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Rational Quasi-Interpolation Approximation of Scattered Data in $\mathbb{R}^3$

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Abstract. This paper is concerned with a piecewise smooth rational quasi-interpolation with algebraic accuracy of degree $(n+1)$ to approximate the scattered data in $\mathbb{R}^3$. We firstly use the modified Taylor expansion to expand the mean value coordinates interpolation with algebraic accuracy of degree one to one with algebraic accuracy of degree $(n+1)$. Then, based on the triangulation of the scattered nodes in $\mathbb{R}^2$, on each triangle a rational quasi-interpolation function is constructed. The constructed rational quasi-interpolation is a linear combination of three different expanded mean value coordinates interpolations and it has algebraic accuracy of degree $(n+1)$. By comparing accuracy, stability, and efficiency with the C^1-Tri-interpolation method of Goodman[16] and the MQ Shepard method, it is observed that our method has some computational advantages.

AMS subject classifications: 41A05, 41A25

Key words: Scattered data, mean value coordinates interpolation, modified Taylor expansion, rational quasi-interpolation, algebraic accuracy.

1. Introduction

The problem of scattered data approximation appears in many fields of science and engineering. For example, geology, geography, reverse engineering, numerical simulation, computer graphics and geometric modeling, etc.. The most commonly used approximation method is the radial basis function interpolation [1-3], which is a kind of global interpolation method and need to solve linear system of equations to determine the coefficients of interpolation basis functions. The system is usually ill-conditioned when scattered data on a large scale, so they can't be solved effectively and stably. One of the ways to solve this problem is to find a better basis function, for example, the basis function in [4]. One way to get around this problem is the quasi-interpolation method. The quasi-interpolation

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method gives an explicit expression of the approximation function using the given data. Thus it avoids solving large-scale systems of linear algebraic equations in the radial basis function interpolation.

For a set of function values $\{f(\mathbf{v}_j)\}_{1\le j\le N}$ taken on a set of nodes $\Xi=\{\mathbf{v}_j\}_{1\le j\le N}\subset\mathbb{R}^d$, the form of quasi-interpolation function $\Phi(f;\mathbf{v})$ corresponding to $f(\mathbf{v})$ is

$$\Phi(f;\mathbf{v})=\sum_{j=1}^{N} f(\mathbf{v}_j)\varphi_j(\mathbf{v}),$$

where $\{\varphi_j\}_{1\le j\le N}$ is a set of quasi-interpolation basis functions. The set of nodes $\{\mathbf{v}_j\}_{1\le j\le N}$ usually has two kinds: the uniform grid node set and the scattered node set. The standard quasi-interpolant based on the uniform grid node set in $\mathbb{Z}^d$ is

$$\sum_{j\in\mathbb{Z}^d} f(jh)\varphi_{j,h}(\mathbf{v}), \tag{1.1}$$

in which Schoenberg model [5]

$$\sum f(jh)\Phi(\frac{\mathbf{v}}{h}-j)\sim f(\mathbf{v}),\quad \mathbf{v}\in\mathbb{R}^d \tag{1.2}$$

has attracted the most attention. Quasi-interpolant (1.2) can be studied via the theory of principal shift-invariant spaces, which has been developed in several articles by de Boor et al. [6,7]. Strang and Fix [8] also give a necessary and sufficient condition for the convergence of such a standard form of quasi-interpolant. The quasi-interpolants based on the uniform grid node set, have been applying in the numerical integration, the numerical solution of integral equation and the differential equation [9,10]. The quasi-interpolant (1.1) is based on the values of $f(\mathbf{v})$ in the uniform grid node set, which limits its range of application. For example, the above mentioned large-scale scattered data approximation, the numerical solution of integral equation and differential equation which are based on the non-uniform grid subdivision, and other solving problems. These problems can be solved, relying on the quasi-interpolants based on the scattered node set. The construction of the quasi-interpolants based on the high dimensional scattered node set, is firstly studied by Dyn and Ron [11]. They proposed the general idea about extending the quasi-interpolant based on the uniform grid node set to the scattered node set. Buhmann et al. [12] extended the scheme based on the uniform node set in [13] to the quasi-uniform distribution of the infinite scattered node set. By constructing the suitable "bell shape" basis function and using the convolution equation, Yoon [14] gives an integral form of the quasi-interpolant which is based on the scattered node set. The constructed quasi-interpolants based on the scattered node set in these papers not only need the function information at the scattered node but also need the function information at uniform node or all the information of the approximated function. This still limits the application of these methods. Wu and Liu [15] use the generalized Strang-Fix condition which is related to non-stationary quasi-interpolation, to extend their constructed quasi-interpolant based

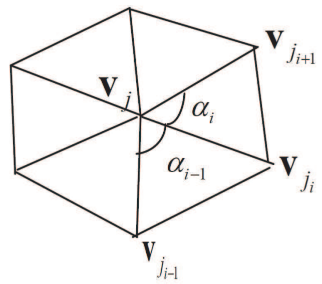

Figure 1: The polygon Ψ_j enclosing node $\mathbf{v}_j$.

on the uniform grid node set to the complete scattered node set. It noted that they do not give a practical and operational form.

In the paper we construct a piecewise smooth rational quasi-interpolant to approximate scattered data in $\mathbb{R}^3$, which only needs the information about the approximated function values or the approximated function values and derivative values at the scattered nodes. The mean value coordinates interpolation is an interpolation which is based on planar polygons and has algebraic accuracy of degree one. We firstly use the modified Taylor expansion to expand the mean value coordinates interpolation to an interpolation with algebraic accuracy of any degree. Then based on the triangulation of the scattered nodes in $\mathbb{R}^2$, on each triangle we construct a rational quasi-interpolation function, which approximates the scattered data in $\mathbb{R}^3$. The constructed rational quasi-interpolation function is a linear combination of three different expanded mean value coordinates interpolations with algebraic accuracy of degree $(n+1)$ and it also has algebraic accuracy of degree $(n+1)$. In the paper we analyse its algebraic accuracy, approximation error and smoothness. The advantage of the piecewise rational quasi-interpolation is that it is completely based on the scattered node set. It is a practical, easy-operation, high-precision and stable local quasi-interpolant. Details of the paper are shown in the subsequent sections.

2. The rational quasi-interpolation function with algebraic accuracy of degree $(n+1)$ on triangle

Given a scattered data set $\Xi = \{(x_j, y_j, f(x_j, y_j))\}_{j=1}^N \subset \mathbb{R}^3$, it can be seen as sampling on a bivariate function $f(x,y)$. The projection point set $\{\mathbf{v}_j = (x_j, y_j)\}_{j=1}^N$ of Ξ onto the two-dimensional plane $\mathbb{R}^2$, is called site set or node set. Here, we use Ω to denote the convex hull of the node set and T is the Delaunay Triangulation of Ω. Then, any node $\mathbf{v}_j$ in the node set $\{\mathbf{v}_j\}_{j=1}^N$ must be a common vertex of some triangles, which locates either on the boundary of T or inside the interior of T. Now, we might as well suppose that $\mathbf{v}_j$ is an interior node of T and the node set $\{\mathbf{v}_{j_i} = (x_{j_i}, y_{j_i})\}_{i=1}^k$ are all other vertices of all triangles with node $\mathbf{v}_j$ as its vertex. The node set is arranged counter-clock and it constructs a polygon enclosing $\mathbf{v}_j$, written as Ψ_j, see Fig. 1.

