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The Weak Galerkin Method for Linear Hyperbolic Equation

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Abstract. The linear hyperbolic equation is of great interest in many branches of physics and industry. In this paper, we use the weak Galerkin method to solve the linear hyperbolic equation. Since the weak Galerkin finite element space consists of discontinuous polynomials, the discontinuous feature of the equation can be maintained. The optimal error estimates are proved. Some numerical experiments are provided to verify the efficiency of the method.

AMS subject classifications: 65N30, 65N15, 65N25, 35B45, 35J50, 35J35

Key words: Weak Galerkin finite element method, linear hyperbolic equation, error estimate.

1 Introduction

The linear hyperbolic equation arises in many branches of physics, including acoustics and fluid mechanics. For example, in the computational fluid dynamics the Lagrangian grids are usually employed, and the physical quantities, like density, velocity and pressure, extend from one medium to another medium through the interface. A linear hyperbolic equation, which is also called the eikonal equation, needs to be solved to verify the physical quantities on the ghost element near the interface. As to the derivation and more applications of the linear hyperbolic equation, readers are referred to [7] and the references therein.

Many numerical methods have been applied to the linear hyperbolic equation, such as the finite difference method [17], the finite element method [18], and the finite volume method [1, 6]. A key issue of the numerical simulation of the linear hyperbolic equation

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is the approximation near the shock. The exact solution of the linear hyperbolic equation may be discontinuous, and it is a challenge for the numerical scheme to avoid oscillation around the discontinuity. In this aspect, the discontinuous Galerkin method [4] is a competitive candidate. The optimal order estimate of the discontinuous Galerkin method is discussed in [11]. The superconvergence phenomenon of the discontinuous Galerkin method is also studied, the $k+2$ order superconvergence and $2k+1$ order superconvergence are investigated in [24] and [2], respectively. There are also many other schemes for the linear hyperbolic equation, such as SUPG [8,9] and least squares method [5,10].

Recently, a numerical method called the weak Galerkin (WG) finite element method is proposed for solving PDEs. The weak Galerkin method has been introduced and analyzed in [20] for the second order elliptic equations. The main idea of the WG method is to use totally discontinuous polynomials as basis functions, and replace the classical derivative operators by specifically defined weak derivative operators in the numerical scheme. It has been applied to a variety of PDEs, including the second order elliptic equation [3,12,21], the biharmonic equation [14,15,27], the Stokes equation [16,22,26], the Brinkman equation [13,23,25] and the linear elasticity equation [19], etc. The weak Galerkin method employs discontinuous polynomials in the finite element space, which can help describe the discontinuity of the solution.

In the computational fluid dynamics, the mesh grid is usually polytopal and unstructured. The WG method can solve this kind of problems efficiently since it utilizes discontinuous elements and suits for polytopal meshes. The numerical simulation of the linear hyperbolic equation is also an important issue in computational fluid dynamics, and we are interested in solving this problem by the WG method.

We consider the linear hyperbolic equation that seeks an unknown function u satisfying

$$\boldsymbol{\beta}\cdot\nabla u+cu=f, \quad \text{in } \Omega, \tag{1.1}$$

$$u=g, \quad \text{on } \Gamma_-, \tag{1.2}$$

where Ω is a polytopal domain in $\mathbb{R}^d$ (polygonal or polyhedral domain for $d=2,3$), the coefficients $\boldsymbol{\beta}$ and c are non-negative functions, and

$$\Gamma_-=\{\mathbf{x}\in\partial\Omega,\boldsymbol{\beta}\cdot\mathbf{n}\le 0 \ \text{ at } \mathbf{x}\}.$$

For the simplicity of analysis, we suppose $\boldsymbol{\beta}$ and c are piecewise constants.

In this paper, we apply the WG method to the linear hyperbolic equation, and give the corresponding estimates.

The rest of paper is structured as follows. In Section 2, we introduce some notations, definitions, and the WG scheme. In Section 3 we derive the error equations for the WG approximations and we give the error estimates. Some numerical experiments are presented in Section 4.

2 The weak Galerkin schemes

In this section, we define some notations, definitions necessary for the WG method, and introduce the WG scheme for (1.1)-(1.2).

Suppose $\mathcal{T}_h$ is a polytopal partition of Ω satisfying regularity assumptions verified in [21]. We denote T to represent an element in $\mathcal{T}_h$ and e to represent an edge/face in $\mathcal{T}_h$. For every element $T\in\mathcal{T}_h$, h_T is its diameter and $h=\max_{T\in\mathcal{T}_h}h_T$. Let $\mathcal{E}_h$ be the set of all the edges/faces in $\mathcal{T}_h$. For any $e\in\mathcal{E}_h$, suppose $T_{1,e}$ and $T_{2,e}$ are two adjoint elements and $\mathbf{n}_e$ is a uniform normal vector of e. Define

$$\mathcal{E}_h^0=\{e\in\mathcal{E}_h,\ \boldsymbol{\beta}|_{T_{1,e}}\cdot\mathbf{n}_e=\boldsymbol{\beta}|_{T_{2,e}}\cdot\mathbf{n}_e=0\}.$$

