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A NONOVERLAPPING DDM FOR GENERAL ELASTIC BODY-PLATE PROBLEMS DISCRETIZED BY THE P_1-NZT FEM

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Abstract. A nonoverlapping domain decomposition method (DDM) is proposed to solve general elastic body-plate problems, discretized by the P_1-NZT finite element method. It is proved in a subtle way that the convergence rate of the method is optimal (independent of the finite element mesh size), even for a regular family of finite element triangulations. This enables us to combine the method with adaptive techniques in practical applications. Some numerical results are included to illustrate the computational performance of the method.

Key words. Domain decomposition method, body-plate structure, new Zienkiewicz-type element (NZT), convergence analysis.

1. Introduction

Elastic multi-structures are composed of a number of substructures that have the same or different dimensions (e.g., bodies, plates, beams, etc.), coupled together with some junction conditions. They are widely used in the fields of aviation, aerospace, civil engineering, mechanical manufacturing, etc. In the past few decades, much work has been done about mathematical modeling, mathematical analysis and numerical solution for elastic multi-structure problems. We refer to [8, 11, 16, 17] and the references therein for details in this subject. Elastic multi-structures have a significant feature, that is, they are very complex from the global view point, but their substructures are quite simple comparably. Therefore, elastic multi-structure problems are particularly suitable for solutions through nonoverlapping domain decomposition methods (DDM). In [14], some substructuring method was proposed for solving the stiffened plate problem, based on conforming element discretization. In [15], two domain decomposition methods were given to solve a regular elastic body-plate problem. However, due to the use of finite element discretization in [27], the body and the plate members must have a cuboid shape, which greatly limits the applicability of these two methods.

In this article, we aim to propose and analyze a nonoverlapping DDM for solving a general elastic body-plate problem, discretized by the finite element method developed in [6], where P_1 conforming elements were used to discretize displacements on the body and longitudinal displacements on the plate, while the NZT element (cf. [28]) was used to discretize the transverse displacement. Hence, this method can apply to any bodies and plates with polyhedral/polygonal shapes. The ideas of constructing the related domain decomposition method are quite natural, similar to the second method in [15]. It can be viewed as a Dirichlet-Neumann type nonoverlapping method due to [19]. We refer the reader to the monograph [21] for a comprehensive understanding of this method and mention that it has been applied to solve a variety of coupling problems (cf. [12, 20, 30, 31]) with high efficiency.

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However, the convergence rate analysis of the proposed DDM is rather involved. We first introduce a Clément-type operator and then develop its error estimates in a subtle way. Then, we apply a new method to derive a certain spectral equivalence lemma. In light of these results, we are able to study the convergence rate of the DDM technically. It is proved that the convergence rate of the method is optimal (independent of the finite element mesh size), even for a regular family of finite element triangulations. It deserves to emphasize that there are few existing results about convergence rate analysis of nonoverlapping DDMs for regular finite element methods. Typically, it is often assumed that the restriction of the finite element triangulation to the interface should be quasi-uniform (cf. [5, 7]), to achieve the required spectral equivalence results (cf. [20, 21, 25, 29]). Hence, our convergence rate analysis developed here might be helpful in investigating convergence rates of some other nonoverlapping DDMs based on regular finite element discretizations. The other benefit is that it enable us to combine the nonverlapping DDM with adaptive techniques (cf. [5, 26]) to numerically solve the general elastic body-plate problem very effectively. Similar to the second method in [15], our method here only requires numerical solution of a pure body problem and a pure plate problem at each iteration step, which can be implemented by existing efficient numerical solvers. The relaxation parameter can be determined by numerical experience or by the power method. We provide an academic example to illustrate the computational performance of our method.

We end this section by introducing some notations and conventions frequently used later on. Throughout this paper, we adopt standard notations for Sobolev spaces [1, 5, 18], e.g., for a given open set G and a non-negative integer k, $H^k(G)$ consists of all $L^2(G)$-integrable functions whose weak derivatives with the total degree $\leq k$ are also $L^2(G)$-integrable, and the norm and seminorm are denoted by $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$, respectively. $H_0^k(G)$ denotes the closure of $C_0^\infty(G)$ with respect to the norm $\|\cdot\|_{k,G}$, and the fractional-order Sobolev spaces are defined by real interpolation of Banach spaces. Moreover, denote by $P_k(G)$ the space of all polynomials over G with the total degree $\leq k$. We use the same index and summation conventions as described in [16, 17]. That means, Latin indices i, j, l take their values in the set $\{1, 2, 3\}$, while the capital Latin indices I, J, L take their values in the set $\{1, 2\}$. The summation is implied when a Latin index (or a capital Latin index) appears exactly twice. We also use the symbol "$\lesssim \cdots$" to denote "$\leq C \cdots$" with a generic positive constant C independent of the finite element mesh size and the functions under consideration, which may take different values in different appearances.

2. The P_1-NZT FEM for general elastic body-plate problems

As shown in Fig. 1, let (x_1, x_2, x_3) be a right-handed orthogonal system in the space $\mathbb{R}^3$, whose orthonormal basis vectors are denoted by $\{\boldsymbol{e}_i\}_{i=1}^3$, respectively. Let Ω be an elastic body-plate structure consisting of an elastic polyhedral body α and an elastic polygonal plate β (precisely speaking, β is the mid-surface of an elastic plate with thickness t_β), which is rigidly connected on the interface β_b, i.e.,

$$\boldsymbol{u}^\alpha = \boldsymbol{u}^\beta \quad \text{on } \beta_b, \tag{1}$$

where $\boldsymbol{u}^\alpha := u_i^\alpha \boldsymbol{e_i}$ and $\boldsymbol{u}^\beta := u_i^\beta \boldsymbol{e_i}$ are the displacement fields in α and β, respectively.

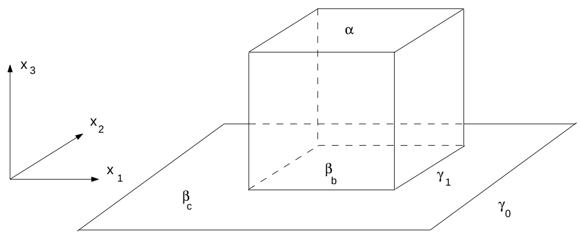

FIGURE 1. The geometric domain of a body-plate elastic structure Ω.

For easy of exposition, we here only consider the case where the plate β is clamped along the exterior boundary $\gamma_0 := \partial\beta$. In other words,

$$u_1^\beta = u_2^\beta = 0, \quad u_3^\beta = \partial_{\boldsymbol{n}^\beta} u_3^\beta = 0 \quad \text{on } \gamma_0,$$

with $\boldsymbol{n}^\beta$ denoting the unit outward normal to $\partial\beta$. Moreover it is imposed the force-free condition on the exterior boundary $\partial\alpha \setminus \beta_b$ of α. Then, under the applied force fields $\boldsymbol{f}^\alpha \in (L^2(\alpha))^3, \boldsymbol{f}^\beta \in (L^2(\beta))^3$, the equilibrium configuration of the elastic body-plate structure Ω is governed by the following problem (cf. [11, 16, 27]):