2.1. Mean value coordinates

Mean value coordinates were firstly proposed by Floater in [18], for the purpose of generalizing the area coordinates of plane triangle. Mean value coordinates have been used in the computer graphics, the finite element and other fields [19-21]. The idea is that node $\mathbf{v}_j$ in the plane polygon Ψ_j can be represented as a linear combination as follows

$$\mathbf{v}_j = \sum_{i=1}^{k}(\lambda_i \cdot \mathbf{v}_{j_i}), \qquad \sum_{i=1}^{k} \lambda_i = 1, \tag{2.1a}$$

$$\lambda_i = \frac{\omega_i}{\sum_{l=1}^{k} \omega_l}, \qquad \omega_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{\| \mathbf{v}_{j_i} - \mathbf{v}_j \|}. \tag{2.1b}$$

The coefficient $\{\lambda_i\}_{i=1}^{k}$ in (2.1a) is called mean value coordinates. Angles α_{i-1} and α_i in (2.1b) can be seen in Fig. 1, and $\| \cdot \|$ is Euclidean distance. We can define a value $\sum_{i=1}^{k}(\lambda_i \cdot f(\mathbf{v}_{j_i}))$ with coefficients in(2.1), it can be seen as a approximation of the value $f(\mathbf{v}_j)$. Now, we fix the vertices $\{\mathbf{v}_{j_1}, \cdots, \mathbf{v}_{j_k}\}$ of polygon Ψ_j unchanged, and let $\mathbf{v}_j$ move on the Ψ_j. We might as well write $\mathbf{v} = (x, y)$, then (2.1) is still set up, namely

$$\mathbf{v} = \sum_{i=1}^{k}(\lambda_i(x, y) \cdot \mathbf{v}_{j_i}). \tag{2.2}$$

The mean value coordinates λ_i, $i = 1, \cdots, k$ are associated with (x, y), which are rational functions. The mean value coordinates $\lambda_i(x, y)$ have the following properties [19]:

(i) Affine precision: $\sum_{i=1}^{k} \lambda_i \varphi(\mathbf{v}_{j_i}) = \varphi$, for any affine function, $\varphi : \mathbb{R}^2 \to \mathbb{R}^d$;

(ii) Lagrange property: $\lambda_i(\mathbf{v}_j) = \delta_{i,j}$;

(iii) Smoothness: λ_i is C^∞, everywhere, except at the vertices $\mathbf{v}_j$, where it is only C^0;

(iv) Partition of unity: $\sum_{i=1}^{k} \lambda_i \equiv 1$;

(v) Linear independence: if $\sum_{i=1}^{k} c_i \lambda_i(\mathbf{v}) = 0$, for all $\mathbf{v} \in \mathbb{R}^2$, then all c_i must be zero;

(vi) Edge property: λ_i is linear along the edges e_j of Ψ_j;

(vii) Positivity: λ_i is positive inside the kernel of star-shaped polygons, in particular, inside convex polygons.

We define a function on Ψ_j with the set of variable coefficients $r_j(x, y) = \sum_{i=1}^{k} \big(\lambda_i(x, y) \cdot f(\mathbf{v}_{j_i})\big)$. According to the Lagrange property of λ_i, we know that $r_j(x, y)$ has interpolation property at $\mathbf{v}_{j_i}, i = 1, \cdots, k$, which is called mean value coordinates interpolation in [19]. It is a rational approximation of function $f(x, y)$ on Ψ_j, and it has been used for Phong Shading, Image Warping and Transfinite Interpolation. R. Alexander etc. have presented the interpolation error estimation for mean value coordinates over the convex polygon in [20].

The mean value coordinates (2.1) by M. S. Floater is not stable enough in geometric computation. Because when $\mathbf{v}_j$ is distributed near $\mathbf{v}_{j_i}$, it will lead to the denominator $\|\mathbf{v}_j - \mathbf{v}_{j_i}\| \to 0$. Feng and Zhao [21] give an equivalent but robust mean value coordinates. Mark s_i and c_i are the signed area of $\Delta_{\mathbf{v}_j\mathbf{v}_{j_{i-1}}\mathbf{v}_{j_i}}$ and $\Delta_{\mathbf{v}_j\mathbf{v}_{j_{i-1}}\mathbf{v}_{j_{i+1}}}$ respectively, l_i is $\|\mathbf{v}_j - \mathbf{v}_{j_i}\|$. Then the homogeneous mean value coordinates given by Feng and Zhao are

$$\omega_i = \frac{1}{2}(l_{i+1}s_i - l_i c_i + l_{i-1}s_{i+1}) \prod_{j\neq i,\, i+1} s_j.$$

2.2. Mean value coordinates interpolation with algebraic accuracy of degree $(n+1)$ on polygon

According to the properties of $\lambda_i(x,y)$, we know that $r_j(x,y)$ only has affine precision. Then how to improve the approximation accuracy of $r_j(x,y)$ to $f(x,y)$ in the case that doesn't add new nodes? In the paper we will use the derivative value of the approximated function $f(x,y)$ at each node to expand $r_j(x,y)$ in order to make it obtain higher algebraic accuracy. Thereby, we achieve the improvement of approximation accuracy of $r_j(x,y)$ to $f(x,y)$.

Suppose that $f:\mathbb{R}^2 \to \mathbb{R}$ is C^{n+1} in a neighbourhood of Ω and $\mathbf{v}=(x,y)\in\Omega$. For each $\mathbf{a}(a_1,a_2)\in\Omega$, suppose that $\mathrm{T}^n_{\mathbf{a}}f(\mathbf{v})$ is the degree n Taylor expansion of f at $\mathbf{a}$, and $\mathrm{R}^n_{\mathbf{a}}f(\mathbf{v})$ is the remainder of $\mathrm{T}^n_{\mathbf{a}}f(\mathbf{v})$. We use the expression of $\mathrm{T}^n_{\mathbf{a}}f(\mathbf{v})$ in [22]

$$\mathrm{T}^n_{\mathbf{a}}f(\mathbf{v}) = f(\mathbf{a}) + [(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]f(\mathbf{u})\,|_{\mathbf{u}=\mathbf{a}} + \cdots + \frac{1}{n!}[(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^n f(\mathbf{u})\,|_{\mathbf{u}=\mathbf{a}},$$

where $\bigtriangledown = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is the gradient operator.

