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Analysis of a Streamline-Diffusion Finite Element Method on Bakhvalov-Shishkin Mesh for Singularly Perturbed Problem

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Abstract. In this paper, a bilinear Streamline-Diffusion finite element method on Bakhvalov-Shishkin mesh for singularly perturbed convection – diffusion problem is analyzed. The method is shown to be convergent uniformly in the perturbation parameter ϵ provided only that $\epsilon \le N^{-1}$. An $\mathcal{O}(N^{-2}(\ln N)^{1/2})$ convergent rate in a discrete streamline-diffusion norm is established under certain regularity assumptions. Finally, through numerical experiments, we verified the theoretical results.

AMS subject classifications: 65N15; 65N30

Key words: singularly perturbed problem, Streamline-Diffusion finite element method, Bakhvalov-Shishkin mesh, error estimate.

1. Introduction

In this paper, we consider a Streamline-Diffusion finite element method (SDFEM) for the singularly perturbed boundary value problem

$$\begin{aligned} &Lu := -\epsilon\Delta u + b\cdot\nabla u + cu = f \quad \text{on} \quad \Omega = (0,1)^2, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $0 < \epsilon \ll 1$ is a small positive parameter, b, c and f are sufficiently smooth functions satisfying

$$b(x,y) = (b_1(x,y), b_2(x,y)) \ge (\beta_1, \beta_2) > (0,0), \quad \forall (x,y) \in \bar{\Omega}, \tag{1.2a}$$

$$c(x,y) \ge 0, \quad c(x,y) - \frac{1}{2}\mathrm{div} b(x,y) \ge c_0 > 0, \quad \forall (x,y) \in \bar{\Omega}, \tag{1.2b}$$

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where β_1, β_2 and c_0 are some constants. These hypotheses ensure that (1.1) has a unique solution in $H_0^1(\Omega) \bigcap H^2(\Omega)$ for all $f \in L^2(\Omega)$. Note that for sufficiently small ϵ, the other hypotheses imply that (1.2b) can always be ensured by the simple change of variable $v(x,y) = e^{-\sigma x} u(x,y)$ where σ is chosen suitably. With the above assumptions, the solution of (1.1) typically has boundary layers of width $\mathcal{O}(\epsilon \ln \frac{1}{\epsilon})$ at the outflow boundary $x = 1$ and $y = 1$.

For small values of ϵ, standard Galerkin discretisation for (1.1) exhibits spurious oscillations and fails to catch the rapid change of the solution in boundary layers, see the numerical results in [15]. Many methods have been developed to overcome the numerical difficulty caused by the boundary layers.

One of the most successful methods is the use of layer-adapted meshes. Provided that some information on the structure of the layers was available, a piecewise uniform Shishkin mesh(S-mesh) could be chosen a priori, see [1, 3]. Linß [8, 9] introduced Bakhvalov-Shishkin mesh(B-S-mesh) which is a modification of S-mesh by using a uniform coarse mesh and a graded fine mesh with Shishkin's simple choice of the transition point. The optimal convergence order $\mathcal{O}(N^{-1})$ on a B-S-mesh had been proved, while on S-mesh it was only convergent of $\mathcal{O}(N^{-1} \ln N)$. Zhang [5] investigated the superconvergence of order $\mathcal{O}(N^{-2}(\ln N)^2)$ in a discrete ϵ-weighted energy norm on a S-mesh.

A powerful method for stabilising convection-diffusion problems is the streamline-diffusion finite element method which was proposed by Hughes and Brooks [16]. This method was known to provide good stability properties and high accuracy in boundary layers. The convergence properties of the SDFEM had been widely studied[3,10-13]. In [13], the error between the SDFEM solution and the interpolation of the solution of (1.1) on S-mesh was of order $\mathcal{O}(N^{-3/2} \ln N)$ in the streamline-diffusion norm(SD norm). In [10], a more careful analysis was performed by using interpolation error identities of Lin, and this error was improved to $\mathcal{O}(N^{-2}(\ln N)^2)$. In order to achieve estimates for the interpolation error in SD norm, Stynes and Tobiska [10] firstly introduced the discrete streamline-diffusion norm, and estimated an error bound of order $\mathcal{O}(N^{-2}(\ln N)^2)$ on S-mesh.

Here we shall analyze a SDFEM on B-S-mesh, and it will give more accurate results than on S-mesh. There are three main results in this paper. First, the interpolation error in discrete SD norm is presented to be convergent of $\mathcal{O}(N^{-2})$. Second, the error between the solution of the discrete problem and the interpolation of the solution of the continuous problem is shown to be bounded in discrete SD norm by $\mathcal{O}(N^{-2}(\ln N)^{1/2})$, uniformly in ϵ. Third, we prove that the error between the solution of the discrete problem and the solution of the continuous problem itself can be estimated in discrete SD norm by $\mathcal{O}(N^{-2}(\ln N)^{1/2})$.

An outline of the paper is as follows. In Section 2 we describe the B-S-mesh and the SDFEM. A decomposition of the solution u and some important preliminaries to the analysis are presented in Section 3, and in Section 4 we analyze the convergence properties of the method. In order to validate our theoretical results, numerical results are presented in Section 5. We end in Section 6 with some concluding remarks.

Notation: Throughout the paper, C will denote a generic positive constant that independent of ϵ and the mesh. Note that C is not necessarily the same at each occurrence.

The standard notation for the Sobolev spaces $W_p^k(D)$ and norms will be used for nonnegative integers k and $1 \le p \le \infty$. An index will be attached to indicate an inner product or a norm on a subdomain, for example, $(\cdot,\cdot)_D$, $|\cdot|_{k,p,D}$ and $\|\cdot\|_{k,p,D}$. When $D = \Omega$, we drop the D from notation for simplicity. We will also simplify the notation in the case $p = 2$ by setting $|\cdot|_{k,D} = |\cdot|_{k,2,D}$ and $\|\cdot\|_{k,D} = \|\cdot\|_{k,2,D}$.

2. The Bakhvalov-Shishkin mesh and the SDFEM

Let N be an even positive integer. We let λ_1 and λ_2 denote two mesh transition parameters that will be used to specify where the changes form coarse to fine; these are defined by

$$\lambda_1 = \min\left(\frac{1}{2}, \frac{2.5\epsilon}{\beta_1}\ln N\right) \quad \text{and} \quad \lambda_2 = \min\left(\frac{1}{2}, \frac{2.5\epsilon}{\beta_2}\ln N\right). \tag{2.1}$$

In fact we make the very mild assumption that $\lambda_1 = \frac{2.5\epsilon}{\beta_1}\ln N$ and $\lambda_2 = \frac{2.5\epsilon}{\beta_2}\ln N$, as otherwise N^{-1} is exponentially small compared with ϵ. We shall also assume throughout the paper that $\epsilon \le N^{-1}$ as is generally the case in practice.

We divide the domain Ω as in Figure 1: $\bar{\Omega} = \Omega_{11} \cup \Omega_{12} \cup \Omega_{21} \cup \Omega_{22}$, where $\Omega_{11} = [0, 1-\lambda_1] \times [0, 1-\lambda_2], \Omega_{12} = [0, 1-\lambda_1] \times [1-\lambda_2, 1], \Omega_{21} = [1-\lambda_1, 1] \times [0, 1-\lambda_2], \Omega_{22} = [1-\lambda_1, 1] \times [1-\lambda_2, 1]$.

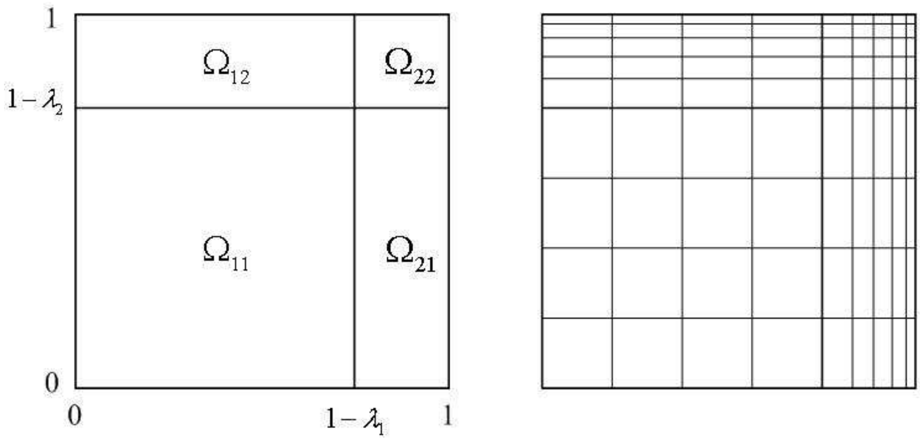

Figure 1: Division of Ω (left) and a corresponding B-S-mesh (right).