Find $\boldsymbol{u} := (\boldsymbol{u}^\alpha, \boldsymbol{u}^\beta) \in \boldsymbol{V}$ such that

$$D(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{2}$$

Here,

$$\boldsymbol{V} := \big\{\boldsymbol{v} = (\boldsymbol{v}^\alpha, \boldsymbol{v}^\beta); \quad \boldsymbol{v}^\alpha \in (H^1(\alpha))^3, \boldsymbol{v}^\beta \in (H_0^1(\beta))^2 \times H_0^2(\beta), \quad \boldsymbol{v} \text{ satisfies } (1)\big\};$$

for $\boldsymbol{v} = (\boldsymbol{v}^\alpha, \boldsymbol{v}^\beta) \in \boldsymbol{V}$,

$$F(\boldsymbol{v}) := F^\alpha(\boldsymbol{v}^\alpha) + F^\beta(\boldsymbol{v}^\beta),$$
$$F^\alpha(\boldsymbol{v}^\alpha) := \int_\alpha \boldsymbol{f}^\alpha \cdot \boldsymbol{v}^\alpha d\alpha, \quad F^\beta(\boldsymbol{v}^\beta) := \int_\beta \boldsymbol{f}^\beta \cdot \boldsymbol{v}^\beta d\beta;$$

for $\boldsymbol{w} = (\boldsymbol{w}^\alpha, \boldsymbol{w}^\beta) \in \boldsymbol{V}$,

$$D(\boldsymbol{v}, \boldsymbol{w}) := D^\alpha(\boldsymbol{v}, \boldsymbol{w}) + D^\beta(\boldsymbol{v}, \boldsymbol{w}),$$

$$D^\alpha(\boldsymbol{v}, \boldsymbol{w}) := \int_\alpha \sigma_{ij}^\alpha(\boldsymbol{v}) \varepsilon_{ij}^\alpha(\boldsymbol{w}) d\alpha, \tag{3}$$

$$\varepsilon_{ij}^\alpha(\boldsymbol{v}) := (\partial_i v_j^\alpha + \partial_j v_i^\alpha)/2, \quad \partial_i v_j^\alpha := \partial v_j^\alpha / \partial x_i, \tag{4}$$

$$\sigma_{ij}^\alpha(\boldsymbol{v}) := \frac{E_\alpha}{1+\nu_\alpha} \varepsilon_{ij}^\alpha(\boldsymbol{v}) + \frac{E_\alpha \nu_\alpha}{(1+\nu_\alpha)(1-2\nu_\alpha)} (\varepsilon_{ll}^\alpha(\boldsymbol{v}))\delta_{ij}, 1 \le i, j, l \le 3, \tag{5}$$

$$D^\beta(\boldsymbol{v}, \boldsymbol{w}) := \int_\beta \mathcal{Q}_{IJ}^\beta(\boldsymbol{v}) \varepsilon_{IJ}^\beta(\boldsymbol{w}) d\beta + \int_\beta \mathcal{M}_{IJ}^\beta(\boldsymbol{v}) \mathcal{K}_{IJ}^\beta(\boldsymbol{w}) d\beta,$$

$$\varepsilon_{IJ}^\beta(\boldsymbol{v}) := (\partial_I v_J^\beta + \partial_J v_I^\beta)/2, \ \partial_I v_J^\beta := \frac{\partial v_J^\beta}{\partial x_I},$$

$$\mathcal{Q}_{IJ}^\beta(\boldsymbol{v}) := \frac{E_\beta t_\beta}{1-\nu_\beta^2}((1-\nu_\beta)\varepsilon_{IJ}^\beta(\boldsymbol{v}) + \nu_\beta(\varepsilon_{LL}^\beta(\boldsymbol{v}))\delta_{IJ}), 1 \le I, J, L \le 2,$$

$$\mathcal{K}^{\beta}_{IJ}(\boldsymbol{v}) := -\partial_{IJ} v^{\beta}_3 = -\frac{\partial^2 v^{\beta}_3}{\partial x_I \partial x_J},$$

$$\mathcal{M}^{\beta}_{IJ}(\boldsymbol{v}) := \frac{E_\beta t^3_\beta}{12(1-\nu^2_\beta)}((1-\nu_\beta)\mathcal{K}^{\beta}_{IJ}(\boldsymbol{v}) + \nu_\beta(\mathcal{K}^{\beta}_{LL}(\boldsymbol{v}))\delta_{IJ}). \tag{6}$$

In addition, $E_\omega > 0$ and $\nu_\omega \in (0, 1/2)$ denote Young's modulus and Poisson's ratio of the elastic member $\omega = \alpha$, β, respectively; t_β is the thickness of plate β; δ_{ij} and δ_{IJ} stand for the usual Kronecker delta.

The unique solvability of the problem (2) was proved in [16], and if the solution $\boldsymbol{u}$ satisfies the regularity conditions:

$$\boldsymbol{u}^\alpha \in (H^2(\alpha))^3, \boldsymbol{u}^\beta \in (H^2(\beta))^2 \times H^3(\beta), \tag{7}$$

then we further have the following equilibrium equations (cf. [16]):

$$-\sigma^{\alpha}_{ij,j}(\boldsymbol{u}) = f^{\alpha}_i \quad \text{in } \alpha,$$
$$\sigma^{\alpha}_{ij,j}(\boldsymbol{u}) n^{\alpha}_j \boldsymbol{e}_i = \boldsymbol{0} \quad \text{on } \partial\alpha \setminus \beta_b,$$
$$-\mathcal{Q}^{\beta}_{IJ,J}(\boldsymbol{u})\boldsymbol{e}_I - \mathcal{M}^{\beta}_{IJ,IJ}(\boldsymbol{u})\boldsymbol{e}_3 = \boldsymbol{f}^{\beta} \quad \text{in } \beta \setminus \beta_b,$$
$$\sigma^{\alpha}_{ij,j}(\boldsymbol{u}) n^{\alpha}_j \boldsymbol{e}_i - \mathcal{Q}^{\beta}_{IJ,J}(\boldsymbol{u})\boldsymbol{e}_I - \mathcal{M}^{\beta}_{IJ,IJ}(\boldsymbol{u})\boldsymbol{e}_3 = \boldsymbol{f}^{\beta_b} \quad \text{in } \beta_b.$$

From the above equations we know that the elastic body-plate problem is a heterogeneous model, i.e., different PDEs are imposed in different regions.

Now, let us recall the P_1-NZT element method for problem (2), proposed in [6]. Let $\{\mathcal{T}^{\alpha}_h\}_{h>0}$ be a regular family of triangulations of α into open tetrahedrons K, with h the mesh size of $\mathcal{T}^{\alpha}_h$. Let $\{\mathcal{T}^{\beta}_h\}$ be a regular family of triangulations of β into open triangles τ. We then obtain a family of total partitions of Ω,

$$\mathcal{T}^{\Omega}_h := \{\mathcal{T}^{\alpha}_h, \mathcal{T}^{\beta}_h\}.$$

For ease of exposition, assume that finite element triangulations over α and β have the same size h, and they are matching across the interface β_b.

Denote by $V^1_h(\alpha)$ (resp. $V^1_h(\beta)$) the space of all piecewise linear continuous functions associated with the triangulation $\mathcal{T}^{\alpha}_h$ (resp. $\mathcal{T}^{\beta}_h$). Let $V^{NZT}_h(\beta)$ be the new Zienkiewicz-type (NZT) finite element space associated with the triangulation $\mathcal{T}^{\beta}_h$ (cf. [28]). That means, for each $\tau \in \mathcal{T}^{\beta}_h$, the local shape function space related to $V^{NZT}_h(\beta)$ is

$$P^{NZT}_\tau = P_2(\tau) + \text{span}\{q_{ij}\ ;\ 1 \le i < j \le 3\},$$

where

$$q_{ij} = \lambda^2_i \lambda_j - \lambda_i \lambda^2_j + \left(2(\lambda_i - \lambda_j) + 3\frac{(\nabla\lambda_i - \nabla\lambda_j)^\top \nabla\lambda_k}{\|\nabla\lambda_k\|^2}(2\lambda_k - 1)\right)\lambda_1\lambda_2\lambda_3,$$

with $\{\lambda_i\}^3_{i=1}$ being the barycentric coordinates of the triangle τ and $\|\cdot\|$ the length of a vector. The corresponding nodal variables are given by

$$\Sigma_\tau := \{v(p^\tau_i), \partial_1 v(p^\tau_i), \partial_2 v(p^\tau_i), 1 \le i \le 3\},$$

where $\{p^\tau_i\}^3_{i=1}$ are three vertices of the triangle τ, respectively. In what follows, p with or without index always stands for a node of some finite element triangulation.