To simplify the formula, we use $([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^n f)(\mathbf{a})$ to replace the term $[(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^n\cdot f(\mathbf{u})\,|_{\mathbf{u}=\mathbf{a}}$. Thus the degree n Taylor polynomial expansion at $\mathbf{a}$ changes into

$$\mathrm{T}^n_{\mathbf{a}}f(\mathbf{v}) = \sum_{j=0}^{n}\frac{1}{j!}([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^j f)(\mathbf{a}).$$

The remainder changes into

$$\mathrm{R}^n_{\mathbf{a}}f(\mathbf{v}) = \frac{1}{(n+1)!}\Big([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^{n+1}f\Big)(\mathbf{z}),$$

where

$$\mathbf{z} = \mathbf{a} + \theta(\mathbf{v}-\mathbf{a}) = (a_1+\theta(x-a_1),\, a_2+\theta(y-a_2)), \quad 0\le\theta\le 1$$

is a point on the line segment joining point $\mathbf{a}$ and point $\mathbf{v}$.

Lemma 2.1. *Let f be $n+1$ times continuously differentiable in a neighborhood of $\mathbf{a}+\theta(\mathbf{v}-\mathbf{a})$. Then*

$$
\begin{aligned}
&[(\mathbf{v}-\mathbf{a})\cdot\triangledown]([[(\mathbf{v}-\mathbf{a})\cdot\triangledown]^n f)(\mathbf{z})\\
=&n\big([[(\mathbf{v}-\mathbf{a})\cdot\triangledown]^n f\big)(\mathbf{z})+\big(\theta+\theta_x(x-a_1)\big)\big([[(\mathbf{v}-\mathbf{a})\cdot\triangledown]^{n+1} f\big)(\mathbf{z})\\
&+\Big(\theta_y(y-a_2)-\theta_x(x-a_1)\Big)\sum_{i=0}^{n+1}\frac{i}{n+1}\binom{n+1}{i}(x-a_1)^{n+1-i}(y-a_2)^i\frac{\partial^{n+1}f}{\partial x^{n+1-i}\partial y^i}(\mathbf{z}),
\end{aligned}
$$

where θ_x, θ_y are the partial functions of θ with respect to x, y respectively.

Proof. Note

$$
\begin{aligned}
&(\mathbf{v}-\mathbf{a})\cdot\triangledown=(x-a_1)\frac{\partial}{\partial x}+(y-a_2)\frac{\partial}{\partial y},\\
&([[(\mathbf{v}-\mathbf{a})\cdot\triangledown]^n f)(\mathbf{z})=\sum_{i=0}^{n}\binom{n}{i}(x-a_1)^{n-i}(y-a_2)^i\frac{\partial^n f}{\partial x^{n-i}y^i}(\mathbf{z}).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&[(\mathbf{v}-\mathbf{a})\cdot\triangledown]([[(\mathbf{v}-\mathbf{a})\cdot\triangledown]^n f)(\mathbf{z})\\
=&\left[(x-a_1)\frac{\partial}{\partial x}+(y-a_2)\frac{\partial}{\partial y}\right]\left[\sum_{i=0}^{n}\binom{n}{i}(x-a_1)^{n-i}(y-a_2)^i\frac{\partial^n f}{\partial x^{n-i}y^i}(\mathbf{z})\right]\\
=&\sum_{i=0}^{n}\binom{n}{i}\bigg[(n-i)(x-a_1)^{n-i}(y-a_2)^i\frac{\partial^n f}{\partial x^{n-i}y^i}(\mathbf{z})+\theta(x-a_1)^{n+1-i}(y-a_2)^i\\
&\cdot\frac{\partial^{n+1}f}{\partial x^{n+1-i}\partial y^i}(\mathbf{z})+\theta_x(x-a_1)^{n+2-i}(y-a_2)^i\frac{\partial^{n+1}f}{\partial x^{n+1-i}\partial y^i}(\mathbf{z})\bigg]+\sum_{i=0}^{n}\binom{n}{i}\\
&\cdot\bigg[i(x-a_1)^{n-i}(y-a_2)^i\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(\mathbf{z})+\theta(x-a_1)^{n-i}(y-a_2)^{i+1}\frac{\partial^{n+1}f}{\partial x^{n-i}y^{i+1}}(\mathbf{z})\\
&+\theta_y(x-a_1)^{n-i}(y-a_2)^{i+2}\frac{\partial^{n+1}f}{\partial x^{n-i}\partial y^{i+1}}(\mathbf{z})\bigg]\\
=&n\sum_{i=0}^{n}\binom{n}{i}(x-a_1)^{n-i}(y-a_2)^i\frac{\partial^n f}{\partial x^{n-i}y^i}(\mathbf{z})+\theta\sum_{i=0}^{n}\bigg[\frac{n+1-i}{n+1}\binom{n+1}{i}\\
&\cdot(x-a_1)^{n+1-i}(y-a_2)^i\frac{\partial^{n+1}f}{\partial x^{n+1-i}\partial y^i}(\mathbf{z})+\frac{i+1}{n+1}\binom{n+1}{i+1}(x-a_1)^{(n+1)-(i+1)}\bigg]\\
&\cdot(y-a_2)^i\frac{\partial^{n+1}f}{\partial x^{n+1-i}\partial y^i}(\mathbf{z})+\theta_x(x-a_1)\sum_{i=0}^{n}\frac{n+1-i}{n+1}\binom{n+1}{i}(x-a_1)^{n+1-i}\\
&+\theta_y(y-a_2)\sum_{i=0}^{n}\frac{i+1}{n+1}\binom{n+1}{i+1}(x-a_1)^{(n+1)-(i+1)}(y-a_2)^{i+1}\frac{\partial^{n+1}f}{\partial x^{(n+1)-(i+1)}\partial y^{i+1}}(\mathbf{z})\\
=&n\big([[(\mathbf{v}-\mathbf{a})\cdot\triangledown]^n f\big)(\mathbf{z})+\big(\theta+\theta_x(x-a_1)\big)\big([[(\mathbf{v}-\mathbf{a})\cdot\triangledown]^{n+1} f\big)(\mathbf{z})+\big(\theta_y(y-a_2)\\
&-\theta_x(x-a_1)\big)\sum_{i=0}^{n+1}\frac{i}{n+1}\binom{n+1}{i}(x-a_1)^{n+1-i}(y-a_2)^i\frac{\partial^{n+1}f}{\partial x^{n+1-i}\partial y^i}(\mathbf{z}).
\end{aligned}
$$

The proof is completed. □

According to Lemma 2.1 with $\theta = 0$, we have

$$[(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]\mathrm{T}_{\mathbf{a}}^{n+1}f(\mathbf{v}) = \sum_{j=0}^{n}\frac{1}{j!}\big([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^{j+1}f\big)(\mathbf{a}). \tag{2.3}$$

