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LINEAR AND QUADRATIC IMMERSED FINITE ELEMENT METHODS FOR THE MULTI-LAYER POROUS WALL MODEL FOR CORONARY DRUG-ELUTING STENTS

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Abstract. In this paper, we consider a multi-layer porous wall model for coronary drug-eluting stents that leads to an interface problem whose coefficients have multiple discontinuous points, and an imperfect contact interface jump condition is imposed at the first discontinuous point where the stent meets the artery. The existence and uniqueness of the solution to the related weak problem are established. A linear and a quadratic immersed finite element (IFE) methods are developed for solving this interface problem. Error estimation is carried out to show that the proposed IFE methods converge optimally. Numerical examples are presented to demonstrate features of these IFE methods.

Key words. Linear immersed interface method, quadratic immersed interface method, multi-layer porous wall model, coronary drug-eluting stents, imperfect contact interface point.

1. Introduction

It is known that alteration of blood flow due to the narrowing or occlusion of an artery is one of the most common occurrences in cardiovascular diseases. A treatment for cardiovascular diseases is alteration of blood flow in which, in order to hold open and to provide structural stability to the damaged vessel, a drug-eluting stent (DES) is inserted in the artery. A number of mathematical models [40, 41, 44] are proposed to simulate the drug transfer in the arterial wall in this kind of treatment. As is well known that the arterial wall consists of many layers with different structural and chemical properties [23]. It is believed that a better modeling of the wall structure brings us a more effective description of the drug release from a DES. One of these complete wall models is the multi-layer wall model that takes into account the heterogeneous properties of the different layers constituting the arterial wall. Because the mass dynamics mainly occurs along the direction normal to the stent's coating, G. Pontrelli and F. Monte proposed a simplified one-dimensional (1D) multi-layer porous wall model in [42], see the illustration in Fig. 1 which is based on Fig. 2 in [42].

First of all, let us review this model briefly. In a general 1D framework, we consider a set of intervals $[\alpha_{i-1}, \alpha_i]$, $i = 0, 1, 2, \ldots, n$, having thickness $l_i = \alpha_i - \alpha_{i-1}$ modeling the drug coating ($i = 0$) and the arterial wall layers ($i = 1, 2, \ldots, n$), as shown in Fig. 1. At the initial time ($t = 0$), the drug is contained only in the coating and it is distributed with maximum concentration u_0 and, subsequently, released into the arterial wall. Here, and throughout this paper, a mass volume-averaged concentration $u(x, t)$ is considered.

We know that the metallic strut is impermeable to the drug, so there is no mass flux passes through the boundary surface at $x = \alpha_{-1}$. Thus, the dynamics of the drug in the coating $[\alpha_{-1}, \alpha_0]$ should satisfy the following 1D diffusion equation and

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related boundary-initial conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(-D_0\dfrac{\partial u}{\partial x}) = 0,\, x \in [\alpha_{-1}, \alpha_0], \\ -D_0\dfrac{\partial u}{\partial x} = 0,\, x = \alpha_{-1}, \\ u(x, 0) = u_0, \end{cases}$$

where D_0 is the drug diffusivity, u_0 the concentration in the coating.

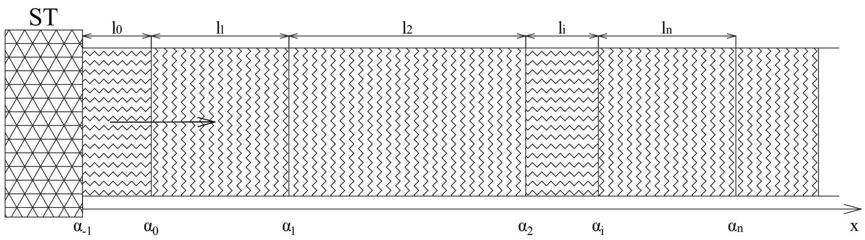

FIGURE 1. A sketch of the layered wall. ST indicates the metallic stent strut bearing the polymeric coating, while $[\alpha_{-1}, \alpha_0]$ means the polymeric coating. The continuous wall layers are defined by $[\alpha_{i-1}, \alpha_i]$, $i = 1, 2, \ldots, n$. This illustration is based on Fig. 2 in [42].

To prolong the drug release time, we need to slow down the drug release rate. To achieve this goal, a permeable membrane (called topcoat) of permeability p is placed at the interface ($x = \alpha_0$) between the coating and the arterial wall. Thus, the mass flux passed through it is continuous while the drug concentration might have a possible jump. In this case, the mass transfer through the topcoat can be described by the following second Kedem-Katchalsky equation:

$$\begin{cases} -D_0\dfrac{\partial u(\alpha_0^-)}{\partial x} = p(\dfrac{u(\alpha_0^-)}{c_0} - \dfrac{u(\alpha_0^+)}{c_1}), \\ D_0\dfrac{\partial u(\alpha_0^-)}{\partial x} = D_1\dfrac{\partial u(\alpha_0^+)}{\partial x} - 2\delta_1 u(\alpha_0^+), \end{cases}$$

where, c_0 and c_1 are two constants relevant to the porosity. Hereafter, D_i is the diffusivity of drug and δ_i denotes for a constant characteristic convection parameter in $[\alpha_{i-1}, \alpha_i]$, $i = 1, 2, \ldots, n$.

Then, we consider the drug transfer in the layers of the arterial wall. In the i-th layer, the drug transfer obeys the following advection-diffusion-reaction equation and related initial conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(-D_i\dfrac{\partial u}{\partial x} + 2\delta_i u) + \beta_i u = 0,\, x \in (\alpha_{i-1}, \alpha_i),\, i = 1, 2, \ldots, n, \\ u(x, 0) = 0,\, x \in (\alpha_{i-1}, \alpha_i), \\ u(\alpha_n, t) = 0, \end{cases}$$

where β_i denotes for the drug reaction coefficient in $[\alpha_{i-1}, \alpha_i]$. To close this mass transfer system, the jump conditions requiring the continuity of flux and concentration are assigned at each interface point $x = \alpha_i$ $(i = 1, 2, \dots, n-1)$:

$$\begin{cases} u(\alpha_i^-) = u(\alpha_i^+), \\ D_i \dfrac{\partial u(\alpha_i^-)}{\partial x} - 2\delta_i u(\alpha_i^-) = D_{i+1} \dfrac{\partial u(\alpha_i^+)}{\partial x} - 2\delta_{i+1} u(\alpha_i^+). \end{cases}$$

In this paper, we develop immersed finite element (IFE) spaces for the steady state multi-layer porous wall model. The steady state model has its own importance and the IFE spaces developed are also applicable to the dynamic multi-layer porous wall model which we plan to address in a follow up article. Specifically, the steady state multi-layer porous wall model is the following interface problem for the mass volume-averaged concentration $u(x)$:

$$(1) \qquad (-Du'(x) + 2\delta u(x))' + \beta u(x) = f(x),\ x \in [\alpha_{i-1}, \alpha_i],\ i = 1, 2, \dots, n,$$

$$(2) \qquad D_0 u'(\alpha_{-1}) = 0,\ u_n(\alpha_n) = 0,$$

with the imperfect contact jump conditions at the first interface point:

$$(3a) \qquad \begin{cases} -\lambda D_0 u'(\alpha_0^-) = u(\alpha_0^-) - u(\alpha_0^+), \\ D_0 u'(\alpha_0^-) = D_1 u'(\alpha_0^+) - 2\delta_1 u(\alpha_0^+), \end{cases}$$

and the usual concentration and flux continuity jump conditions at other interface points:

$$(3b) \qquad \begin{cases} u(\alpha_i^-) = u(\alpha_i^+), \\ D_i u'(\alpha_i^-) - 2\delta_i u(\alpha_i^-) = D_{i+1} u'(\alpha_i^+) - 2\delta_{i+1} u(\alpha_i^+), \end{cases} \qquad 1 \le i \le n-1,$$

where $\lambda = \frac{1}{p}$, $D(x)$, $\delta(x)$, $\beta(x)$ and $f(x)$ are functions on (α_{-1}, α_n) such that

$$(4) \qquad \begin{cases} D(x) = D_i \text{ when } x \in (\alpha_{i-1}, \alpha_i),\ 0 \le i \le n, \\ \delta(x) = \delta_i \text{ when } x \in (\alpha_{i-1}, \alpha_i),\ 0 \le i \le n,\ \delta_0 = 0, \\ \beta(x) = \beta_i \text{ when } x \in (\alpha_{i-1}, \alpha_i),\ 0 \le i \le n,\ \beta_0 = 0, \\ f(x) = f_i(x) \text{ when } x \in (\alpha_{i-1}, \alpha_i),\ 0 \le i \le n. \end{cases}$$

We also assume that $D_i > 0$ for $0 \le i \le n$ and δ_i, $\beta_i > 0$ for $1 \le i \le n$. It is clearly that this is an elliptic interface problem with two types of interface points: the imperfect contact interface point at α_0 and the rough coefficient interface points at α_j, $1 \le j \le n-1$.

Generally speaking, two groups of numerical methods have been introduced for the interface problems. Methods in one group are based on the finite difference formulation, such as the immersed interface method [25], the matched interface and boundary methods [49], and the ghost fluid method [13]. More details about immersed interface methods based on the finite difference formulation can be found in [27]. Methods in another group are based on finite element (FE) formulation, such as the penalty discontinuous method [3, 8], the unfitted finite element method [17], the discontinuous Galerkin method [9, 16]. These methods modify the weak formulation of finite element methods when treating with the element near the interface.

Proposed in [6], another idea of solving interface problems with rough coefficients is to use a finite element space constructed specifically according to the problem to be solved. One of the popular technique is to modify the approximating functions around the interface, such as the general finite element method [4, 5], the multiscale finite element method [11, 12], and the partition of unity method [7, 39]. Along this idea, the immersed finite element (IFE) method has be developed to solve interface problems with meshes independent of the interface [1, 2, 18, 19, 21, 22, 26, 28, 31, 38, 43, 45], see [10, 14, 20, 32, 33, 34, 35, 36, 37, 46] for more details. The basic idea of IFE method is to construct special basis functions according to the jump conditions on interface elements while using standard basis functions on the non-interface elements. There have been publications about solving second order elliptic interface problems [1, 2, 15, 18, 19, 22, 24, 26, 28, 29, 30, 38, 47, 48], but, to our best knowledge, no IFE methods have been developed for dealing with the the imperfect contact condition with convection and reaction such as the jump condition imposed at α_0 in the interface problem described by (1)-(3) for the mathematical modeling of the drug transfer from the stent coat to the arterial wall.

This paper is organized as follows. In Section 2, we will consider the weak formulation of the interface problem described by (1)-(3). Then we will develop and analyze a linear and a quadratic IFE methods in Section 3 and Section 4, respectively. In Section 5, we present numerical examples for the IFE method developed in this article. Finally, some conclusions are given in Section 6.

2. Setting and Weak Solution

In this section, we study the existence and uniqueness of the weak solution for the interface problem described by (1)-(3). We start from the following space for the weak problem:

$$H^k_\alpha(\alpha_{-1},\alpha_n) = \left\{v \in L^2(\alpha_{-1},\alpha_n) \mid v|_{\Omega^\pm} \in H^k(\Omega^\pm),\, v(\alpha_n)=0\right\}, \tag{5}$$

where $k \geq 1$ is an integer and $\Omega^- = (\alpha_{-1},\alpha_0)$, $\Omega^+ = (\alpha_0,\alpha_n)$. On $H^k_\alpha(\alpha_{-1},\alpha_n)$, we will use the following norm:

$$\|u\|_{k,(\alpha_{-1},\alpha_n)} = \sqrt{\|u\|^2_{k,\Omega^-} + \|u\|^2_{k,\Omega^+}}. \tag{6}$$

We also use $||\cdot||_{0,(\alpha_{-1},\alpha_n)}$ to denote the L2 norm throughout this paper. By the standard procedure, the interface problem described by (1)-(3) leads to the following weak problem: find $u \in H^1_\alpha(\alpha_{-1},\alpha_n)$ such that

$$a(u,v) = (f,v),\ \forall\, v \in H^1_\alpha(\alpha_{-1},\alpha_n), \tag{7}$$

where the bilinear form $a(u,v)$ is defined as

$$\begin{aligned} a(u,v) &= \frac{[v]_{\alpha_0}[u]_{\alpha_0}}{\lambda} \\ &+ \int_{\alpha_{-1}}^{\alpha_n} \left(v'(x)(Du'(x) - 2\delta u(x)) + \beta v(x)u(x)\right) dx,\ \forall\, u,\, v \in H^1_\alpha(\alpha_{-1},\alpha_n). \end{aligned} \tag{8}$$

We now consider a bilinear form related with $a(u,v)$:

(9)

$$a_0(u,v) = \frac{[v]_{\alpha_0}[u]_{\alpha_0}}{\lambda} + \int_{\alpha_{-1}}^{\alpha_n} \left(Dv'(x)u'(x) + \beta v(x)u(x)\right) dx,\ \forall\, u,\, v \in H^1_\alpha(\alpha_{-1},\alpha_n).$$

It is obvious that $a_0(u,v)$ is a symmetric semi-positive-definite bilinear form on $H^1_\alpha(\alpha_{-1},\alpha_n)$. Now, let $u \in H^1_\alpha(\alpha_{-1},\alpha_n)$ be such that $a_0(u,u) = 0$. Then

$$\frac{([u]_{\alpha_0})^2}{\lambda} + \int_{\alpha_{-1}}^{\alpha_n} \left(D(u'(x))^2 + \beta(u(x))^2\right) dx = a_0(u,u) = 0, \tag{10}$$

which, because of the positiveness of D and β described in (4), leads to $\|u\|_{1,(\alpha_0,\alpha_n)} = 0$. This further implies $u(\alpha_0^+) = 0$. Then, by (10), we have

$$\frac{(u(\alpha_0^-))^2}{\lambda} + \int_{\alpha_{-1}}^{\alpha_0} D_0(u'(x))^2\,dx = 0. \tag{11}$$