On partition $\mathcal{T}_h$, the weak Galerkin finite element space is defined as follows

$$V_h=\{(v_0,v_b): v_0\in P_k(T),\ \forall T\in\mathcal{T}_h,\ \text{ and }\ v_b\in P_k(e),\ \forall e\in\mathcal{E}_h\}.$$

It should be noted that v_0 and v_b can be totally discontinuous, i.e. v_b is not supposed to be the trace of v_0. We can also define

$$V_h^0=\{v\in V_h,\ v_b=0\ \text{ on } \Gamma_-\}.$$

Now we introduce some projection operators. For each element T, we denote by Q_0 the L^2 projection operator onto $P_k(T)$, and $\mathbb{Q}_h$ be the L^2 projection operator onto $[P_{k-1}(T)]^d$. On each edge/face e, denote by Q_b the L^2 projection operator onto $P_k(e)$. Combining Q_0 and Q_b, we have $Q_h=\{Q_0,Q_b\}$ the projection operator onto V_h.

One of key feature of the weak Galerkin method is to use the weak derivative operator instead of the classical derivative operator. On the weak Galerkin finite element space, we define the weak gradient operator as follows.

Definition 2.1. [20] For any $v=\{v_0,v_b\}\in V_h$, on each element T, define $\nabla_w v$ the unique polynomial in $[P_{k-1}(T)]^d$ satisfying

$$(\nabla_w v,\mathbf{q})_T=-(v_0,\nabla\cdot\mathbf{q})_T+\langle v_b,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T},\quad \forall\mathbf{q}\in[P_{k-1}(T)]^d,$$

where $\mathbf{n}$ is the unit outward normal vector of ∂T.

Let $v,w\in V_h$. Introduce the following two bilinear forms on V_h,

$$\begin{aligned} a_w(v,w)&=(\boldsymbol{\beta}\cdot\nabla_w v+cv_0,\boldsymbol{\beta}\cdot\nabla_w w+cw_0),\\ s(v,w)&=\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle v_0-v_b,w_0-w_b\rangle_{\partial T}. \end{aligned}$$

For any $v\in V_h$, define

$$|||v|||^2=a_w(v,v)+s(v,v).$$

We claim that $|||\cdot|||$ defines a norm on V_h^0. First we need the following identity.

Lemma 2.1. *For any $v=\{v_0,v_b\}\in V_h$, $\varphi\in P_{k-1}(T)$, the following identity holds true on each T,*

$$(\boldsymbol{\beta}\cdot\nabla_w v,\varphi)_T=(\boldsymbol{\beta}\cdot\nabla v_0,\varphi)_T-\langle(\boldsymbol{\beta}\cdot\mathbf{n})(v_0-v_b),\varphi\rangle_{\partial T}.$$

Proof. From the definition of weak gradient and the integration by parts, we can obtain

$$\begin{aligned}(\boldsymbol{\beta}\cdot\nabla_w v,\varphi)_T&=-(v_0,\nabla\cdot(\boldsymbol{\beta}\varphi))_T+\langle v_b,(\boldsymbol{\beta}\cdot\mathbf{n})\varphi\rangle_{\partial T}\\&=-(v_0,\nabla\cdot(\boldsymbol{\beta}\varphi))_T+\langle v_0,(\boldsymbol{\beta}\cdot\mathbf{n})\varphi\rangle_{\partial T}-\langle v_0-v_b,(\boldsymbol{\beta}\cdot\mathbf{n})\varphi\rangle_{\partial T}\\&=(\boldsymbol{\beta}\cdot\nabla v_0,\varphi)_T-\langle(\boldsymbol{\beta}\cdot\mathbf{n})(v_0-v_b),\varphi\rangle_{\partial T},\end{aligned}$$

which completes the proof. □

Lemma 2.2. *$|||\cdot|||$ defines a norm on V_h^0.*

Proof. We need to prove that $|||v|||=0$ implies $v=0$ in V_h^0. From the definition of $|||\cdot|||$, $a_w(v,v)\geq 0$ and $s(v,v)\geq 0$, which implies $a_w(v,v)=s(v,v)=0$. Therefore on each element, we have $\boldsymbol{\beta}\cdot\nabla_w v+cv_0=0$ in T, and $v_0-v_b=0$ on ∂T.

From Lemma 2.1, for each element T and $\varphi\in P_{k-1}(T)$,

$$\begin{aligned}(\boldsymbol{\beta}\cdot\nabla_w v,\varphi)_T&=(\boldsymbol{\beta}\cdot\nabla v_0,\varphi)_T-\langle(\boldsymbol{\beta}\cdot\mathbf{n})(v_0-v_b),\varphi\rangle_{\partial T}\\&=(\boldsymbol{\beta}\cdot\nabla v_0,\varphi)_T,\end{aligned}$$

which implies $\boldsymbol{\beta}\cdot\nabla_w v=\boldsymbol{\beta}\cdot\nabla v_0$. It follows that,

$$\begin{aligned}\boldsymbol{\beta}\cdot\nabla v_0+cv_0&=0, \quad \text{in } \Omega,\\ v_0&=0, \quad \text{on } \Gamma_-.\end{aligned}$$

From the uniqueness of solution of Eqs. (1.1)-(1.2), we can obtain $v_0=0$ on Ω. That implies $v_b=0$ which completes the proof. □

Now we can introduce the following weak Galerkin method for the linear convection equation (1.1)-(1.2).