Again it follow from Lemma 2.1, we also can get

$$\begin{aligned}
&\big[(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown\big]f(\mathbf{v}) - \big[(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown\big]\mathrm{T}_{\mathbf{a}}^{n+1}f(\mathbf{v})\\
=&\big[(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown\big]\mathrm{R}_{\mathbf{a}}^{n+1}f(\mathbf{v})\\
=&\big[(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown\big]\frac{1}{(n+2)!}\big([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^{n+2}f\big)(\mathbf{z})\\
=&(n+2)\mathbf{R}_{\mathbf{a}}^{n+1}f(\mathbf{v}) + (n+3)\big(\theta+\theta_x(x-a_1)\big)\mathbf{R}_{\mathbf{a}}^{n+2}f(\mathbf{v})\\
&\quad+\frac{1}{(n+2)!}(\theta_y(y-a_2)-\theta_x(x-a_1))\sum_{i=0}^{n+3}\frac{i}{n+3}\binom{n+3}{i}\\
&\quad\cdot(x-a_1)^{n+3-i}(y-a_2)^i\frac{\partial^{n+3}f}{\partial x^{n+3-i}\partial y^i}(\mathbf{z}).
\end{aligned}\tag{2.4}$$

Definition 2.1. *For* $\mathbf{a}\in\mathbb{R}^2$, *let* $\mathrm{L}_{\mathbf{a}}^n$ *be the linear mapping given by*

$$\mathrm{L}_{\mathbf{a}}^n f(\mathbf{v}) = \sum_{j=0}^{n}\frac{n+1-j}{n+1}\frac{1}{j!}\big([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^{j}f\big)(\mathbf{a}).$$

Lemma 2.2.

$$\mathrm{T}_{\mathbf{a}}^{n+1}f(\mathbf{v}) - \mathrm{L}_{\mathbf{a}}^n f(\mathbf{v}) = \frac{1}{n+1}[(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]\mathrm{T}_{\mathbf{a}}^{n+1}f(\mathbf{v}). \tag{2.5}$$

Proof. Note that

$$\begin{aligned}
&\mathrm{T}_{\mathbf{a}}^{n+1}f(\mathbf{v}) - \mathrm{L}_{\mathbf{a}}^n f(\mathbf{v})\\
=&\sum_{j=0}^{n+1}\frac{1}{j!}\big([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^{j}f\big)(\mathbf{a}) - \sum_{j=0}^{n+1}\frac{n+1-j}{n+1}\frac{1}{j!}\big([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^{j}f\big)(\mathbf{a})\\
=&\frac{1}{n+1}\sum_{j=1}^{n+1}\frac{1}{(j-1)!}\big([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^{j}f\big)(\mathbf{a})\\
=&\frac{1}{n+1}\sum_{j=0}^{n}\frac{1}{j!}\big([(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]^{j+1}f\big)(\mathbf{a})\\
=&\frac{1}{n+1}[(\mathbf{v}-\mathbf{a})\cdot\bigtriangledown]\mathrm{T}_{\mathbf{a}}^{n+1}f(\mathbf{v}). \quad (by\ (2.3))
\end{aligned}$$

The proof is completed. □

Definition 2.2. *The interpolation* I_j^n *is defined as follows*

$$\mathrm{I}_j^n(\mathbf{v})=\sum_{i=1}^{k}\mathrm{L}_{\mathbf{v}_{j_i}}^n f(\mathbf{v})\lambda_i(\mathbf{v}),\quad \mathbf{v}\in\Psi_j. \tag{2.6}$$

Theorem 2.1. *If f is a polynomial with degree $\le n+1$, then we have $\mathrm{I}_j^n(\mathbf{v})\equiv f(\mathbf{v})$ on Ω.*

Proof. According to the property (iv) of $\lambda_i(x,y)$ and Eq. (2.3), we have

$$\sum_{i=1}^{k}(\mathbf{v}-\mathbf{v}_{j_i})\lambda_i(\mathbf{v})=\mathbf{0}.$$

Then we get

$$\sum_{i=1}^{k}\Big([(\mathbf{v}-\mathbf{v}_{j_i})\cdot\bigtriangledown]f(\mathbf{v})\Big)\lambda_i(\mathbf{v})=0. \tag{2.7}$$

Consequently, we have

$$\begin{aligned}
f(\mathbf{v})-\mathrm{I}_j^n(\mathbf{v})&=\sum_{i=1}^{k}\big(f(\mathbf{v})-\mathrm{L}_{\mathbf{v}_{j_i}}^n f(\mathbf{v})\big)\lambda_i(\mathbf{v})\\
&=\sum_{i=1}^{k}\Big(f(\mathbf{v})-\mathrm{T}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})+\mathrm{T}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})-\mathrm{L}_{\mathbf{v}_{j_i}}^n f(\mathbf{v})\Big)\lambda_i(\mathbf{v})\\
&=\sum_{i=1}^{k}\Big(\mathrm{R}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})+\big(\mathrm{T}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})-\mathrm{L}_{\mathbf{v}_{j_i}}^n f(\mathbf{v})\big)-\frac{1}{n+1}[(\mathbf{v}-\mathbf{v}_{j_i})\cdot\bigtriangledown]f(\mathbf{v})\Big)\lambda_i(\mathbf{v})\\
&=\sum_{i=1}^{k}\Big(\mathrm{R}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})+\frac{1}{n+1}[(\mathbf{v}-\mathbf{v}_{j_i})\cdot\bigtriangledown]\mathrm{T}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})-\frac{1}{n+1}[(\mathbf{v}\\
&\qquad-\mathbf{v}_{j_i})\cdot\bigtriangledown]f(\mathbf{v})\Big)\lambda_i(\mathbf{v})\quad (by\ \text{Lemma 2.2})\\
&=-\frac{1}{n+1}\sum_{i=1}^{k}\mathrm{R}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})\lambda_i(\mathbf{v})-\frac{n+3}{n+1}\sum_{i=1}^{k}\big(\theta+\theta_x(x-a_{1,j_i})\big)\mathbf{R}_{\mathbf{v}_{j_i}}^{n+2}f(\mathbf{v})\lambda_i(\mathbf{v})\\
&\qquad-\frac{1}{(n+1)(n+2)!}\sum_{i=1}^{k}\Bigg(\big(\theta_y(y-a_{2,j_i})-\theta_x(x-a_{1,j_i})\big)\sum_{l=0}^{n+3}\frac{l}{n+3}\\
&\qquad\cdot\binom{n+3}{l}(x-a_{1,j_i})^{n+3-l}(y-a_{2,j_i})^l\frac{\partial^{n+3}f}{\partial x^{n+3-l}\partial y^l}(\mathbf{z})\Bigg)\lambda_i(\mathbf{v}).\quad (by\ (2.4))
\end{aligned}$$

In the case that $f(\mathbf{v})$ is a polynomial with degree $\le n+1$, we know that

$$\mathrm{R}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})=\mathrm{R}_{\mathbf{v}_{j_i}}^{n+2}f(\mathbf{v})\equiv 0,\qquad \frac{\partial^{n+3}f}{\partial x^{n+3-l}\partial y^l}(\mathbf{v})=0.$$

Then we have $f(\mathbf{v})-\mathrm{I}_j^n(\mathbf{v})\equiv 0$. The proof is completed. $\square$

Remark 2.1. The result of Theorem 2.1 illustrates $\mathrm{I}_j^n(x,y)$ has algebraic accuracy of degree $(n+1)$ on Ω.