Thus, $u(\alpha_0^-) = 0$ and $u'(x) = 0, \forall x \in (\alpha_{-1}, \alpha_0)$. Therefore, $u = 0$ on (α_{-1}, α_0) and we have $u = 0$ on (α_{-1}, α_n). All of these show that $a_0(u, v)$ is a symmetric positive-definite bilinear form on $H^1_\alpha(\alpha_{-1}, \alpha_n)$; hence, we can use it to define a norm as follows:

$$\|u\|_{a_0} = \left(\frac{([u]_{\alpha_0})^2}{\lambda} + \int_{\alpha_{-1}}^{\alpha_n} \left(D(u'(x))^2 + \beta(u(x))^2\right) dx\right)^{1/2}, \forall\, u \in H^1_\alpha(\alpha_{-1}, \alpha_n).$$

Lemma 2.1. *There exist positive constants $C_i(D, \beta, \lambda)$, $i = 1, 2$ such that*

$$C_1 \|u\|_{1,(\alpha_{-1},\alpha_n)} \le \|u\|_{a_0} \le C_2 \|u\|_{1,(\alpha_{-1},\alpha_n)}, \forall\, u \in H^1_\alpha(\alpha_{-1}, \alpha_n). \tag{12}$$

Proof. The existence of $C_2(D, \lambda, \beta)$ for (12) follows from the definition of $\|\cdot\|_{a_0}$ and the Sobolev imbedding theorem; hence, we prove the first inequality by showing the continuity of identity mapping:

$$I : \left(H^1_\alpha(\alpha_{-1}, \alpha_n), \|\cdot\|_{a_0}\right) \to \left(H^1_\alpha(\alpha_{-1}, \alpha_n), \|\cdot\|_{1,(\alpha_{-1},\alpha_n)}\right),$$

here, by $\left(H^1_\alpha(\alpha_{-1}, \alpha_n), \|\cdot\|_{a_0}\right)$ and $\left(H^1_\alpha(\alpha_{-1}, \alpha_n), \|\cdot\|_{1,(\alpha_{-1},\alpha_n)}\right)$, we consider the linear space $H^1_\alpha(\alpha_{-1}, \alpha_n)$ as normed spaces with norm $\|\cdot\|_{a_0}$ and norm $\|\cdot\|_{1,(\alpha_{-1},\alpha_n)}$, respectively. Let $u \in H^1_\alpha(\alpha_{-1}, \alpha_n)$ and let $\{u_m\}_{i=1}^\infty \subseteq H^1_\alpha(\alpha_{-1}, \alpha_n)$ be a sequence such that $\lim_{m\to\infty} \|u_m - u\|_{a_0} = 0$. Then, by

$$\min_{1\le i\le n} \{D_i, \beta_i\} \|u_m - u\|^2_{1,(\alpha_0,\alpha_n)} \le \|u_m - u\|^2_{a_0},$$

we have $\lim_{m\to\infty} \|u_m - u\|_{1,(\alpha_0,\alpha_n)} = 0$. This implies that $\lim_{m\to\infty} \left|u_m(\alpha_0^+) - u(\alpha_0^+)\right| = 0$. Then, from

$$\begin{aligned}
&\left|u_m(\alpha_0^-) - u(\alpha_0^-)\right| \\
\le\ & \left|u_m(\alpha_0^+) - u(\alpha_0^+)\right| + \lambda^{1/2} \frac{\left|(u_m(\alpha_0^+) - u(\alpha_0^+)) - (u_m(\alpha_0^-) - u(\alpha_0^-))\right|}{\lambda^{1/2}} \\
\le\ & \left|u_m(\alpha_0^+) - u(\alpha_0^+)\right| + \lambda^{1/2} \|u_m - u\|_{a_0},
\end{aligned}$$

we have $\lim_{m\to\infty} \left|u_m(\alpha_0^-) - u(\alpha_0^-)\right| = 0$. In addition, from

$$D_0 \|u'_m - u'\|^2_{0,(\alpha_{-1},\alpha_0)} \le \|u_m - u\|_{a_0},$$

we know that $\lim_{m\to\infty} \|u'_m - u'\|_{0,(\alpha_{-1},\alpha_0)} = 0$. Finally, by

$$u_m(x) - u(x) = u_m(\alpha_0^-) - u(\alpha_0^-) - \int_x^{\alpha_0} \left(u'_m(s) - u'(s)\right) ds,$$

it holds $\lim_{m\to\infty} \|u_m - u\|_{0,(\alpha_{-1},\alpha_0)} = 0$. Therefore, $\lim_{m\to\infty} \|u_m - u\|_{1,(\alpha_{-1},\alpha_0)} = 0$ which, together with $\lim_{m\to\infty} \|u_m - u\|_{1,(\alpha_0,\alpha_n)} = 0$, leads to $\lim_{m\to\infty} \|u_m - u\|_{1,(\alpha_{-1},\alpha_n)} = 0$, and the continuity of the identity mapping I is proven. □

Theorem 2.2. *The bilinear form $a(u, v)$ defined by (8) is continuous and coercive under the assumption that $(C_1(D, \lambda, \beta))^2 > 2 \max_{1\le i\le n} \delta_i$. Thus, the weak problem (7) admits a unique solution $u \in H^1_\alpha(\alpha_{-1}, \alpha_n)$ for every $f \in L^2(\alpha_{-1}, \alpha_n)$.*

Proof. Again, the continuity of $a(u,v)$ follows directly from its definition and the Sobolev imbedding theorem. Then, by Lemma 2.1, we have

$$\begin{aligned} a(u,u) &= a_0(u,u) - 2\int_{\alpha_{-1}}^{\alpha_n} \delta u'(x)u(x)\,dx = \|u\|_{a_0}^2 - 2\int_{\alpha_{-1}}^{\alpha_n} \delta u'(x)u(x)\,dx \\ &\geq (C_1(D,\lambda,\beta))^2\,\|u\|_{1,(\alpha_{-1},\alpha_n)}^2 - 2\max_{1\leq i\leq n}\delta_i\,\|u\|_{1,(\alpha_{-1},\alpha_n)}^2 \\ &= \left((C_1(D,\lambda,\beta))^2 - 2\max_{1\leq i\leq n}\delta_i\right)\|u\|_{1,(\alpha_{-1},\alpha_n)}^2, \end{aligned}$$

from which the coercivity follows because of the assumption that $(C_1(D,\lambda,\beta))^2 > 2\max\limits_{1\leq i\leq n}\delta_i$. Finally, the existence and uniqueness of the weak solution to (7) follows from the Lax-Milgram theorem. □

3. Linear IFE Method

In this section, we develop a linear IFE method for solving the interface problem (1)-(3). We start from the construction of a linear IFE space for the interface problem. Form a uniform partition $\mathcal{T}_h$ for the solution domain $[\alpha_{-1},\alpha_n]$ as follows:

$$\begin{aligned} &\alpha_{-1} = x_0 < x_1 < x_2 < \ldots < x_i < x_{i+1} < \ldots < x_{N-1} < x_N = \alpha_n, \\ &h_i = x_i - x_{i-1} = h,\ i = 1,\,2,\ldots,N, \qquad (13) \\ &\mathcal{T}_h = \{[x_i,x_{i+1}],\ i = 0,\,1,\,2,\ldots,N-1\}. \end{aligned}$$

As usual, we call $\mathcal{N}_h = \{x_i\}_{i=0}^N$ the set of nodes. For each element $T\in\mathcal{T}_h$, we call it an interface element if $T\cap\{\alpha_i\}_{i=0}^{n-1}\neq\emptyset$; otherwise, we name it a non-interface element. Without loss of generality, we assume that each interface element contains only one interface point. Throughout this paper, $\mathcal{T}_h^{int}$ denotes the collection of interface elements, and $\mathcal{T}_h^{non} = \mathcal{T}_h/\mathcal{T}_h^{int}$ denotes the collection of non-interface elements. On each element $T = [x_i,x_{i+1}]$, whether it is an interface element or non-interface element, we let $L_{i,0}^{(1)}(x)$, $L_{i,1}^{(1)}(x)$ be the two linear Lagrange shape functions associated with nodes x_i and x_{i+1}, respectively, i.e.,

$$L_{i,j}^{(1)}(x_k) = \delta_{i+j,k},\ k = i,\ i+1,\ j = 0,\ 1.$$

Following the general framework of IFE, on each non-interface element $T = [x_i,x_{i+1}]$, the local linear IFE space is the standard linear finite element space, i.e.,

$$S_h^{(1)}(T) = span\{L_{i,0}^{(1)},\,L_{i,1}^{(1)}\},\ T = [x_i,x_{i+1}],\ T\cap\{\alpha_i\}_{i=0}^{n-1} = \emptyset,$$

hereinafter, the superscript (p) with $p = 1,\,2$ in $S_h^{(p)}$ and $L_{i,k}^{(p)}$, $k = 0,\,1$ emphasizes the involved space is constructed with p-th degree polynomials.

Our main effort is to develop local linear IFE spaces on all the interface elements. Then, all the local IFE spaces are put together to form a conforming IFE space for solving the interface problem. For the construction of the local linear IFE space on an interface element $T = [x_i,x_{i+1}]$ containing the j-th interface point α_j for $j = 0,\,1,\,2\ldots,n-1$, we will use the following linear polynomials:

$$\begin{cases} \check{L}_{i,\alpha_j}(x) = \frac{\alpha_j - x}{\alpha_j - x_i},\ \check{L}_{\alpha_j,i}(x) = \frac{x - x_i}{\alpha_j - x_i}, \\ \\ \check{L}_{\alpha_j,i+1}(x) = \frac{x_{i+1} - x}{x_{i+1} - \alpha_j},\ \check{L}_{i+1,\alpha_j}(x) = \frac{x - \alpha_j}{x_{i+1} - \alpha_j}. \end{cases} \qquad (14)$$

3.1. Local Linear IFE Space for the First Interface. Let us consider the local linear IFE space in the interface element $T = [x_i, x_{i+1}]$ such that $\alpha_0 \in T$. We note that the interface jump conditions across α_0 are different from those at other interface points. Let

$$(15)\qquad \begin{aligned} \psi_i^0(x) &= \begin{cases} \check{L}_{i,\alpha_0}(x) + a_i \check{L}_{\alpha_0,i}(x),\ x \in [x_i, \alpha_0], \\ d_i \check{L}_{\alpha_0,i+1}(x),\ x \in [\alpha_0, x_{i+1}], \end{cases} \\ \psi_{i+1}^0(x) &= \begin{cases} a_{i+1} \check{L}_{\alpha_0,i}(x),\ x \in [x_i, \alpha_0], \\ d_{i+1} \check{L}_{\alpha_0,i+1}(x) + \check{L}_{i+1,\alpha_0}(x),\ x \in [\alpha_0, x_{i+1}]. \end{cases} \end{aligned}$$

Then $\psi_i^0(x)$ and $\psi_{i+1}^0(x)$ are piecewise linear polynomials such that $\psi_k^0(x_j) = \delta_{k,j}$, $j,\ k = i,\ i+1$. The coefficients in these piecewise linear functions can be further determined by the interface jump conditions across α_0 as stated in the following theorem.

Theorem 3.1. *Coefficients of $\psi_i^0(x)$ and $\psi_{i+1}^0(x)$ defined in* (15) *are uniquely determined by interface jump conditions in* (3a) *across α_0 such that the coefficients $a_i,\ a_{i+1},\ d_i,\ d_{i+1}$ of $\psi_i^0(x)$ and $\psi_{i+1}^0(x)$ are as follows:*

$$(16)\qquad \begin{aligned} a_i &= \frac{1}{\Delta}(-D_0 \check{L}'_{i,\alpha_0} - 2\lambda D_0 \delta_1 \check{L}'_{i,\alpha_0} + D_0 D_1 \lambda \check{L}'_{i,\alpha_0} \check{L}'_{\alpha_0,i+1}), \\ d_i &= \frac{-1}{\Delta} D_0 \check{L}'_{i,\alpha_0},\ a_{i+1} = \frac{1}{\Delta}(D_1 \check{L}'_{i+1,\alpha_0}), \\ d_{i+1} &= \frac{1}{\Delta}(D_1 \check{L}'_{i+1,\alpha_0} + D_0 D_1 \lambda \check{L}'_{\alpha_0,i} \check{L}'_{i+1,\alpha_0}), \\ \Delta &= 2\delta_1 + D_0 \check{L}'_{\alpha_0,i} - D_1 \check{L}'_{\alpha_0,i+1} + 2\delta_1 \lambda D_0 \check{L}'_{\alpha_0,i} - \lambda D_0 D_1 \check{L}'_{\alpha_0,i} \check{L}'_{\alpha_0,i+1}. \end{aligned}$$

Proof. Applying the jump conditions (3a) across α_0 to $\psi_i^0(x)$ and $\psi_{i+1}^0(x)$, by direct calculations, we can see that a_i, d_i satisfy

$$\begin{cases} d_i - a_i = \lambda D_0 (\check{L}'_{i,\alpha_0} + a_i \check{L}'_{\alpha_0,i}), \\ d_i - a_i = \lambda D_1 d_i \check{L}'_{\alpha_0,i+1} - 2\lambda \delta_1 d_i, \end{cases}$$

while a_{i+1}, d_{i+1} satisfy

$$\begin{cases} d_{i+1} - a_{i+1} = \lambda D_0 a_{i+1} \check{L}'_{\alpha_0,i}, \\ d_{i+1} - a_{i+1} = \lambda D_1 (d_{i+1} \check{L}'_{\alpha_0,i+1} + \check{L}'_{i+1,\alpha_0}) - 2\lambda \delta_1 d_{i+1}. \end{cases}$$

It is easy to see these are two linear systems about the coefficients in $\psi_i^0(x)$ and $\psi_{i+1}^0(x)$, and the determinant of their coefficient matrices both are Δ. In addition, we can easily verify that $\Delta > 0$ since it is a sum of positive terms. Hence each of these two systems must have a unique solution, and $a_i,\ a_{i+1},\ d_i,\ d_{i+1}$ are uniquely determined. In addition, they can be expressed as (16). □

Theorem 3.1 indicates that $\psi_i^0(x)$ and $\psi_{i+1}^0(x)$ are well defined IFE shape functions on the interface element $T = [x_i, x_{i+1}]$ containing the first interface points α_0, and we can then use them to form a local linear IFE space

$$S_h^{(1)}(T) = span\{\psi_i^0,\ \psi_{i+1}^0\},$$

where the superscript in ψ_i^0 and ψ_{i+1}^0 emphasizes that this local linear IFE space is for the element $T = [x_i, x_{i+1}]$ containing the first interface points α_0.