Introduce the following finite element spaces:

$$\boldsymbol{W}_h(\alpha) := (V^1_h(\alpha))^3;$$
$$\boldsymbol{W}_h(\beta) := (V^1_{0h}(\beta))^2 \times V^{NZT}_{0h}(\beta),$$

where

$$V^1_{0h}(\beta) := \{v_h \in V^1_h(\beta);\ v_h(p) = 0\ \forall\ p \in \gamma_0\}\ ,$$

$$V_{0h}^{NZT}(\beta) := \{v_h \in V_h^{NZT}(\beta);\ v_h(p) = \partial_1 v_h(p) = \partial_2 v_h(p) = 0\ \forall\ p \in \gamma_0\}.$$

The discrete rigid junction related to (1) is given by

$$v_i^\alpha(p) = v_i^\beta(p) \quad \forall\ p \in \beta_b, \quad 1 \le i \le 3, \tag{8}$$

from which we can construct a total finite element space by

$$\boldsymbol{V}_h := \Big\{\boldsymbol{v}_h \in \boldsymbol{W}_h(\alpha) \times \boldsymbol{W}_h(\beta),\ \boldsymbol{v}_h \text{ satisfies (8)}\Big\}.$$

Then the P_1-NZT element method for problem (2) reads as follows:
Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$D_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = F(\boldsymbol{v}_h) \quad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{9}$$

where

$$\begin{aligned}
&D_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := D_h^\alpha(\boldsymbol{u}_h, \boldsymbol{v}_h) + D_h^\beta(\boldsymbol{u}_h, \boldsymbol{v}_h), \qquad (10)\\
&D_h^\alpha(\boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{K \in \mathcal{T}_h^\alpha} \int_K \sigma_{ij}^\alpha(\boldsymbol{u}_h)\varepsilon_{ij}^\alpha(\boldsymbol{v}_h) dK,\\
&D_h^\beta(\boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{\tau \in \mathcal{T}_h^\beta} \int_\tau \mathcal{Q}_{IJ}^\beta(\boldsymbol{u}_h)\varepsilon_{IJ}^\beta(\boldsymbol{v}_h) d\tau + Z_h^\beta(\boldsymbol{u}_h, \boldsymbol{v}_h),\\
&Z_h^\beta(\boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{\tau \in \mathcal{T}_h^\beta} \int_\tau \mathcal{M}_{IJ}^\beta(\boldsymbol{u}_h)\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h) d\tau.
\end{aligned}$$

As shown in [6], there exists a unique solution $\boldsymbol{u}_h$ to the problem (9), and the error $\boldsymbol{u}-\ \boldsymbol{u}_h$ in the discrete energy norm is of the size $O(h)$ provided the regularity assumption (7) holds. In what follows, we intend to propose certain nonoverlapping DDM for solving problem (9) and then develop its convergence rate analysis.

3. A Clément-type interpolation operator with error estimates and a spectral equivalence lemma

Define

$$V_h^1(\alpha; \beta_b) = \{v \in V_h^1(\alpha);\ v(p) = 0 \ \ \forall\ p \in \beta_b\}.$$

We have the following result corresponding to the body problem, which is known as the discrete extension theorem in the context of domain decomposition methods (cf. [21, 25, 29]).

Lemma 3.1. *For any vector-valued function $\boldsymbol{v} \in (V_h^1(\alpha))^3$ satisfying*

$$D_h^\alpha(\boldsymbol{v}, \boldsymbol{w}) = 0 \quad \forall\ \boldsymbol{w} \in (V_h^1(\alpha; \beta_b))^3,$$

there holds

$$D_h^\alpha(\boldsymbol{v}, \boldsymbol{v}) \lesssim \|\boldsymbol{v}\|_{1/2,\beta_b}^2.$$

Proof. We first introduce an auxiliary problem by finding $\boldsymbol{u} \in (H^1(\alpha))^3$ such that

$$\begin{cases} \boldsymbol{u} = \boldsymbol{v} & \text{on } \beta_b,\\ D(\boldsymbol{u}, \boldsymbol{w}) = 0 & \forall\ \boldsymbol{w} \in (H^1(\alpha; \beta_b))^3. \end{cases}$$

Then, according to the regularity theory of elliptic equations in non-smooth domains (cf. [13]), it follows that

$$\|\boldsymbol{u}\|_{1,\alpha} \lesssim \|\boldsymbol{v}\|_{1/2,\beta_b}. \tag{11}$$

On the other hand, by the minimum energy principle,

$$D_h^\alpha(\boldsymbol{v}, \boldsymbol{v}) \le D_h^\alpha(\boldsymbol{w}, \boldsymbol{w}) \tag{12}$$

for all $\boldsymbol{w} \in (V_h^1(\alpha))^3$ with $\boldsymbol{w} = \boldsymbol{v}$ on β_b. Let $\boldsymbol{Q}_h^\alpha$ be the weak interpolation operator keeping traces of functions on β_b (cf. [23]). Then, recalling the definitions (3)-(5), we have by the Cauchy-Schwarz inequality that

$$D_h^\alpha(\boldsymbol{Q}_h^\alpha \boldsymbol{u}, \boldsymbol{Q}_h^\alpha \boldsymbol{u}) = D^\alpha(\boldsymbol{Q}_h^\alpha \boldsymbol{u}, \boldsymbol{Q}_h^\alpha \boldsymbol{u}) \lesssim \|\boldsymbol{Q}_h^\alpha \boldsymbol{u}\|_{1,\alpha}^2. \tag{13}$$

Therefore, choosing $\boldsymbol{w} = \boldsymbol{Q}_h^\alpha \boldsymbol{u}$ in the estimate (12), and using (11), (13) and the stability estimates for $\boldsymbol{Q}_h^\alpha$, we find

$$D_h^\alpha(\boldsymbol{v}, \boldsymbol{v}) \le D_h^\alpha(\boldsymbol{Q}_h^\alpha \boldsymbol{u}, \boldsymbol{Q}_h^\alpha \boldsymbol{u}) \lesssim \|\boldsymbol{Q}_h^\alpha \boldsymbol{u}\|_{1,\alpha}^2 \lesssim \|\boldsymbol{u}\|_{1,\alpha}^2 \lesssim \|\boldsymbol{v}\|_{1/2,\beta_b}^2,$$

as required. □

Now, we introduce some notations for later uses. Let G be any open subset of β, which is aligned with the triangulation $\mathcal{T}_h^\beta$. Write $\mathcal{T}_h^G$ and $V_h^{NZT}(G)$ as the restriction of $\mathcal{T}_h^\beta$ and $V_h^{NZT}(\beta)$ to G, respectively. It is mentioned that the same convention also applies to other finite element spaces. If v is a piecewise H^k-smooth function with respect to the triangulation $\mathcal{T}_h^G$, define

$$\|v\|_{k,h,G} = \left\{ \sum_{\tau \in \mathcal{T}_h^G} \|v\|_{k,\tau}^2 \right\}^{1/2}, \ |v|_{k,h,G} = \left\{ \sum_{\tau \in \mathcal{T}_h^G} |v|_{k,\tau}^2 \right\}^{1/2}.$$

The next result follows readily from the technique in [5] for deriving the discrete Poincaré-Friedrichs inequality for piecewise Sobolev functions.

Lemma 3.2. *There holds*

$$\|v\|_{2,h,\beta} \lesssim |v|_{2,h,\beta} \quad \forall\, v \in V_{0h}^{NZT}(\beta).$$

Let $\beta_c := \beta \backslash \bar{\beta}_b$. Now, let us construct a Clément-type interpolation operator $\Pi_h^{\beta_c}$ from $H^2(\beta_c)$ into $V_h^{NZT}(\beta_c)$ and then establish the corresponding error estimates (cf. [9]). It is emphasized that all the results can be extended to any subset G of β, which is aligned with the triangulation $\mathcal{T}_h^\beta$.

For any subset $G \subset \beta_c$, define

$$\triangle_G^{\beta_c} = \{(\cup \bar{\tau})^0;\ \tau \in \mathcal{T}_h^{\beta_c}, \bar{\tau} \cap \bar{G} \neq \emptyset\}.$$

For $p \in \mathcal{T}_h^{\beta_c}$, $\triangle_p^{\beta_c}$ is exactly the macro-element consisting of all triangles in $\mathcal{T}_h^{\beta_c}$ that share the node p. It was shown in [2, 9] that there exist positive constants $\{A_i\}_{i=1}^4$ which rely only on the parameter describing the shape-regularity of the triangulation $\mathcal{T}_h^{\beta_c}$ (cf. [5, 7]), such that

(1) for all $p \in \mathcal{T}_h^{\beta_c}$, the number of triangles contained in $\triangle_p^{\beta_c}$ is no more than A_1;

(2) for all $p \in \mathcal{T}_h^{\beta_c}$,

$$\forall\, \tau \subset \triangle_p^{\beta_c}, \quad A_2 h_\tau \le h_{\triangle_p^{\beta_c}} \le A_3 h_\tau,$$

where, for an open set G, h_G denotes the diameter of G;

(3) for all $\tau \subset \mathcal{T}_h^{\beta_c}$, the number of macro-elements that contain τ is no more than A_4.

