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THE l^1-STABILITY OF A HAMILTONIAN-PRESERVING SCHEME FOR THE LIOUVILLE EQUATION WITH DISCONTINUOUS POTENTIALS*

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Abstract

We study the l^1-stability of a Hamiltonian-preserving scheme, developed in [Jin and Wen, *Comm. Math. Sci.,* 3 (2005), 285-315], for the Liouville equation with a discontinuous potential in one space dimension. We prove that, for suitable initial data, the scheme is stable in the l^1-norm under a hyperbolic CFL condition which is in consistent with the l^1-convergence results established in [Wen and Jin, *SIAM J. Numer. Anal.,* 46 (2008), 2688-2714] for the same scheme. The stability constant is shown to be independent of the computational time. We also provide a counter example to show that for other initial data, in particular, the measure-valued initial data, the numerical solution may become l^1-unstable.

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1. Introduction

In [7], we constructed a class of numerical schemes for the d-dimensional Liouville equation in classical mechanics:

$$f_t + \mathbf{v}\cdot\nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} V\cdot\nabla_{\mathbf{v}} f = 0, \quad t>0, \quad \mathbf{x},\mathbf{v}\in R^d, \tag{1.1}$$

where $f(t,\mathbf{x},\mathbf{v})$ is the density distribution of a classical particle at position $\mathbf{x}$, time t and traveling with velocity $\mathbf{v}$. $V(\mathbf{x})$ is the potential. The main interest is in the case of a discontinuous potential $V(\mathbf{x})$, corresponding to a potential barrier. When V is discontinuous, the Liouville equation (1.1) is a linear hyperbolic equation with a measure-valued coefficient. One needs to provide additional condition in order to select a unique, physically relevant solution across the barrier. The main idea of the Hamiltonian-preserving schemes developed in [7] was to build into the numerical flux the particle behavior at the barrier. See also the related work on Hamiltonian-preserving schemes [2, 3, 5, 6, 8–13].

The Liouville equation (1.1) is a different formulation of Newton's second law:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = -\nabla_{\mathbf{x}} V, \tag{1.2}$$

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which is a Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}|\mathbf{v}|^2 + V(\mathbf{x}). \tag{1.3}$$

It is known from classical mechanics that the Hamiltonian remains constant across a potential barrier. By using this mechanism in the numerical flux, the schemes developed in [7] provide a physically relevant solution to the underlying problem. It was proved that the two schemes developed in [7], under a hyperbolic CFL condition, are positive, and stable under both l^∞ and l^1 norms in one space dimension except the l^1-stability of Scheme I. Scheme I uses a finite difference approach involving interpolations in the phase space and the l^1-stability of this scheme is more sophisticated. In this paper we consider this issue in details. We will prove that Scheme I is l^1-stable with the stability constant independent of the computational time if the initial data satisfy certain condition, but can be l^1-unstable if the initial data condition is violated. The initial data condition is satisfied when applying the decomposition technique proposed in [4] for solving the Liouville equation with measure-valued initial data arisen from the semiclassical limit of the linear Schrödinger equation. Recently the l^1-convergence of the same scheme under certain initial data condition has been established in [19] by applying the l^1-error estimates developed in [16, 18] for the immersed interface upwind scheme to the linear advection equations with piecewise constant coefficients. We show that the results established in this paper is in consistent with the convergence results established in [19] since the initial data condition considered in this paper is more general than that in [19].

The paper is organized as follows. In Sect. 2, we first present Scheme I developed in [7]. In Sect. 3, we prove the l^1-stability of this scheme for suitable initial data. We give a counter example in Sect. 4 to show that for more general initial data, in particular the measure-valued initial data, the numerical solution may become unbounded. We conclude the paper in Sect. 5.

2. A Hamiltonian-Preserving Scheme

Consider the Liouville equation in one space dimension:

$$f_t + \xi f_x - V_x f_\xi = 0 \tag{2.1}$$

with a discontinuous potential $V(x)$.