For the approximation capability, we consider the IFE interpolation in $S_h^{(1)}(T)$. For every $u \in L^2(T)$ such that $u|_{[x_i,\alpha_0]} \in C^0[x_i, \alpha_0]$ and $u|_{[\alpha_0,x_{i+1}]} \in C^0[\alpha_0, x_{i+1}]$, its linear IFE interpolation is

$$\check{I}_h^0 u(x) := u(x_i)\psi_i^0(x) + u(x_{i+1})\psi_{i+1}^0(x). \tag{17}$$

Then, by Theorem 3.1, we can write the IFE interpolation as follows

$$\check{I}_h^0 u(x) = \begin{cases} u(x_i)\check{L}_{i,\alpha_0}(x) + u_I^-\check{L}_{\alpha_0,i}(x),\ x \in [x_i, \alpha_0], \\ u_I^+\check{L}_{\alpha_0,i+1}(x) + u(x_{i+1})\check{L}_{i+1,\alpha_0}(x),\ x \in (\alpha_0, x_{i+1}], \end{cases} \tag{18}$$

where u_I^- and u_I^+ have the following expression:

$$\begin{aligned} u_I^- =& \frac{1}{\Delta}(-D_0 u(x_i)\check{L}'_{i,\alpha_0} + D_1 u(x_{i+1})\check{L}'_{i+1,\alpha_0} - 2\lambda u(x_i) D_0 \check{L}'_{i,\alpha_0}\delta_1 \\ &+ D_0 D_1 \lambda u(x_i)\check{L}'_{i,\alpha_0}\check{L}'_{\alpha_0,i+1}), \\ u_I^+ =& \frac{1}{\Delta}(D_0 u(x_i)\check{L}'_{i,\alpha_0} + D_1 u(x_{i+1})\check{L}'_{i+1,\alpha_0} + D_0 D_1 \lambda u(x_{i+1})\check{L}'_{\alpha_0,i}\check{L}'_{i+1,\alpha_0}). \end{aligned}$$

On the other hand, we can also form a standard linear finite element interpolation of u on $T = [x_i, x_{i+1}]$ as follows

$$\tilde{I}_h^0 u(x) := \begin{cases} u(x_i)\check{L}_{i,\alpha_0}(x) + u(\alpha_0^-)\check{L}_{\alpha_0,i}(x),\ x \in [x_i, \alpha_0], \\ u(\alpha_0^+)\check{L}_{\alpha_0,i+1}(x) + u(x_{i+1})\check{L}_{i+1,\alpha_0}(x),\ x \in [\alpha_0, x_{i+1}]. \end{cases} \tag{19}$$

Then, we have the following standard error estimates for linear finite element interpolation:

$$\begin{aligned} &\|u - \tilde{I}_h^0\|_{0,(x_i,\alpha_0)} + h\|u - \tilde{I}_h^0 u\|_{1,(x_i,\alpha_0)} \le Ch^2\|u\|_{2,(x_i,\alpha_0)}, \\ &\|u - \tilde{I}_h^0\|_{0,(\alpha_0,x_{i+1})} + h\|u - \tilde{I}_h^0 u\|_{1,(\alpha_0,x_{i+1})} \le Ch^2\|u\|_{2,(\alpha_0,x_{i+1})}, \end{aligned} \tag{20}$$

provided further that $u|_{(x_i,\alpha_0)} \in H^2(x_i,\alpha_0)$ and $u|_{(\alpha_0,x_{i+1})} \in H^2(\alpha_0,x_{i+1})$.

We can then estimate the error in $\check{I}_h^0 u$ by the splitting $u - \check{I}_h^0 u = u - \tilde{I}_h^0 u + \tilde{I}_h^0 u - \check{I}_h^0 u$.

Theorem 3.2. *Let $T = [x_i, x_{i+1}]$ be an interface element containing the first interface point α_0. Then, for every $u \in L^2(T)$ such that $u|_{(x_i,\alpha_0)} \in H^2(x_i,\alpha_0)$ and $u|_{(\alpha_0,x_{i+1})} \in H^2(\alpha_0,x_{i+1})$, we have*

$$\|u - \check{I}_h^0 u\|_{0,(x_i,x_{i+1})} + h\|u - \check{I}_h^0 u\|_{1,(x_i,x_{i+1})} \le Ch^2\|u\|_{2,(x_i,x_{i+1})}, \tag{21}$$

where C is a constant independent of $\alpha_0 \in [x_i, x_{i+1}]$.

Proof. By (18) and (19), we have

$$\tilde{I}_h^0 u - \check{I}_h^0 u = \begin{cases} (u(\alpha^-) - u_I^-)\check{L}_{\alpha_0,i}(x),\ x \in [x_i, \alpha_0], \\ (u(\alpha^+) - u_I^+)\check{L}_{\alpha_0,i+1}(x),\ x \in [\alpha_0, x_{i+1}]. \end{cases} \tag{22}$$

By simple calculations and the jump conditions, we have

$$\begin{aligned} |u(\alpha^+) - u_I^+| =& \frac{1}{\Delta}| - D_0(\tilde{I}_h^0 u - u)'(\alpha_0^-) + D_1(\tilde{I}_h^0 u - u)'(\alpha_0^+) \\ &+ D_0 D_1 \lambda \check{L}'_{\alpha_0,i}(\tilde{I}_h^0 u - u)'(\alpha_0^+)| \\ \le& \frac{1}{\Delta}(|J_1| + |J_2| + |J_3|), \end{aligned} \tag{23}$$

where

$$J_1 = D_0 e'(\alpha_0^-),\ J_2 = D_1 e'(\alpha_0^+),\ J_3 = D_0 D_1 \lambda \check{L}'_{\alpha_0,i} e'(\alpha_0^+),$$

with $e(x) = \tilde{I}_h^0 u(x) - u(x)$. Because $e(x_i) = e(\alpha_0^-) = e(\alpha_0^+) = e(x_{i+1}) = 0$, we have

$$|e'(\alpha_0^-)| \le (\alpha_0 - x_i)^{1/2}\|u\|_{2,(x_i,\alpha_0)},\ |e'(\alpha_0^+)| \le (x_{i+1} - \alpha_0)^{1/2}\|u\|_{2,(x_i,\alpha_0)}.$$

Then, following the standard procedure, we obtain

$$|J_1| \le D_0(\alpha_0 - x_i)^{\frac{1}{2}}\|u\|_{2,(x_i,\alpha_0)},$$
$$|J_2| \le D_1(x_{i+1} - \alpha_0)^{\frac{1}{2}}\|u\|_{2,(\alpha_0,x_{i+1})},$$
$$|J_3| \le D_0 D_1(\alpha_0 - x_i)^{-1}(x_{i+1} - \alpha_0)^{\frac{1}{2}}\|u\|_{2,(\alpha_0,x_{i+1})}.$$

In addition, we have the following estimate:

$$\frac{1}{\Delta} \le C\min\{(\alpha_0 - x_i), (x_{i+1} - \alpha_0), (\alpha_0 - x_i)(x_{i+1} - \alpha_0)\}.$$

Then, by applying these estimates to (23), we obtain

$$|u_I^+ - u(\alpha^+)| \le Ch^{\frac{3}{2}}\|u\|_{2,(x_i,x_{i+1})}.$$

Hence, by (22) and the fact that

$$\|\check{L}_{\alpha_0,i+1}(x)\|_{0,(\alpha_0,x_{i+1})} + h\|\check{L}_{\alpha_0,i+1}(x)\|_{1,(\alpha_0,x_{i+1})} \le Ch^{1/2},$$

we have

$$\|\tilde{I}_h^0 u - \check{I}_h^0 u\|_{0,(\alpha_0,x_{i+1})} + h\|\tilde{I}_h^0 u - \check{I}_h^0 u\|_{1,(\alpha_0,x_{i+1})} \le Ch^2\|u\|_{2,(x_i,x_{i+1})}. \tag{24}$$

Since $\|u - \check{I}_h^0 u\|_{k,(\alpha_0,x_{i+1})} \le \|u - \tilde{I}_h^0 u\|_{k,(\alpha_0,x_{i+1})} + \|\tilde{I}_h^0 u - \check{I}_h^0 u\|_{k,(\alpha_0,x_{i+1})},\ k = 0,\,1,$ by (20) and (24), we have the following estimate:

$$\|u - \check{I}_h^0 u\|_{0,(\alpha_0,x_{i+1})} + h\|u - \check{I}_h^0 u\|_{1,(\alpha_0,x_{i+1})} \le Ch^2\|u\|_{2,(x_i,x_{i+1})}. \tag{25}$$

Similarly, we can show

$$\|u - \check{I}_h^0 u\|_{0,(x_i,\alpha_0)} + h\|(u - \check{I}_h u)\|_{1,(x_i,\alpha_0)} \le Ch^2\|u\|_{2,(x_i,x_{i+1})}. \tag{26}$$

Finally, estimate given in (21) follows from (25) together with (26). □

3.2. Local Linear IFE Space for Other Interfaces. We now consider the local linear IFE space on the interface element $T = [x_i, x_{i+1}]$ such that $\alpha_j \in T$ for an integer $j \in \{1, 2, \ldots, n-1\}$. As before, we let

$$\begin{aligned} \psi_i^j(x) &= \begin{cases} \check{L}_{i,\alpha_j}(x) + b_i\check{L}_{\alpha_j,i}(x),\ x \in [x_i,\alpha_j],\\ c_i\check{L}_{\alpha_j,i+1}(x),\ x \in [\alpha_j, x_{i+1}],\end{cases}\\ \psi_{i+1}^j(x) &= \begin{cases} b_{i+1}\check{L}_{\alpha_j,i}(x),\ x \in [x_i,\alpha_j],\\ c_{i+1}\check{L}_{\alpha_j,i+1}(x) + \check{L}_{i+1,\alpha_j}(x),\ x \in [\alpha_j, x_{i+1}].\end{cases}\end{aligned} \tag{27}$$

We note that the interface jump conditions at α_j, $j \in \{1, 2, \ldots, n-1\}$ are the same type and they can be used to determine coefficients in $\psi_i^j(x)$ and $\psi_{i+1}^j(x)$ as stated in the following theorem.

Theorem 3.3. *Assume that $\delta_j \le \delta_{j+1}$, then coefficients of $\psi_i^j(x)$ and $\psi_{i+1}^j(x)$ defined by* (27) *are uniquely determined by interface jump conditions in* (3b) *across α_j such that the coefficients b_i, b_{i+1}, c_i, c_{i+1} of $\psi_i^j(x)$ and $\psi_{i+1}^j(x)$ are as follows:*

$$\begin{aligned} &b_i = c_i = \frac{-1}{\Delta}D_j\check{L}'_{i,\alpha_j},\ b_{i+1} = c_{i+1} = \frac{1}{\Delta}D_{j+1}\check{L}'_{i+1,\alpha_j},\\ &\Delta = -D_{j+1}\check{L}'_{\alpha_j,i+1} + D_j\check{L}'_{\alpha_j,i} + 2\delta_{j+1} - 2\delta_j.\end{aligned} \tag{28}$$

Proof. By applying the jump conditions in (3b) across α_j to $\psi_i^j(x)$ and $\psi_{i+1}^j(x)$, we can see that coefficients b_i and c_i satisfy

$$\begin{cases} b_i = c_i,\\ D_j(\check{L}'_{i,\alpha_j} + b_i\check{L}'_{\alpha_j,i}) - 2\delta_j b_i = D_{j+1}c_i\check{L}'_{\alpha_j,i+1} - 2\delta_{j+1}c_i,\end{cases}$$

while b_{i+1} and c_{i+1} satisfy

$$\begin{cases} b_{i+1} = c_{i+1}, \\ D_j b_{i+1} \check{L}'_{\alpha_j,i} - 2\delta_j b_{i+1} = D_{j+1}(c_{i+1}\check{L}'_{\alpha_j,i+1} + \check{L}'_{i+1,\alpha_j}) - 2\delta_{j+1}c_{i+1}. \end{cases}$$

The determinant of coefficient matrices in these two linear systems is Δ. Since $\delta_{i+1} \geq \delta_i$, it can be easily shown that $\Delta > 0$. Hence each of these two systems must have a unique solution, respectively, and b_i, b_{i+1}, c_i, c_{i+1} are uniquely determined and they can expressed as in (28). □

Remark: It seems to the authors that the assumption $\delta_j \leq \delta_{j+1}$ is a reasonable assumption for practical applications. Recall that δ_j is the convection parameter in the layer $[\alpha_{j-1}, \alpha_j]$. According to the model setup, a larger j means the layer $[\alpha_{j-1}, \alpha_j]$ is closer to the blood flow in the artery; hence, it is reasonable to assume a larger convection there.

It is obvious that functions $\psi_k^j(x)$, $k = i,\, i+1$ defined by (27) are such that $\psi_k^j(x_l) = \delta_{k,l}$, $k,\, l = i,\, i+1$. Again, these functions are linear IFE shape functions and they can be used to define a local linear IFE space:

$$S_h^{(1)}(T) = span\{\psi_i^j, \psi_{i+1}^j\},$$

where the superscript in ψ_i^j and ψ_{i+1}^j emphasizes that this local linear IFE space is for the $T = [x_i, x_{i+1}]$ containing the j-th interface point α_j for $j \in \{1,\, 2,\, \ldots, n-1\}$.