The interval $[0, 1-\lambda_1]$ is uniformly dissected into $N/2$ subintervals, while $[1-\lambda_1, 1]$ is partitioned into the same number of mesh intervals by inverting the function $exp(-\beta_1(1-x)/(2.5\epsilon))$. We specify the x_i, for $i = N/2, \cdots, N$, so that $\{e^{-\beta_1(1-x_i)/(2.5\epsilon)}\}_i$

is a linear function in i, i.e., we set

$$e^{-\beta_1(1-x_i)/(2.5\epsilon)} = Ai + B,$$

and choose the unknowns A and B so that $x_{N/2} = 1 - \lambda_1$ and $x_N = 1$. An analogous formula can be given for the mesh points y_j, for $j = 0, \cdots, N$.

The mesh points $\Omega^N = \{(x_i, y_j) \in \bar{\Omega} : i, j = 0, 1, \cdots, N\}$ are the rectangular lattices defined by

$$x_i = \begin{cases} (1 - \frac{2.5\epsilon}{\beta_1} \ln N)\frac{2i}{N} & i = 0, \cdots, \frac{N}{2}, \\ 1 + \frac{2.5\epsilon}{\beta_1} \ln\left(\frac{N^2-2(N-i)(N-1)}{N^2}\right) & i = \frac{N}{2} + 1, \cdots, N, \end{cases}$$

$$y_j = \begin{cases} (1 - \frac{2.5\epsilon}{\beta_2} \ln N)\frac{2j}{N} & j = 0, \cdots, \frac{N}{2}, \\ 1 + \frac{2.5\epsilon}{\beta_2} \ln\left(\frac{N^2-2(N-j)(N-1)}{N^2}\right) & j = \frac{N}{2} + 1, \cdots, N. \end{cases}$$

Our mesh is constructed by drawing lines parallel to the coordinate axes through these mesh points. This divides Ω into a set $\mathcal{T}_N$ of mesh rectangles K whose sides are parallel to the axes (see Figure 1). Given an element $K = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$, its dimensions are written as $h_{x,i}(h_{x,K}) = x_i - x_{i-1}, h_{y,j}(h_{y,K}) = y_j - y_{j-1}$, and its barycenter is denoted by (x_K, y_K).

We now describe the SDFEM on the rectangular mesh. The bilinear form $B_{SD}(\cdot,\cdot)$ used in the SDFEM is defined by

$$B_{SD}(u, v) = B_{GAL}(u, v) + B_{STAB}(u, v),$$

where

$$\begin{aligned} B_{GAL}(u, v) &= \epsilon(\nabla u, \nabla v) + (b \cdot \nabla u + cu, v), \\ B_{STAB}(u, v) &= \sum_{K\subset\Omega_{11}} \rho_K(-\epsilon\triangle u + b \cdot \nabla u + cu, b \cdot \nabla v)_K. \end{aligned}$$

The weak formulation of the model (1.1) is: Find $u \in H_0^1(\Omega)$ such that

$$B_{SD}(u, v) = (f, v) + \sum_{K\subset\Omega_{11}} \rho_K(f, b \cdot \nabla v)_K, \qquad \forall v \in H_0^1(\Omega).$$

Let $V^N \subset H_0^1(\Omega)$ be the continuous piecewise bilinear finite element space on the B-S-mesh:

$$V^N = \Big\{v \in C(\bar{\Omega}) : v|_{\partial\Omega} = 0,\ v|_K \in Q_1(K) \quad \forall K \subset \mathcal{T}_N\Big\}.$$

Then the SDFEM is defined as follows: Find $u_h \in V^N$ such that

$$B_{SD}(u_h, v_h) = (f, v_h) + \sum_{K\subset\Omega_{11}} \rho_K(f, b \cdot \nabla v_h)_K, \qquad \forall v_h \in V^N. \tag{2.2}$$

The orthogonality property holds clearly

$$B_{SD}(u-u_h,v_h)=0,\quad \forall v_h\in V^N.$$

For each $K\in\mathcal{T}_N$, set $h_K=\min\{h_{x,K},h_{y,K}\}$. Let C_{inv} be a constant such that the inverse inequality

$$\|\triangle v_h\|_{0,K}\le C_{inv}h_K^{-1}\|\nabla v_h\|_{0,K},\quad \forall v_h\in V^N,\ \forall K\subset\Omega_{11}$$

is valid. Similarly to [3], we set

$$\rho_K=\begin{cases} C_1N^{-1} & K\subset\Omega_{11}\\ 0 & \text{otherwise,}\end{cases}$$

where the positive constant C_1 is chosen (independent of ϵ) such that

$$0\le\rho_K\le\frac{1}{2}\min\left(\frac{c_0}{\max_K|c(x,y)|^2},\frac{h_K^2}{C_{inv}^2\epsilon}\right),\qquad \forall K\subset\Omega_{11}.$$

Then the argument of [3] shows that the inequality

$$B_{SD}(v_h,v_h)\ge\frac{1}{2}\|v_h\|_{SD}^2,\quad \forall v_h\in V^N$$

holds, where SD norm $\|\cdot\|_{SD}$ is given by

$$\|v\|_{SD}=\left\{c_0\|v\|_0^2+\epsilon|v|_1^2+\sum_{K\subset\Omega_{11}}\rho_K\|b\cdot\nabla v\|_{0,K}^2\right\}^{1/2}.$$

It follows that (2.2) has a unique solution $u_h\in V^N$.

We shall use the discrete SD norm $\|\cdot\|_{SD,d}$ defined by

$$\|v\|_{SD,d}$$
$$=\left\{c_0\|v\|_0^2+\sum_{K\subset\Omega}\epsilon(areaK)|\nabla v(x_K,y_K)|^2+\sum_{K\subset\Omega_{11}}\rho_K(areaK)|b\cdot\nabla v(x_K,y_K)|^2\right\}^{1/2}.$$

3. Solution decomposition and preliminary

Our subsequent analysis will rely on the precise knowledge of the behaviour of the solution u of the convection-diffusion problem (1.1). The typical behaviour of u is given in the following assumption.

Assumption 3.1. Assume that

$$u=S+E_1+E_2+E_{12},$$

where there exists a constant C such that for all $(x,y) \in \Omega$ we have

$$\left| \frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y) \right| \le C \tag{3.1}$$

for $0 \le i+j \le 3$ and

$$\left| \frac{\partial^{i+j} E_1}{\partial x^i \partial y^j}(x,y) \right| \le C\epsilon^{-i} e^{-\frac{\beta_1(1-x)}{\epsilon}}, \tag{3.2}$$

$$\left| \frac{\partial^{i+j} E_2}{\partial x^i \partial y^j}(x,y) \right| \le C\epsilon^{-j} e^{-\frac{\beta_2(1-y)}{\epsilon}}, \tag{3.3}$$

$$\left| \frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(x,y) \right| \le C\epsilon^{-(i+j)} e^{-\frac{\beta_1(1-x)+\beta_2(1-y)}{\epsilon}} \tag{3.4}$$

for $0 \le i+j \le 3$.

Remark 3.1. In [4], a proof was given that under certain compatibility conditions on the date f of problem (1.1), the bounds (3.2)-(3.4) and $0 \le i+j \le 2$ in (3.1) of Assumption 3.1 hold true. The extension of this result to the case $0 \le i+j \le 3$ in (3.1) as needed in our case seems to be possible but tedious. The number of these sufficient conditions will increase rapidly with increasing differentiation order.