Without loss of generality, we employ a uniform mesh with grid points at $x_{i+\frac{1}{2}}, i = 0, \cdots, N$, in the x-direction and $\xi_{j+\frac{1}{2}}, j = 0, \cdots, M$ in the ξ-direction. The cells are centered at $(x_i, \xi_j), i = 1, \cdots, N, j = 1, \cdots, M$ with

$$x_i = \frac{1}{2}(x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}), \quad \xi_j = \frac{1}{2}(\xi_{j+\frac{1}{2}} + \xi_{j-\frac{1}{2}}).$$

The mesh size is denoted by $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \Delta\xi = \xi_{j+\frac{1}{2}} - \xi_{j-\frac{1}{2}}$. We also assume a uniform time step Δt and the discrete time is given by $0 = t_0 < t_1 < \cdots < t_L = T$. We introduce mesh ratios $\lambda_x^t = \Delta t/\Delta x, \lambda_\xi^t = \Delta t/\Delta\xi, \lambda_x^\xi = \Delta\xi/\Delta x$, assumed to be fixed. We define the cell averages of f as

$$f_{ij} = \frac{1}{\Delta x \Delta\xi} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{\xi_{j-\frac{1}{2}}}^{\xi_{j+\frac{1}{2}}} f(x, \xi, t)\, d\xi\, dx.$$

The 1-d average quantity $f_{i+1/2,j}$ is defined as

$$f_{i+1/2,j} = \frac{1}{\Delta\xi} \int_{\xi_{j-\frac{1}{2}}}^{\xi_{j+\frac{1}{2}}} f(x_{i+1/2}, \xi, t) d\xi\,.$$

$f_{1,j+1/2}$ is defined similarly.

A typical semi-discrete finite difference method for this equation is

$$\partial_t f_{ij} + \xi_j \frac{f_{i+\frac{1}{2},j} - f_{i-\frac{1}{2},j}}{\Delta x} - DV_i \frac{f_{i,j+\frac{1}{2}} - f_{i,j-\frac{1}{2}}}{\Delta \xi} = 0, \tag{2.2}$$

where the numerical fluxes $f_{i+\frac{1}{2},j}, f_{i,j+\frac{1}{2}}$ are defined by the upwind scheme, and DV_i is some numerical approximation of V_x at $x = x_i$.

Such a discretization suffers from at least two problems:

- The above discretization in general does not preserve a constant Hamiltonian $H = \frac{1}{2}\xi^2 + V$ across the discontinuities of V. Such a numerical approximation may lead to unphysical solutions or poor numerical resolution.

- If an explicit time discretization is used, the CFL condition for this scheme requires the time step to satisfy

$$\Delta t \left[\frac{\max_j |\xi_j|}{\Delta x} + \frac{\max_i |DV_i|}{\Delta \xi} \right] \le 1. \tag{2.3}$$

Since the potential $V(x)$ is discontinuous at some points, $\max_i |DV_i| = \mathcal{O}(1/\Delta x)$, so the CFL condition (2.3) requires $\Delta t = \mathcal{O}(\Delta x \Delta \xi)$.

In classical mechanics, a particle will either cross a potential barrier with a changing momentum, or be reflected, depending on its momentum and on the strength of the potential barrier. *The Hamiltonian* $H = \frac{1}{2}\xi^2 + V$ *should be preserved across the potential barrier:*

$$\frac{1}{2}(\xi^+)^2 + V^+ = \frac{1}{2}(\xi^-)^2 + V^- \tag{2.4}$$

where the superscripts $\pm$ indicate the right and left limits of the quantity at the potential barrier.