For the approximation capability of $S_h^{(1)}(T)$ with $\alpha_j \in T$ for some $j \in \{1,\, 2,\, \ldots, n-1\}$, we consider the error in the linear IFE interpolation in $S_h^{(1)}(T)$. For every $u \in C^0(T) = C^0([x_i, x_{i+1}])$, its linear IFE interpolation is

$$\check{I}_h^j u(x) := u(x_i)\psi_i^j(x) + u(x_{i+1})\psi_{i+1}^j(x). \tag{29}$$

By Theorem 3.3, the linear IFE interpolation on $T = [x_i, x_{j+1}]$ can be written as

$$\check{I}_h^j u(x) = \begin{cases} u(x_i)\check{L}_{i,\alpha_j}(x) + u_{\alpha_j}^I \check{L}_{\alpha_j,i}(x),\ x \in [x_i, \alpha_j], \\ u_{\alpha_j}^I \check{L}_{\alpha_j,i+1}(x) + u(x_{i+1})\check{L}_{i+1,\alpha_j}(x),\ x \in [\alpha_j, x_{i+1}], \end{cases} \tag{30}$$

here,

$$u_{\alpha_j}^I = \frac{1}{\Delta}(-u(x_i)D_j\check{L}'_{i,\alpha_j} + u(x_{i+1})D_{j+1}\check{L}'_{i+1,\alpha_j}).$$

We can also form the standard linear Lagrange interpolation of u on $T = [x_i, x_{i+1}]$ as follows:

$$\tilde{I}_h^j u(x) := \begin{cases} u(x_i)\check{L}_{i,\alpha_j}(x) + u(\alpha_j)\check{L}_{\alpha_j,i}(x),\ x \in [x_i, \alpha_j], \\ u(\alpha_j)\check{L}_{\alpha_j,i+1}(x) + u(x_{i+1})\check{L}_{i+1,\alpha_j}(x),\ x \in [\alpha_j, x_{i+1}]. \end{cases} \tag{31}$$

The standard finite element approximation theory provides the following error bounds for $\tilde{I}_h^j u$:

$$\begin{aligned} &\|u - \tilde{I}_h^j u\|_{0,(x_i,\alpha_j)} + h\|u - \tilde{I}_h^j u\|_{1,(x_i,\alpha_j)} \leq Ch^2\|u\|_{2,(x_i,\alpha_j)}, \\ &\|u - \tilde{I}_h^j u\|_{0,(\alpha_j,x_{i+1})} + h\|u - \tilde{I}_h^j u\|_{1,(\alpha_j,x_{i+1})} \leq Ch^2\|u\|_{2,(\alpha_j,x_{i+1})}. \end{aligned} \tag{32}$$

We now turn to the error estimation for $u - \check{I}_h^j u$.

Theorem 3.4. *Let $T = [x_i, x_{i+1}]$ be an interface element containing the j-th interface point α_j, $j \in \{1,\, 2,\, \ldots, n-1\}$ and assume that $\delta_j \leq \delta_{j+1}$. Then, for every $u \in C^0(T)$ such that $u|_{(x_i,\alpha_j)} \in H^2(x_i, \alpha_j)$ and $u|_{(\alpha_j,x_{i+1})} \in H^2(\alpha_j, x_{i+1})$, we have*

$$\|u - \check{I}_h^j u\|_{0,(x_i,x_{i+1})} + h\|u - \check{I}_h^j u\|_{1,(x_i,x_{i+1})} \leq Ch^2\|u\|_{2,(x_i,x_{i+1})}, \tag{33}$$

where C is a constant independent of $\alpha_j \in (x_i, x_{i+1})$.

Proof. Let us consider the estimation on the subelement $[x_i, \alpha_j]$. By the jump conditions, we have

(34)
$$\tilde{I}_h^j u(x) - \check{I}_h^j u(x) = (u(\alpha_j) - u_{\alpha_j}^I)\check{L}_{\alpha_j,i}(x) = \frac{x_i - x}{\Delta(x_{i+1}-\alpha_j)(\alpha_j - x_i)}(I + II + III),$$

here

$$I = D_j \check{L}_{i,\alpha_j}(x_{i+1})(u(\alpha_j) - u(x_i)),$$
$$II = D_{j+1}(u(x_{i+1}) - u(\alpha_j)),$$
$$III = -2u(\alpha_j)(x_{i+1} - \alpha_j)(\delta_{j+1} - \delta_j).$$

Clearly, we have

$$\int_{x_i}^{\alpha_j}\int_x^{\alpha_j} u''(y)\,dydx = u'(\alpha_j^-)(\alpha_j - x_i) + (u(x_i) - u(\alpha_j^-)),$$

and

$$\int_{\alpha_j}^{x_{i+1}}\int_{\alpha_j}^{x} u''(y)\,dydx = (u(x_{i+1}) - u(\alpha_j^+)) - u'(\alpha_j^+)(x_{i+1} - \alpha_j),$$

which means

$$\begin{aligned}
I + II &= -D_j\check{L}_{i,\alpha_j}(x_{i+1})\left(\int_{x_i}^{\alpha_j}\int_x^{\alpha_j} u''(y)\,dydx - u'(\alpha_j^-)(\alpha_j - x_i)\right)\\
&\quad + D_{j+1}\left(\int_{\alpha_j}^{x_{i+1}}\int_{\alpha_j}^{x} u''(y)\,dydx + u'(\alpha_j^+)(x_{i+1} - \alpha_j)\right)\\
&= -D_j\check{L}_{i,\alpha_j}(x_{i+1})\int_{x_i}^{\alpha_j}\int_x^{\alpha_j} u''(y)\,dydx + D_{j+1}\int_{\alpha_j}^{x_{i+1}}\int_{\alpha_j}^{x} u''(y)\,dydx\\
&\quad + D_{j+1}u'(\alpha_j^+)(x_{i+1} - \alpha_j) - D_j(x_{i+1} - \alpha_j)u'(\alpha_j^-),\\
&= J_1 + J_2,
\end{aligned} \tag{35}$$

where

$$J_1 = -D_j\check{L}_{i,\alpha_j}(x_{i+1})\int_{x_i}^{\alpha_j}\int_x^{\alpha_j} u''(y)\,dydx + D_{j+1}\int_{\alpha_j}^{x_{i+1}}\int_{\alpha_j}^{x} u''(y)\,dydx,$$
$$J_2 = D_{j+1}u'(\alpha_j^+)(x_{i+1} - \alpha_j) - D_j(x_{i+1} - \alpha_j)u'(\alpha_j^-).$$

By the jump condition at α_j, we have

$$\begin{aligned}
&J_2 + III\\
=&(D_{j+1}u'(\alpha_j^+) - 2\delta_{j+1}u(\alpha_j^+) - D_ju'(\alpha_j^-) + 2\delta_j u(\alpha_j^-))(x_{i+1} - \alpha_j) = 0.
\end{aligned} \tag{36}$$

Then, it remains to estimate J_1. Noted that

$$\begin{aligned}
|J_1| &\le D_j|\check{L}_{i,\alpha_j}(x_{i+1})|\int_{x_i}^{\alpha_j}\int_{x_i}^{\alpha_j} |u''(y)|\,dydx + D_{j+1}\int_{\alpha_j}^{x_{i+1}}\int_{\alpha_j}^{x_{j+1}} |u''(y)|\,dydx\\
&\le Ch^{3/2}\|u\|_{2,(x_i,x_{i+1})},
\end{aligned}$$

by the assumption that $\delta_{j+1} \ge \delta_j$, we have

$$\Delta(x_{i+1} - \alpha_j)(\alpha_j - x_i) \ge D_j(x_{i+1} - \alpha_j) + D_{j+1}(\alpha_j - x_i) \ge \min\{D_j, D_{j+1}\}h.$$

Therefore,

$$\Big|\frac{J_1}{\Delta(x_{i+1-\alpha_j})(\alpha_j - x_i)}\Big| \le Ch^{1/2}\|u\|_{2,(x_i,x_{i+1})}. \tag{37}$$

Then, applying (35), (36), and (37) to (34), we obtain

$$\begin{aligned}
&\|\tilde{I}_h^j u - \check{I}_h^j u\|_{0,(x_i,\alpha_j)} \\
&\leq Ch^{1/2}\|x - x_i\|_{0,(x_i,\alpha_j)}\|u\|_{2,(x_i,x_{i+1})} \\
&\leq Ch^{1/2}\Big(\int_{x_i}^{\alpha_j} |x - x_i|^2\, dx\Big)^{1/2}\|u\|_{2,(x_i,x_{i+1})} \leq Ch^2\|u\|_{2,(x_i,x_{i+1})},
\end{aligned}$$

and $\|\tilde{I}_h^j u - \check{I}_h^j u\|_{1,(x_i,\alpha_j)} \leq Ch\|u\|_{2,(x_i,x_{i+1})}$. These two estimates lead to

$$\|\tilde{I}_h^j u - \check{I}_h^j u\|_{0,(x_i,\alpha_j)} + h\|\tilde{I}_h^j u - \check{I}_h^j u\|_{1,(x_i,\alpha_j)} \leq Ch^2\|u\|_{2,(x_i,x_{i+1})}.$$

Applying the similar arguments to the subelement $[\alpha_j, x_{i+1}]$, we obtain

$$\|\tilde{I}_h^j u - \check{I}_h^j u\|_{0,(\alpha_j,x_{i+1})} + h\|\tilde{I}_h^j u - \check{I}_h^j u\|_{1,(\alpha_j,x_{i+1})} \leq Ch^2\|u\|_{2,(x_i,x_{i+1})}.$$

Hence, we have

$$\|\tilde{I}_h^j u - \check{I}_h^j u\|_{0,(x_i,x_{i+1})} + h\|\tilde{I}_h^j u - \check{I}_h^j u\|_{1,(x_i,x_{i+1})} \leq Ch^2\|u\|_{2,(x_i,x_{i+1})}.$$

Finally, using the triangle inequality and the classical approximation result (32), we can derive the estimate in (33) as follows:

$$\begin{aligned}
&\|u - \check{I}_h^j u\|_{0,(x_i,x_{i+1})} + h\|u - \check{I}_h^j u\|_{1,(x_i,x_{i+1})} \\
&\leq \|u - \tilde{I}_h^j u\|_{0,(x_i,x_{i+1})} + h\|u - \tilde{I}_h^j u\|_{1,(x_i,x_{i+1})} \\
&\quad + \|\tilde{I}_h^j u - \check{I}_h^j u\|_{0,(x_i,x_{i+1})} + h\|\tilde{I}_h^j u - \check{I}_h^j u\|_{1,(x_i,x_{i+1})} \\
&\leq Ch^2\|u\|_{2,(x_i,x_{i+1})}.
\end{aligned} \tag{38}$$

□

3.3. Convergence of the Linear IFE Space. Using the local linear IFE spaces on each element $T \in \mathcal{T}_h$, we can define a linear IFE space globally on whole solution domain (α_{-1}, α_n) as follows:

$$S_h^{(1)}(\alpha_{-1}, \alpha_n) = \{v \in L^2(\alpha_{-1}, \alpha_n) \mid v|_{\Omega^\pm} \in C^0(\Omega^\pm), v|_T \in S_h^{(1)}(T), \forall\, T \in \mathcal{T}_h\},$$

where, we recall that the local linear IFE space $S_h^{(1)}(T)$ is defined by

$$S_h^{(1)}(T) = \begin{cases} span\{L_{i,0}^{(1)}, L_{i,1}^{(1)}\},\ T = [x_i, x_{i+1}],\ T \cap \{\alpha_i\}_{i=0}^{n-1} = \emptyset, \\ span\{\psi_i^j, \psi_{i+1}^j\},\ T = [x_i, x_{i+1}],\ \alpha_j \in T, j = 0,\, 1,\, 2, \ldots, n-1. \end{cases}$$

For a function $u \in H_\alpha^1(\alpha_{-1}, \alpha_n)$, we define its linear IFE interpolation $\check{I}_h u \in S_h^{(1)}(\alpha_{-1}, \alpha_n)$ piecewisely such that, for every element $T = [x_i, x_{i+1}]$,

$$\check{I}_h u|_T = \begin{cases} u(x_i)L_{i,0}^{(1)}(x) + u(x_{i+1})L_{i,1}^{(1)}(x),\ \text{when } T \cap \{\alpha_i\}_{i=0}^{n-1} = \emptyset, \\ u(x_i)\psi_i^j(x) + u(x_{i+1})\psi_i^j(x),\ \text{when } \alpha_j \in T, j = 0,\, 1,\, 2, \ldots, n-1. \end{cases}$$

Then, we derive an error bound for the linear IFE interpolation in the following theorem.

Theorem 3.5. *Assume that $\delta_j \leq \delta_{j+1}$, $j = 1, 2, \ldots, n-1$, then there exists a positive constant C independent of h and the position of α_j, $j = 0, 1, \ldots, n-1$ such that*

$$\|u - \check{I}_h u\|_{0,(\alpha_{-1},\alpha_n)} + h\|u - \check{I}_h u\|_{1,(\alpha_{-1},\alpha_n)} \leq Ch^2\|u\|_{2,(\alpha_{-1},\alpha_n)},\ \forall\, u \in H_\alpha^2(\alpha_{-1}, \alpha_n). \tag{39}$$

Proof. By the definition of $\check{I}_h u$, we have for $k = 0$, 1,

$$\|u - \check{I}_h u\|_{k,(\alpha_{-1},\alpha_n)} = \sum_{T\in\mathcal{T}_h} \|u - \check{I}_h^j u\|_{k,T}$$
$$= \sum_{T\in\mathcal{T}_h^{non}} \|u - \check{I}_h^j u\|_{k,T} + \sum_{T\in\mathcal{T}_h^{int}} \|u - \check{I}_h^j u\|_{k,T},$$
$$= \sum_{T\in\mathcal{T}_h^{non}} \|u - \tilde{I}_h u\|_{k,T} + \sum_{T\in\mathcal{T}_h^{int}} \|u - \check{I}_h^j u\|_{k,T}.$$

Then, estimates for the standard linear finite element interpolation error $\|u - \tilde{I}_h u\|_{k,T}$, $T \in \mathcal{T}_h^{non}$ and the estimates for the linear IFE interpolation error $\|u - \check{I}_h u\|_{k,T}$, $T \in \mathcal{T}_h^{int}$ given in Theorems 3.2 and 3.4 imply that

$$\|u - \check{I}_h u\|_{k,(\alpha_{-1},\alpha_n)} \le Ch^{2-k}\|u\|_{2,(\alpha_{-1},\alpha_n)},\ k = 0,\ 1,$$

which further leads to (39). □

We now discuss the linear IFE solution to the interface problem described by (1)-(3). Let

$$S_{h,0}^{(1)}(\alpha_{-1},\alpha_n) = \{v | v \in S_h^{(1)}(\alpha_{-1},\alpha_n),\ v|_{x=\alpha_n} = 0\},$$

and the linear IFE solution $u_h \in S_{h,0}^{(1)}(\alpha_{-1},\alpha_n)$ is then defined to be such that

$$a(u_h, v_h) = (f, v_h),\ \forall\, v_h \in S_{h,0}^{(1)}(\alpha_{-1},\alpha_n). \tag{40}$$

The error bound for the linear IFE solution u_h is given in the following theorem.