Weak Galerkin Algorithm 1. Find $u_h\in V_h$, such that $u_b=Q_b g$ on Γ_-, and

$$a_w(u_h,v)+s(u_h,v)=(f,\boldsymbol{\beta}\cdot\nabla_w v+cv_0), \quad \forall v\in V_h^0. \tag{2.1}$$

3 Error analysis

In this section, we derive an error equation and analyze the order of convergence of Weak Galerkin Algorithm 1.

3.1 Error equation

The following communicative property plays an essential role in the analysis, which reveals the relationship between the classical gradient operator and the weak gradient operator.

Lemma 3.1. *For any $w\in H^1(\Omega)$,*

$$\nabla_w(Q_h w)=\mathbb{Q}_h(\nabla w),\quad \forall T\in\mathcal{T}_h.$$

Proof. For any $\mathbf{q}\in[P_{k-1}(T)]^d$, from the definition of weak gradient and the integration by parts, we can obtain

$$\begin{aligned}(\nabla_w(Q_h w),\mathbf{q})_T&=-(Q_0 w,\nabla\cdot\mathbf{q})_T+\langle Q_b w,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}\\&=-(w,\nabla\cdot\mathbf{q})_T+\langle w,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}\\&=(\nabla w,\mathbf{q})_T\\&=(\mathbb{Q}_h(\nabla w),\mathbf{q})_T,\end{aligned}$$

which completes the proof. □

On the regular polytopal partition, the following trace inequality and inverse inequality hold true. The proof can be found in [21].

Lemma 3.2 (Trace inequality)**.** *Suppose $\varphi\in H^1(T)$, then the following inequality holds on each element $T\in\mathcal{T}_h$,*

$$\|\varphi\|_{\partial T}^2\le C(h_T^{-1}\|\varphi\|_T^2+h_T\|\varphi\|_{1,T}^2).$$

Lemma 3.3 (Inverse inequality)**.** *Suppose $\psi\in P_k(T)$, then there exists a constant C such that on each element $T\in\mathcal{T}_h$,*

$$\|\nabla\psi\|_T\le Ch_T^{-1}\|\psi\|_T.$$

The well posedness of the Weak Galerkin Algorithm 1 is a direct corollary of Lemma 2.2.

Lemma 3.4. *The Weak Galerkin Algorithm* 1 *has a unique solution.*

Proof. Since the Weak Galerkin Algorithm 1 is a linear system, we just need to verify the uniqueness of the homogeneous problem. Suppose $f=g=0$, from the scheme (2.1) we can obtain

$$a_w(u_h,v)+s(u_h,v)=0,\quad \forall v\in V_h.$$

By letting $v=u_h$, it follows that

$$|||u_h|||^2=a_w(u_h,u_h)+s(u_h,u_h)=0.$$

From Lemma 2.2 we have $u_h=0$, which completes the proof. □

Lemma 3.5. *Suppose* $u\in H^1(\Omega)$ *is the exact solution of* (1.1)-(1.2)*,* u_h *is the numerical solution of the Weak Galerkin Algorithm* 1*, and* $e_h=Q_hu-u_h$*. Then for any* $v\in V_h$*, the following equation holds*

$$a_w(e_h,v)+s(e_h,v)=(\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),\boldsymbol{\beta}\cdot\nabla_w v+cv_0)+s(Q_hu,v).$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}
&a_w(Q_hu,v)+s(Q_hu,v)\\
=&(\boldsymbol{\beta}\cdot\nabla_w(Q_hu)+cQ_0u,\boldsymbol{\beta}\cdot\nabla_w v+cv_0)+s(Q_hu,v)\\
=&(\boldsymbol{\beta}\cdot\mathbb{Q}_h(\nabla u)+cQ_0u,\boldsymbol{\beta}\cdot\nabla_w v+cv_0)+s(Q_hu,v).
\end{aligned}\tag{3.1}$$

Testing Eq. (1.1) by $(\boldsymbol{\beta}\cdot\nabla_w v+cv_0)$ yields

$$\begin{aligned}
(\boldsymbol{\beta}\cdot\nabla u+cu,\boldsymbol{\beta}\cdot\nabla_w v+cv_0)&=(f,\boldsymbol{\beta}\cdot\nabla_w v+cv_0)\\
&=a_w(u_h,v)+s(u_h,v).
\end{aligned}\tag{3.2}$$

Subtracting (3.2) from (3.1),

$$\begin{aligned}
&a_w(e_h,v)+s(e_h,v)\\
=&(\boldsymbol{\beta}\cdot\mathbb{Q}_h(\nabla u)+cQ_0u,\boldsymbol{\beta}\cdot\nabla_w v+cv_0)+s(Q_hu,v)-(\boldsymbol{\beta}\cdot\nabla u+cu,\boldsymbol{\beta}\cdot\nabla_w v+cv_0)\\
=&(\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),\boldsymbol{\beta}\cdot\nabla_w v+cv_0)+s(Q_hu,v),
\end{aligned}$$

which completes the proof. □

3.2 Error estimate

The remainders in error equation can be estimate by the following lemma.