The main ingredient in the Hamiltonian-preserving schemes developed in [7], like the early work for shallow-water equations [15], was to build into the numerical flux the particle behavior at the barrier. Since the density distribution f remains unchanged across the potential barrier, thus

$$f(t, x^+, \xi^+) = f(t, x^-, \xi^-) \tag{2.5}$$

at a discontinuous point x of $V(x)$, where ξ^+ and ξ^- are related by the constant Hamiltonian condition (2.4). This was used in constructing the numerical flux in [7].

We now present the first Hamiltonian-preserving scheme, called *Scheme I* in [7].

Assume that the discontinuous points of the potential V are located at the grid points. Let the left and right limits of V at point $x_{i+1/2}$ be $V^+_{i+\frac{1}{2}}$ and $V^-_{i+\frac{1}{2}}$ respectively. Note that if V is continuous at $x_{j+1/2}$, then $V^+_{i+\frac{1}{2}} = V^-_{i+\frac{1}{2}}$. We approximate V by a piecewise linear function

$$V(x) \approx V^+_{i-1/2} + \frac{V^-_{i+1/2} - V^+_{i-1/2}}{\Delta x}(x - x_{i-1/2}) .$$

The flux-splitting, semidiscrete scheme (with time continuous) reads

$$\partial_t f_{ij} + \xi_j \frac{f^-_{i+\frac{1}{2},j} - f^+_{i-\frac{1}{2},j}}{\Delta x} - \frac{V^-_{i+\frac{1}{2}} - V^+_{i-\frac{1}{2}}}{\Delta x} \frac{f_{i,j+\frac{1}{2}} - f_{i,j-\frac{1}{2}}}{\Delta \xi} = 0, \tag{2.6}$$

where the numerical fluxes $f_{i,j+\frac{1}{2}}$ are defined using the upwind discretization. Since the characteristics of the Liouville equation may be different on the two sides of a potential discontinuity, the corresponding numerical fluxes should also be different. The essential part of the algorithm is to define the split numerical fluxes $f^-_{i+\frac{1}{2},j}, f^+_{i-\frac{1}{2},j}$ at each cell interface. (2.5) will be used to define these fluxes.

Assume V is discontinuous at $x_{i+1/2}$. Consider the case $\xi_j > 0$. Using upwind scheme, $f^-_{i+\frac{1}{2},j} = f_{ij}$. However,

$$f^+_{i+1/2,j} = f\left(x^+_{i+1/2}, \xi^+_j\right) = f\left(x^-_{i+1/2}, \xi^-_j\right)$$

while ξ^- is obtained from $\xi^+_j = \xi_j$ from (2.4). Since ξ^- may not be a grid point, we have to define it approximately. The first approach is to locate the two cell centers that bound this velocity, then use a linear interpolation to evaluate the needed numerical flux at ξ^-. The case of $\xi_j < 0$ is treated similarly. The algorithm to generate the numerical flux is given in [7]. Here we present the simplified algorithm for the case $V^-_{i+\frac{1}{2}} > V^+_{i+\frac{1}{2}}$ being considered in this paper.

Algorithm 2.1.

- $\xi_j > 0$

  $$f^-_{i+\frac{1}{2},j} = f_{ij},$$

  ❏ *if* $\xi_j > \sqrt{2\left(V^-_{i+\frac{1}{2}} - V^+_{i+\frac{1}{2}}\right)}$,

  $$\xi^- = \sqrt{\xi_j^2 + 2\left(V^+_{i+\frac{1}{2}} - V^-_{i+\frac{1}{2}}\right)}$$

  if $\xi_k \le \xi^- < \xi_{k+1}$ *for some* k, *then*

  $$f^+_{i+\frac{1}{2},j} = \frac{\xi_{k+1} - \xi^-}{\Delta\xi} f_{ik} + \frac{\xi^- - \xi_k}{\Delta\xi} f_{i,k+1}.$$