Remark 2.2. According to the property (iii) of $\lambda_i(x,y)$ we know interpolation function $\mathrm{I}_j^n(x,y)$ is C^∞ in $\mathbb{R}^2$ everywhere, except at the vertices $\mathbf{v}_{j_i}$ of Ψ_j, where it is only C^0.

Theorem 2.2. *Suppose f is C^{n+3} in a neighbourhood of Ω ,θ is C^1 in Ψ_j and h_j is the minimum size of radius of the circle which contains the polygon Ψ_j. Let*

$$
\begin{aligned}
&M_{3,l}=\max_{\mathbf{v}\in\Omega}\left|\frac{\partial^{n+3}f}{\partial x^{n+3-l}\partial y^l}(\mathbf{v})\right|, \quad l=0,\cdots,n+3,\\
&M_{2,l}=\max_{\mathbf{v}\in\Omega}\left|\frac{\partial^{n+2}f}{\partial x^{n+2-l}\partial y^l}(\mathbf{v})\right|, \quad l=0,\cdots,n+2,\\
&M=\max_l\{M_{2,l},\ M_{3,l}\} \ \ \textit{and} \ \ \Theta=\max\{\max_{\mathbf{v}\in\Psi_j}|\theta(\mathbf{v})|,\ \max_{\mathbf{v}\in\Psi_j}|\theta_x(\mathbf{v})|,\ \max_{\mathbf{v}\in\Psi_j}|\theta_y(\mathbf{v})|\}.
\end{aligned}
$$

Then

$$
\left|f(\mathbf{v})-\mathrm{I}_j^n(\mathbf{v})\right|\le\frac{M(1+\Theta(1+4h_j)4h_j)}{(n+1)(n+2)!}(4h_j)^{n+2}\sum_{i=1}^k|\lambda_i(\mathbf{v})|, \quad \mathbf{v}\in\Psi_j. \tag{2.8}
$$

Proof. Note that

$$
\begin{aligned}
\left|\mathbf{R}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})\right|&=\frac{1}{(n+2)!}\left|\sum_{l=0}^{n+2}\binom{n+2}{l}(x-a_{1,j_i})^{n+2-l}(y-a_{2,j_i})^l\frac{\partial^{n+2}f}{\partial x^{n+2-l}\partial y^l}(\mathbf{z})\right|\\
&\le\frac{M}{(n+2)!}(4h_j)^{n+2},\\
\left|\mathbf{R}_{\mathbf{v}_{j_i}}^{n+2}f(\mathbf{v})\right|&\le\frac{M}{(n+3)!}(4h_j)^{n+3}.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\left|-\frac{1}{n+1}\sum_{i=1}^k\mathbf{R}_{\mathbf{v}_{j_i}}^{n+1}f(\mathbf{v})\lambda_i(\mathbf{v})\right|\\
\le&\frac{1}{n+1}\sum_{i=1}^k\frac{M}{(n+2)!}(4h_j)^{n+2}|\lambda_i(\mathbf{v})|=\frac{M(4h_j)^{n+2}}{(n+1)(n+2)!}\sum_{i=1}^k|\lambda_i(\mathbf{v})|, \qquad (2.9a)\\
&\left|-\frac{n+3}{n+1}\sum_{i=1}^k\left(\theta+\theta_x(x-a_{1,j_i})\right)\mathbf{R}_{\mathbf{v}_{j_i}}^{n+2}f(\mathbf{v})\lambda_i(\mathbf{v})\right|\\
\le&\frac{M\Theta}{(n+1)(n+2)!}(1+2h_j)(4h_j)^{n+3}\sum_{i=1}^k|\lambda_i(\mathbf{v})|, \qquad (2.9b)
\end{aligned}
$$

$$\Big| -\frac{1}{(n+1)(n+2)!}\sum_{i=1}^{k}\Big((\theta_y(y-a_{2,j_i})-\theta_x(x-a_{1,j_i})\Big)$$
$$\sum_{l=0}^{n+3}\frac{l}{n+3}\binom{n+3}{l}(x-a_{1,j_i})^{n+3-l}\cdot(y-a_{2,j_i})^l\frac{\partial^{n+3}f}{\partial x^{n+3-l}\partial y^l}(\mathbf{z})\Big)\lambda_i(\mathbf{v})\Big|$$
$$\leq\frac{M\Theta}{(n+1)(n+2)!}(2h_j)(4h_j)^{n+3}\sum_{i=1}^{k}|\lambda_i(\mathbf{v})|. \tag{2.9c}$$

According to the above three inequalities and the proof of Theorem 2.1, we have

$$\begin{aligned}|f(\mathbf{v})-\mathrm{I}_j^n(\mathbf{v})|\leq&\frac{M(4h_j)^{n+2}}{(n+1)(n+2)!}\sum_{i=1}^{k}|\lambda_i(\mathbf{v})|+\frac{M\Theta}{(n+1)(n+2)!}(1+2h_j)(4h_j)^{n+3}\\&\cdot\sum_{i=1}^{k}|\lambda_i(\mathbf{v})|+\frac{M\Theta}{(n+1)(n+2)!}(2h_j)(4h_j)^{n+3}\sum_{i=1}^{k}|\lambda_i(\mathbf{v})|\\=&\frac{M(1+\Theta(1+4h_j)4h_j)}{(n+1)(n+2)!}(4h_j)^{n+2}\sum_{i=1}^{k}|\lambda_i(\mathbf{v})|.\end{aligned}\tag{2.10}$$

The proof is completed. □

Remark 2.3. The expression $\sum_{i=1}^{k}|\lambda_i(\mathbf{v})|$ in the right side of inequality (2.10) that determine the convergence rate can be regarded as the Lebesgue constant of interpolation $I_j^n(\mathbf{v})$. When the polygon Ψ_j is a convex polygon, there is $\sum_{i=1}^{k}|\lambda_i(\mathbf{v})|=\sum_{i=1}^{k}\lambda_i(\mathbf{v})=1$.

Remark 2.4. $\mathrm{I}_j^n(\mathbf{v})$ $(n\geq 1)$ is a generalization of $r_j(\mathbf{v})$, called as expanded mean value coordinates interpolation. Compared with the approximation accuracy of $r_j(\mathbf{v})$ on Ψ_j(see [20]), the approximation accuracy of $\mathrm{I}_j^n(\mathbf{v})$ has improved.