Next we introduce some equalities and inequalities that will be used in the analysis. inverse inequalities: for $v \in V^N$,

$$\int_K \left(\frac{\partial v}{\partial x} \right)^2 \mathrm{d}x\mathrm{d}y \le \frac{C}{h_{x,K}^2} \int_K v^2 \mathrm{d}x\mathrm{d}y, \tag{3.5a}$$

$$\int_K \left(\frac{\partial v}{\partial y} \right)^2 \mathrm{d}x\mathrm{d}y \le \frac{C}{h_{y,K}^2} \int_K v^2 \mathrm{d}x\mathrm{d}y. \tag{3.5b}$$

Let v_I be the piecewise bilinear interpolation of function $v \in W_\infty^1(\Omega)$, then [5]

$$\frac{\partial v_I}{\partial x}(x_K, y_K) = \frac{1}{2h_{x,K}} \int_{x_{i-1}}^{x_i} \left(\frac{\partial v}{\partial x}(x, y_{j-1}) + \frac{\partial v}{\partial x}(x, y_j) \right) \mathrm{d}x, \tag{3.6}$$

$$\frac{\partial v_I}{\partial y}(x_K, y_K) = \frac{1}{2h_{y,K}} \int_{y_{j-1}}^{y_j} \left(\frac{\partial v}{\partial y}(x_{i-1}, y) + \frac{\partial v}{\partial y}(x_i, y) \right) \mathrm{d}y. \tag{3.7}$$

and if $v \in W_\infty^3(\Omega)$, we have

$$\begin{aligned} &\frac{\partial (v_I - v)}{\partial x}(x_K, y_K) \qquad\qquad (3.8)\\ =& \frac{1}{2h_{x,K}} \int_{-h_{x,K}/2}^{h_{x,K}/2} \left[\left(\frac{t^2}{2} \frac{\partial^3 v}{\partial x^3} - \frac{t h_{y,K}}{2} \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{h_{y,K}^2}{8} \frac{\partial^3 v}{\partial x \partial y^2} \right) \left(x_K + s_1 t, y_K - s_1 \frac{h_{y,K}}{2} \right) \right. \\ &\left. + \left(\frac{t^2}{2} \frac{\partial^3 v}{\partial x^3} + \frac{t h_{y,K}}{2} \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{h_{y,K}^2}{8} \frac{\partial^3 v}{\partial x \partial y^2} \right) \left(x_K + s_2 t, y_K + s_2 \frac{h_{y,K}}{2} \right) \right] \mathrm{d}t, \end{aligned}$$

$$
\begin{aligned}
&\frac{\partial(v_I-v)}{\partial y}(x_K,y_K) \qquad (3.9)\\
=&\frac{1}{2h_{y,K}}\int_{-h_{y,K}/2}^{h_{y,K}/2}\Bigg[\bigg(\frac{h_{x,K}^2}{8}\frac{\partial^3 v}{\partial x^2\partial y}-\frac{th_{x,K}}{2}\frac{\partial^3 v}{\partial x\partial y^2}+\frac{t^2}{2}\frac{\partial^3 v}{\partial y^3}\bigg)\Big(x_K-s_3\frac{h_{x,K}}{2},y_K+s_3t\Big)\\
&+\bigg(\frac{h_{x,K}^2}{8}\frac{\partial^3 v}{\partial x^2\partial y}+\frac{th_{x,K}}{2}\frac{\partial^3 v}{\partial x\partial y^2}+\frac{t^2}{2}\frac{\partial^3 v}{\partial y^3}\bigg)\Big(x_K+s_4\frac{h_{x,K}}{2},y_K+s_4t\Big)\Bigg]\mathrm{d}t,
\end{aligned}
$$

where $0<s_i<1$ for $i=1,2,3,4$; see [5].

Lemma 3.1. ([6]) *Let $K\in\mathcal{T}_N$ and $p\in[1,+\infty]$. Assume that $v\in W_p^2(K)$, We denote by v_I the bilinear function that interpolations to v. Then*

$$\|v-v_I\|_{0,p,K}\le C\left\{h_{x,K}^2\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{0,p,K}+h_{x,K}h_{y,K}\left\|\frac{\partial^2 v}{\partial x\partial y}\right\|_{0,p,K}+h_{y,K}^2\left\|\frac{\partial^2 v}{\partial y^2}\right\|_{0,p,K}\right\}, \qquad (3.10)$$

$$\left\|\frac{\partial(v-v_I)}{\partial x}\right\|_{0,p,K}\le C\left\{h_{x,K}\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{0,p,K}+h_{y,K}\left\|\frac{\partial^2 v}{\partial x\partial y}\right\|_{0,p,K}\right\}, \qquad (3.11)$$

$$\left\|\frac{\partial(v-v_I)}{\partial y}\right\|_{0,p,K}\le C\left\{h_{x,K}\left\|\frac{\partial^2 v}{\partial x\partial y}\right\|_{0,p,K}+h_{y,K}\left\|\frac{\partial^2 v}{\partial y^2}\right\|_{0,p,K}\right\}. \qquad (3.12)$$

Lemma 3.2. (Lin identities) ([7]) *Let K be a mesh rectangle, and we denote the south, east, north, west edges of K by $l_{i,K}$ for $i=1,2,3,4$, respectively. Let $v\in H^3(K)$, and let $v_I\in Q_1(K)$ be its bilinear interpolation. Then for each $v_N\in Q_1(K)$, we have*

$$\int_K\frac{\partial(v-v_I)}{\partial x}\frac{\partial v_N}{\partial x}\mathrm{d}x\mathrm{d}y=\int_K\frac{\partial^3 v}{\partial x\partial y^2}F(y)\left(\frac{\partial v_N}{\partial x}-\frac{2}{3}(y-y_K)\frac{\partial^2 v_N}{\partial x\partial y}\right)\mathrm{d}x\mathrm{d}y, \qquad (3.13)$$

$$\int_K\frac{\partial(v-v_I)}{\partial y}\frac{\partial v_N}{\partial y}\mathrm{d}x\mathrm{d}y=\int_K\frac{\partial^3 v}{\partial x^2\partial y}G(x)\left(\frac{\partial v_N}{\partial y}-\frac{2}{3}(x-x_K)\frac{\partial^2 v_N}{\partial x\partial y}\right)\mathrm{d}x\mathrm{d}y, \qquad (3.14)$$

$$\int_K\frac{\partial(v-v_I)}{\partial x}v_N\mathrm{d}x\mathrm{d}y=\int_K R(v,v_N)\mathrm{d}x\mathrm{d}y+\frac{h_{x,K}^2}{12}\left(\int_{l_{2,K}}-\int_{l_{4,K}}\right)\frac{\partial^2 v}{\partial x^2}v_N\mathrm{d}y, \qquad (3.15)$$

where

$$G(x)=\frac{1}{2}\left[(x-x_K)^2-\left(\frac{h_{x,K}}{2}\right)^2\right],\qquad F(y)=\frac{1}{2}\left[(y-y_K)^2-\left(\frac{h_{y,K}}{2}\right)^2\right],$$

$$
\begin{aligned}
R(v,v_N)=&\frac{1}{3}G(x)(x-x_K)\frac{\partial^3 v}{\partial x^3}\frac{\partial v_N}{\partial x}-\frac{h_{x,K}^2}{12}\frac{\partial^3 v}{\partial x^3}v_N+F(y)\frac{\partial^3 v}{\partial x\partial y^2}\bigg(v_N\\
&-(x-x_K)\frac{\partial v_N}{\partial x}-\frac{2}{3}(y-y_K)\frac{\partial v_N}{\partial y}+\frac{2}{3}(x-x_K)(y-y_K)\frac{\partial^2 v_N}{\partial x\partial y}\bigg). \qquad (3.16)
\end{aligned}
$$

Lemma 3.3. ([8]) *Let $\gamma_1 = i - \frac{N}{2}, \gamma_2 = j - \frac{N}{2}$, then the step sizes of the mesh $\mathcal{T}_N$ satisfy: for $i, \gamma_1 = 1, \cdots, N/2$, and for $j, \gamma_2 = 1, \cdots, N/2$,*

$$h_{x,i} \le 2N^{-1}, \qquad h_{x,\gamma_1} \le \frac{5\epsilon}{\beta_1 \gamma_1} \le CN^{-1}; \tag{3.17}$$

$$h_{y,j} \le 2N^{-1}, \qquad h_{y,\gamma_2} \le \frac{5\epsilon}{\beta_2 \gamma_2} \le CN^{-1}. \tag{3.18}$$

Finally, we list some inequalities regarding the exponential boundary layer function which will be used in the next section: for $i = N/2+1, \cdots, N$ and for $j = N/2 + 1, \cdots, N$,

$$\sum_{i=1}^{N/2} h_{x,i} e^{-\frac{2\beta_1(1-x_i)}{\epsilon}} \le CN^{-5}(\epsilon + N^{-1}), \tag{3.19}$$

$$\sum_{j=1}^{N/2} h_{y,j} e^{-\frac{2\beta_2(1-y_j)}{\epsilon}} \le CN^{-5}(\epsilon + N^{-1}), \tag{3.20}$$

$$\int_{x_{i-1}}^{x_i} e^{-\frac{2\beta_1(1-x)}{\epsilon}} \mathrm{d}x \le \frac{2.5\epsilon}{\beta_1} N^{-1} e^{-\frac{8\beta_1(1-x_i)}{5\epsilon}}; \tag{3.21}$$

$$\int_{y_{j-1}}^{y_j} e^{-\frac{2\beta_2(1-y)}{\epsilon}} \mathrm{d}y \le \frac{2.5\epsilon}{\beta_2} N^{-1} e^{-\frac{8\beta_2(1-y_j)}{5\epsilon}}. \tag{3.22}$$

4. Analysis and main results

Lemma 4.1. ([9]) *If Assumption* 3.1 *holds, then we have the interpolation error estimates*

$$\|u - u_I\|_{0,\infty} \le CN^{-2}. \tag{4.1}$$

Lemma 4.2. ([10]) *There exists a positive constant C such that*

$$\|v_N\|_{SD,d} \le C\|v_N\|_{SD}, \quad \forall v_N \in V^N.$$

Lemma 4.3. *Let $E = E_1 + E_2 + E_{12}$ satisfy the regularity* (3.2)-(3.4). *Then there is a constant C, such that*

$$\sum_{K \subset \Omega} \epsilon(areaK)|\nabla(E - E_I)(x_K, y_K)|^2 \le CN^{-4}.$$

Proof. Based on the boundary layer behaviour of E_1, we separate the discussion into the case of $\Omega_{21} \bigcup \Omega_{22}$ and $\Omega_{11} \bigcup \Omega_{12}$.