  ❏ *else*

  $$f^+_{i+\frac{1}{2},j} = f_{i+1,k} \text{ where } \xi_k = -\xi_j$$

  ❏ *end*

- $\xi_j < 0$

  $$f^+_{i+\frac{1}{2},j} = f_{i+1,j},$$

  $$\xi^+ = -\sqrt{\xi_j^2 + 2\left(V^-_{i+\frac{1}{2}} - V^+_{i+\frac{1}{2}}\right)}$$

  if $\xi_k \le \xi^+ < \xi_{k+1}$ *for some* k, *then*

  $$f^-_{i+\frac{1}{2},j} = \frac{\xi_{k+1} - \xi^+}{\Delta\xi} f_{i+1,k} + \frac{\xi^+ - \xi_k}{\Delta\xi} f_{i+1,k+1}.$$

Remark 2.1. In the case $V^-_{i+\frac{1}{2}} > V^+_{i+\frac{1}{2}}$, the following situation needs to be specially dealt with.

- For

  $$\xi_j > \sqrt{2\left(V^-_{i+\frac{1}{2}} - V^+_{i+\frac{1}{2}}\right)},$$

assume 0 is not located at a mesh point in ξ-direction. Denote the index $k^{0,+}$ such that $\xi_{k^{0,+}-1} < 0$, $\xi_{k^{0,+}} > 0$. If

$$\xi^- = \sqrt{\xi_j^2 + 2\left(V^+_{i+\frac{1}{2}} - V^-_{i+\frac{1}{2}}\right)}$$

belongs to $(0, \xi_{k^{0,+}})$, we set the numerical flux as $f^+_{i+\frac{1}{2},j} = f_{i,k^{0,+}}$ instead of the average of $f_{i,k^{0,+}-1}$ and $f_{i,k^{0,+}}$ as described in the algorithm.

After the spatial discretization is specified, one can use any time discretization for the time derivative.

In [7] we proved that, when the first order upwind scheme is used spatially, and the forward Euler method is used in time, and the potential V has a single jump, Scheme I is positive and l^∞-contracting under the CFL condition:

$$\Delta t \left[\frac{\max_j |\xi_j|}{\Delta x} + \frac{\max_i \left| V^-_{i+\frac{1}{2}} - V^+_{i-\frac{1}{2}} \right| / \Delta x}{\Delta \xi} \right] \le 1. \tag{2.7}$$

Note that the quantity $\left| V^-_{i+\frac{1}{2}} - V^+_{i-\frac{1}{2}} \right| / \Delta x$ represents the gradient of potential at its *smooth* point, which has a *finite* upper bound. Thus the scheme satisfies a hyperbolic CFL condition.

3. The l^1-stability of Scheme I

In this section we prove the l^1-stability of Scheme I (with the first order numerical flux and the forward Euler method in time) under a suitable condition on the initial data. We consider the simple case when $V(x)$ is a step function, with a jump $-D, D > 0$ at $x_{m+\frac{1}{2}}$. Namely

$$V^-_{m+\frac{1}{2}} - V^+_{m+\frac{1}{2}} = D, \quad V^\pm_{i+\frac{1}{2}} = V^-_{m+\frac{1}{2}}, i < m, \quad V^\pm_{i+\frac{1}{2}} = V^+_{m+\frac{1}{2}}, i > m.$$

We consider the typical situation that $\xi_1 < -\sqrt{2D}, \xi_M > \sqrt{2D}$, so that all possible particle behaviors are included. We also choose the mesh such that 0 is a grid point in the ξ-direction, and $\pm\sqrt{2D}$ are *not* located at cell center points in ξ-direction.