Lemma 3.6. *Suppose* $u\in H^{k+1}(\Omega)$*, then for any* $v\in V_h$*, the following estimates hold*

$$|(\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),\boldsymbol{\beta}\cdot\nabla_w v+cv_0)|\le Ch^k\|u\|_{k+1}|||v|||,\tag{3.3}$$

$$|s(Q_hu,v)|\le Ch^k\|u\|_{k+1}|||v|||.\tag{3.4}$$

Proof. To prove the inequality (3.3), we use the property of L^2 projection operator to obtain

$$\begin{aligned}
&|(\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),\boldsymbol{\beta}\cdot\nabla_w v+cv_0)|\\
\le& C\|\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u)\|.\|\boldsymbol{\beta}\cdot\nabla_w v+cv_0\|\\
\le& C\left(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u)\|_T^2\right)^{\frac{1}{2}}|||v|||\\
\le& Ch^k\|u\|_{k+1}|||v|||.
\end{aligned}$$

As to (3.4), it follows from the trace inequality and the inverse inequality,

$$
\begin{aligned}
|s(Q_hu,v)| &= \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_0u-Q_bu, v_0-v_b\rangle_{\partial T}\\
&= \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_0u-u, v_0-v_b\rangle_{\partial T}\\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0u-u\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0-v_b\|_{\partial T}^2\right)^{\frac12}\\
&\le C\left(\sum_{T\in\mathcal{T}_h} (h_T^{-2}\|Q_0u-u\|_T^2+\|Q_0u-u\|_{1,T}^2)\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0-v_b\|_{\partial T}^2\right)^{\frac12}\\
&\le Ch^k\|u\|_{k+1}|||v|||,
\end{aligned}
$$

which completes the proof. □

Theorem 3.1. *Suppose $u\in H^{k+1}(\Omega)$ is the exact solution of* (1.1)-(1.2)*, $u_h\in V_h$ is the solution of the weak Galerkin scheme* (2.1)*. Then the following estimate holds.*

$$
|||Q_hu-u_h|||\le Ch^k\|u\|_{k+1}.
$$

Proof. Denote $e_h=Q_hu-u_h$. Letting $v=e_h$ in Lemma 3.5 we have

$$
\begin{aligned}
|||e_h|||^2 &= a_w(e_h,e_h)+s(e_h,e_h)\\
&= (\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),\boldsymbol{\beta}\cdot\nabla_w e_h+ce_0)+s(Q_hu,e_h).
\end{aligned}
$$

Applying Lemma 3.6 yields

$$
\begin{aligned}
|||e_h|||^2 &= (\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),\boldsymbol{\beta}\cdot\nabla_w e_h+ce_0)+s(Q_hu,e_h)\\
&\le Ch^k\|u\|_{k+1}|||e_h|||,
\end{aligned}
$$

which completes the proof. □

Lemma 3.7. *For any $\phi\in H^1(T)$ and $v\in V_h$, the following identity is true on each element T,*

$$
(\boldsymbol{\beta}\cdot\nabla\phi,\boldsymbol{\beta}\cdot\nabla v_0)_T=(\boldsymbol{\beta}\cdot\nabla_w Q_h\phi,\boldsymbol{\beta}\cdot\nabla_w v)_T+\langle\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\phi,(\boldsymbol{\beta}\cdot\mathbf{n})(v_0-v_b)\rangle_{\partial T}.
$$

Proof. Let $\phi\in H^1(T)$ and $v\in V_h$. Using Lemma 3.1,

$$
\begin{aligned}
(\boldsymbol{\beta}\cdot\nabla_w Q_h\phi,\boldsymbol{\beta}\cdot\nabla_w v)_T &= (\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\phi,\boldsymbol{\beta}\cdot\nabla_w v)_T\\
&= -(v_0,\nabla\cdot(\boldsymbol{\beta}^T\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\phi))_T+\langle(\boldsymbol{\beta}\cdot\mathbf{n})v_b,\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\phi\rangle_{\partial T}\\
&= (\nabla v_0,(\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\phi))_T-\langle(\boldsymbol{\beta}\cdot\mathbf{n})(v_0-v_b),\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\phi\rangle_{\partial T}\\
&= (\boldsymbol{\beta}\cdot\nabla v_0,\boldsymbol{\beta}\cdot(\nabla\phi))_T-\langle\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\phi,(\boldsymbol{\beta}\cdot\mathbf{n})(v_0-v_b)\rangle_{\partial T},
\end{aligned}
$$

which completes the proof. □

Lemma 3.8. *Suppose $u\in H^{k+1}(\Omega)$, then for any $v\in V_h$, the following estimate holds*

$$\left|\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),\boldsymbol{\beta}\cdot(v_0-v_b)\rangle_{\partial T}\right|\le Ch^k\|u\|_{k+1}|||v|||. \tag{3.5}$$