(a) $K \subset \Omega_{21} \bigcup \Omega_{22}$. Applying the regularity results (3.2) to (3.8),(3.9), we derive

$$|\nabla(E_1 - E_{1,I})(x_K, y_K)|$$

$$\le Ce^{-\frac{\beta_1(1-x_i)}{\epsilon}}\left(\frac{h_{x,K}^2}{8}(\epsilon^{-3}+\epsilon^{-2})+\frac{h_{x,K}h_{y,K}}{4}(\epsilon^{-2}+\epsilon^{-1})+\frac{h_{y,K}^2}{8}(\epsilon^{-1}+1)\right)$$
$$\le Ce^{-\frac{\beta_1(1-x_i)}{\epsilon}}\left(h_{x,K}^2\epsilon^{-3}+2h_{x,K}h_{y,K}\epsilon^{-2}+h_{y,K}^2\epsilon^{-1}\right).$$

Adding all elements on $\Omega_{21}\bigcup\Omega_{22}$ yields

$$\begin{aligned}
&\sum_{K\subset\Omega_{21}\bigcup\Omega_{22}}\epsilon(areaK)|\nabla(E_1-E_{1,I})(x_K,y_K)|^2\\
\le\, & C\epsilon\sum_{j=1}^{N}h_{y,j}\sum_{i=N/2+1}^{N}h_{x,i}e^{-\frac{2\beta_1(1-x_i)}{\epsilon}}\left(h_{x,i}^2\epsilon^{-3}+2h_{x,i}h_{y,j}\epsilon^{-2}+h_{y,j}^2\epsilon^{-1}\right)^2\\
\le\, & C\sum_{i=N/2+1}^{N}h_{x,i}^5\epsilon^{-5}e^{-\frac{2\beta_1(1-x_i)}{\epsilon}}+CN^{-2}\sum_{i=N/2+1}^{N}h_{x,i}^3\epsilon^{-3}e^{-\frac{2\beta_1(1-x_i)}{\epsilon}}\\
&+CN^{-4}\sum_{i=N/2+1}^{N}h_{x,i}\epsilon^{-1}e^{-\frac{2\beta_1(1-x_i)}{\epsilon}}\\
\le\, & C\sum_{\gamma_1=1}^{N/2}\left(\frac{5\epsilon}{\beta_1\gamma_1}\right)^5\epsilon^{-5}\left(\frac{2\gamma_1(N-1)+N}{N^2}\right)^5\\
&+CN^{-2}\sum_{\gamma_1=1}^{N/2}\left(\frac{5\epsilon}{\beta_1\gamma_1}\right)^3\epsilon^{-3}\left(\frac{2\gamma_1(N-1)+N}{N^2}\right)^5\\
&+CN^{-4}\sum_{\gamma_1=1}^{N/2}\left(\frac{5\epsilon}{\beta_1\gamma_1}\right)\epsilon^{-1}\left(\frac{2\gamma_1(N-1)+N}{N^2}\right)^5\\
\le\, & CN^{-4},
\end{aligned}\tag{4.2}$$

where we used Lemma 3.3 and the expression of x_i.

(b) $K\subset\Omega_{11}\bigcup\Omega_{12}$. By the regularity (3.2), we have

$$(areaK)|\nabla E_1(x_K,y_K)|^2\le Ch_{x,K}h_{y,K}(\epsilon^{-2}+1)e^{-\frac{2\beta_1(1-x_K)}{\epsilon}}.$$

Summing up all elements on $\Omega_{11}\bigcup\Omega_{12}$ yields

$$\begin{aligned}
&\sum_{K\subset\Omega_{11}\bigcup\Omega_{12}}\epsilon(areaK)|\nabla E_1(x_K,y_K)|^2\\
\le\, & C(\epsilon^{-1}+\epsilon)\sum_{j=1}^{N}h_{y,j}\sum_{i=1}^{N/2}h_{x,i}e^{-\frac{2\beta_1(1-x_K)}{\epsilon}}\\
\le\, & C\epsilon^{-1}e^{-\frac{\beta_1 h_{x,K}}{\epsilon}}\sum_{i=1}^{N/2}h_{x,i}e^{-\frac{2\beta_1(1-x_i)}{\epsilon}}
\end{aligned}$$

$$\leq CN^{-5}e^{-\frac{\beta_1}{\epsilon N}}\left(1+(\epsilon N)^{-1}\right) \leq CN^{-5}, \tag{4.3}$$

where we used (3.19) and the boundedness of $e^{-t}(1+t)$ on $\mathbb{R}_+$. It remains only to estimate $|\nabla E_{1,I}(x_K, y_K)|$. Invoking the Cauchy-Schwarz inequality and (3.6), (3.7), we have

$$\frac{\partial E_{1,I}}{\partial x}(x_K, y_K) \leq \frac{C}{h_{x,K}}\int_{x_{i-1}}^{x_i} \epsilon^{-1}e^{-\frac{\beta_1(1-x)}{\epsilon}}\mathrm{d}x \leq \frac{C\epsilon^{-1}}{\sqrt{h_{x,K}}}\left(\int_{x_{i-1}}^{x_i} e^{-\frac{2\beta_1(1-x)}{\epsilon}}\mathrm{d}x\right)^{1/2},$$
$$\frac{\partial E_{1,I}}{\partial y}(x_K, y_K) \leq \frac{C}{h_{y,K}}\int_{y_{j-1}}^{y_j}\left(e^{-\frac{\beta_1(1-x_i)}{\epsilon}} + e^{-\frac{\beta_1(1-x_{i-1})}{\epsilon}}\right)\mathrm{d}y \leq Ce^{-\frac{\beta_1(1-x_i)}{\epsilon}}.$$

Summing up, we obtain

$$\begin{aligned}&\sum_{K\subset\Omega_{11}\bigcup\Omega_{12}} \epsilon(areaK)|\nabla E_{1,I}(x_K, y_K)|^2\\ \leq\ & C\epsilon^{-1}\sum_{j=1}^{N}h_{y,j}\sum_{i=1}^{N/2}\int_{x_{i-1}}^{x_i} e^{-\frac{2\beta_1(1-x)}{\epsilon}}\mathrm{d}x + C\epsilon\sum_{j=1}^{N}h_{y,j}\sum_{i=1}^{N/2}h_{x,i}e^{-\frac{2\beta_1(1-x_i)}{\epsilon}}\\ \leq\ & CN^{-5},\end{aligned} \tag{4.4}$$

where we used (3.19). Combining (4.3) with (4.4), we get

$$\sum_{K\subset\Omega_{11}\bigcup\Omega_{12}} \epsilon(areaK)|\nabla(E_1 - E_{1,I})(x_K, y_K)|^2 \leq CN^{-5}.$$

This, combined with (4.2), established the conclusion

$$\sum_{K\subset\Omega} \epsilon(areaK)|\nabla(E_1 - E_{1,I})(x_K, y_K)|^2 \leq CN^{-4}. \tag{4.5}$$

The argument for E_2 is similar.

The proof for E_{12} is separated into the case of Ω_{22}, $\Omega_{11}\bigcup\Omega_{12}$ and Ω_{21}. Using the similar techniques, we have

$$\sum_{K\subset\Omega} \epsilon(areaK)|\nabla(E_{12} - E_{12,I})(x_K, y_K)|^2 \leq CN^{-5}. \tag{4.6}$$

Thus, collecting (4.5) and (4.6), we get the statement of the Lemma. □

Now we have a look at the interpolation error.