Define an index set

$$D_l^4 = \left\{ (i,j) | x_i \le x_m, \xi_j < -\sqrt{\xi_1^2 - 2D} \right\}.$$

Due to velocity change across the potential jump at $x_{m+\frac{1}{2}}$, D_l^4 represents the area where particles come from outside of the domain $[x_1, x_N] \times [\xi_1, \xi_M]$. In order to implement Scheme I conveniently, we need to choose the computational domain as

$$E_d = \left\{ (i,j) | i = 1, \cdots, N, j = 1, \cdots, M \right\} \setminus D_l^4. \tag{3.1}$$

Figure 3.1 depicts E_d and D_l^4. We define the l^1-norm of a numerical solution f_{ij} to be

$$|f|_1 = \frac{1}{N_d} \sum_{(i,j) \in E_d} |f_{ij}| \tag{3.2}$$

with N_d being the number of elements in E_d. We consider f satisfying the zero boundary condition at incoming boundary in which case the true solution is l^1-stable. Denote

$$\mu_j = \lambda_x^t |\xi_j|, \quad 1 \le j \le M. \tag{3.3}$$

Under the CFL condition (2.7), $\mu_j \le 1,\ 1 \le j \le M$.

Since $V_x(x) = 0$ except at $x = x_{m+1/2}$, Scheme I is given by:

1) if $\xi_j > 0, i \neq m+1$,

$$f_{ij}^{n+1} = (1-\mu_j) f_{ij} + \mu_j f_{i-1,j}\,; \tag{3.4}$$

2) if $\xi_j < 0, i \neq m$,

$$f_{ij}^{n+1} = (1-\mu_j) f_{ij} + \mu_j f_{i+1,j}\,; \tag{3.5}$$

3) if $\xi_j > \sqrt{2D}$,

$$f_{m+1,j}^{n+1} = (1-\mu_j) f_{m+1,j} + \mu_j (c_{j,k} f_{m,k} + c_{j,k+1} f_{m,k+1})\,; \tag{3.6}$$

4) if $0 < \xi_j \le \sqrt{2D}$,

$$f_{m+1,j}^{n+1} = (1-\mu_j) f_{m+1,j} + \mu_j f_{m+1,k}\,; \tag{3.7}$$

5) if $\xi_j < 0$,

$$f_{mj}^{n+1} = (1-\mu_j) f_{mj} + \mu_j (c_{jk} f_{m+1,k} + c_{j,k+1} f_{m+1,k+1}), \tag{3.8}$$

where $0 \le c_{jk} \le 1$ and $c_{jk} + c_{j,k+1} = 1$. In (3.6), k is determined by

$$\xi_k \le \sqrt{\xi_j^2 - 2D} < \xi_{k+1};$$

in (3.7), $\xi_k = -\xi_j$, and in (3.8)

$$\xi_k \le -\sqrt{\xi_j^2 + 2D} < \xi_{k+1}.$$

We omit the superscript n of f_{ij} on the right hand side.

We now impose an assumption under which we will establish the l^1-stability of Scheme I:

Assumption 3.1. *There exists a positive constant ξ_z such that*

$$\forall (i,j) \in S_z = \big\{(i,j) \;\big|\; x_i < x_{m+\frac12},\ 0 < \xi_j < \xi_z\big\}, \tag{3.9}$$

it holds that

$$|f_{ij}^0| \le C_1 |f^0|_1\,. \tag{3.10}$$

Remark 3.1. When arisen from the semiclassical limit of the linear Schrödinger equation, the Liouville equation is supplied with measure-valued initial data [1, 14], which does not satisfy this assumption. Thus Scheme I, when directly applied to this problem, may have stability problems, as shown in the next subsection. However, in [4], a decomposition of the initial data was introduced, which allows one to solve the semiclassical limit problem with only bounded initial data. Thus Scheme I is still suitable by using this decomposition. Recently in [19] we have established the l^1-convergence of Scheme I with a step function potential and Dirichlet incoming boundary condition when the initial data satisfy the following assumption