Proof. It follows from Cauchy-Schwartz inequality,

$$\begin{aligned}&\left|\sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),\boldsymbol{\beta}\cdot(v_0-v_b))_{\partial T}\right|\\ \le\,& C\left(\sum_{T\in\mathcal{T}_h}h_T\|\boldsymbol{\beta}\cdot(\mathbb{Q}_h(\nabla u)-\nabla u)\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\boldsymbol{\beta}\cdot(v_0-v_b)\|_{\partial T}^2\right)^{\frac12}\\ \le\,& Ch^k\|u\|_{k+1}|||v|||.\end{aligned}$$

This completes the proof. □

The L^2 error estimate can be obtained by using a dual argument. Consider an auxiliary problem,

$$\begin{aligned}-\boldsymbol{\beta}\cdot\nabla\varphi&=e_0, \quad \text{in } \Omega, &(3.6)\\ \boldsymbol{\beta}\cdot\nabla\psi+c^2\psi&=\varphi, \quad \text{in } \Omega, &(3.7)\\ \varphi&=0, \quad \text{on } \partial\Omega\backslash\Gamma_-, &(3.8)\\ \psi&=0, \quad \text{on } \Gamma_-. &(3.9)\end{aligned}$$

Suppose $\boldsymbol{\beta}$ and c are continuous on Ω, and the dual problem (3.6)-(3.9) has H^2-regularity, i.e.

$$\|\psi\|_2\le C\|\varphi\|_1\le C\|e_0\|.$$

Theorem 3.2. *Suppose $u\in H^{k+1}(\Omega)$ is the exact solution of (1.1)-(1.2), $u_h\in V_h$ is the solution of the weak Galerkin scheme (2.1), and the dual problem (3.6)-(3.9) has H^2-regularity. When h is sufficiently small, the following estimate holds*

$$\|Q_0u-u_0\|\le Ch^{k+1}\|u\|_{k+1}.$$

Proof. By testing (3.6) by e_0, we obtain,

$$\begin{aligned}\|e_0\|^2&=(-\boldsymbol{\beta}\cdot\nabla\varphi,e_0)\\ &=\sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla\psi,\boldsymbol{\beta}\cdot\nabla e_0)_T-\sum_{T\in\mathcal{T}_h}\langle(\boldsymbol{\beta}\cdot\mathbf{n})(e_0-e_b),\boldsymbol{\beta}\cdot\nabla\psi\rangle_{\partial T}+\sum_{T\in\mathcal{T}_h}(c\psi,ce_0)_T.\end{aligned}$$

Setting $\phi=\psi$ and $v=e_0$ in Lemma 3.7 we get,

$$(\boldsymbol{\beta}\cdot\nabla\psi,\boldsymbol{\beta}\cdot\nabla e_0)_T=(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi,\boldsymbol{\beta}\cdot\nabla_w e_h)_T+\langle\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\psi,(\boldsymbol{\beta}\cdot\mathbf{n})(e_0-e_b)\rangle_{\partial T}.$$

Then,

$$
\begin{aligned}
\|e_0\|^2 = & \sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi,\boldsymbol{\beta}\cdot\nabla_w e_h)_T+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\psi,(\boldsymbol{\beta}\cdot\mathbf{n})(e_0-e_b)\rangle_{\partial T}\\
& +\sum_{T\in\mathcal{T}_h}(c\psi,ce_0)_T-\sum_{T\in\mathcal{T}_h}\langle(\boldsymbol{\beta}\cdot\mathbf{n})(e_0-e_b),\boldsymbol{\beta}\cdot\nabla\psi\rangle_{\partial T}.
\end{aligned}
$$

Adding and subtracting the terms $\sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi,ce_0)_T$, $\sum_{T\in\mathcal{T}_h}(cQ_h\psi,\boldsymbol{\beta}\cdot\nabla_w e_h)_T$, and $s(Q_h\psi,e_h)$, we get,