Theorem 3.6. *Assume the condition required by Theorem 2.2 holds, $\delta_j \le \delta_{j+1}$, $j = 1, 2, \ldots, n-1$, and that the solution u to the weak problem* (7) *is such that $u \in H_\alpha^2(\alpha_{-1},\alpha_n)$. Then the linear IFE solution u_h defined by* (40) *satisfies the following estimate:*

$$\|u - u_h\|_{0,(\alpha_{-1},\alpha_n)} + h\|u - u_h\|_{1,(\alpha_{-1},\alpha_n)} \le Ch^2\|u\|_{2,(\alpha_{-1},\alpha_n)}. \tag{41}$$

Proof. It is easy to see that

$$a(u - u_h, v_h) = 0,\ \forall\, v_h \in V_h.$$

Hence, by using Theorem 2.2, we have

$$\|u - u_h\|_{1,(\alpha_{-1},\alpha_n)}^2 \le Ca(u - u_h, u - u_h) = a(u - u_h, u - v_h)$$
$$\le C\|u - u_h\|_{1,(\alpha_{-1},\alpha_n)}\|u - v_h\|_{1,(\alpha_{-1},\alpha_n)}.$$

The above inequality together with Theorem 3.5 imply

$$\|u - u_h\|_{1,(\alpha_{-1},\alpha_n)} \le C \inf_{v\in S_{h,0}^{(1)}(\alpha_{-1},\alpha_n)} \|u - v_h\|_{1,(\alpha_{-1},\alpha_n)}$$
$$\le C\|u - \check{I}_h u\|_{1,(\alpha_{-1},\alpha_n)} \le Ch\|u\|_{2,(\alpha_{-1},\alpha_n)}.$$

Using the usual duality argument, we get

$$\|u - u_h\|_{0,(\alpha_{-1},\alpha_n)} \le Ch^2\|u\|_{2,(\alpha_{-1},\alpha_n)}.$$

Then, (41) is proven. □

4. Quadratic IFE Method

In this section, we develop a quadratic IFE method for solving the interface problem (1)-(3). Let $\mathcal{T}_h$ be the partition of the solution domain (α_{-1}, α_n) defined in (13). As usual, on each element $T = [x_i, x_{i+1}]$, we introduce another node $x_{i+1/2} = \frac{x_i + x_{i+1}}{2}$ and the standard local quadratic finite element shape functions $L_{i,k}^{(2)}(x)$, $k = 0, 1, 2$ associated with the three local nodes: x_i, $x_{i+1/2}$ and x_{i+1}. On each non-interface element $T = [x_i, x_{i+1}]$, the local quadratic IFE space is the standard quadratic finite element space:

$$S_h^{(2)}(T) = span\{L_{i,0}^{(2)}, L_{i,1}^{(2)}, L_{i,2}^{(2)}(x)\}, \text{ when } T \cap \{\alpha_i\}_{i=0}^{n-1} = \emptyset.$$

We need to construc quadratic IFE shape functions on interface elements. Let $T = [x_i, x_{i+1}]$ be an interface element containing α_j, $j \in \{0, 1, 2, \dots, n-1\}$. Without loss of generality, our discussions in this section mainly focus on the case: $x_i < x_{i+1/2} < \alpha_j < x_{i+1}$ in which we will use the following quadratic polynomials as the building blocks for the quadratic IFE shape functions:

(42)

$$L_{i,\alpha_j}(x) = \frac{(x - x_{i+1/2})(x - \alpha_j)}{(x_i - x_{i+1/2})(x_i - \alpha_j)}, \qquad L_{i+1/2,\alpha_j}(x) = \frac{(x - x_i)(x - \alpha_j)}{(x_{i+1/2} - x_i)(x_{i+1/2} - \alpha_j)},$$
$$L_{\alpha_j,i}(x) = \frac{(x - x_{i+1/2})(x - x_i)}{(\alpha_j - x_{i+1/2})(\alpha_j - x_i)}, \qquad H_{i+1,\alpha_j}(x) = \frac{(x - \alpha_j)^2}{(x_{i+1} - \alpha_j)^2},$$
$$H_{\alpha_j,0} = 2\frac{x - x_{i+1}}{\alpha_j - x_{i+1}} - \frac{(x - x_{i+1})^2}{(\alpha_j - x_{i+1})^2}, \quad H_{\alpha_j,1} = \frac{(x - \alpha_j)(x - x_{i+1})}{(\alpha_j - x_{i+1})}.$$

All the ideas and results in this section can be readily extended to the other case: $x_i < \alpha_j < x_{i+1/2} < x_{i+1}$ in which we will use three Hermite type basis functions in the subinterval $[x_i, \alpha_j]$ and three Lagrange type basis functions in the subinterval $[\alpha_j, x_{i+1}]$ in forms similar to (42) as building blocks for the corresponding quadratic IFE shape functions.

The quadratic polynomials in (42) have the following properties:

(43)

$$L'_{i,\alpha_j}(\alpha_j) = \frac{-1}{(x_i - x_{i+1/2})} + \frac{1}{(x_i - \alpha_j)}, \quad L'_{\alpha_j,i}(\alpha_j) = \frac{1}{(\alpha_j - x_{i+1/2})} + \frac{1}{(\alpha_j - x_i)},$$
$$L''_{i,\alpha_j}(x) = \frac{2}{(x_i - x_{i+1/2})(x_i - \alpha_j)}, \quad L''_{\alpha_j,i}(x) = \frac{2}{(\alpha_j - x_{i+1/2})(\alpha_j - x_i)},$$
$$H''_{i+1,\alpha_j}(x) = \frac{2}{(x_{i+1} - \alpha_j)^2}, \quad H''_{\alpha_j,0}(x) = \frac{-2}{(\alpha_j - x_{i+1})^2}, \quad H''_{\alpha_j,1}(x) = \frac{2}{(\alpha_j - x_{i+1})}.$$

In addition, by using the standard interpolation error analysis procesure, we derive the following estimates about these quadratic polynomials:

$$\begin{aligned} &\|L_{\alpha_j,i}\|_{0,(x_i,\alpha_j)} + h\|L_{\alpha_j,i}\|_{1,(x_i,\alpha_j)} \le h^{3/2}(\alpha_j - x_{i+1/2})^{-1}, \\ &\|H_{\alpha_j,1}\|_{0,(\alpha_j,x_{i+1})} + h\|H_{\alpha_j,1}\|_{1,(\alpha_j,x_{i+1})} \le Ch^{3/2}, \\ &\|H_{\alpha_j,0}\|_{0,(\alpha_j,x_{i+1})} + h\|H_{\alpha_j,0}\|_{1,(\alpha_j,x_{i+1})} \le Ch^{1/2}. \end{aligned} \tag{44}$$

4.1. Local Quadratic IFE Space for the First Interface. Let $T = [x_i, x_{i+1}]$ be an interface element containing the first interface point α_0. Firstly, We propose three quadratic IFE shape functions on this interface element in the following formats:

$$\psi_i^0(x) = \begin{cases} L_{i,\alpha_0}(x) + c_i^0 L_{\alpha_0,i}(x),\ x \in [x_i, \alpha_0], \\ a_i^0 H_{\alpha_0,0}(x) + b_i^0 H_{\alpha_0,1}(x),\ x \in [\alpha_0, x_{i+1}], \end{cases}$$

$$\psi_{i+1/2}^0(x) = \begin{cases} L_{i+1/2,\alpha_0}(x) + c_{i+1/2}^0 L_{\alpha_0,i}(x),\ x \in [x_i, \alpha_0], \\ a_{i+1/2}^0 H_{\alpha_0,0}(x) + b_{i+1/2}^0 H_{\alpha_0,1}(x),\ x \in [\alpha_0, x_{i+1}], \end{cases} \tag{45}$$

$$\psi_{i+1}^0(x) = \begin{cases} c_{i+1}^0 L_{\alpha_0,i}(x),\ x \in [x_i, \alpha_0], \\ a_{i+1}^0 H_{\alpha_0,0}(x) + b_{i+1}^0 H_{\alpha_0,1}(x) + H_{i+1,\alpha_0}(x),\ x \in [\alpha_0, x_{i+1}], \end{cases}$$

whose coefficients a_k^0, b_k^0 and c_k^0, $k = i,\ i+1/2,\ i+1$ are to be chosen so that these piecewise quadratic functions can satisfy the interface jump conditions (3a) across α_0. However, the two equations in the interface jump conditions (3a) are obviously not enough to determine all three coefficients in each of the proposed quadratic IFE shape functions. Following the ideas in [2, 30], we therefore propose to impose one extra jump condition for the unique determination of the quadratic IFE shape functions:

$$D_0(\psi_k^0)''(\alpha_0^-) = D_1(\psi_k^0)''(\alpha_0^+) - 2\delta_1(\psi_k^0)'(\alpha_0^+),\ k = i,\ i+1/2,\ i+1. \tag{46}$$

Other types of extra jump conditions can be considered, but the related error estimation confirms that the one given in (46) leads to an optimally convergent quadratic IFE space.

Theorem 4.1. *Functions $\psi_i^0(x)$, $\psi_{i+1/2}^0(x)$ and $\psi_{i+1}^0(x)$ in (45) are uniquely determined by interface jump conditions in* (3a) *and* (46). *In addition, the coefficients in these functions are as follows:*

$$c_k^0 = \begin{cases} \dfrac{1}{\Delta}[\Theta' L'_{k,\alpha_0}(\alpha_0) + D_0 D_1 L''_{k,\alpha_0}],\ k = i, i+1/2, \\ \dfrac{-1}{\Delta} D_1^2 H''_{i+1,\alpha_0},\ k = i+1, \end{cases}$$

$$a_k^0 = \begin{cases} \lambda D_0 L'_{k,\alpha_0}(\alpha_0) + [1 + \lambda D_0 L'_{\alpha_0,k}(\alpha_0)] c_k^0,\ k = i, i+1/2, \\ [1 + \lambda D_0 L'_{\alpha_0,i}(\alpha_0)] c_{i+1}^0,\ k = i+1, \end{cases} \tag{47}$$

$$b_k^0 = \begin{cases} \dfrac{1}{D_1}[D_0 L'_{k,\alpha_0}(\alpha_0) + D_0 L'_{\alpha_0,k}(\alpha_0) c_i + 2\delta_1 a_k^0],\ k = i, i+1/2, \\ \dfrac{1}{D_1}[D_0 L'_{\alpha_0,i}(\alpha_0) c_{i+1}^0 + 2\delta_1 a_{i+1}^0],\ k = i+1, \end{cases}$$

with

$$\begin{aligned} \Delta &= D_1^2 H''_{\alpha_0,0} - D_0 D_1 L''_{\alpha_0,i} + 2D_1\delta_1 H''_{\alpha_0,1} - 4\delta_1^2 - \Theta' L'_{\alpha_0,i}(\alpha_0), \\ \Theta' &= -D_0 D_1 H''_{\alpha_0,1} + 2\delta_1 D_0 - 2D_0 D_1 \lambda\delta_1 H''_{\alpha_0,1} + 4D_0\delta_1^2\lambda - \lambda D_0 D_1^2 H''_{\alpha_0,0}. \end{aligned} \tag{48}$$

Proof. Applying jump conditions in (3a) and (46) to $\psi_k^0(x)$, $k = i,\ i+1/2,\ i+1$, we have

$$\begin{cases} \psi_k^0(\alpha_0^+) - \psi_k^0(\alpha_0^-) = \lambda D_0(\psi_k^0)'(\alpha_0^-), \\ D_0(\psi_k^0)'(\alpha_0^-) = D_1(\psi_k^0)'(\alpha_0^+) - 2\delta_1\psi_k^0(\alpha_0^+), \\ D_0(\psi_k^0)''(\alpha_0^-) = D_1(\psi_k^0)''(\alpha_0^+) - 2\delta_1(\psi_k^0)'(\alpha_0^+),\ k = i,\ i+1/2,\ i+1. \end{cases}$$

We know that this is a linear system for coefficients a_k^0, b_k^0, and c_k^0, and the determinant of its coefficient matrix is Δ. It can be verified that $\Delta < 0$ under the assumption $x_i < x_{i+1/2} < \alpha_0 < x_{i+1}$. Hence this linear system has a unique solution which yields the formulas in (47) for a_k^0, b_k^0, and c_k^0, $k = i, \, i+1/2, \, i+1$. $\square$

It is easy to verify that the quadratic IFE shape functions given in Theorem 4.1 have the following property:

$$\psi_j^{(0)}(x_k) = \delta_{j,\,k}, \; j, k = i, \, i+1/2, \, i+1.$$

Hence, we can use these quadratic IFE shape functions to define a local quadratic IFE space on the interface element $T = [x_i, x_{i+1}]$ containing α_0 as follows:

$$S_h^{(2)}(T) = span\{\psi_i^{(0)}, \, \psi_{i+1/2}^{(0)}, \, \psi_{i+1}^{(0)}\}.$$