For all $p \in \mathcal{T}_h^{\beta_c}$, let Q_h^{p,β_c} be the $L^2(\triangle_p^{\beta_c})$-orthogonal projection operator from $L^2(\triangle_p^{\beta_c})$ onto $P_1(\triangle_p^{\beta_c})$, which admits the following error estimates (cf. [9]).

$$\sum_{k=0}^{2} h_{\triangle_p^{\beta_c}}^k |v - Q_h^{p,\beta_c} v|_{k,\triangle_p^{\beta_c}} \lesssim h_{\triangle_p^{\beta_c}}^2 |v|_{2,\triangle_p^{\beta_c}} \quad \forall\, v \in H^2(\triangle_p^{\beta_c}). \tag{14}$$

Moreover, let ψ_p, $\psi_{p,1}$, $\psi_{p,2}$ be the nodal basis functions in $V_h^{NZT}(\beta_c)$ associated with the nodal variables $v(p)$, $\partial_1 v(p)$, and $\partial_2 v(p)$, respectively. Then our Clément-type interpolation operator $\Pi_h^{\beta_c}$ is defined by

$$\Pi_h^{\beta_c} v = \sum_{p\in\mathcal{T}_h^{\beta_c}} \big((Q_h^{p,\beta_c}v)(p)\psi_p + \partial_1(Q_h^{p,\beta_c}v)(p)\psi_{p,1} + \partial_2(Q_h^{p,\beta_c}v)(p)\psi_{p,2}\big). \tag{15}$$

Lemma 3.3. *Let $\Pi_h^{\beta_c}$ be the Clément-type interpolation operator defined by (15). Then for all $\tau\in\mathcal{T}_h^{\beta_c}$, there holds*

$$\sum_{k=0}^{2} h_\tau^k |v-\Pi_h^{\beta_c}v|_{k,\tau} \lesssim h_\tau^2 |v|_{2,\triangle_\tau^{\beta_c}} \quad \forall\, v\in H^2(\triangle_\tau^{\beta_c}). \tag{16}$$

Proof. Let P^τ be the $L^2(\tau)$-orthogonal projection from $L^2(\tau)$ onto $P_1(\tau)$. It is well-known that

$$\sum_{k=0}^{2} h_\tau^k |v-P^\tau v|_{k,\tau} \lesssim h_\tau^2 |v|_{2,\tau} \quad \forall\, v\in H^2(\tau). \tag{17}$$

By the scaling argument we know, if p is a vertex of the triangle τ, then

$$|\psi_p|_{k,\tau} \lesssim h_\tau^{1-k}, \quad |\psi_{p,I}|_{k,\tau} \lesssim h_\tau^{2-k},\ 0\le k\le 2,\ 1\le I\le 2. \tag{18}$$

On the other hand, we have from (15) that

$$\begin{aligned} &P^\tau v - \Pi_h^{\beta_c} v \\ =&\sum_{p\in\tau} \big((P^\tau v - Q_h^{p,\beta_c}v)(p)\psi_p + \partial_1(P^\tau v - Q_h^{p,\beta_c}v)(p)\psi_{p,1} + \partial_2(P^\tau v - Q_h^{p,\beta_c}v)(p)\psi_{p,2}\big). \end{aligned}$$

Here and in what follows, for a given set G, $p\in G$ means that $p\in\bar G$ and it is a vertex of a triangle in $\mathcal{T}_h^{\beta}$ as well. Therefore, from (14), (17)-(18), the local inverse inequality for finite elements and the properties for macro-elements given before, we arrive at

$$\begin{aligned} |P^\tau v - \Pi_h^{\beta_c} v|_{k,\tau} \le& \sum_{p\in\tau} \big(|(P^\tau v - Q_h^{p,\beta_c}v)(p)||\psi_p|_{k,\tau} + |\partial_1(P^\tau v - Q_h^{p,\beta_c}v)(p)||\psi_{p,1}|_{k,\tau} \\ &+ |\partial_2(P^\tau v - Q_h^{p,\beta_c}v)(p)||\psi_{p,2}|_{k,\tau}\big) \\ \lesssim& h_\tau^{-1}\big(\|P^\tau v - Q_h^{p,\beta_c}v\|_{0,\tau} h_\tau^{1-k} + h_\tau^{2-k}|P^\tau v - Q_h^{p,\beta_c}v|_{1,\tau}\big) \\ \lesssim& h_\tau^{-k}(\|v-P^\tau v\|_{0,\tau} + \|v - Q_h^{p,\beta_c}v\|_{0,\tau}) + h_\tau^{1-k}(|v-P^\tau v|_{1,\tau} \\ &+ |v-Q_h^{p,\beta_c}v|_{1,\tau}) \\ \lesssim& h_\tau^{2-k}|v|_{2,\triangle_p^{\beta_c}} \lesssim h_\tau^{2-k}|v|_{2,\triangle_\tau^{\beta_c}}, \end{aligned}$$

which, in conjunction with (17), gives (16) readily. $\square$

Denote by $V_h^B(\beta_b)$ the Bell element space with respect to the triangulation $\mathcal{T}_h^{\beta_b}$ (cf. [4, 5, 7]). That means, for each $\tau\in\mathcal{T}_h^{\beta_b}$ with three vertices $\{p_i^\tau\}_{i=1}^3$, the local shape function space is

$$P_\tau := \{v\in P_5(\tau);\ \partial_{\boldsymbol{n}^\tau} v\big|_{F^\tau}\in P_3(F^\tau) \quad \forall\, F^\tau\subset\partial\tau\},$$

where $\boldsymbol{n}^\tau$ stands for the unit outward normal to τ and F^τ is an edge of τ; the nodal variables are given by

$$\Sigma_\tau := \{v(p_i^\tau),\, \partial_1 v(p_i^\tau),\, \partial_2 v(p_i^\tau),\, \partial_{11} v(p_i^\tau),\, \partial_{12} v(p_i^\tau),\, \partial_{22} v(p_i^\tau), \quad 1\le i\le 3\}.$$

We next introduce a connection operator $E_h^{\beta_b}$ from $V_h^{NZT}(\beta_b)$ into the Bell conforming element space $V_h^B(\beta_b)$ as follows. For all $v \in V_h^{NZT}(\beta_b)$, $E_h^{\beta_b}v$ is uniquely determined by the conditions:

$$\left\{\begin{array}{ll} (E_h^{\beta_b}v)(p) = v(p) & \forall\ p \in \beta_b, \\ (\partial_I E_h^{\beta_b}v)(p) = (\partial_I v)(p) & \forall\ p \in \beta_b,\ 1 \le I \le 2, \\ (\partial_{IJ} E_h^{\beta_b}v)(p) = 0 & \forall\ p \in \beta_b,\ 1 \le I, J \le 2, \\ (\partial_{\boldsymbol{n}^\tau} E_h^{\beta_b}v)\big|_{F^\tau} \in P_3(F^\tau) & \forall\ F^\tau \subset \tau \in \mathcal{T}_h^{\beta_b}. \end{array}\right. \tag{19}$$

Employing the technique for proving Lemma 5.1 in [4], we can derive the following result.