$$
\begin{aligned}
\|e_0\|^2 = & \sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi+cQ_0\psi,\boldsymbol{\beta}\cdot\nabla_w e_h+ce_0)_T-\sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi,ce_0)_T+s(Q_h\psi,e_h)\\
& -\sum_{T\in\mathcal{T}_h}(cQ_h\psi,\boldsymbol{\beta}\cdot\nabla_w e_h)_T+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{\beta}\cdot(\mathbb{Q}_h\nabla\psi-\nabla\psi),(\boldsymbol{\beta}\cdot\mathbf{n})(e_0-e_b)\rangle_{\partial T}-s(Q_h\psi,e_h)\\
= & \sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi+cQ_0\psi,\boldsymbol{\beta}\cdot\nabla_w e_h+ce_0)_T+s(Q_h\psi,e_h)-\sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi,ce_0)_T\\
& -\sum_{T\in\mathcal{T}_h}(cQ_0\psi,\boldsymbol{\beta}\cdot\nabla_w e_h)_T-s(Q_h\psi,e_h)+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{\beta}\cdot(\mathbb{Q}_h\nabla\psi-\nabla\psi),(\boldsymbol{\beta}\cdot\mathbf{n})(e_0-e_b)\rangle_{\partial T},
\end{aligned}
$$

from Eq. (3.1), we have

$$
\begin{aligned}
\|e_0\|^2 = & -\sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi,ce_0)_T-\sum_{T\in\mathcal{T}_h}(cQ_h\psi,\boldsymbol{\beta}\cdot\nabla_w e_h)_T-s(Q_h\psi,e_h)\\
& +\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{\beta}\cdot(\mathbb{Q}_h\nabla\psi-\nabla\psi),(\boldsymbol{\beta}\cdot\mathbf{n})(e_0-e_b)\rangle_{\partial T}+a_w(Q_h\psi,e_h)+s(Q_h\psi,e_h).
\end{aligned}
$$

Consider the first two terms

$$
\begin{aligned}
& \sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi,ce_0)+\sum_{T\in\mathcal{T}_h}(cQ_h\psi,\boldsymbol{\beta}\cdot\nabla_w e_h)_T\\
= & \sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\nabla_w Q_h\psi,ce_0)+\sum_{T\in\mathcal{T}_h}(c\psi,\boldsymbol{\beta}\cdot\nabla_w e_h)_T\\
= & \sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\psi,ce_0)-\sum_{T\in\mathcal{T}_h}(ce_0,\nabla\cdot(\boldsymbol{\beta}\psi))_T+\sum_{T\in\mathcal{T}_h}\langle ce_b,(\boldsymbol{\beta}\cdot\mathbf{n})\psi\rangle_{\partial T}\\
= & \sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot\mathbb{Q}_h\nabla\psi,ce_0)-\sum_{T\in\mathcal{T}_h}(ce_0,\nabla\cdot(\boldsymbol{\beta}\psi))_T\\
= & \sum_{T\in\mathcal{T}_h}(\boldsymbol{\beta}\cdot(\mathbb{Q}_h\nabla\psi-\nabla\psi),ce_0)_T\\
\le & Ch\|\psi\|_2\|e_0\|.
\end{aligned}
\tag{3.10}
$$

Denote $\hat{Q}_0$ the L^2 projection onto $P_{k-1}(T)$. From Lemma 3.5 we have

$$\begin{aligned}
&a_w(Q_h\psi,e_h)+s(Q_h\psi,e_h)\\
=&(\beta\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),\beta\cdot\nabla_w Q_h\psi+cQ_0\psi)+s(Q_hu,Q_h\psi)\\
=&(\beta\cdot(\mathbb{Q}_h(\nabla u)-\nabla u),c(Q_0\psi-\hat{Q}_0\psi))+\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle Q_0u-Q_bu,Q_0\psi-Q_b\psi\rangle_{\partial T}\\
\le&C\|\mathbb{Q}_h(\nabla u)-\nabla u\|\|Q_0\psi-\hat{Q}_0\psi\|\\
&+\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|Q_0u-Q_bu\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|Q_0\psi-Q_b\psi\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le&Ch^{k+1}\|u\|_{k+1}\|\psi\|_2.
\end{aligned}\tag{3.11}$$

Letting $k=1$, $u=\psi$ and $v=e_h$ in Lemma 3.8 yields

$$\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{\beta}\cdot(\mathbb{Q}_h\nabla\psi-\nabla\psi),(\boldsymbol{\beta}\cdot\mathbf{n})(e_0-e_b)\rangle_{\partial T}\le Ch\|\psi\|_2|||e_h|||.\tag{3.12}$$

Therefore by using inequalities (3.4), (3.10), (3.11), and (3.12), we get

$$\|e_0\|^2\le Ch\|\psi\|_2|||e_h|||+Ch^{k+1}\|u\|_{k+1}\|\psi\|_2+Ch\|\psi\|_2\|e_0\|,$$

which implies

$$\|e_0\|\le Ch^{k+1}\|u\|_{k+1}.$$

This completes the proof. □

4 Numerical experiments

In this section, we present some numerical results to verify the efficiency and robustness of the weak Galerkin method.

Example 4.1. Consider the problem (1.1)-(1.2) on a unit square domain $(0,1)\times(0,1)$, where the exact solution is set to be

$$u=\sin(\pi x)\sin(\pi y).$$

The coefficients are $\boldsymbol{\beta}=(1,2)^T$ and $c=1$. The homogenous boundary is applied on $\Gamma^-=\{(x,y):x=0,y\in[0,1]\}\cup\{(x,y):y=0,x\in[0,1]\}$. The right-hand side function f is calculated accordingly. The uniform triangle mesh is employed.