2.3. Quasi-interpolation function with algebraic accuracy of degree $(n+1)$ on triangles in T

Now, we suppose $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ are three adjacent interior nodes of Delaunay triangulation T and three vertices of some triangle $\triangle_{123}$ in T. Ψ_1,Ψ_2,Ψ_3 are three polygons enclosing points $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ respectively, $\mathrm{I}_1^n(x,y),\mathrm{I}_2^n(x,y),\mathrm{I}_3^n(x,y)$ are the expanded mean value coordinate interpolation functions on polygons Ψ_1,Ψ_2,Ψ_3 respectively, see Fig. 2. We define a function on $\triangle_{123}$

$$Q_{123}^n(x,y)=\alpha_1(x,y)\mathrm{I}_1^n(x,y)+\alpha_2(x,y)\mathrm{I}_2^n(x,y)+\alpha_3(x,y)\mathrm{I}_3^n(x,y), \tag{2.11}$$

where $\alpha_1(x,y),\alpha_2(x,y),\alpha_3(x,y)$ are basis functions of the bivariate linear Lagrange interpolation on $\triangle_{123}$

$$\alpha_1(x,y)=\frac{(y-y_2)(x_3-x_2)-(x-x_2)(y_3-y_2)}{(y_1-y_2)(x_3-x_2)-(x_1-x_2)(y_3-y_2)},$$

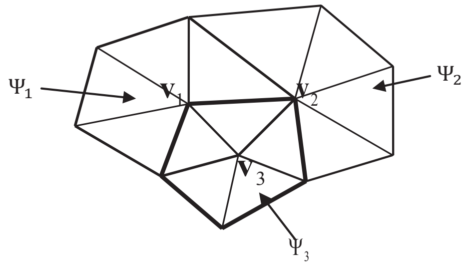

Figure 2: Three vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of a triangle included in triangulation T.

$$\alpha_2(x,y) = \frac{(y-y_1)(x_3-x_1)-(x-x_1)(y_3-y_1)}{(y_2-y_1)(x_3-x_1)-(x_2-x_1)(y_3-y_1)},$$
$$\alpha_3(x,y) = \frac{(y-y_1)(x_2-x_1)-(x-x_1)(y_2-y_1)}{(y_3-y_1)(x_2-x_1)-(x_3-x_1)(y_2-y_1)}.$$

Theorem 2.3. *If f is a polynomial degree $\leq n+1$, then we have $Q^n_{123}(\mathbf{v}) \equiv f(\mathbf{v})$ on Ω.*

Proof. According to the result of Theorem 2.1, we have $\mathrm{I}^n_1(\mathbf{v}) \equiv \mathrm{I}^n_2(\mathbf{v}) \equiv \mathrm{I}^n_3(\mathbf{v}) \equiv f(\mathbf{v})$, $\mathbf{v} \in \Omega$, when $f(\mathbf{v})$ is a polynomial with degree $\leq n+1$. Because $\triangle_{123}$ is the intersection of three adjacent polygons Ψ_1, Ψ_2, Ψ_3, so we get

$$Q^n_{123}(x,y) = (\alpha_1(x,y)+\alpha_2(x,y)+\alpha_3(x,y)) \cdot f(x,y) = f(x,y), \quad (x,y) \in \Omega.$$

The proof is completed. □

For $Q^n_{123}(x,y)$, we have the following error estimate.

Theorem 2.4. *Suppose that in a neighbourhood of Ω, f is C^{n+3}, then we have*

$$Q^n_{123}(\mathbf{v}) - f(\mathbf{v}) = \mathcal{O}(h^{n+2}), \quad \mathbf{v} \in \triangle_{123}, \quad where \ \mathrm{h} = \max_{\mathrm{j}} \mathrm{h_j}. \tag{2.12}$$

According to Theorem 2.2, the conclusion is obvious, so the process of proof is omitted.

Remark 2.5. If $\mathbf{v}_i, i = 1,2,3$ are the interior nodes of T, then, in general,

$$Q^n_{123}(\mathbf{v}_i) = \alpha_1(\mathbf{v}_i)\mathrm{I}^n_1(\mathbf{v}_i) + \alpha_2(\mathbf{v}_i)\mathrm{I}^n_2(\mathbf{v}_i) + \alpha_3(\mathbf{v}_i)\mathrm{I}^n_3(\mathbf{v}_i) = \mathrm{I}^n_i(\mathbf{v}_i) \neq f(\mathbf{v}_i).$$

This illustrates Q^n_{123} does not interpolate the function value $f(\mathbf{v}_i), i = 1,2,3$, at nodes $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, in general. Thus,$Q_{123}$ is a rational quasi-interpolation function.

Remark 2.6. Q^n_{123} is C^∞ everywhere on Ω, except at the vertices of the polygons Ψ_1, Ψ_2, Ψ_3, where it is only C^0.

3. The construction of piecewise rational approximation function with algebraic accuracy of degree $(n+1)$ on the bounded domain Ω

The function $Q_f^n(\mathbf{v})$, which approximates to the scattered data $\Xi \subset \Omega$ in $\mathbb{R}^3$ to be constructed, will have different expressions with different locations of point $\mathbf{v}(x,y) \in \Omega$. We divide the location of point $\mathbf{v}$ on Ω into two types according to the triangulation T. Thereby $Q_f^n(\mathbf{v})$ has two different expressions, and the concrete expressions are as follows:

(1) If point $\mathbf{v}(x,y)$ lies on a triangle, which is an interior triangle of T, or to say, vertices of the triangle are interior nodes of T, see Fig. 2. Then we assign the value of $Q_{123}^n(x,y)$ at $\mathbf{v}$ to the approximate function $Q_f^n(x,y)$, namely, $Q_f^n(x,y) = Q_{123}^n(x,y)$;

(2) If point $\mathbf{v}(x,y)$ lies on a triangle $\widehat{\triangle}_{123}$ close to the boundary of T, see Fig. 3, there is at least one vertex of $\widehat{\triangle}_{123}$ lying on the boundary of T. For example, the vertex $\mathbf{v}_3$ in Fig. 3 is a boundary node, and there doesn't exist a polygon enclosing it. But from Fig. 3, we see that point $\mathbf{v}$ lies in the polygon $\{\mathbf{v}_4, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_4\}$, so we might as well write the polygon as Ψ_3. Then we construct an expanded mean value coordinates interpolation function according to the method in Section 2, denoted by $\widetilde{\mathrm{I}}_3^n(x,y)$. Then at $\mathbf{v}$, we use the following function value

$$\widetilde{Q}_{123}^n(x,y) = \alpha_1(x,y)\cdot \mathrm{I}_1^n(x,y) + \alpha_2(x,y)\cdot \mathrm{I}_2^n(x,y) + \alpha_3(x,y)\cdot \widetilde{\mathrm{I}}_3^n(x,y) \tag{3.1}$$

to the approximate function $Q_f^n(x,y)$. If point $\mathbf{v}$ lies on $\widehat{\triangle}_{253}$ close to the boundary of T, see Fig. 3, then the approximation value at $\mathbf{v}$ is

$$\widetilde{Q}_{253}^n(x,y) = \alpha_2(x,y)\cdot \mathrm{I}_2^n(x,y) + \alpha_5(x,y)\cdot \widetilde{\mathrm{I}}_5^n(x,y) + \alpha_3(x,y)\cdot \widetilde{\mathrm{I}}_3^n(x,y), \tag{3.2}$$

where the construction of expanded mean value coordinates interpolation function $\widetilde{\mathrm{I}}_5^n(x,y)$ is similar to $\widetilde{\mathrm{I}}_3^n(x,y)$ in (3.1). If the three vertices of $\widehat{\triangle}_{123}$ are all boundary nodes of T, then all the expanded mean value coordinates functions in the approximation value at $\mathbf{v}$ are constructed as $\widetilde{\mathrm{I}}_3^n(x,y)$ in (3.1).