Theorem 4.1. *If Assumption* 3.1 *holds, then we have the interpolation error estimates*

$$\|u - u_I\|_{SD,d} \leq CN^{-2}.$$

Proof. By Lemma 4.1, we obtain

$$\|u-u_I\|_0 \le CN^{-2}. \tag{4.7}$$

Next, we now analyze $\sum\limits_{K\subset\Omega} \epsilon(areaK)|\nabla(u-u_I)(x_K,y_K)|^2$.

$$\begin{aligned}
&\frac{\partial(S-S_I)}{\partial x}(x_K,y_K)\\
\le&\frac{h_{x,K}^2}{8}\left\|\frac{\partial^3 S}{\partial x^3}\right\|_{0,\infty}+\frac{h_{x,K}h_{y,K}}{4}\left\|\frac{\partial^3 S}{\partial x^2\partial y}\right\|_{0,\infty}+\frac{h_{y,K}^2}{8}\left\|\frac{\partial^3 S}{\partial x\partial y^2}\right\|_{0,\infty}\\
\le& C\left(\frac{h_{x,K}^2}{8}+\frac{h_{x,K}h_{y,K}}{4}+\frac{h_{y,K}^2}{8}\right)\le CN^{-2},\\
&\frac{\partial(S-S_I)}{\partial y}(x_k,y_k)\\
\le&\frac{h_{x,K}^2}{8}\left\|\frac{\partial^3 S}{\partial x^2\partial y}\right\|_{0,\infty}+\frac{h_{x,K}h_{y,K}}{4}\left\|\frac{\partial^3 S}{\partial x\partial y^2}\right\|_{0,\infty}+\frac{h_{y,K}^2}{8}\left\|\frac{\partial^3 S}{\partial y^3}\right\|_{0,\infty}\\
\le& C\left(\frac{h_{x,K}^2}{8}+\frac{h_{x,K}h_{y,K}}{4}+\frac{h_{y,K}^2}{8}\right)\le CN^{-2},
\end{aligned}$$

where we used Lemma 3.3 and (3.1),(3.8),(3.9). So

$$\sum_{K\subset\Omega} \epsilon(areaK)|\nabla(S-S_I)(x_K,y_K)|^2 \le CN^{-5}. \tag{4.8}$$

Moving on to the layer part E of u, it is given in Lemma 4.3. Finally, we analyze $\sum\limits_{K\subset\Omega_{11}} \rho_K(areaK)|b\cdot\nabla(u-u_I)(x_K,y_K)|^2$. It is shown in [10, Lemma 5.2] that

$$\sum_{K\subset\Omega_{11}} \rho_K(areaK)|b\cdot\nabla(u-u_I)(x_K,y_K)|^2 \le CN^{-5}. \tag{4.9}$$

Combining (4.7)-(4.9) and Lemma 4.3, Theorem 4.1 is proved. □

Remark 4.1. It is difficult to bound the term

$$\sum_{K\subset\Omega_{11}} \rho_K \|b\cdot\nabla(u-u_I)\|_{0,K}^2$$

independent of ϵ. In order to avoid this dilemma, we estimate the interpolation error in discrete SD norm instead of SD norm.

Next, we discuss the error bound for u_I-u_h. In order to obtain the bound, We firstly give Lemma 4.4, Lemma 4.5 and Lemma 4.6 by invoking the sharp superconvergence results of Lin [7].

Lemma 4.4. *Let S satisfy the regularity* (3.1). *Then there exists a constant C, such that*

$$|(b\cdot\nabla(S-S_I),v_N)|\le CN^{-2}\|v_N\|_{SD},\qquad \forall v_N\in V^N.$$

Proof. We define a piecewise constant approximation $\bar{b}$ of b by

$$\bar{b}|_K=\frac{1}{areaK}\int_K bdxdy,\qquad \forall K\in\mathcal{T}_N.$$

Then we decompose

$$(b\cdot\nabla(S-S_I),v_N)=((b-\bar{b})\cdot\nabla(S-S_I),v_N)+(\bar{b}\cdot\nabla(S-S_I),v_N)$$

The first term can be bounded using standard interpolation error estimate and the property of b:

$$\begin{aligned}&\big|((b-\bar{b})\cdot\nabla(S-S_I),v_N)\big|\\ \le\,& CN^{-2}\,\|b\|_{1,\infty}\,|S|_2\|v_N\|_0\le CN^{-2}\|v_N\|_{SD}.\end{aligned}\tag{4.10}$$

For the second term, we write

$$\begin{aligned}&\Big|(\bar{b}\cdot\nabla(S-S_I),v_N)\Big|\\ \le\,&\Big|\sum_{K\subset\Omega}\bar{b}_1\int_K\frac{\partial(S-S_I)}{\partial x}v_N\mathrm{d}x\mathrm{d}y\Big|+\Big|\sum_{K\subset\Omega}\bar{b}_2\int_K\frac{\partial(S-S_I)}{\partial y}v_N\mathrm{d}x\mathrm{d}y\Big|.\end{aligned}\tag{4.11}$$

Through the inverse inequalities (3.5) and the expression of $R(S,v_N)$, we are able to get

$$\begin{aligned}&\Big|\sum_{K\subset\Omega}\bar{b}_1\int_K R(S,v_N)\mathrm{d}x\mathrm{d}y\Big|\\ \le\,& CN^{-2}\sum_{K\subset\Omega}|S|_{3,K}\|v_N\|_{0,K}\le CN^{-2}\|v_N\|_{SD}.\end{aligned}\tag{4.12}$$

Set

$$\bar{b}_1(x,y)=b_{i,j}\quad\text{for}\quad x_{i-1}<x<x_i\quad\text{and}\quad y_{j-1}<y<y_j.$$

Since $v_N(0,y)=v_N(1,y)=0$, we have

$$\begin{aligned}&\left|\sum_{K\subset\Omega}h_{x,K}^2\left(\int_{l_{2,K}}\bar{b}_1\frac{\partial^2S}{\partial x^2}v_N\mathrm{d}y-\int_{l_{4,K}}\bar{b}_1\frac{\partial^2S}{\partial x^2}v_N\mathrm{d}y\right)\right|\\ \le CN^{-2}&\left|\sum_{i=1}^N\sum_{j=1}^N\int_{y_{j-1}}^{y_j}b_{i,j}\left[\left(\frac{\partial^2S}{\partial x^2}v_N\right)(x_i,y)-\left(\frac{\partial^2S}{\partial x^2}v_N\right)(x_{i-1},y)\right]\mathrm{d}y\right|\end{aligned}$$

$$\begin{aligned}
&\leq CN^{-2}\left|\sum_{j=1}^{N}\sum_{i=1}^{N-1}\int_{y_{j-1}}^{y_j}(b_{i,j}-b_{i+1,j})\left(\frac{\partial^2 S}{\partial x^2}v_N\right)(x_i,y)\,\mathrm{d}y\right| \\
&\leq CN^{-3}\sum_{j=1}^{N}\sum_{i=1}^{N-1}\int_{y_{j-1}}^{y_j}|v_N(x_i,y)|\,\mathrm{d}y \\
&\leq CN^{-3}\sum_{j=1}^{N}\sum_{i=1}^{N-1}\sqrt{\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}|v_N(x,y)|^2\,\mathrm{d}x\mathrm{d}y} \\
&\leq CN^{-2}\|v_N\|_{SD}. \qquad (4.13)
\end{aligned}$$

Substituting (4.12) and (4.13) into (3.15), we finish the estimate for the first term on the right-hand side of (4.11). The estimate of the second term on the right-hand side of (4.11) is similar. Hence, the proof of Lemma 4.4 is completed. □

Lemma 4.5. *Let* $E=E_1+E_2+E_{12}$ *satisfy the regularity* (3.2)-(3.4). *Then there is a constant* C*, such that*

$$|\epsilon(\nabla(E-E_I),\nabla v_N)|\leq CN^{-2}\|v_N\|_{SD}, \qquad \forall v_N\in V^N.$$

Proof. Using the expression of $G(x)$ and $F(y)$, the identity (3.13) is estimated

$$\begin{aligned}
&\epsilon\left|\int_K\frac{\partial(E-E_I)}{\partial x}\frac{\partial v_N}{\partial x}\mathrm{d}x\mathrm{d}y\right| \\
&\leq\epsilon\int_K\left|\frac{\partial^3 E}{\partial x\partial y^2}\right||F(y)|\left(\left|\frac{\partial v_N}{\partial x}\right|+\frac{2}{3}\left|(y-y_K)\frac{\partial^2 v_N}{\partial x\partial y}\right|\right)\mathrm{d}x\mathrm{d}y \\
&\leq C\epsilon h_{y,K}^2\left\|\frac{\partial^3 E}{\partial x\partial y^2}\right\|_{0,K}\left\|\frac{\partial v_N}{\partial x}\right\|_{0,K}, \qquad (4.14)
\end{aligned}$$

where we used the inverse inequality (3.5) and the Cauchy-Schwarz inequality in the last step. In the y-direction, we have

$$\begin{aligned}
&\epsilon\left|\int_K\frac{\partial(E-E_I)}{\partial y}\frac{\partial v_N}{\partial y}\mathrm{d}x\mathrm{d}y\right| \\
&\leq\epsilon\int_K\left|\frac{\partial^3 E}{\partial x^2\partial y}\right||G(x)|\left(\left|\frac{\partial v_N}{\partial y}\right|+\frac{2}{3}\left|(x-x_K)\frac{\partial^2 v_N}{\partial x\partial y}\right|\right)\mathrm{d}x\mathrm{d}y \\
&\leq C\epsilon h_{x,K}^2\left\|\frac{\partial^3 E}{\partial x^2\partial y}\right\|_{0,K}\left\|\frac{\partial v_N}{\partial y}\right\|_{0,K}. \qquad (4.15)
\end{aligned}$$

Now, we estimate E_1.