We now consider the approximation capability of this local quadratic IFE space. For every $u \in L^2(T)$ such that $u|_{[x_i,\alpha_0]} \in C^0[x_i, \alpha_0]$ and $u|_{[\alpha_0,x_{i+1}]} \in C^0[\alpha_0, x_{i+1}]$, we define its quadratic IFE interpolation as

$$\check{I}_h^0 u(x) := u(x_i)\psi_i^0(x) + u(x_{i+1/2})\psi_{i+1/2}^0(x) + u(x_{i+1})\psi_{i+1}^0(x). \tag{49}$$

Using the formulas for the coefficients of ψ_k^0, $k = i, \, i+1/2, \, i+1$ given in Theorem 4.1, we show that

(50)

$$\check{I}_h^0 u(x) = \begin{cases} u(x_i)L_{i,\alpha_0}(x) + u(x_{i+1/2})L_{i+1/2,\alpha_0}(x) + \bar{u}_{\alpha_0}^- L_{\alpha_{0,i}}(x), \; x \in [x_i, \alpha_0], \\ \bar{u}_{\alpha_0}^+ H_{\alpha_0,0}(x) + \bar{u}'_{\alpha_0} H_{\alpha_0,1}(x) + u(x_{i+1})H_{i+1,\alpha_0}(x), \; x \in [\alpha_0, x_{i+1}], \end{cases}$$

where, constants $\bar{u}_{\alpha_0}^-$, $\bar{u}_{\alpha_0}^+$ and $\bar{u}'_{\alpha_0}$ are defined by

$$\begin{aligned} \bar{u}_{\alpha_0}^- &= \frac{1}{\Delta}[\Theta' l_0'(\alpha_0) + D_0 D_1 l_0''(\alpha_0) - D_1^2 u(x_{i+1})H''_{i+1,\alpha_0}], \\ \bar{u}_{\alpha_0}^+ &= \lambda D_0 l_0'(\alpha_0) + [1 + \lambda D_0 L'_{\alpha_0}(\alpha_0)]\bar{u}_{\alpha_0}^-, \\ \bar{u}'_{\alpha_0} &= \frac{1}{D_1}[D_0 l_0'(\alpha_0) + D_0 L'_{\alpha_0,i}(\alpha_0)\bar{u}_{\alpha_0}^- + 2\delta_1 \bar{u}_{\alpha_0}^+], \end{aligned} \tag{51}$$

with

$$l_j(x) = u(x_i)L_{i,\alpha_j}(x) + u(x_{i+1/2})L_{i+1/2,\alpha_j}(x), \; j = 0, \, 1, \, 2\ldots, n-1. \tag{52}$$

In addition, using (42) and the definition of Δ given in (48), we derive the following estimate about $\frac{1}{\Delta}$:

$$\begin{aligned} \frac{1}{|\Delta|} \leq C \min\big\{&(\alpha_0 - x_i), (\alpha_0 - x_i)(\alpha_0 - x_{i+1/2}), \\ &(\alpha_0 - x_{i+1/2})(x_{i+1} - \alpha_0)^2, (\alpha_0 - x_i)(\alpha_0 - x_{i+1})^2\big\}. \end{aligned} \tag{53}$$

On the other hand, we can also interpolate u by those quadratic polynomials in (42) as follows:

(54)

$$\tilde{I}_h^0 u(x) := \begin{cases} u(x_i)L_{i,\alpha_0}(x) + u(x_{i+1/2})L_{i+1/2,\alpha_0}(x) + u(\alpha_0^-)L_{\alpha_0,i}(x), \; x \in [x_i, \alpha_0], \\ u(\alpha_0^+)H_{\alpha_0,0}(x) + u'(\alpha_0^+)H_{\alpha_0,1}(x) + u(x_{i+1})H_{i+1,\alpha_0}(x), \; x \in [\alpha_0, x_{i+1}]. \end{cases}$$

The above Lagrange-Hermit interpolation have the following standard error estimates for $\tilde{I}_h^0 u$:

$$\begin{aligned} \|u - \tilde{I}_h^0 u\|_{0,(x_i,\alpha_0)} + h\|u - \tilde{I}_h^0 u\|_{1,(x_i,\alpha_0)} &\leq Ch^3 \|u\|_{3,(x_i,\alpha_0)}, \\ \|u - \tilde{I}_h^0 u\|_{0,(\alpha_0,x_{i+1})} + h\|u - \tilde{I}_h^0 u\|_{1,(\alpha_0,x_{i+1})} &\leq Ch^3 \|u\|_{3,(\alpha_0,x_{i+1})}, \end{aligned} \tag{55}$$

provided further that $u|_{(x_i,\alpha_0)} \in H^3(x_i,\alpha_0)$ and $u|_{(\alpha_0,x_{i+1})} \in H^3(\alpha_0,x_{i+1})$. For the point-wise estimates, we have the following results:

$$
\begin{aligned}
&|u''(\alpha_0^-) - (\tilde{I}_h^0 u)''(\alpha_0^-)| \le (\alpha_0 - x_i)^{1/2}\|u\|_{3,(x_i,\alpha_0)},\\
&|u''(\alpha_0^+) - (\tilde{I}_h^0 u)''(\alpha_0^+)| \le (x_{i+1} - \alpha_0)^{1/2}\|u\|_{3,(\alpha_0,x_{i+1})},\\
&|u'(\alpha_0^-) - (\tilde{I}_h^0 u)'(\alpha_0^-)| \le (\alpha_0 - x_{i+1/2})(\alpha_0 - x_i)^{1/2}\|u\|_{3,(x_i,\alpha_0)},\\
&|u'(\alpha_0^+) - (\tilde{I}_h^0 u)'(\alpha_0^+)| \le (x_{i+1} - \alpha_0)^{3/2}\|u\|_{3,(\alpha_0,x_{i+1})}.
\end{aligned} \tag{56}
$$

The next theorem shows that the local quadratic IFE space has the expected optimal convergence.

Theorem 4.2. *Let $T = [x_i, x_{i+1}]$ be an interface element such that $\alpha_0 \in [x_i, x_{i+1}]$. Then, for every $u \in L^2(T)$ such that $u|_{(x_i,\alpha_0)} \in H^3(x_i,\alpha_0)$ and $u|_{(\alpha_0,x_{i+1})} \in H^3(\alpha_0,x_{i+1})$, we have the following estimate for the quadratic IFE interpolation:*

$$\|u - \check{I}_h^0 u\|_{0,(x_i,x_{i+1})} + h\|u - \check{I}_h^0 u\|_{1,(x_i,x_{i+1})} \le Ch^3\|u\|_{3,(x_i,x_{i+1})}, \tag{57}$$

where C is a constant independent of $\alpha_0 \in [x_i, x_{i+1}]$.

Proof. Subtracting (50) from (54), we have

$$
\tilde{I}_h^0 u - \check{I}_h^0 u = \begin{cases} \big(u(\alpha_0^-) - \bar{u}_{\alpha_0}^-\big)L_{\alpha_0,i}(x),\ x \in [x_i,\alpha_0],\\ \big(u(\alpha_0^+) - \bar{u}_{\alpha_0}^+\big)H_{\alpha_0,0}(x) + \big(u'(\alpha_0^+) - \bar{u}'_{\alpha_0}\big)H_{\alpha_0,1},\ x \in [\alpha_0, x_{i+1}]. \end{cases} \tag{58}
$$

Using (51) and the jump conditions in (3a) and (46), we have

$$u(\alpha_0^-) - \bar{u}_{\alpha_0}^- = \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5, \tag{59}$$

where

$$
\begin{aligned}
&\tau_1 = -\frac{1}{\Delta}(D_0D_1 + 2D_0D_1\delta_1\lambda)H''_{\alpha_0,1}e'(\alpha_0^-),\ \tau_2 = \frac{1}{\Delta}(2D_0\delta_1 + 4D_0\lambda\delta_1^2)e'(\alpha_0^-),\\
&\tau_3 = \frac{1}{\Delta}D_0D_1e''(\alpha_0^-),\ \tau_4 = -\frac{1}{\Delta}\lambda D_0D_1^2H''_{\alpha_0,0}e'(\alpha_0^-),\ \tau_5 = -\frac{1}{\Delta}D_1^2e''(\alpha_0^+),
\end{aligned} \tag{60}
$$

with $e(x) = u(x) - \tilde{I}_h^0 u(x)$. (53) implies

$$\frac{1}{|\Delta|} \le C(\alpha_0 - x_i)(x_{i+1} - \alpha_0).$$

Substituting the above inequality, $H''_{\alpha_0,1}$ given in (43), and the estimate of $|e'(\alpha_0^-)|$ given in (56) into (60), we obtain

$$|\tau_1| \le Ch^{3/2}(\alpha_0 - x_{i+1/2})\|u\|_{3,(x_i,x_{i+1})}.$$

Similarly, using (53), the bounds for $|e'(\alpha_0^-)|$, $|e''(\alpha_0^-)|$, $|e''(\alpha_0^+)|$ given in (56) and the $H''_{\alpha_0,0}$ given in (43), we get

$$|\tau_k| \le Ch^{3/2}(\alpha_0 - x_{i+1/2})\|u\|_{3,(x_i,x_{i+1})},\ k = 2,\ 3,\ 4,\ 5.$$

Then, putting these estimates for $\tau_k, 1 \le k \le 5$ in (59), we obtain

$$|u(\alpha_0^-) - \bar{u}_{\alpha_0}^-| \le Ch^{3/2}(\alpha_0 - x_{i+1/2})\|u\|_{3,(x_i,x_{i+1})}. \tag{61}$$

Substituting (61) and the estimate about $L_{\alpha_0,i}$ given in (44) into (58), we obtain the following estimate on the left subelement (x_i, α_0):

$$\|\tilde{I}_h^0 u - \check{I}_h^0 u\|_{0,(x_i,\alpha_0)} + h\|\tilde{I}_h^0 u - \check{I}_h^0 u\|_{1,(x_i,\alpha_0)} \le Ch^3\|u\|_{3,(x_i,x_{i+1})}. \tag{62}$$

Now we aim to derive the estimates on the right subelement $[\alpha_j, x_{i+1}]$. Using (51), the jump conditions in (3a), and (46), we have

$$u^+_{\alpha_0} - u(\alpha_0^+) = \frac{1}{\Delta}\Big[D_0D_1e''(\alpha_0^-) + 2D_0\delta_1e'(\alpha_0^-) - D_0D_1H''_{\alpha_0,1}e'(\alpha_0^-) - D_1^2e''(\alpha_0^+) + \lambda D_0^2D_1(L'_{\alpha_0,i}(\alpha_0^-)e''(\alpha_0^-) - L''_{\alpha_0,i}e'(\alpha_0^-)) - D_0D_1^2L'_{\alpha_0,i}(\alpha_0-)\lambda e''(\alpha_0^+)\Big],$$

$$\bar{u}'_{\alpha_0} - u'(\alpha_0^+) = \frac{D_0L'_{\alpha_0,i}(\alpha_0-)}{D_1}(\bar{u}^-_{\alpha_0} - u(\alpha_0^-)) + \frac{D_0}{D_1}e'(\alpha_0^-) + \frac{2\delta_1}{D_1}(\bar{u}^+_{\alpha_0} - u(\alpha_0^+)).$$

Using arguments similar to those we used in (61), we have

$$|\bar{u}^+_{\alpha_0} - u(\alpha_0^+)| \le Ch^{3/2}(\alpha_0 - x_{i+1/2})\|u\|_{3,(x_i,x_{i+1})}, \tag{63}$$

and

(64)

$$|\bar{u}'_{\alpha_0} - u'(\alpha_0^+)| \le |\frac{D_0L'_{\alpha_0,i}(\alpha_0-)}{D_1}(\bar{u}^-_{\alpha_0} - u(\alpha_0^-))| + |\frac{D_0}{D_1}e'(\alpha_0^-)| + |\frac{2\delta_1}{D_1}(\bar{u}^+_{\alpha_0} - u(\alpha_0^+))| \le Ch^{3/2}\|u\|_{3,(x_i,x_{i+1})}.$$

Then, applying (63), (64), and the estimates about $H_{\alpha_0,k}, k = 0, 1$ given in (44) to (58), we have the following estimate on the right subelement $[\alpha_0, x_{i+1}]$:

$$\|\tilde{I}_h^0u - \check{I}_h^0u\|_{0,(\alpha_0,x_{i+1})} + h\|\tilde{I}_h^0u - \check{I}_h^0u\|_{1,(\alpha_0,x_{i+1})} \le Ch^3\|u\|_{3,(x_i,x_{i+1})}. \tag{65}$$

Thus, the combination of (62) and (65) yields

$$\|\tilde{I}_h^0u - \check{I}_h^0u\|_{0,(x_i,x_{i+1})} + h\|\tilde{I}_h^0u - \check{I}_h^0u\|_{1,(x_i,x_{i+1})} \le Ch^3\|u\|_{3,(x_i,x_{i+1})}. \tag{66}$$

Finally, estimate given in (57) follows from

$$\begin{aligned}&\|u - \check{I}_h^0u\|_{0,(x_i,x_{i+1})} + h\|u - \check{I}_h^0u\|_{1,(x_i,x_{i+1})}\\ \le\ &\|u - \tilde{I}_h^0u\|_{0,(x_i,x_{i+1})} + h\|u - \tilde{I}_h^0u\|_{1,(x_i,x_{i+1})}\\ &+ \|\tilde{I}_h^0u - \check{I}_h^0u\|_{0,(x_i,x_{i+1})} + h\|\tilde{I}_h^0u - \check{I}_h^0u\|_{1,(x_i,x_{i+1})},\end{aligned}$$

and then applying estimates in (55) and (66). □

4.2. Local Quadratic IFE Space for Other Interfaces. We now develop a local quadratic IFE space on the interface elements $T = [x_i, x_{i+1}]$ that contains the interface point α_j, for an integer $j \in \{1, 2, \ldots, n-1\}$. We propose three shape functions $\psi_i^j(x)$, $\psi_{i+1/2}^j(x)$ and $\psi_{i+1}^j(x)$ in the following piecewise quadratic polynomial formats:

$$\psi_i^j(x) = \begin{cases} L_{i,\alpha_j}(x) + a_i^jL_{\alpha_j,i}(x),\ x \in [x_i, \alpha_j],\\ a_i^jH_{\alpha_j,0}(x) + b_i^jH_{\alpha_j,1}(x),\ x \in [\alpha_j, x_{i+1}],\end{cases}$$

$$\psi_{i+1/2}^j(x) = \begin{cases} L_{i+1/2,\alpha_j}(x) + a_{i+1/2}^jL_{\alpha_j,i}(x),\ x \in [x_i, \alpha_j],\\ a_{i+1/2}^jH_{\alpha_j,0}(x) + b_{i+1/2}^jH_{\alpha_j,1}(x),\ x \in [\alpha_j, x_{i+1}],\end{cases} \tag{67}$$

$$\psi_{i+1}^j(x) = \begin{cases} a_{i+1}^jL_{\alpha_j,i}(x),\ x \in [x_i, \alpha_j],\\ a_{i+1}^jH_{\alpha_j,0}(x) + b_{i+1}^jH_{\alpha_j,1}(x) + H_{i+1,\alpha_j}(x),\ x \in [\alpha_j, x_{i+1}].\end{cases}$$