Based on the above construction idea, the algorithm to generate approximating the sampling function $f(x,y)$ for a given set of the scattered data Ξ in $\mathbb{R}^3$ is summarized as follows.

From the above algorithm, we obtain an approximation function of $f(x,y)$ on Ω

$$Q_f^n(x,y) = \begin{cases} Q_{123}^n(x,y), & (x,y) \in \triangle_{123} \subset \mathrm{T}, \\ \widetilde{Q}_{123}^n(x,y), & (x,y) \in \widehat{\triangle}_{123} \subset \mathrm{T}. \end{cases} \tag{3.3}$$

Remark 3.1. The approximation order of $Q_f^n(x,y)$ to $f(x,y)$, see Theorem 2.4.

Remark 3.2. According to Theorem 2.3, $Q_f^n(x,y)$ is a rational quasi-interpolation function with algebraic accuracy of degree $(n+1)$ on Ω.

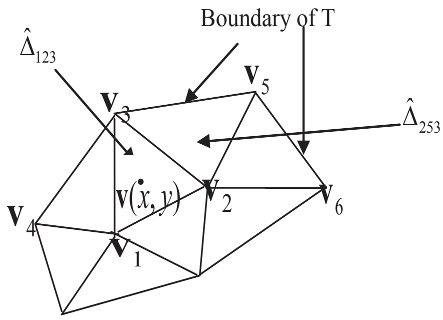

Figure 3: Point $\mathbf{v}(x,y)$ lies on some triangle $\widehat{\triangle}_{123}$ close to the boundary of T.

Algorithm 3.1 The algorithm to generate approximating the sampling function $f(x,y)$.

1. Project the set of the scattered data $\Xi = \{(x_j, y_j, f_j)\}_{j=1}^N$ onto plane to obtain a set of scattered nodes $\{(x_j, y_j)\}_{j=1}^N$;
2. Use Delaunay triangulation method to generate a triangulation T of $\{(x_j, y_j)\}_{j=1}^N$;
3. We choose different expressions of the approximation function $Q_f^n(x,y)$ to calculate the approximation value of $f(x,y)$, according to the location of $\mathbf{v}(x,y)$ on Ω: If point $\mathbf{v}(x,y)$ lies on a triangle $\triangle_{123}$, whose three vertices are interior nodes of T. Then we use the rational function $Q_{123}^n(x,y)$ in (2.6) to calculate the approximation value of $f(x,y)$; If point $\mathbf{v}(x,y)$ lies on a triangle $\widehat{\triangle}_{123}$ close to the boundary of T, namely, at least one of whose three vertices belongs to the boundary of T. Then we use $\widetilde{Q}_{123}^n(x,y)$ or $\widetilde{Q}_{253}^n(x,y)$ in (3.1-3.2) to calculate the approximation value of $f(x,y)$.

4. Numerical experiments

In this section, we use a bivariate quadratic polynomial function $f_1(x,y)$ and Franke function $f_2(x,y)$, $0 \le x,\ y \le 1$:

$$\begin{aligned} f_1(x,y) =& 3x^2+4y^2+5xy+6x+7y+8, \quad 0\le x, \quad y\le 1,\\ f_2(x,y) =& \frac{3}{4}\exp\left[-\frac{(9x-2)^2+(9y-2)^2}{4}\right]+\frac{3}{4}\exp\left[-\frac{(9x+1)^2}{49}-\frac{(9y+1)}{10}\right]\\ &-\frac{1}{5}\exp\left[-(9x-4)^2-(9y-7)^2\right]+\frac{1}{2}\exp\left[-\frac{(9x-7)^2+(9y-3)^2}{4}\right], \end{aligned}$$

as the approximated functions to get the scattered sampling data set. Fig. 4 present the Franke function $f_2(x,y)$. Fig. 5 displays a set of 300 scattered data points sampled from

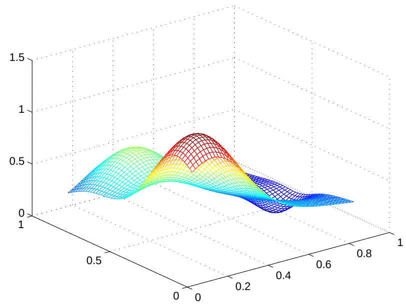

Figure 4: The figure of Franke function.

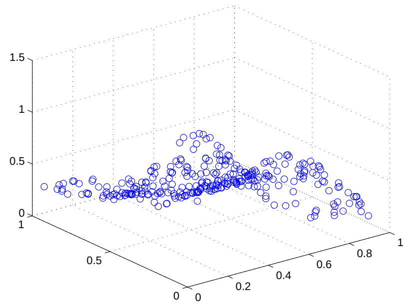

Figure 5: A set of 300 scattered data points from Franke function.

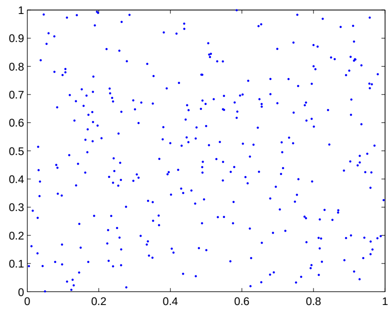

Figure 6: The projection node set of 300 scattered data points in Fig. 5.

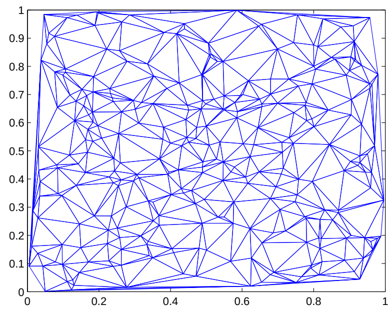

Figure 7: The triangulation of 300 scattered nodes in Fig. 6.

Franke function. Fig. 6 is the projected node set of 300 scattered data points in Fig. 5. Fig. 7 is the triangulation T of 300 scattered nodes in Fig. 6. Fig. 8 the reconstruction function $Q_f^1(x,y)$ using 300 scattered data and the partial derivatives of order one of Franke function. In the section all different sets of scattered nodes are generated by function rand() of MATLAB. Using the algorithm provided in Sections 2-3 and the sampling data $\{(x_j,y_j,f_j)\}_{j=1}^N$ and derivatives of order one, we generate quasi-interpolation functions $Q_f^n(x,y)(n=0,1)$ to approximate $f_1(x,y), f_2(x,y)$; According to different scattered data sets, we calculate mean absolute errors (MAE) and max errors(L_∞ error) of $Q_f^n(x,y)$ to the two approximated functions on 50*50 test points, and investigate its quadratic reproducing and approximation capability. Meanwhile, we compare the MAE and L_∞ errors of $Q_f^1(x,y)$ with piecewise C^1 smooth triangular interpolation suggested by Goodman [16](abbr. C^1-Tri-interpolation) and MQ modified quadratic Shepard method [17] on the same scattered point sets. Finally, we compare CPU execution times of these three methods on the same type of computer (Inter Q8400 2.66GHz, Memory: 4GB).