(a) $K\subset\Omega_{21}\bigcup\Omega_{22}$. Recall the regularity (3.2), apply (4.14) to E_1, summing over K, we have

$$\epsilon\sum_{K\subset\Omega_{21}\bigcup\Omega_{22}}\left|\int_K\frac{\partial(E_1-E_{1,I})}{\partial x}\frac{\partial v_N}{\partial x}\mathrm{d}x\mathrm{d}y\right|$$

$$
\begin{aligned}
&\leq C\epsilon^{1/2}N^{-2}\sum_{i=N/2+1}^{N}\sum_{j=1}^{N}\left(\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}e^{-\frac{2\beta_1(1-x)}{\epsilon}}\mathrm{d}x\mathrm{d}y\right)^{1/2}\left\|\frac{\partial v_N}{\partial x}\right\|_{0,K}\\
&\leq C\epsilon^{1/2}N^{-2}\left\|\frac{\partial v_N}{\partial x}\right\|_0\leq CN^{-2}\|v_N\|_{SD},
\end{aligned}
\tag{4.16}
$$

where we used Lemma 3 and the Cauchy-Schwarz inequality. Applying (4.15) to E_1, and summing over K, we have

$$
\begin{aligned}
&\epsilon\sum_{K\subset\Omega_{21}\bigcup\Omega_{22}}\left|\int_K\frac{\partial(E_1-E_{1,I})}{\partial y}\frac{\partial v_N}{\partial y}\mathrm{d}x\mathrm{d}y\right|\\
&\leq C\epsilon^{-1}\sum_{i=N/2+1}^{N}\sum_{j=1}^{N}\left(h_{x,i}^4\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}e^{-\frac{2\beta_1(1-x)}{\epsilon}}\mathrm{d}x\mathrm{d}y\right)^{1/2}\left\|\frac{\partial v_N}{\partial y}\right\|_{0,K}\\
&\leq C\epsilon^{-1}\left(\sum_{i=N/2+1}^{N}\left(\frac{5\epsilon}{\beta_1\gamma_1}\right)^4\epsilon N^{-1}e^{-\frac{8\beta_1(1-x_i)}{5\epsilon}}\right)^{1/2}\left\|\frac{\partial v_N}{\partial y}\right\|_0\\
&\leq C\epsilon^{1.5}N^{-2}\left\|\frac{\partial v_N}{\partial y}\right\|_0\leq CN^{-3}\|v_N\|_{SD}.
\end{aligned}
\tag{4.17}
$$

Here we used (3.21) and the Cauchy-Schwarz inequality. Putting together the above two estimates yields

$$
\left|\epsilon\int_{\Omega_{21}\bigcup\Omega_{22}}\nabla(E_1-E_{1,I})\nabla v_N\mathrm{d}x\mathrm{d}y\right|\leq CN^{-2}\|v_N\|_{SD}. \tag{4.18}
$$

(b) $K\subset\Omega_{11}\bigcup\Omega_{12}$. From (3.2), by integration, we obtain

$$
\left\|\frac{\partial E_1}{\partial x}\right\|^2_{0,\Omega_{11}\bigcup\Omega_{12}}\leq C\epsilon^{-1}N^{-5}.
$$

On the other hand, apply the inverse inequality (3.5) to $E_{1,I}$,

$$
\begin{aligned}
\left\|\frac{\partial E_{1,I}}{\partial x}\right\|^2_{0,\Omega_{11}\bigcup\Omega_{12}}&\leq CN^2\|E_{1,I}\|^2_{0,\Omega_{11}\bigcup\Omega_{12}}\\
&\leq CN^2\sum_{i=1}^{N/2}h_{x,i}e^{-\frac{2\beta_1(1-x_i)}{\epsilon}}\sum_{j=1}^{N}h_{y,j}\leq CN^{-4},
\end{aligned}
$$

where we used (3.19). Summing up all $K\subset\Omega_{11}\bigcup\Omega_{12}$, applying the triangle inequality and the Cauchy-Schwarz inequality, we have

$$
\epsilon\sum_{K\subset\Omega_{11}\bigcup\Omega_{12}}\left|\int_K\frac{\partial(E_1-E_{1,I})}{\partial x}\frac{\partial v_N}{\partial x}\mathrm{d}x\mathrm{d}y\right|
$$

$$
\begin{aligned}
&\leq C\epsilon\left(\left\|\frac{\partial E_1}{\partial x}\right\|_{0,\Omega_{11}\bigcup\Omega_{12}}+\left\|\frac{\partial E_{1,I}}{\partial x}\right\|_{0,\Omega_{11}\bigcup\Omega_{12}}\right)\left\|\frac{\partial v_N}{\partial x}\right\|_{0,\Omega_{11}\bigcup\Omega_{12}}\\
&\leq C(\epsilon^{0.5}N^{-2.5}+\epsilon N^{-2})\left\|\frac{\partial v_N}{\partial x}\right\|_{0,\Omega_{11}\bigcup\Omega_{12}}\\
&\leq CN^{-2.5}\|v_N\|_{SD}. \qquad (4.19)
\end{aligned}
$$

Furthermore, by (3.12) in Lemma 3.1 with $p=2$, we get

$$
\begin{aligned}
&\left\|\frac{\partial(E_1-E_{1,I})}{\partial y}\right\|^2_{0,\Omega_{11}\bigcup\Omega_{12}}\\
\leq\,& C\sum_{K\subset\Omega_{11}\bigcup\Omega_{12}}\left(h^2_{x,K}\left\|\frac{\partial^2 E_1}{\partial x\partial y}\right\|^2_{0,K}+h^2_{y,K}\left\|\frac{\partial^2 E_1}{\partial y^2}\right\|^2_{0,K}\right)\\
\leq\,& C\epsilon^{-2}N^{-2}\sum_{i=1}^{N/2}\int_{x_{i-1}}^{x_i}e^{-\frac{2\beta_1(1-x)}{\epsilon}}\mathrm{d}x\leq C\epsilon^{-1}N^{-7}.
\end{aligned}
$$

Thus, by the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\epsilon\sum_{K\subset\Omega_{11}\bigcup\Omega_{12}}\left|\int_K\frac{\partial(E_1-E_{1,I})}{\partial y}\frac{\partial v_N}{\partial y}\mathrm{d}x\mathrm{d}y\right|\\
\leq\,& C\epsilon\left\|\frac{\partial(E_1-E_{1,I})}{\partial y}\right\|_{0,\Omega_{11}\bigcup\Omega_{12}}\left\|\frac{\partial v_N}{\partial y}\right\|_{0,\Omega_{11}\bigcup\Omega_{12}}\\
\leq\,& C\epsilon^{0.5}N^{-3.5}\left\|\frac{\partial v_N}{\partial y}\right\|_{0,\Omega_{11}\bigcup\Omega_{12}}\\
\leq\,& CN^{-3.5}\|v_N\|_{SD}. \qquad (4.20)
\end{aligned}
$$

Collecting (4.19), (4.20) and (4.18) yields

$$
\left|\epsilon\int_\Omega\nabla(E_1-E_{1,I})\nabla v_N\mathrm{d}x\mathrm{d}y\right|\leq CN^{-2}\|v_N\|_{SD}.
$$

Using similar arguments, we can show that

$$
\begin{aligned}
&\left|\epsilon\int_\Omega\nabla(E_2-E_{2,I})\nabla v_N\mathrm{d}x\mathrm{d}y\right|\leq CN^{-2}\|v_N\|_{SD},\\
&\left|\epsilon\int_\Omega\nabla(E_{12}-E_{12,I})\nabla v_N\mathrm{d}x\mathrm{d}y\right|\leq CN^{-2.5}\|v_N\|_{SD}.
\end{aligned}
$$

The proof is complete. □

Lemma 4.6. *Let $E=E_1+E_2+E_{12}$ satisfy the regularity* (3.2)-(3.4). *Then there is a constant C, such that*

$$
|((E-E_I),b\cdot\nabla v_N)|\leq CN^{-2}\|v_N\|_{SD},\qquad \forall v_N\in V^N. \qquad (4.21)
$$

Proof. We estimate E_1 which is discussed into the case of $\Omega_{11}\bigcup\Omega_{12}$ and $\Omega_{21}\bigcup\Omega_{22}$.