For the unique determination of the above three quadratic IFE shape functions, we propose to use the following extra jump condition because it leads to an optimally convergent quadratic IFE space:

(68)

$$D_j(\psi_k^j)''(\alpha_j^-) - 2\delta_j(\psi_k^j)'(\alpha_j^-) = D_{j+1}(\psi_k^j)''(\alpha_j^+) - 2\delta_{j+1}(\psi_k^j)'(\alpha_j^+),\ k = i,\ i+1/2,\ i+1.$$

Theorem 4.3. *Assume that $\delta_{j+1} \geq \delta_j$ and $D_j\delta_{j+1} \geq D_{j+1}\delta_j$, then, the shape functions proposed in* (67) *are uniquely determined by the jump conditions* (3b) *and* (68)*. In addition, the coefficients in these functions have the following representations:*

$$a_k^j = \begin{cases} -\dfrac{1}{\Delta}[\Theta L'_{k,\alpha_j}(\alpha_j) + D_jD_{j+1}L''_{k,\alpha_j}],\ k = i,\ i+1/2 \\ -\dfrac{1}{\Delta}D_{j+1}^2 H''_{i+1,\alpha_j},\ k = i+1, \end{cases}$$

$$b_k^j = \begin{cases} \dfrac{1}{D_{j+1}}\left[\hat{\Theta}a_k^j + D_jL'_{k,\alpha_j}(\alpha_j)\right],\ k = i,\ i+1/2, \\ \dfrac{1}{D_{j+1}}\hat{\Theta}a_{i+1}^j,\ k = i+1, \end{cases}$$

with

(69)

$$\Theta = -D_jD_{j+1}H''_{\alpha_j,1} + 2\delta_{j+1}D_j - 2\delta_jD_{j+1},\ \hat{\Theta} = D_jL'_{\alpha_j,i}(\alpha_j) - 2\delta_j + 2\delta_{j+1},$$
$$\Delta = -\Theta L'_{\alpha_j,i}(\alpha_j) - D_jD_{j+1}L''_{\alpha_j} + D_{j+1}^2H''_{\alpha_j,0} + 2(\delta_{j+1} - \delta_j)(D_{j+1}H''_{\alpha_j,1} - 2\delta_{j+1}).$$

Proof. The proof of the uniqueness is similar to Theorem 4.1 by using jump conditions (3b) and (46), and the coefficients are obtained by solving the related linear system. □

By direct verification, we can see that the quadratic shape functions given in (67) have the following property:

$$\psi_k^j(x_l) = \delta_{k,l},\ k,\ l = i,\ i+1/2,\ i+1.$$

Hence, they can be used to define the local quadratic IFE space on the interface element $T = [x_i, x_{i+1}]$ containing the interface point α_j, $j \in \{1, 2, \ldots, n-1\}$ as follows:

$$S_h^{(2)}(T) = span\{\psi_i^j,\ \psi_{i+1/2}^j,\ \psi_{i+1}^j\}.$$

In order to get approximation capability of the local quadratic IFE space $S_h^{(2)}(T)$, we firstly consider the error in the quadratic IFE interpolation in $S_h^{(2)}(T)$. For every $u \in C^0(T) = C^0([x_i, x_{i+1}])$, we define its quadratic IFE interpolation as

$$\check{I}_h^j u(x) := u(x_i)\psi_i^j(x) + u(x_{i+1/2})\psi_{i+1/2}^j(x) + u(x_{i+1})\psi_{i+1}^j(x). \tag{70}$$

By using the formulas given in Theorem 4.3, the quadratic interpolation on $T = [x_i, x_{i+1}]$ can be written as

(71)

$$\check{I}_h^j u(x) = \begin{cases} u(x_i)L_{i,\alpha_j}(x) + u(x_{i+1/2})L_{i+1/2,\alpha_j}(x) + \bar{u}_{\alpha_j}L_{\alpha_j,i}(x),\ x \in [x_i, \alpha_j], \\ \bar{u}_{\alpha_j}H_{\alpha_j,0}(x) + \bar{u}'_{\alpha_j}H_{\alpha_j,1}(x) + u(x_{i+1})H_{i+1,\alpha_j}(x),\ x \in [\alpha_j, x_{i+1}], \end{cases}$$

here,

$$\begin{aligned} \bar{u}_{\alpha_j} &= \frac{1}{\Delta}\left(\Theta l'_j(\alpha_j) + D_jD_{j+1}l''_j(\alpha_j) - D_{j+1}^2u(x_{i+1})H''_{i+1,\alpha_j}\right), \\ \bar{u}'_{\alpha_j} &= \frac{1}{D_{j+1}}\left(\hat{\Theta}\bar{u}_{\alpha_j} + D_jl'_j(\alpha_j)\right). \end{aligned} \tag{72}$$

In addition, by (42) and the definition of Δ given in (69), we can derive the following estimation about $\frac{1}{\Delta}$:

$$\begin{aligned}\frac{1}{\Delta} \quad &\le \quad C\min\{(x_{i+1}-\alpha_j)(\alpha_j-x_i),(\alpha_j-x_i),\\ &(\alpha_j-x_i)(\alpha_j-x_{i+1/2}),(\alpha_j-x_i)^{3/2}(\alpha_j-x_{i+1/2})^{1/2}\}.\end{aligned} \tag{73}$$

The Lagrange-Hermit interpolation of u using the quadratic polynomials in (42) has the following expression:

$$\tilde{I}_h^j u = \begin{cases} u(x_i)L_{i,\alpha_j}(x)+u(x_{i+1/2})L_{i+1/2,\alpha_j}(x)+u(\alpha_j)L_{\alpha_j,i}(x),\ x\in[x_i,\alpha_j],\\ u(\alpha_j)H_{\alpha_j,0}(x)+u'(\alpha_j^+)H_{\alpha_j,1}(x)+u(x_{i+1})H_{i+1,,\alpha_j}(x),\ x\in[\alpha_j,x_{i+1}].\end{cases} \tag{74}$$

If $u|_{(x_i,\alpha_j)}\in H^3(x_i,\alpha_j)$ and $u|_{(\alpha_j,x_{i+1})}\in H^3(\alpha_j,x_{i+1})$, the standard finite element approximation theory provides the following estimates for $\tilde{I}_h^j u$:

$$\|u-\tilde{I}_h^j u\|_{0,(x_i,x_{i+1})}+h\|u-\tilde{I}_h^j u\|_{1,(x_i,x_{i+1})}\le Ch^3\|u\|_{3,(x_i,x_{i+1})}. \tag{75}$$

In addition, point-wise estimates in (56) can be readily extended to α_j as follows:

$$\begin{aligned}&|u''(\alpha_j^-)-(\tilde{I}_h^j u)''(\alpha_j^-)|\le(\alpha_j-x_i)^{1/2}\|u\|_{3,(x_i,\alpha_j)},\\ &|u''(\alpha_j^+)-(\tilde{I}_h^j u)''(\alpha_j^+)|\le(x_{i+1}-\alpha_j)^{1/2}\|u\|_{3,(\alpha_j,x_{i+1})},\\ &|u'(\alpha_j^-)-(\tilde{I}_h^j u)'(\alpha_j^-)|\le(\alpha_j-x_{i+1/2})(\alpha_j-x_i)^{1/2}\|u\|_{3,(x_i,\alpha_j)}.\end{aligned} \tag{76}$$

We now turn to the estimation for $u-\check{I}_h^j u$.

Theorem 4.4. *Let $[x_i,x_{i+1}]$ be an interface element containing the interface point α_j, $j=1,2,\dots,n-1$, and assume that $\delta_{j+1}\ge\delta_j$ and $D_j\delta_{j+1}\ge D_{j+1}\delta_j$, then for every $u\in C^0(T)$ such that $u|_{(x_i,\alpha_j)}\in H^3(x_i,\alpha_j)$ and $u|_{(\alpha_j,x_{i+1})}\in H^3(\alpha_j,x_{i+1})$, we have*

$$\|u-\check{I}_h^j u\|_{0,(x_i,x_{i+1})}+h\|u-\check{I}_h^j u\|_{1,(x_i,x_{i+1})}\le Ch^3\|u\|_{3,(x_i,x_{i+1})}, \tag{77}$$

where C is a constant independent of $\alpha_j\in[x_i,x_{i+1}]$.

Proof. Let us firstly consider the difference between $\check{I}_h^j u$ and $\tilde{I}_h^j u$. Subtracting (71) from (74), we have

$$\tilde{I}_h^j u-\check{I}_h^j u = \begin{cases}(u(\alpha_j)-\bar{u}_{\alpha_j})L_{\alpha_j,i}(x),\ x\in[x_i,\alpha_j],\\ (u(\alpha_j)-\bar{u}_{\alpha_j})H_{\alpha_j,0}(x)+[u'(\alpha_j^+)-\bar{u}'_{\alpha_j}]H_{\alpha_j,1}(x),\ x\in[\alpha_j,x_{i+1}].\end{cases} \tag{78}$$

By using jump conditions in (3b) and (68), we have

$$\bar{u}_{\alpha_j}-u(\alpha_j)=\sigma_1+\sigma_2+\sigma_3+\sigma_4,$$

here,

$$\begin{aligned}&\sigma_1=-\frac{1}{\Delta}D_jD_{j+1}H''_{\alpha_j,1}e'(\alpha_j^-),\ \sigma_2=2\frac{1}{\Delta}(D_j\delta_{j+1}e'(\alpha_j^-)-2D_{j+1}\delta_je'(\alpha_j^-)),\\ &\sigma_3=\frac{1}{\Delta}D_jD_{j+1}e''(\alpha_j^-),\ \sigma_4=-\frac{1}{\Delta}D_{j+1}^2e''(\alpha_j^+),\end{aligned}$$

with $e(x)=\tilde{I}_h^j u(x)-u(x)$. Using (73), we have

$$\left|\frac{1}{\Delta}\right|\le C(x_{i+1}-\alpha_j)(\alpha_j-x_i).$$

The last equation in (43), together with the estimate for $e'(\alpha_j^-)$ given in (76) imply that

$$|\sigma_1| \leq Ch^{3/2}(\alpha_j - x_{i+1/2})\|u\|_{3,(x_i,x_{i+1})}.$$

Similarly, by using (73) and the bounds for $|e'(\alpha_j^-)|$, $|e''(\alpha_j^-)|$, $|e''(\alpha_j^+)|$ given in (76), we obtain

$$|\sigma_k| \leq Ch^{3/2}(\alpha_j - x_{i+1/2})\|u\|_{3,(x_i,x_{i+1})},\ k = 2,\ 3,\ 4.$$

Summing up σ_k from 1 to 4, we obtain

$$|u(\alpha_j) - \bar{u}_{\alpha_j}| \leq Ch^{3/2}(\alpha_j - x_{i+1/2})\|u\|_{3,(x_i,x_{i+1})}. \tag{79}$$

Applying (79) and the estimate about $L_{\alpha_j,i}$ given in (44) to (78) leads to the following estimate on the left subelement $[x_i, \alpha_j]$:

$$\|\tilde{I}_h^j u - \check{I}_h^j u\|_{0,(x_i,\alpha_j)} + h\|\tilde{I}_h^j u - \check{I}_h^j u\|_{1,(x_i,\alpha_j)} \leq Ch^3\|u\|_{3,(x_i,x_{i+1})}. \tag{80}$$

Similar, using (72) and the jump conditions in (3b) and (68), we have

$$|u'(\alpha_j^+) - \bar{u}'_{\alpha_j}| \leq Ch^{3/2}\|u\|_{3,(x_i,x_{i+1})}. \tag{81}$$

Then, using (79), (81), and the estimates of $H_{\alpha_j,k}$, $k = 0,\ 1$ given in (44), we have the following estimate on the right subelement $[\alpha_j, x_{i+1}]$:

$$\|\tilde{I}_h^j u - \check{I}_h^j u\|_{0,(\alpha_j,x_{i+1})} + h\|\tilde{I}_h^j u - \check{I}_h^j u\|_{1,(\alpha_j,x_{i+1})} \leq Ch^3\|u\|_{3,(x_i,x_{i+1})}. \tag{82}$$

(80) and (82) imply

$$\|\tilde{I}_h^j u - \check{I}_h^j u\|_{0,(x_i,x_{i+1})} + h\|\tilde{I}_h^j u - \check{I}_h^j u\|_{1,(x_i,x_{i+1})} \leq Ch^3\|u\|_{3,(x_i,x_{i+1})}. \tag{83}$$

Finally, (75) together with (83) imply (77) directly:

$$\|u - \check{I}_h^j u\|_{0,(x_i,x_{i+1})} + h\|u - \check{I}_h^j u\|_{1,(x_i,x_{i+1})} \leq Ch^3\|u\|_{3,(x_i,x_{i+1})}.$$

□

4.3. Convergence of quadratic IFE space. We can define a quadratic IFE space globally on the whole solution domain (α_{-1}, α_n) as follows:

$$S_h^{(2)}(\alpha_{-1}, \alpha_n) = \{v \in L^2(\alpha_{-1}, \alpha_n) \mid v|_{\Omega^\pm} \in C^0(\Omega^\pm), v|_T \in S_h^{(2)}(T),\ \forall T \in \mathcal{T}_h\},$$

where, we recall that the local quadratic IFE space $S_h^{(2)}(T)$ is defined by

$$S_h^{(2)}(T) = \begin{cases} span\{L_{i,0}^{(2)},\ L_{i,1}^{(2)},\ L_{i,2}^{(2)}\},\ T = [x_i, x_{i+1}],\ T \cap \{\alpha_i\}_{i=0}^{n-1} = \emptyset, \\ span\{\psi_i^j,\ \psi_{i+1/2}^j,\ \psi_{i+1}^j\},\ T = [x_i, x_{i+1}],\ \alpha_j \in T,\ j = 0,\ 1,\ 2, \ldots, n-1. \end{cases}$$

For a function $u \in H_\alpha^1(\alpha_{-1}, \alpha_n)$, we define its quadratic IFE interpolation $\check{I}_h u \in S_h^{(2)}(\alpha_{-1}, \alpha_n)$ piecewisely such that, for every element $T = [x_i, x_{i+1}]$,

$$\check{I}_h u|_T = \begin{cases} u(x_i)L_{i,0}^{(2)}(x) + u(x_{i+1/2})L_{i,1}^{(2)} + u(x_{i+1})L_{i,2}^{(2)}(x),\ T \cap \{\alpha_i\}_{i=0}^{n-1} = \emptyset, \\ u(x_i)\psi_i^j(x) + u(x_{i+1/2})\psi_{i+1/2}^j(x) + u(x_{i+1})\psi_i^j(x),\ \alpha_j \in T, j = 0,\ 1,\ 2, \ldots, n-1. \end{cases}$$

Then, we get the error bound for the quadratic IFE interpolation which is given in the following theorem.