The results for $k=1$, $k=2$ and $k=3$ are listed in Table 1-Table 3, respectively. The numerical results coincide with the theoretical analysis in the previous section.

Table 1: Convergence orders for $k=1$.

h	$\|\|\|Q_h u - u_h\|\|\|$	order	$\|Q_0 u - u_0\|$	order
1/4	2.55044e-1		6.65463e-2	
1/8	9.32122e-2	1.4522	1.56324e-2	2.0898
1/16	3.33496e-2	1.4828	3.83604e-3	2.0268
1/32	1.18564e-2	1.4920	9.54415e-4	2.0069
1/64	4.20351e-3	1.4960	2.38440e-4	2.0010
1/128	1.48825e-3	1.4980	5.96209e-5	1.9997

Table 2: Convergence orders for $k=2$.

h	$\|\|\|Q_h u - u_h\|\|\|$	order	$\|Q_0 u - u_0\|$	order
1/4	3.24396e-2		5.93643e-3	
1/8	5.89965e-3	2.4591	7.12894e-4	3.0578
1/16	1.05244e-3	2.4869	8.74500e-5	3.0272
1/32	1.86678e-4	2.4951	1.08500e-5	3.0108
1/64	3.30470e-5	2.4980	1.35175e-6	3.0048
1/128	5.84565e-6	2.4991	1.68697e-7	3.0023

Table 3: Convergence orders for $k=3$.

h	$\|\|\|Q_h u - u_h\|\|\|$	order	$\|Q_0 u - u_0\|$	order
1/4	3.17560e-3		4.33191e-4	
1/8	2.86789e-4	3.4690	2.74788e-5	3.9786
1/16	2.55011e-5	3.4914	1.72617e-6	3.9927
1/32	2.25826e-6	3.4973	1.07870e-7	4.0002
1/64	1.99716e-7	3.4992	6.73036e-9	4.0025
1/128	1.76994e-8	3.4962	4.30970e-10	3.9650

Example 4.2. Consider the problem (1.1)-(1.2) on a unit square domain $(0,1)\times(0,1)$, where the exact solution is set to be

$$u=\sin(\pi x)\sin(\pi y).$$

The coefficients are

$$\boldsymbol{\beta}(x,y)=\begin{cases}(1,2)^T, & \text{when } x+y\leq 1,\\ (2,4)^T, & \text{when } x+y>1,\end{cases}$$

and

$$c(x,y)=\begin{cases}1, & \text{when } x+y\leq 1,\\ 2, & \text{when } x+y>1.\end{cases}$$

The homogenous boundary is applied on $\Gamma^- = \{(x,y) : x=0, y\in[0,1]\}\cup\{(x,y) : y=0, x\in [0,1]\}$. The right-hand side function f is calculated accordingly. The non-uniform triangle mesh generated by Matlab is employed.

The results for $k=3$ are listed in Table 4. The numerical results coincide with the theoretical analysis in the previous section.

Table 4: Convergence orders for $k=3$ on non-uniform mesh.

h	$\|\|\|Q_h u - u_h\|\|\|$	order	$\|\|Q_0 u - u_0\|\|$	order
1/4	2.5186e-3		2.8637e-4	
1/8	2.2017e-4	3.5159	2.1295e-5	3.7493
1/16	2.0802e-5	3.4038	3.3490e-6	2.6687
1/32	1.8593e-6	3.4840	2.6066e-7	3.6835
1/64	1.9035e-7	3.2880	1.2111e-8	4.4278
1/128	1.5253e-8	3.6415	8.5772e-10	3.8196

Example 4.3. Consider the problem (1.1)-(1.2) on a unit square domain $(0,1)\times(0,1)$. The coefficients are $\boldsymbol{\beta}=(1,\tan(35^\circ))^T$ and $c=0$. The right-hand side function f is set to be 0. The inflow boundary is $\Gamma^-=\{(x,y) : x=0, y\in[0,1]\}\cup\{(x,y) : y=0, x\in[0,1]\}$, and the boundary condition is

$$g(x,y)=\begin{cases} 2, & \text{on } \{0\}\times(0,1), \\ 1, & \text{on } (0,1)\times\{0\}. \end{cases}$$

The uniform triangle mesh is employed and $k=1$.