(a) $K\subset\Omega_{21}\bigcup\Omega_{22}$. Using (3.10) in Lemma 3.1 with $p=2$, we get

$$\begin{aligned}&\|E_1-E_{1,I}\|_0^2 \\ \le& C\sum_{K\subset\Omega_{21}\bigcup\Omega_{22}}\left\{h_{x,K}^4\left\|\frac{\partial^2 E_1}{\partial x^2}\right\|_{0,K}^2+h_{x,K}^2h_{y,K}^2\left\|\frac{\partial^2 E_1}{\partial x\partial y}\right\|_{0,K}^2+h_{y,K}^4\left\|\frac{\partial^2 E_1}{\partial y^2}\right\|_{0,K}^2\right\}.\end{aligned} \tag{4.22}$$

By the bound (3.2), the first term is estimated

$$\begin{aligned}&\sum_{K\subset\Omega_{21}\bigcup\Omega_{22}}h_{x,K}^4\left\|\frac{\partial^2 E_1}{\partial x^2}\right\|_{0,K}^2\\ \le& C\sum_{j=1}^N h_{y,j}\sum_{i=N/2+1}^N\left(\frac{5\epsilon}{\beta_1\gamma_1}\right)^4\int_{x_{i-1}}^{x_i}\epsilon^{-4}e^{-\frac{2\beta_1(1-x)}{\epsilon}}\mathrm{d}x\\ \le& C\epsilon N^{-1}\sum_{i=N/2+1}^N\gamma_1^{-4}e^{-\frac{8\beta_1(1-x_i)}{5\epsilon}}\le C\epsilon N^{-4},\end{aligned} \tag{4.23}$$

where we used (3.21) and Lemma 3.3.
For the second term in (4.22), we obtain by using Lemma 3.3

$$\begin{aligned}&\sum_{K\subset\Omega_{21}\bigcup\Omega_{22}}h_{x,K}^2h_{y,K}^2\left\|\frac{\partial^2 E_1}{\partial x\partial y}\right\|_{0,K}^2\\ \le& C\sum_{j=1}^N h_{y,j}^3\sum_{i=N/2+1}^N\left(\frac{5\epsilon}{\beta_1\gamma_1}\right)^2\int_{x_{i-1}}^{x_i}\epsilon^{-2}e^{-\frac{2\beta_1(1-x)}{\epsilon}}\mathrm{d}x\\ \le& C\epsilon N^{-3}\sum_{i=N/2+1}^N\gamma_1^{-2}e^{-\frac{8\beta_1(1-x_i)}{5\epsilon}}\le C\epsilon N^{-4}.\end{aligned} \tag{4.24}$$

By integration and Lemma 3.3, we calculate

$$\begin{aligned}&\sum_{K\subset\Omega_{21}\bigcup\Omega_{22}}h_{y,K}^4\left\|\frac{\partial^2 E_1}{\partial y^2}\right\|_{0,K}^2\le C\sum_{j=1}^N h_{y,j}^5\sum_{i=N/2+1}^N\int_{x_{i-1}}^{x_i}e^{-\frac{2\beta_1(1-x)}{\epsilon}}\mathrm{d}x\\ \le& CN^{-4}\int_{1-\lambda_1}^1 e^{-\frac{2\beta_1(1-x)}{\epsilon}}\mathrm{d}x\le C\epsilon N^{-4},\end{aligned}$$

which, combined (4.23) and (4.24), proves

$$\begin{aligned}&|((E_1-E_{1,I}),b\cdot\nabla v_N)_{\Omega_{21}\bigcup\Omega_{22}}|\\ \le& C\|E_1-E_{1,I}\|_{0,\Omega_{21}\bigcup\Omega_{22}}\|\nabla v_N\|_{0,\Omega_{21}\bigcup\Omega_{22}}\le CN^{-2}\|v_N\|_{SD},\end{aligned} \tag{4.25}$$

where we used the Cauchy-Schwarz inequality and the continuity of b.
(b) $K \subset \Omega_{11} \bigcup \Omega_{12}$. By the bound (3.2), we obtain by integrating

$$\|E_1\|^2_{0,\Omega_{11}\bigcup\Omega_{12}} \le C \int_0^{1-\lambda_1} e^{-\frac{2\beta_1(1-x)}{\epsilon}} \mathrm{d}x \le CN^{-6}.$$

We notice that on an element K, $|E_{1,I}| \le |E_1| \le Ce^{-\frac{2\beta_1(1-x_i)}{\epsilon}}$. By using (3.19), we have

$$\|E_{1,I}\|^2_{0,\Omega_{11}\bigcup\Omega_{12}} \le C \sum_{j=1}^{N} h_{y,j} \sum_{i=1}^{N/2} h_{x,i} e^{-\frac{2\beta_1(1-x_i)}{\epsilon}} \le CN^{-6}.$$

Therefore, by the inverse inequality (3.5), we have

$$\begin{aligned}
&|((E_1 - E_{1,I}), b\cdot\nabla v_N)_{\Omega_{11}\bigcup\Omega_{12}}| \\
\le\, & C\|E_1 - E_{1,I}\|_{0,\Omega_{11}\bigcup\Omega_{12}} \|\nabla v_N\|_{0,\Omega_{11}\bigcup\Omega_{12}} \le CN^{-2}\|v_N\|_{SD}.
\end{aligned}$$

This, together with the estimate (4.25), proves

$$|((E_1 - E_{1,I}), b\cdot\nabla v_N)| \le CN^{-2}\|v_N\|_{SD}.$$

The estimates for E_2 and E_{12} are similar. Hence, (4.21) is obtained. □

We will estimate the term B_{GAL}.

Lemma 4.7. *Let u be the solution of the continuous problems* (1.1) *and let u_I be the bilinear interpolation of u on the B-S-mesh. If Assumption* 3.1 *holds, then there is a constant C, such that*

$$|B_{GAL}(u - u_I, v_N)| \le CN^{-2}\|v_N\|_{SD}, \qquad \forall v_N \in V^N.$$

Proof. We rewrite the bilinear form for $w \in H_0^1(\Omega)$:

$$\begin{aligned}
B_{GAL}(w, v) &= \epsilon(\nabla w, \nabla v) + (b\cdot\nabla w, v) + (cw, v) \\
&= \epsilon(\nabla w, \nabla v) - (w, b\cdot\nabla v) + ((c - \mathrm{div} b)w, v).
\end{aligned}$$

We shall use whichever of these expressions is more convenient.

$$|B_{GAL}(u - u_I, v_N)| \le |B_{GAL}(E - E_I, v_N)| + |B_{GAL}(S - S_I, v_N)|.$$

Now we estimate E-term. In the light of Lemma 4.5 and Lemma 4.6, for any $v_N \in V^N$ we have

$$\begin{aligned}
&|\epsilon(\nabla(E - E_I), \nabla v_N)| \le CN^{-2}\|v_N\|_{SD}, \\
&|((E - E_I), b\cdot\nabla v_N)| \le CN^{-2}\|v_N\|_{SD}, \\
&|((c - \mathrm{div} b)(E - E_I), v_N)| \le C\|E - E_I\|_0\|v_N\|_0 \le CN^{-2}\|v_N\|_{SD},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality and (4.7). Hence,

$$|B_{GAL}(E-E_I,v_N)| \le CN^{-2}\|v_N\|_{SD}. \tag{4.26}$$

Next, we analyze S-term. Applying (4.14) and (4.15) to S, we get

$$\epsilon\left|\int_K \frac{\partial(S-S_I)}{\partial x}\frac{\partial v_N}{\partial x}\mathrm{d}x\mathrm{d}y\right| \le C\epsilon N^{-2}\left\|\frac{\partial^3 S}{\partial x\partial y^2}\right\|_{0,K}\left\|\frac{\partial v_N}{\partial x}\right\|_{0,K},$$
$$\epsilon\left|\int_K \frac{\partial(S-S_I)}{\partial y}\frac{\partial v_N}{\partial y}\mathrm{d}x\mathrm{d}y\right| \le C\epsilon N^{-2}\left\|\frac{\partial^3 S}{\partial x^2\partial y}\right\|_{0,K}\left\|\frac{\partial v_N}{\partial y}\right\|_{0,K}.$$