From the results of Table 1, we see that $Q_f^1(x,y)$ using exact derivatives of one order can regenerate the bivariate quadratic polynomial.

In Table 2, we compare the errors of $Q_f^1(x,y)$ (expanded scheme using the first-order estimate derivatives, which use Goodman method [23]) and $Q_f(x,y)$ $(n=0)$ (unex-

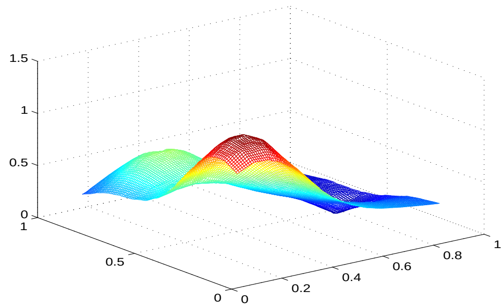

Figure 8: The figure of $Q_f^1(x,y)$ using 300 scattered data and exact derivative.

Table 1: The approximation error of $Q_f^1(x,y)$ to $f_1(x,y)$.

Number of scattered nodes	Max error	Mean absolute error
300	2.8422e-14	2.1092e-015
500	4.6190e-14	2.1646e-015
800	1.7760e-14	2.0586e-015
1500	1.0840e-14	2.0657e-015

Table 2: The comparison of approximation errors of $Q_f^1(x,y)$ and $Q_f(x,y)(n=0)$ to $f_2(x,y)$ when using the approximate derivative in $Q_f^1(x,y)$.

Number of scattered nodes	$Q_f^1(x,y)$		$Q_f(x,y)(n=0)$	
	Max error	Mean absolute error	Max error	Mean absolute error
300	0.0753	0.0048	0.0982	0.0113
500	0.0320	0.0025	0.0633	0.0071
800	0.0184	0.0013	0.0328	0.0039
1000	0.0165	0.0011	0.0306	0.0034
1500	0.0129	8.0566e-004	0.0259	0.0023

panded scheme) to the approximated function $f_2(x,y)$. From the results we see that the errors of $Q_f^1(x,y)$ are less than $Q_f(x,y)$ $(n=0)$, especially the L_∞ error. This shows that the approximation accuracy of $Q_f^1(x,y)$ using approximate derivative is better than $Q_f(x,y)$ $(n=0)$. It reaches our purpose to improve the approximation accuracy of $Q_f(x,y)$ $(n=0)$ without increasing any information and nodes.

Table 3 shows the approximation errors of $Q_f^1(x,y)$ to $f_2(x,y)$ when using the exact derivative and the approximate derivative generated by Goodman method. The results indicate that due to the error of the estimate derivative itself leads to the addition of the error of $Q_f^1(x,y)$.

Table 4 shows, under a few sets of scattered data, the approximation errors of three different methods: $Q_f^1(x,y)$, C^1-Tri-interpolation and MQ Shepard to $f_2(x,y)$. Through comparison we find when using the estimate derivative, the approximation accuracy of

Table 3: The approximation errors of $Q_f^1(x,y)$ to $f_2(x,y)$ when using the exact derivative and approximate derivative in $Q_f^1(x,y)$.

Number of scattered nodes	Exact derivatives		Estimation derivatives	
	Max error	Mean absolute error	Max error	Mean absolute error
300	0.0752	0.0024	0.0753	0.0048
500	0.0296	8.6497e-004	0.0320	0.0025
1000	0.0109	2.4011e-004	0.0165	0.0011
2000	0.0028	7.1501e-005	0.0110	6.1149e-004
4000	4.166e-04	1.6933e-005	0.0094	2.8367e-004

Table 4: The approximation errors of $Q_f^1(x,y)$, C^1-Tri-interpolation and MQShepard to $f_2(x,y)$.

Number of scattered nodes	$Q_f^1(x,y)$		C^1 -Tri-interpolation		MQShepard	
	L_∞ error	MAE	L_∞ error	MAE	L_∞ error	MAE
300	0.0753	0.0048	0.0819	0.0038	0.0620	0.0087
500	0.0320	0.0025	0.0288	0.0020	0.0622	0.0066
1000	0.0165	0.0011	0.0213	0.0011	0.0442	0.0044
2000	0.0110	6.12e-4	0.0155	5.65e-4	0.0287	0.0029
4000	0.0094	2.84e-4	0.0135	2.56e-4	0.0263	0.0021

Table 5: The CPU execution time (unit: second) of $Q_f^1(x,y)$, C^1-Tri-interpolation and MQShepard.

Number of scattered nodes	$Q_f^1(x,y)$	C^1 -Tri-interpolation	MQShepard
300	3.375325s	3.60	3.14
500	5.293426s	5.51	4.89
1000	9.814830s	11.14	9.60
2000	19.594339s	22.26	19.10
4000	36.449786s	44.56	38.64
6000	56.362534s	68.13	59.82

$Q_f^1(x,y)$ is close to C^1-Tri-interpolation, and they are better than MQ Shepard method. Table 5 shows the execution time of the three method in Table 4. Table 5 shows the execution time of our method is the least, and it is less than C^1-Tri-interpolation method and MQ Shepard method. In addition, MQ Shepard method needs user to specify the influence radius of the weight function and the node basis function, and still needs solve a small-scale algebraic system to obtain coefficients of the node basis function. When the scattered data points are dense, some of these small-scale algebraic systems are often ill-conditioned. So in this case the computation is instability and loses accuracy. So from the three aspects of computation error, execution time and computation stability, we see our method is better than C^1-Tri-interpolation method and MQ Shepard method.

5. Conclusions

In the paper we propose an expanded mean value coordinates interpolation method with any degree of algebraic accuracy. Then based on the expanded mean value coordinates interpolation and triangulation of node set we construct a piecewise rational quasi-interpolant with any degree of algebraic accuracy to approximate the scattered data in $\mathbb{R}^3$. This method has a larger improvement in accuracy compared with the unexpanded one. By comparison of approximation accuracy, execution time and computation stability we see the method proposed in the paper is better than the two local approximation methods-C^1-Tri-interpolation method and MQ Shepard method. However, from Table 3 we can see that in case that the exact derivative information is unavailable, the key to get a higher approximation accuracy lies in the approximation accuracy of approximation derivatives. So how to use the given data to obtain an approximate derivative with a higher order approximation accuracy, and how to expand our method to a higher dimension are our future research topic.

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