Lemma 3.4. *For the connection operator $E_h^{\beta_b}$ defined by (19),*

$$\|E_h^{\beta_b}v\|_{2,h,\beta_b} \lesssim \|v\|_{2,h,\beta_b} \quad \forall\ v \in V_h^{NZT}(\beta_b). \tag{20}$$

For any function $v \in V_h^{NZT}(\beta_b)$, we can extend it as a unique function $\hat{v} \in V_{0h}^{NZT}(\beta)$ such that

$$\left\{\begin{array}{l} \hat{v}(p) = v(p), \partial_I \hat{v}(p) = \partial_I v(p) \quad \forall\ p \in \gamma_1,\ I = 1, 2, \\ Z_h^{\beta_c}(\hat{v}, w) = 0 \quad \forall\ w \in V_h^{NZT}(\beta_c; \gamma_2), \end{array}\right.$$

where $\gamma_1 := \partial\beta_b$, $\gamma_2 := \gamma_0 \cup \gamma_1$,

$$V_h^{NZT}(\beta_c; \gamma_2) := \{v_h \in V_h^{NZT}(\beta_c);\ v_h(p) = \partial_1 v_h(p) = \partial_2 v_h(p) = 0\ \forall\ p \in \gamma_2\},$$

and for any open subset G of β which is aligned with the triangulation $\mathcal{T}_h^\beta$,

$$Z_h^G(v, w) := \sum_{\tau \in \mathcal{T}_h^G} \int_\tau \mathcal{M}_{IJ}^\beta(v) \mathcal{K}_{IJ}^\beta(w) d\tau.$$

It should be emphasized that only the third component of a vector-valued function $\boldsymbol{v}$ is used in the definitions of $\mathcal{M}_{IJ}(\boldsymbol{v})$ and $\mathcal{K}_{IJ}(\boldsymbol{v})$ (cf. (6)). So we simply write them as $\mathcal{M}_{IJ}(v)$ and $\mathcal{K}_{IJ}(v)$, with v denoting the third component of $\boldsymbol{v}$, where there is no confusion caused.

Lemma 3.5. *There exist two positive constants C_1 and C_2, independent of the finite element mesh size h, such that the following results hold.*

$$C_1\|v\|_{2,h,\beta_b}^2 \le Z_h^\beta(\hat{v}, \hat{v}) \le C_2\|v\|_{2,h,\beta_b}^2 \quad \forall\, v \in V_h^{NZT}(\beta_b).$$

Proof. Observe that there exist two positive constants B_1 and B_2, which depend only on the physical parameters of plate β, such that (cf. [4])

$$B_1|w|_{2,h,G}^2 \le Z_h^G(w, w) \le B_2|w|_{2,h,G}^2 \quad \forall\, w \in V_h^{NZT}(G), \tag{21}$$

for any open subset G of β, which is aligned with $\mathcal{T}_h^\beta$. Hence, it suffices for us to verify the following results

$$C_1\|v\|_{2,h,\beta_b}^2 \le |\hat{v}|_{2,h,\beta}^2 \le C_2\|v\|_{2,h,\beta_b}^2. \tag{22}$$

The left side inequality comes readily from Lemma 3.2. It is rather involved to achieve the right side inequality of (22). At first, by the minimum energy principle and (21), we know $v_c := \hat{v}|_{\beta_c}$ satisfies that

$$|v_c|_{2,h,\beta_c}^2 \lesssim Z_h^{\beta_c}(v_c, v_c) \le Z_h^{\beta_c}(w_c, w_c) \lesssim |w_c|_{2,h,\beta_c}^2, \tag{23}$$

for all $w_c \in V_h^{NZT}(\beta_c)$ with

$$\begin{array}{l} w_c(p) = v(p),\ \partial_I(w_c)(p) = (\partial_I v)(p)\ \forall\, p \in \gamma_1,\ 1 \le I \le 2, \\ w_c(p) = 0,\ \partial_I(w_c)(p) = 0\ \forall\, p \in \gamma_0,\ 1 \le I \le 2. \end{array} \tag{24}$$

Recalling the definition of $E_h^{\beta_b}$ (cf. (19)), we know $E_h^{\beta_b} v \in H^2(\beta_b)$ and

$$(E_h^{\beta_b} v)(p) = v(p),\ \partial_I (E_h^{\beta_b} v)(p) = (\partial_I v)(p)\ \forall\, p \in \gamma_1,\ 1 \le I \le 2.$$

Then, using the extension theorem for Sobolev spaces (cf. [24]) and the technique of cut-off functions (cf. [1]), there exists an extension operator $\tilde{E}$ from $H^2(\beta_b)$ into $H_0^2(\beta)$ such that

$$\|\tilde{E} w\|_{2,\beta} \lesssim \|w\|_{2,\beta_b} \quad \forall\, w \in H^2(\beta_b), \tag{25}$$

where the generic constant depends only on the geometric nature of β_b and β.

Now, let $\bar{w}_c$ be the restriction of $\tilde{E}(E_h^{\beta_b} v)$ to β_c. It is easy to check that $\bar{w}_c$ is in $H^2(\beta_c)$ and satisfies the conditions (24). We next construct a function w_c in $V_h^{NZT}(\beta_c)$ based on $\bar{w}_c$ as follows. If $p \in \gamma_2$,

$$w_c(p) := \bar{w}_c(p), \quad (\partial_I w_c)(p) := (\partial_I \bar{w}_c)(p), \quad 1 \le I \le 2;$$

if $p \in \beta_c \backslash \gamma_2$,

$$w_c(p) := (\Pi_h^{\beta_c} \bar{w}_c)(p), \quad (\partial_I w_c)(p) := (\partial_I (\Pi_h^{\beta_c} \bar{w}_c))(p), \quad 1 \le I \le 2,$$

where $\Pi_h^{\beta_c}$ is the Clément-type interpolation operator defined by (15).

We can write the finite element function w_c via the nodal basis functions in the form

$$\begin{aligned} w_c = &\sum_{p\in\gamma_2} \big(\bar{w}_c(p)\psi_p + (\partial_I \bar{w}_c)(p)\psi_{p,I}\big) \\ &+ \sum_{p\in\beta_c\backslash\gamma_2} \big((\Pi_h^{\beta_c}\bar{w}_c)(p)\psi_p + (\partial_I(\Pi_h^{\beta_c}\bar{w}_c)(p)\psi_{p,I}\big), \end{aligned}$$

which implies that

$$w_c = \Pi_h^{\beta_c}\bar{w}_c + \sum_{p\in\gamma_2} \big(\bar{w}_c(p) - (\Pi_h^{\beta_c}\bar{w}_c)(p)\big)\psi_p + \sum_{p\in\gamma_2} \big((\partial_I\bar{w}_c)(p) - \partial_I(\Pi_h^{\beta_c}\bar{w}_c)(p)\big)\psi_{p,I}.$$

Hence, using error estimate for the interpolation operator $\Pi_h^{\beta_c}$ (cf. (16)), the estimate (18), the local inverse inequality for finite elements and the property of macro-elements shown before, we see that

$$\begin{aligned} |w_c|_{2,h,\beta_c}^2 \lesssim &|\Pi_h^{\beta_c}\bar{w}_c|_{2,h,\beta_c}^2 + \sum_{p\in\gamma_2} \big|\bar{w}_c(p) - (\Pi_h^{\beta_c}\bar{w}_c)(p)\big|^2 |\psi_p|_{2,\beta_c}^2 \\ &+ \sum_{p\in\gamma_2} \big|(\partial_I\bar{w}_c)(p) - \partial_I(\Pi_h^{\beta_c}\bar{w}_c)(p)\big|^2 |\psi_{p,I}|_{2,\beta_c}^2 \\ \lesssim &|\bar{w}_c|_{2,\beta_c}^2 + \sum_{\tau\in\mathcal{T}_h^{\beta_c}} |\bar{w}_c - \Pi_h^{\beta_c}\bar{w}_c|_{0,\infty,\tau}^2 h_\tau^{-2} + \sum_{\tau\in\mathcal{T}_h^{\beta_c}} |\bar{w}_c - \Pi_h^{\beta_c}\bar{w}_c|_{1,\infty,\tau}^2 \\ \lesssim &|\bar{w}_c|_{2,\beta_c}^2 + \sum_{\tau\in\mathcal{T}_h^{\beta_c}} |\bar{w}_c - \Pi_h^{\beta_c}\bar{w}_c|_{0,\tau}^2 h_\tau^{-4} + \sum_{\tau\in\mathcal{T}_h^{\beta_c}} |\bar{w}_c - \Pi_h^{\beta_c}\bar{w}_c|_{1,\tau}^2 h_\tau^{-2} \\ \lesssim &|\bar{w}_c|_{2,\beta_c}^2 + \sum_{\tau\in\mathcal{T}_h^{\beta_c}} |\bar{w}_c|_{2,\triangle_p^\tau}^2 \lesssim |\bar{w}_c|_{2,\beta_c}^2, \end{aligned}$$

from which, (20) and (25) we are led to

$$|w_c|_{2,h,\beta_c} \lesssim |\bar{w}_c|_{2,\beta_c} = |\tilde{E}(E_h^{\beta_b} v)|_{2,\beta_c} \le \|\tilde{E}(E_h^{\beta_b} v)\|_{2,\beta_c} \lesssim \|E_h^{\beta_b} v\|_{2,\beta_b} \lesssim \|v\|_{2,h,\beta_b}.$$