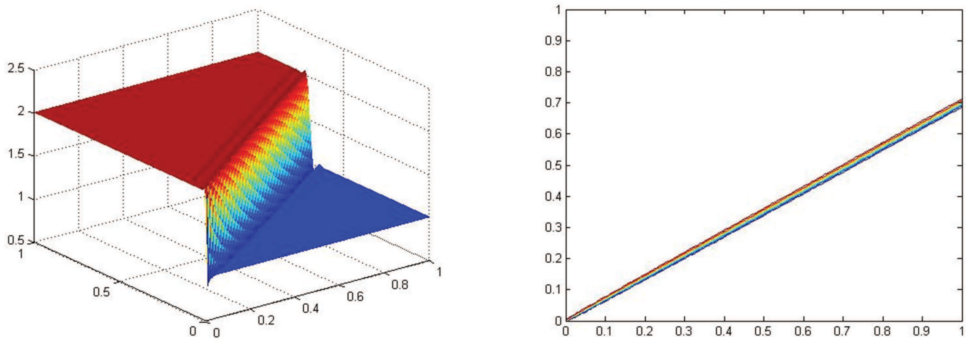
The figures of solution and contour curve are presented in Fig. 1(a)-Fig. 1(b), respectively.

Example 4.4. Consider the problem (1.1)-(1.2) on a unit square domain $(0,1)\times(0,1)$. The coefficients are $\boldsymbol{\beta}=(-y,x)^T$ and $c=0$. The right hand side function f is set to be 0. The inflow boundary is $\Gamma^-=\{(x,y):x=1, y\in[0,1]\}\cup\{(x,y):y=0, x\in[0,1]\}$, and the boundary condition is

$$g(x,y)=\begin{cases} -1, & \text{on } \left(0,\frac{43}{64}\right)\times\{0\}, \\ 1, & \text{on } \left(\frac{43}{64},1\right)\times\{0\}, \\ 1, & \text{on } \{1\}\times(0,1). \end{cases}$$

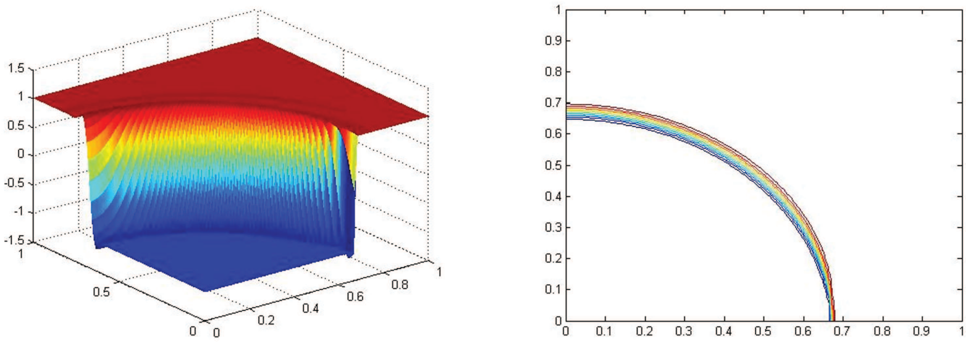
The uniform triangle mesh is employed and $k=1$.

The figures of solution and contour curve are presented in Fig. 2(a)-Fig. 2(b), respectively.



(a) The plot of numerical solution. (b) The contour of numerical solution.

Figure 1: The numerical solution of Example 4.3.



(a) The plot of numerical solution. (b) The contour of numerical solution.

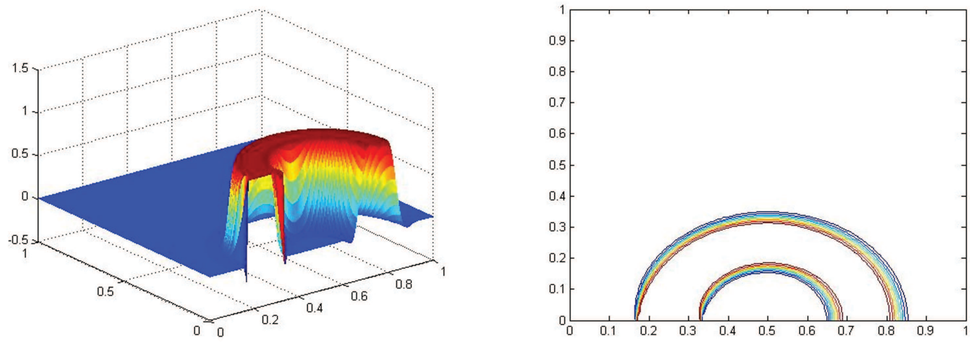
Figure 2: The numerical solution of Example 4.4.

Example 4.5. Consider the problem (1.1)-(1.2) on a unit square domain $(0,1)\times(0,1)$. The coefficients are $\boldsymbol{\beta}=(y,0.5-x)^T$ and $c=0$. The right-hand side function f is set to be 0. The inflow boundary is $\Gamma^-=\{(x,y):x=0,y\in[0,1]\}\cup\{(x,y):y=0,x\in[0,0.5]\}\cup\{(x,y):y=1,x\in[0.5,1]\}$, and the boundary condition is

$$g(x,y)=\begin{cases}1, & \text{on } (0.17,0.33)\times\{0\},\\ 0, & \text{otherwise.}\end{cases}$$

The uniform triangle mesh is employed and $k=1$.

The figures of solution and contour curve are presented in Fig. 3(a)-Fig. 3(b), respectively.



(a) The plot of numerical solution. (b) The contour of numerical solution.

Figure 3: The numerical solution of Example 4.5.

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