Theorem 4.5. *Assume that $\delta_{j+1} \geq \delta_j$ and $D_j\delta_{j+1} \geq D_{j+1}\delta_j$, then there exists a positive constant C independent of h and the position of α_j, $j = 0,\ 1, \ldots, n-1$ such that*

$$\|u - \check{I}_h u\|_{0,(\alpha_{-1},\alpha_n)} + h\|u - \check{I}_h u\|_{1,(\alpha_{-1},\alpha_n)} \leq Ch^3\|u\|_{3,(\alpha_{-1},\alpha_n)}.$$

Proof. Since we already have the interpolation error estimate on each element, the proof follows from arguments similar to those used in Theorem 3.5. □

We now consider the quadratic IFE solution to the weak problem defined by (7). Let

$$S_{h,0}^{(2)}(\alpha_{-1},\alpha_n) = \{v|v \in S_h^{(2)}(\alpha_{-1},\alpha_n),\ v|_{x=\alpha_n} = 0\}.$$

The quadratic IFE solution $u_h \in S_{h,0}^{(2)}(\alpha_{-1},\alpha_n)$ to the interface problem described by (1)-(3) is defined to be such that

$$a(u_h, v_h) = (f, v_h),\ \forall\, v_h \in S_{h,0}^{(2)}(\alpha_{-1},\alpha_n). \tag{84}$$

Then the error bound for the quadratic IFE solution u_h can be derived as before which is stated in the following theorem.

Theorem 4.6. *Assume the condition required by Theorem 2.2 holds, $\delta_{j+1} \geq \delta_j$ and $D_j\delta_{j+1} \geq D_{j+1}\delta_j$. Let $u \in H_\alpha^3([\alpha_{-1},\alpha_n])$ be the solution to the weak problem* (7), *then the quadratic IFE solution u_h defined by* (84) *satisfies the following estimate:*

$$\|u-u_h\|_{0,(\alpha_{-1},\alpha_n)} + h\|u-u_h\|_{1,(\alpha_{-1},\alpha_n)} \leq Ch^3\|u\|_{3,(\alpha_{-1},\alpha_n)}.$$

Proof. The proof follows form arguments similar to those used in Theorem 3.6. □

5. Numerical Experiments

In this section we present numerical examples for demonstrating the convergence of the linear and quadratic IFE methods. We let the simulation interval be $[0,1]$, and assume there are three interface points: $\alpha_0 = 1/9$, $\alpha_1 = 1/3$ and $\alpha_2 = 2/3$. These interface points separate the interval into four sub-intervals $[0,1/9]$, $[1/9,1/3]$, $[1/3,2/3]$ and $[2/3,1]$. We set the exact solution u for this problem to be

$$u(x) = \begin{cases} u_0(x),\ x \in [0,1/9], \\ u_1(x),\ x \in [1/9,1/3], \\ u_2(x),\ x \in [1/3,2/3], \\ u_3(x),\ x \in [2/3,1], \end{cases} \tag{85}$$

with

$$u_0(x) = \frac{1}{30}x^{n-1},\ u_1(x) = \frac{1}{3}x^n,\ u_2(x) = x^{n+1},\ u_3(x) = 3(1-x)x^{n+1},$$

here, n is an integer. We also let

$$\begin{aligned} D_1 &= \frac{18(n-1)D_0}{10n}, & \delta_1 &= \frac{1}{2}(9nD_1 - 8.1(n-1)D_0), \\ D_2 &= \frac{6nD_1 - 2\delta_1}{3(n+1)}, & \delta_2 &= \frac{1}{2}(3(n+1)D_2 - 3nD_1 + 2\delta_1), \\ D_3 &= \frac{8\delta_2 - 3(n+1)D_2}{3(n+5)}, & \delta_3 &= \frac{1}{4}(3(n-1)D_3 - 3(n+1)D_2 + 4\delta_2), \\ \lambda &= \frac{1}{81(n-1)D_0}. \end{aligned}$$

Then we can verify that $u(x)$ satisfies the jump conditions (3). The right hand side term $f_i(x)$, $i = 0, 1, 2, 3$, is determined by (1). We now report numerical results generated by applying the IFE methods developed in Section 3 and Section 4 to the interface problem described by (1)-(3) whose exact solution is $u(x)$ defined in (85).

Example 1. *In this group of numerical experiments, we observe that the proposed IFE methods work well for large values of* β_i, $i = 1, 2, 3$ *emphasizing a stronger reaction. We have tested these IFE methods with* D_0 *arbitrarily chosen between* 10^{-2} *and* 2×10^{-2} *and* $\beta_1 \in (1, 2)$, $\beta_2 \in (10^2, 2 \times 10^2)$, $\beta_3 \in (10^4, 2 \times 10^4)$.

Table 1 presents typical numerical results for errors and the convergence rates for the linear IFE method when $n = 3$ and $n = 6$, Table 2 contains correspondingly numerical results for the quadratic IFE method. In related computations for these data, we used the following parameters:

$$D_0 = 0.010616229584014,\ \beta_0 = 0,\ \beta_1 = 1.72333131659,$$
$$\beta_2 = 140.10036408113,\ \beta_3 = 16730.88858585343$$

Data in these tables clearly show that the linear and quadratic IFE methods converge optimally in both the L^2 and H^1 norm.

TABLE 1. Errors and convergence rates of the linear IFE method when $n = 3$ (left) and $n = 6$ (right) with large values for $\beta_j, j = 1, 2, 3$.

N	L^2norm	rate	H^1norm	rate
10	4.3701e-03		2.2745e-01	
20	9.1751e-04	2.2519	1.1101e-01	1.0348
40	2.0358e-04	2.1721	5.4916e-02	1.0154
80	4.7784e-05	2.0910	2.7257e-02	1.0106
160	1.1416e-05	2.0655	1.3569e-02	1.0063
320	2.8087e-06	2.0231	6.7707e-03	1.0030
640	6.9626e-07	2.0122	3.3820e-03	1.0014
1280	1.7385e-07	2.0018	1.6906e-03	1.0004
2560	4.3403e-08	2.0019	8.4523e-04	1.0001

N	L^2norm	rate	H^1norm	rate
10	5.8815e-03		2.7493e-01	
20	1.3050e-03	2.1721	1.4092e-01	0.9642
40	2.8071e-04	2.2169	6.9235e-02	1.0253
80	6.3358e-05	2.1475	3.4282e-02	1.0140
160	1.4811e-05	2.0968	1.6990e-02	1.0128
320	3.5812e-06	2.0482	8.4612e-03	1.0058
640	8.8460e-07	2.0173	4.2231e-03	1.0026
1280	2.2033e-07	2.0054	2.1106e-03	1.0006
2560	5.5041e-08	2.0011	1.0552e-03	1.0002

TABLE 2. Errors and convergence rates of the quadratic IFE method when $n = 3$ (left) and $n = 6$ (right) with large values for $\beta_j, j = 1, 2, 3$.

N	L^2norm	rate	H^1norm	rate
10	2.6661e-04		1.8276e-02	
20	3.3866e-05	2.9768	4.5238e-03	2.0143
40	3.7201e-06	3.1865	9.6740e-04	2.2254
80	4.6298e-07	3.0063	2.4106e-04	2.0047
160	5.7719e-08	3.0038	5.9870e-05	2.0095
320	7.2261e-09	2.9977	1.4985e-05	1.9983
640	9.0293e-10	3.0005	3.7433e-06	2.0011
1280	1.1287e-10	2.9999	9.3613e-07	1.9995
2560	1.4107e-11	3.0002	2.3399e-07	2.0003

N	L^2norm	rate	H^1norm	rate
10	5.3218e-04		3.7440e-02	
20	6.8757e-05	2.9523	9.2084e-03	2.0236
40	8.7238e-06	2.9785	2.2819e-03	2.0127
80	1.0967e-06	2.9918	5.6983e-04	2.0016
160	1.3729e-07	2.9978	1.4243e-04	2.0003
320	1.7170e-08	2.9993	3.5611e-05	1.9998
640	2.1465e-09	2.9999	8.9031e-06	1.9999
1280	2.6833e-10	2.9999	2.2259e-06	1.9999
2560	3.3541e-11	3.0000	5.5647e-07	2.0000

Example 2. *We have observed that the IFE methods also work well for small values of* β_i, $i = 1, 2, 3$ *which let the model described by the interface problem* (1)-(3) *emphasize the diffusion or convection more than the reaction. In this group of numerical experiments, we have tested the IFE methods for* β_i, $i = 1, 2, 3$ *randomly chosen such that* $\beta_1 \in (0.01, 0.02)$, $\beta_2 \in (0.001, 0.002)$, $\beta_3 \in (0.0001, 0.0002)$.

TABLE 3. Errors and convergence rates of the linear IFE method when $n = 3$ (left) and $n = 6$ (right) with small values for $\beta_j, j = 1, 2, 3$.

N	L^2norm	rate	H^1norm	rate	N	L^2norm	rate	H^1norm	rate
10	4.9674e-03		2.1563e-01		10	6.4165e-03		2.6721e-01	
20	1.2271e-03	2.0172	1.0790e-01	0.9989	20	1.6163e-03	1.9891	1.3491e-01	0.9860
40	3.0870e-04	1.9910	5.3986e-02	0.9990	40	3.9695e-04	2.0257	6.7431e-02	1.0005
80	7.6854e-05	2.0060	2.7025e-02	0.9983	80	1.0065e-04	1.9796	3.3759e-02	0.9982
160	1.9307e-05	1.9930	1.3516e-02	0.9997	160	2.4837e-05	2.0188	1.6876e-02	1.0003
320	4.8130e-06	2.0041	6.7603e-03	0.9994	320	6.2946e-06	1.9803	8.4407e-03	0.9996
640	1.2293e-06	1.9691	3.3804e-03	0.9999	640	1.5535e-06	2.0186	4.2202e-03	1.0001
1280	3.0329e-07	2.0191	1.6904e-03	0.9999	1280	3.9354e-07	1.9810	2.1102e-03	0.9999
2560	7.5839e-08	1.9997	8.4520e-04	1.0000	2560	9.7113e-08	2.0188	1.0551e-03	1.0000

TABLE 4. Errors and convergence rates of the quadratic IFE method when $n = 3$ (left) and $n = 6$ (right) with small values for $\beta_j, j = 1, 2, 3$.

N	L^2norm	rate	H^1norm	rate	N	L^2norm	rate	H^1norm	rate
10	2.3101e-04		1.5134e-02		10	5.4047e-04		3.5801e-02	
20	2.9875e-05	2.9510	3.8832e-03	1.9624	20	6.9574e-05	2.9576	9.0744e-03	1.9801
40	3.6881e-06	3.0180	9.5535e-04	2.0232	40	8.7780e-06	2.9866	2.2754e-03	1.9957
80	4.6321e-07	2.9932	2.4015e-04	1.9921	80	1.0982e-06	2.9987	5.6951e-04	1.9983
160	5.7849e-08	3.0013	5.9856e-05	2.0044	160	1.3749e-07	2.9978	1.4242e-04	1.9996
320	7.2484e-09	2.9966	1.4984e-05	1.9981	320	1.7173e-08	3.0011	3.5611e-05	1.9997
640	9.0585e-10	3.0003	3.7433e-06	2.0010	640	2.1486e-09	2.9987	8.9031e-06	1.9999
1280	1.1333e-10	2.9987	9.3613e-07	1.9995	1280	2.6835e-10	3.0012	2.2259e-06	1.9999
2560	1.5537e-11	2.8669	2.3399e-07	2.0002	2560	3.4181e-11	2.9728	5.5647e-07	2.0000

Typical numerical results are presented in Tables 3-4 which, again, clearly demonstrate the optimal convergence of the proposed linear and quadratic IFE methods. In related computations for these data, we used the following parameters:

$$D_0 = 0.010616229584014,\ \beta_0 = 0,\ \beta_1 = 0.015482995717499,$$
$$\beta_2 = 0.001133907855848\ \beta_3 = 0.000162651593614.$$

6. Conclusions

In this article, we have developed linear and quadratic IFE methods for the steady state interface problem about a multi-layer wall model for the drug-eluting stent. Error estimation have been carried out to establish the optimal convergence of the proposed IFE methods and numerical examples are provided to corroborate the theoretical analysis. The developed linear and quadratic IFE spaces can be applied to the related time dependent multi-layer wall model for the drug-eluting stent.

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