Furthermore, it is easy to check that w_c is in $V_h^{NZT}(\beta_c)$ and satisfies the conditions (24), so the last estimate and (23) together allows

$$|\hat{v}|_{2,h,\beta}^2 = |v_c|_{2,h,\beta_c}^2 + |v|_{2,h,\beta_b}^2 \lesssim |w_c|_{2,h,\beta_c}^2 + |v|_{2,h,\beta_b}^2 \lesssim \|v\|_{2,h,\beta_b}^2,$$

leading to the right side inequality of (22). The proof is complete. □

Remark 3.1. *It is easy to check that even for any function $v \in V_{0h}^{NZT}(\beta)$, there holds*

$$\|v\|_{2,h,\beta_b}^2 \le C Z_h^\beta(v,v),$$

which is a direct consequence of Lemma 3.2.

Remark 3.2. *Lemma 3.5 can be viewed as a spectral equivalence lemma corresponding to the finite element space $V_{0h}^{NZT}(\beta)$. As shown in* [20, 21]*, such kind of results play important roles in convergence rate analysis of nonoverlapping DDMs and are usually derived by the discrete extension theorems under some additional conditions over triangulations. Here, we derive the above lemma in view of the extension theorem for Sobolev spaces, avoiding the use of the so-called discrete extension theorem. So, it holds even for a regular family of triangulations.*

4. A nonoverlapping DDM and convergence rate analysis

Similar to the second algorithm in [15], we can construct a nonoverlapping domain decomposition method for solving problem (9), described as follows.

Algorithm 1 The Body-Plate Alternating Method.

Let $\boldsymbol{\lambda}_h^0 \in (V_h^1(\beta_b))^2 \times V_h^{NZT}(\beta_b)$ be any given vector-valued function, and $\theta \in (0,1)$ a fixed parameter. Set $n = 0$.

Step 1. Sequentially solve the problems

$$\left\{ \begin{array}{l} \boldsymbol{u}_h^{\alpha,n+1} \in (V_h^1(\alpha))^3, \\ \boldsymbol{u}_h^{\alpha,n+1}(p) = \boldsymbol{\lambda}_h^n(p) \quad \forall\, p \in \beta_b, \\ D_h^\alpha(\boldsymbol{u}_h^{\alpha,n+1}, \boldsymbol{v}_h^\alpha) = F^\alpha(\boldsymbol{v}_h^\alpha) \quad \forall\, \boldsymbol{v}_h^\alpha \in (V_h^1(\alpha;\beta_b))^3 \end{array} \right. \tag{26}$$

and

$$\left\{ \begin{array}{l} \boldsymbol{u}_h^{\beta,n+1} \in (V_{0h}^1(\beta))^2 \times V_{0h}^{NZT}(\beta), \\ D_h^\beta(\boldsymbol{u}_h^{\beta,n+1}, \boldsymbol{v}_h^\beta) = F^\beta(\boldsymbol{v}_h^\beta) + [F^\alpha(\boldsymbol{v}_h^\alpha) - D_h^\alpha(\boldsymbol{u}_h^{\alpha,n+1}, \boldsymbol{v}_h^\alpha)], \end{array} \right. \tag{27}$$

where $\boldsymbol{v}_h^\beta \in (V_{0h}^1(\beta)^2 \times V_{0h}^{NZT}(\beta)$ and $\boldsymbol{v}_h^\alpha \in (V_h^1(\alpha))^3$ satisfy $\boldsymbol{v}_h^\alpha(p) = \boldsymbol{v}_h^\beta(p)$ $\forall\, p \in \beta_b$.

Step 2. Let

$$\boldsymbol{\lambda}_h^{n+1} = \theta \boldsymbol{\lambda}_h^n + (1-\theta)\boldsymbol{u}_h^{\beta,n+1}\,|_{\beta_b}\,. \tag{28}$$

Step 3. Set $n := n+1$. Goto Step 1 and repeat the above iteration until convergence.

Remark 4.1. *The above algorithm can be naturally extended to deal with the numerical solution of more complicated elastic multi-structure problems as given in* [16, 17]*. However, it is a very difficult issue to develop the convergence rate analysis in this case. It is our forthcoming work to attack this difficulty.*

Let $\boldsymbol{\mu}_h^n := \boldsymbol{\lambda}_h^n - \boldsymbol{u}_h^\beta|_{\beta_b}$, $\boldsymbol{\delta}_h^{\alpha,n} := \boldsymbol{u}_h^{\alpha,n} - \boldsymbol{u}_h^\alpha$ and $\boldsymbol{\delta}_h^{\beta,n} := \boldsymbol{u}_h^{\beta,n} - \boldsymbol{u}_h^\beta$. From (9)-(10), and (26)-(28), we find that $\boldsymbol{\delta}_h^{\alpha,n}$ and $\boldsymbol{\delta}_h^{\beta,n}$ must satisfy the following conditions

$$\left\{ \begin{array}{l} \boldsymbol{\delta}_h^{\alpha,n+1} \in (V_h^1(\alpha))^3, \\ \boldsymbol{\delta}_h^{\alpha,n+1}(p) = \boldsymbol{\mu}_h^n(p) \quad \forall\, p \in \beta_b, \\ D_h^\alpha(\boldsymbol{\delta}_h^{\alpha,n+1}, \boldsymbol{v}_h^\alpha) = 0 \quad \forall\, \boldsymbol{v}_h^\alpha \in (V_h^1(\alpha;\beta_b))^3, \end{array} \right. \tag{29}$$

$$
\left\{ \begin{array}{l} \boldsymbol{\delta}_h^{\beta,n+1} \in (V_{0h}^1(\beta))^2 \times V_{0h}^{NZT}(\beta), \\ D_h^\beta(\boldsymbol{\delta}_h^{\beta,n+1}, \boldsymbol{v}_h^\beta) + D_h^\alpha(\boldsymbol{\delta}_h^{\alpha,n+1}, \boldsymbol{v}_h^\alpha) = 0, \end{array} \right. \tag{30}
$$

$$
\widetilde{\boldsymbol{\mu}}_h^{n+1} = \theta \widetilde{\boldsymbol{\mu}}_h^n + (1-\theta)\boldsymbol{\delta}_h^{\beta,n+1}. \tag{31}
$$

where $\boldsymbol{v}_h^\alpha$ and $\boldsymbol{v}_h^\beta$ are defined as in (27), and $\widetilde{\boldsymbol{\mu}}_h^n$ denotes the extension of the vector-valued function $\boldsymbol{\mu}_h^n$ on β_b such that

$$
\left\{ \begin{array}{l} \widetilde{\boldsymbol{\mu}}_h^n(p) = \boldsymbol{\mu}_h^n(p),\ \partial_I \widetilde{\mu}_{3,h}^n(p) = \partial_I \mu_{3,h}^n(p),\ I = 1,2, \forall\ p \in \beta_b, \\ D_h^{\beta_c}(\widetilde{\boldsymbol{\mu}}_h^n, \boldsymbol{v}_h^{\beta_c}) = 0 \quad \forall\ \boldsymbol{v}_h^{\beta_c} \in (V_h^1(\beta_c;\gamma_2))^2 \times V_h^{NZT}(\beta_c;\gamma_2), \end{array} \right.
$$

where

$$
V_h^1(\beta_c;\gamma_2) := \{v_h \in V_h^1(\beta_c);\ v_h(p) = 0\ \forall\ p \in \gamma_2\},
$$

and $V_h^{NZT}(\beta_c;\gamma_2)$ is defined as in the last section.