These, together with the Cauchy-Schwarz inequality, prove

$$\begin{aligned}&|\epsilon(\nabla(S-S_I),\nabla v_N)|\\ \le\, & C\epsilon N^{-2}\sum_{K\subset\Omega}\left(\left\|\frac{\partial^3 S}{\partial x\partial y^2}\right\|_{0,K}+\left\|\frac{\partial^3 S}{\partial x^2\partial y}\right\|_{0,K}\right)\|\nabla v_N\|_{0,K}\\ \le\, & CN^{-2.5}\|v_N\|_{SD}.\end{aligned}$$

The second term of $B_{GAL}(S-S_I,v_N)$ is handled in Lemma 4.4 that

$$|(b\cdot\nabla(S-S_I),v_N)| \le CN^{-2}\|v_N\|_{SD}.$$

Furthermore, standard approximation theory gives us

$$|(c(S-S_I),v_N)| \le C\|S-S_I\|_0\|v_N\|_0 \le CN^{-2}\|v_N\|_{SD}.$$

Hence, we obtain

$$B_{GAL}(S-S_I,v_N) \le CN^{-2}\|v_N\|_{SD}$$

by collecting the above three bounds. Recalling (4.26), Lemma 10 is proved. □

It is now straightforward to prove the second main result.

Theorem 4.2. *Let u_h be the solution of the discrete problem* (2.2) *and let u_I be the bilinear interpolation of u on the B-S-mesh. If Assumption* 3.1 *holds, then there is a constant C, such that*

$$\|u_I-u_h\|_{SD} \le CN^{-2}(\ln N)^{1/2}.$$

Proof. From the inequality

$$\begin{aligned}&\frac{1}{2}\|u_I-u_h\|_{SD}^2 \le B_{SD}(u_I-u_h,u_I-u_h)\\ \le\, & B_{GAL}(u_I-u,u_I-u_h)+B_{STAB}(u_I-u,u_I-u_h).\end{aligned}$$

Setting $v_N=u_I-u_h$, the second part is shown in [10, Lemma 4.4] that

$$|B_{STAB}(u_I-u,v_N)| \le CN^{-2}(\ln N)^{1/2}\|v_N\|_{SD}.$$

Table 1: Error in the discrete SD norm and maximum norm.

N	$\epsilon = 1.0e-05$ $\|u-u_h\|_{SD,d}$	$\epsilon = 1.0e-05$ $\|u-u_h\|_{0,\infty}$	$\epsilon = 1.0e-06$ $\|u-u_h\|_{SD,d}$	$\epsilon = 1.0e-06$ $\|u-u_h\|_{0,\infty}$	$\epsilon = 1.0e-07$ $\|u-u_h\|_{SD,d}$	$\epsilon = 1.0e-07$ $\|u-u_h\|_{0,\infty}$
4	6.7574e-02	7.7052e-02	6.7574e-02	7.7055e-02	6.7575e-02	7.7056e-02
8	2.4086e-02	2.6539e-02	2.4087e-02	2.6539e-02	2.4087e-02	2.6539e-02
16	7.0152e-03	7.7200e-03	7.0154e-03	7.7196e-03	7.0154e-03	7.7196e-03
32	1.8805e-03	2.1284e-03	1.8805e-03	2.1281e-03	1.8805e-03	2.1281e-03
64	4.8593e-04	5.6323e-04	4.8594e-04	5.6307e-04	4.8594e-04	5.6306e-04
128	1.2344e-04	1.4536e-04	1.2344e-04	1.4530e-04	1.2344e-04	1.4529e-04
256	3.1103e-05	3.7039e-05	3.1104e-05	3.7014e-05	3.1104e-05	3.7012e-05
512	7.8059e-06	9.3709e-06	7.8061e-06	9.3616e-06	7.8059e-06	9.3604e-06

Invoking Lemma 4.7, we get

$$\|v_N\|_{SD}^2 \leq C|B_{SD}(u_I - u, v_N)| \leq CN^{-2}(\ln N)^{1/2}\|v_N\|_{SD}.$$

This proves the statement of the theorem. □

The combination of Theorem 4.1, Theorem 4.2 and Lemma 4.2 leads to our main result directly, i.e.,

Theorem 4.3. *Let u be the solution of the continuous problems* (1.1) *and let u_h be the solution of the discrete problem* (2.2). *If Assumption* 3.1 *holds, then*

$$\|u - u_h\|_{SD,d} \leq CN^{-2}(\ln N)^{1/2}.$$

5. Numerical results

We study the performance of the method when applied to the test problem

$$\begin{aligned} &-\epsilon \triangle u + u_x + u_y + u = f \quad \text{in} \quad \Omega = (0,1)^2, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where the right-hand side f is chosen such that

$$u(x,y) = xy\left(1 - e^{-(1-x)/\epsilon}\right)\left(1 - e^{-(1-y)/\epsilon}\right).$$

This function exhibits typical boundary layer behaviour. For our tests we take $\epsilon = 10^{-5}, \cdots, 10^{-7}$ and $N = 2^2, \cdots, 2^9$. The computation was performed by Matlab 7 and a five-points Gauss-Legendre formula was used to estimate the error. Table 1 lists the error in the discrete SD norm and maximum norm. When $\epsilon = 10^{-7}$, the errors in the discrete SD norm are plotted on log-log chart in Figure 2. Another two reference curves with different decay rates have been plotted in Figure 2. From this Figure, the rate of convergence can be observed directly. From Table 1 and Figure 2, they are clear illustrations of the convergence results of Theorem 4.3. Table 2 displays the errors for the S-mesh and B-S-mesh. It is obvious that the method on B-S-mesh outperforms the S-mesh for all N.

Table 2: Error on the B-S-mesh and S-mesh.

N	$\epsilon = 1.0e-05$ $B-S-mesh$	$\epsilon = 1.0e-05$ $S-mesh$	$\epsilon = 1.0e-06$ $B-S-mesh$	$\epsilon = 1.0e-06$ $S-mesh$	$\epsilon = 1.0e-07$ $B-S-mesh$	$\epsilon = 1.0e-07$ $S-mesh$
4	6.7574e-02	1.2637e-01	6.7574e-02	1.2637e-01	6.7575e-02	1.2637e-01
8	2.4086e-02	7.9554e-02	2.4087e-02	7.9554e-02	2.4087e-02	7.9554e-02
16	7.0152e-03	3.8041e-02	7.0154e-03	3.8041e-02	7.0154e-03	3.8041e-02
32	1.8805e-03	1.5407e-02	1.8805e-03	1.5407e-02	1.8805e-03	1.5407e-02
64	4.8593e-04	5.6306e-03	4.8594e-04	5.6306e-03	4.8594e-04	5.6306e-03
128	1.2344e-04	1.9267e-03	1.2344e-04	1.9267e-03	1.2344e-04	1.9267e-03
256	3.1103e-05	6.3037e-04	3.1104e-05	6.3036e-04	3.1104e-05	6.3036e-04
512	7.8059e-06	1.9958e-04	7.8061e-06	1.9958e-04	7.8059e-06	1.9958e-04

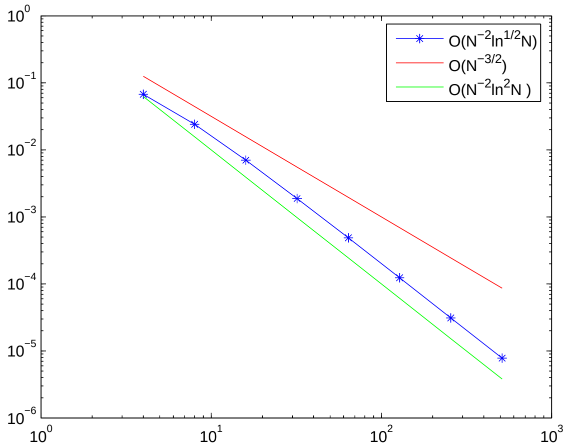

Figure 2: Error: the discrete SD norm, $\epsilon = 10^{-7}$.

6. Conclusion

In this paper, the Streamline-Diffusion finite element method is applied to a singularly perturbed convection-diffusion problem posed on the unit square, using a Bakhvalov-Shishkin mesh with piecewise bilinear trial function. The method is shown to be convergent, uniformly in the perturbation parameter ϵ, of convergent rate $\mathcal{O}(N^{-2}(\ln N)^{1/2})$ in a discrete Streamline-Diffusion norm. It is obvious that the method on Bakhvalov-Shishkin mesh yields more accurate results than on Shishkin's mesh.

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