=1,\dots,N-1.\end{aligned}\tag{5.4}$$

It can be easily established that the homogeneous linear system corresponding to (5.2)-(5.4) has only the trivial solution, from which it follows that the system of linear equations governed by (5.2)-(5.4) has a unique solution.

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