To establish convergence analysis for the above method, we require the next two lemmas. The first lemma can be derived following the proof of Lemma 4 in [15], combined with Lemmas 3.1 and 3.5 developed in Section 3. The second lemma is a consequence of the first one (cf. Lemma 4.1). We omit the details for simplicity.

Lemma 4.1. *There exists a positive constants C^*, independent of the finite element mesh size h, such that, for any vector-valued function $\boldsymbol{v}^\beta \in (V_{0h}^1(\beta))^2 \times V_{0h}^{NZT}(\beta)$ we have*

$$
D_h^\alpha(\boldsymbol{v}_\alpha^\beta, \boldsymbol{v}_\alpha^\beta) \leq C^* D_h^\beta(\boldsymbol{v}^\beta, \boldsymbol{v}^\beta),
$$

where $\boldsymbol{v}_\alpha^\beta \in (V_h^1(\alpha))^3$ is uniquely determined by the conditions

$$
\left\{ \begin{array}{ll} \boldsymbol{v}_\alpha^\beta(p) = \boldsymbol{v}^\beta(p) & \forall\ p \in \beta_b, \\ D_h^\alpha(\boldsymbol{v}_\alpha^\beta, \boldsymbol{w}) = 0 & \forall\ \boldsymbol{w} \in (V_h^1(\alpha;\beta_b))^3. \end{array} \right.
$$

Lemma 4.2. *For any vector-valued function $\boldsymbol{v}^\alpha \in (V_h^1(\alpha))^3$ and $\boldsymbol{v}^\beta \in (V_{0h}^1(\beta))^2 \times V_{0h}^{NZT}(\beta)$ which satisfy the conditions*

$$
\left\{ \begin{array}{ll} D_h^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{w}) = 0 & \forall\ \boldsymbol{w} \in (V_h^1(\alpha,\beta_b))^3, \\ D_h^\alpha(\boldsymbol{v}^\alpha, [\boldsymbol{w}]^\alpha) + D_h^\beta(\boldsymbol{v}^\beta, \boldsymbol{w}) = 0 & \forall\ \boldsymbol{w} \in (V_{0h}^1(\beta))^2 \times V_{0h}^{NZT}(\beta), \end{array} \right.
$$

we have the estimate

$$
D_h^\beta(\boldsymbol{v}^\beta, \boldsymbol{v}^\beta) \leq C^* D_h^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{v}^\alpha),
$$

where C^* is given as in Lemma 4.1.

We are now ready to state and prove the main theorem in this article.

Theorem 4.1. *There exists a fixed parameter $\theta^* \in (0,1)$, independent of the finite element mesh size h, such that, if $\theta^* \leq \theta < 1$, we have the following estimates for the Body-Plate Alternating Method (cf. Algorithm 1):*

$$
\begin{array}{l} \|\boldsymbol{\mu}_h^n\|_h \leq C\rho(\theta)^n \|\boldsymbol{\mu}_h^0\|_h, \\ D_h^\alpha(\boldsymbol{\delta}_h^{\alpha,n}, \boldsymbol{\delta}_h^{\alpha,n}) + D_h^\beta(\boldsymbol{\delta}_h^{\beta,n}, \boldsymbol{\delta}_h^{\beta,n}) \leq C\rho(\theta)^{2n} \|\boldsymbol{\mu}_h^0\|_h^2. \end{array} \tag{32}
$$

Here the explicit form of $\rho(\theta)$ and θ^* are given in the proof below and $\rho(\theta^*) = \sqrt{\theta^*}$, and for $\boldsymbol{v} \in (V_h^1(\beta_b))^2 \times V_h^{NZT}(\beta_b)$,

$$
\|\boldsymbol{v}\|_h := \left(\sum_{i=1}^2 \|v_i\|_{1,\beta_b}^2 + \|v_3\|_{2,h,\beta_b}^2 \right)^{1/2}.
$$

Proof. We proceed by similar arguments for proving Theorem 2 in [15]. In fact, from (29),

$$\boldsymbol{\delta}_h^{\alpha,n+1}(p) = \boldsymbol{\mu}_h^n(p) \quad \forall\, p \in \beta_b,$$

so we can choose $\boldsymbol{v}_h^\beta = \widetilde{\boldsymbol{\mu}}_h^n$, $\boldsymbol{v}_h^\alpha = \boldsymbol{\delta}_h^{\alpha,n+1}$ in (30) to get

$$D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^n, \boldsymbol{\delta}_h^{\beta,n+1}) = -D_h^\alpha(\boldsymbol{\delta}_h^{\alpha,n+1}, \boldsymbol{\delta}_h^{\alpha,n+1}) \le 0. \tag{33}$$

Since $\boldsymbol{\delta}_h^{\beta,n+1}$ and $\boldsymbol{\delta}_h^{\alpha,n+1}$ satisfy (29) and (30), an application of Lemma 4.2 implies

$$D_h^\beta(\boldsymbol{\delta}_h^{\beta,n+1}, \boldsymbol{\delta}_h^{\beta,n+1}) \le C^* D_h^\alpha(\boldsymbol{\delta}_h^{\alpha,n+1}, \boldsymbol{\delta}_h^{\alpha,n+1}) \tag{34}$$

with the constant C^* the same as in Lemma 4.1. Using (33)-(34) and the Cauchy-Schwarz inequality, we find

$$\begin{aligned} D_h^\alpha(\boldsymbol{\delta}_h^{\alpha,n+1}, \boldsymbol{\delta}_h^{\alpha,n+1}) &\le \Big(D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^n, \widetilde{\boldsymbol{\mu}}_h^n)\Big)^{1/2} \Big(D_h^\beta(\boldsymbol{\delta}_h^{\beta,n+1}, \boldsymbol{\delta}_h^{\beta,n+1})\Big)^{1/2} \\ &\le \sqrt{C^*}\Big(D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^n, \widetilde{\boldsymbol{\mu}}_h^n)\Big)^{1/2} \Big(D_h^\alpha(\boldsymbol{\delta}_h^{\alpha,n+1}, \boldsymbol{\delta}_h^{\alpha,n+1})\Big)^{1/2}, \end{aligned}$$

and with (34) we further have

$$D_h^\beta(\boldsymbol{\delta}_h^{\beta,n+1}, \boldsymbol{\delta}_h^{\beta,n+1}) \le C^* D_h^\alpha(\boldsymbol{\delta}_h^{\alpha,n+1}, \boldsymbol{\delta}_h^{\alpha,n+1}) \le C^{*2} D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^n, \widetilde{\boldsymbol{\mu}}_h^n).$$

On the other hand, from (31), (33) and the last estimate it follows that

$$\begin{aligned} D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^{n+1}, \widetilde{\boldsymbol{\mu}}_h^{n+1}) &\le \theta^2 D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^n, \widetilde{\boldsymbol{\mu}}_h^n) + (1-\theta)^2 D_h^\beta(\boldsymbol{\delta}_h^{\beta,n+1}, \boldsymbol{\delta}_h^{\beta,n+1}) \\ &\le [\theta^2 + C^{*2}(1-\theta)^2] D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^n, \widetilde{\boldsymbol{\mu}}_h^n) = \rho(\theta)^2 D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^n, \widetilde{\boldsymbol{\mu}}_h^n), \end{aligned}$$

where $\rho(\theta) := [\theta^2 + C^{*2}(1-\theta)^2]^{1/2}$, which takes its minimum at $\theta^* = C^{*2}/(1+C^{*2}) \in (0,1)$ with $\rho(\theta^*) = \sqrt{\theta^*}$. Moreover, $0 < \rho(\theta) < 1$ whenever $\theta^* \le \theta < 1$. Therefore,

$$D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^n, \widetilde{\boldsymbol{\mu}}_h^n) \le \rho(\theta)^{2n} D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^0, \widetilde{\boldsymbol{\mu}}_h^0). \tag{35}$$

Arguing as in the proof of Lemma 3.5, we know

$$\|\boldsymbol{\mu}_h^n\|_h^2 \lesssim D_h^\beta(\widetilde{\boldsymbol{\mu}}_h^n, \widetilde{\boldsymbol{\mu}}_h^n) \lesssim \|\boldsymbol{\mu}_h^n\|_h^2,$$

which with (35) implies

$$\|\boldsymbol{\mu}_h^n\|_h \le C\rho(\theta)^n \|\boldsymbol{\mu}_h^0\|_h. \tag{36}$$

The second inequality in (32) follows readily from (34), (36) and Lemma 4.2. □

5. Numerical examples

Consider an elastic body member $\alpha := (-1/2, 1/2)^2 \times (0,1)$ and an elastic plate member $\beta := (-1,1)^2 \times \{0\}$. They are rigidly connected along the interface $\beta_b := (-1/2,1/2)^2 \times \{0\}$ to form an elastic body-plate structure Ω.

Let the displacement field $\boldsymbol{u}^\alpha := u_i^\alpha \boldsymbol{e}_i$ on α and the displacement field $\boldsymbol{u}^\beta := u_i^\beta \boldsymbol{e}_i$ on β be given respectively by

$$\begin{cases} u_1^\alpha := (1-x_1^2)(1-x_2^2)(1-4x_1^2)^2(1-4x_2^2)^2(1-x_3)^2, \\ u_2^\alpha := (1-x_1^2)(1-x_2^2)(1-4x_1^2)^2(1-4x_2^2)^2(1-x_3)^2, \\ u_3^\alpha := (1-x_1^2)^2(1-x_2^2)^2(1-4x_1^2)^2(1-4x_2^2)^2(1-x_3)^2, \end{cases}$$

where $-1/2 < x_1,\, x_2 < 1/2$ and $0 < x_3 < 1$,

$$\begin{cases} u_1^\beta := (1-x_1^2)(1-x_2^2)(1-4x_1^2)^2(1-4x_2^2)^2, \\ u_2^\beta := (1-x_1^2)(1-x_2^2)(1-4x_1^2)^2(1-4x_2^2)^2, \\ u_3^\beta := (1-x_1^2)^2(1-x_2^2)^2(1-4x_1^2)^2(1-4x_2^2)^2, \end{cases}$$

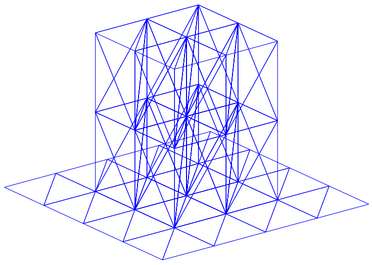

FIGURE 2. The triangulation $\mathcal{T}_h^{\Omega}$ for the body-plate problem.

TABLE 1. The related energy errors vs h for the first example when $\theta = 0.75$.

h	DOF	$R_{n,h}$	n
1/4	780	9.7597E-07	55
1/8	3632	7.4515E-07	31
1/16	20184	8.1101E-07	32
1/32	128936	9.4642E-07	32

where $-1 < x_1, x_2 < 1$. Since $\boldsymbol{u}$ is available, $\boldsymbol{f}^{\alpha}$ and $\boldsymbol{f}^{\beta}$ can be computed explicitly by equilibrium equations.

As shown in Fig. 2, introduce a family of triangulations for the body-plate structure, whose mesh size is denoted by h. Concretely speaking, we partition β into $(2N)^2$ equal squares with the length $h = 1/N$, and then divide each square into two triangles in the same direction, so that we get the triangulation $\mathcal{T}_h^{\beta}$. The triangulation $\mathcal{T}_h^{\alpha}$ is obtained similarly. We use the cubature over tetrahedrons proposed in [10] for assembling stiffness matrix on the body and the PCG with classic AMG preconditioners (cf. [3, 22]) for solving the discrete problems (26) and (27). The algorithm terminates with the error criterion $R_{n,h} < 10^{-6}$, and all the initial functions are taken as zero functions.

Define the energy error and the relative energy error at the n-th iteration step by

$$
\begin{aligned}
E_{n,h} &:= (D_h^{\alpha}(\boldsymbol{u}_h^{\alpha,n} - \boldsymbol{u}_h^{\alpha}, \boldsymbol{u}_h^{\alpha,n} - \boldsymbol{u}_h^{\alpha}) + D_h^{\beta}(\boldsymbol{u}_h^{\beta,n} - \boldsymbol{u}_h^{\beta}, \boldsymbol{u}_h^{\beta,n} - \boldsymbol{u}_h^{\beta}))^{1/2},\\
R_{n,h} &:= E_{n,h}/(D_h^{\alpha}(\boldsymbol{u}_h^{\alpha}, \boldsymbol{u}_h^{\alpha}) + D_h^{\beta}(\boldsymbol{u}_h^{\beta}, \boldsymbol{u}_h^{\beta}))^{1/2}.
\end{aligned}
$$

In the first example, we assume the elastic body-plate structure is made of grey cast iron. Thus, we choose $E_{\alpha} = 120$, $\nu_{\alpha} = 0.25$, $E_{\beta} = 120$, $\nu_{\beta} = 0.25$, and $t_{\beta} = 0.2$. The computational results of $E_{n,h}$ and $R_{n,h}$ with the parameter $\theta = 0.75$ for different choices of h are given in Fig. 3 and Table 1, respectively. The computational results of $R_{n,h}$ with the parameter $\theta = 0.9$ for different choices of h are given in Table 2. From these data we see that the convergence rate of the method is insensitive to the finite element mesh size, which coincides with our theoretical estimate. When $h = 1/16$ and θ is taken as different values, the numerical results for $E_{n,h}$ and $R_{n,h}$ are listed in Table 3, which indicate that the convergence rate is greatly influenced by the choice of the parameter θ. As shown in [15], the desired parameter θ can be obtained by the power method or by numerical experience to avoid more computational cost.

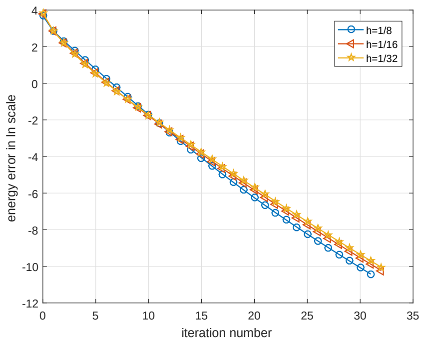

FIGURE 3. Variation of the energy error in ln scale vs the iteration number for the first example when $\theta = 0.75$.

TABLE 2. The related energy errors vs h for the first example when $\theta = 0.9$.

h	DOF	$R_{n,h}$	n
1/4	780	9.8498E-07	78
1/8	3632	9.4591E-07	83
1/16	20184	9.8024E-07	86
1/32	128936	8.8077E-07	88

TABLE 3. The energy errors and related energy errors vs θ for the first example when $h = 1/16$.

θ	0.7	0.75	0.8	0.85	0.9
n	63	32	41	56	86
$E_{n,h}$	4.0581E-05	3.6072E-05	4.9592E-05	4.1327E-05	4.3598E-05
$R_{n,h}$	9.1239E-07	8.1101E-07	8.5619E-07	9.2917E-07	9.8024E-07

TABLE 4. The related energy errors vs h for the second example when $\theta = 0.9$.

h	DOF	$R_{n,h}$	n
1/4	780	9.3426E-07	86
1/8	3632	9.5565E-07	76
1/16	20184	9.2949E-07	81
1/32	128936	9.1357E-07	83

In the second example, we consider the case where the elastic material is made of nickel-chromium steel. Thus, we choose $E_\alpha = 206$, $\nu_\alpha = 0.3$, $E_\beta = 206$, $\nu_\beta = 0.3$, and $t_\beta = 0.14$. To simplify the presentation, we only list in Table 4 the computational results of $R_{n,h}$ for $\theta = 0.9$ with different choices of h. We may find again from these numerical results that the convergence rate of our domain decomposition method is insensitive to the finite element mesh size. Furthermore, comparing the numerical results in Tables 2 and 4, we may conclude that the convergence rate of the method depends on the choice of physical parameters of the underlying problem, but the influence is not very sensitive.

In summary, all of the above numerical results have demonstrated that the Body-Plate Alternating Method proposed here performs well in solving the general elastic body-plate problem.

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