Assumption 3.2. *The initial data $f(x,\xi,0)$ have bounded variation in the x-direction and is Lipschitz continuous in the ξ-direction. Namely*

$$\|f(.,\xi,0)\|_{BV([x_{\frac12},x_{N+\frac12}])} \le A, \quad \forall \xi \in [\xi_{\frac12}, \xi_{M+\frac12}], \tag{3.11}$$

$$|f(x,\xi',0) - f(x,\xi'',0)| \le B|\xi' - \xi''|,$$

$$\forall x \in \big[x_{\frac12}, x_{N+\frac12}\big], \quad \xi', \xi'' \in \big[\xi_{\frac12}, \xi_{M+\frac12}\big]. \tag{3.12}$$

The initial data satisfying Assumption 3.2 is bounded in both l^∞ and l^1-norms on E_d. Thus its cell averages satisfy Assumption 3.1. Therefore the results established in the following Theorem 3.1 imply Scheme I is l^1-stable also for initial data satisfying Assumption 3.2. This is in consistent with the convergence results given in [19] since a convergent scheme for the Liouville equation with the zero incoming boundary condition should be l^1-stable.

We give the following lemma.

Lemma 3.1. *Under the hyperbolic CFL condition* (2.7)*, the mesh size restriction*

$$\Delta\xi < \frac{\sqrt{2D}}{2} \tag{3.13}$$

and the zero incoming boundary condition, Scheme I given by (3.4)-(3.8) *satisfies*

$$|f^L|_1 \le |f^0|_1 + \frac{S_1}{N_d} + \frac{\frac{1}{2}+\frac{1}{2}\lambda_x^t\Delta\xi}{N_d} S_2, \tag{3.14}$$

where

$$S_1 = \sum_{n=0}^{L-1}\Big\{\sum_{(i,j)\in D_m^2}|f_{ij}^n|\Big\}, \quad S_2 = \sum_{n=0}^{L-1}\Big\{\sum_{(i,j)\in D_{m+1}^4}|f_{ij}^n|\Big\}, \tag{3.15}$$

$$D_m^2 = \big\{(m,j)|j\in S_m^2\big\}, \tag{3.16}$$

$$S_m^2 = \left\{k\middle|\xi_k>0,\, \exists\, \xi_j>\sqrt{2D}, \ \text{ s.t. } \ \left|\xi_k - \sqrt{{\xi_j}^2-2D}\right| < \Delta\xi\right\}, \tag{3.17}$$

$$D_{m+1}^4 = \big\{(m+1,j)|\xi_j < -\sqrt{2D}+\Delta\xi\big\}. \tag{3.18}$$

The sets D_m^2, D_{m+1}^4 are sketched in Figure 3.1.

Proof. Applying the triangle inequality to (3.4)-(3.8) and using the zero incoming boundary condition, one typically gets the following

$$|f^{n+1}|_1 \le \frac{1}{N_d}\sum_{(i,j)\in E_d}\alpha_{ij}|f_{ij}^n|, \tag{3.19}$$

where the coefficients α_{ij} are positive. One can check that, under the hyperbolic CFL condition (2.7),

$$\alpha_{ij}\le 1 \text{ except for possibly } (i,j)\in D_m^2\cup D_{m+1}^4. \tag{3.20}$$

Denote

$$M_1 = \max_{(i,j)\in D_m^2}\alpha_{ij}, \quad M_2 = \max_{(i,j)\in D_{m+1}^4}\alpha_{ij}. \tag{3.21}$$

We then estimate these two bounds M_1, M_2. We begin with examining M_1. Define the set

$$S_j^m = \left\{j'\middle|\xi_{j'}>\sqrt{2D}, \left|\sqrt{\xi_{j'}^2-2D}-\xi_j\right| < \Delta\xi\right\} \quad \text{for} \quad (m,j)\in D_m^2.$$

Let the number of elements in S_j^m be N_j^m. One can check that $N_j^m \le 2$ because every two elements $j_1', j_2' \in S_j^m$ satisfy

$$\left|\sqrt{\xi_{j_1'}^2-2D}-\sqrt{\xi_{j_2'}^2-2D}\right| > |\xi_{j_1'}-\xi_{j_2'}| \ge \Delta\xi.$$

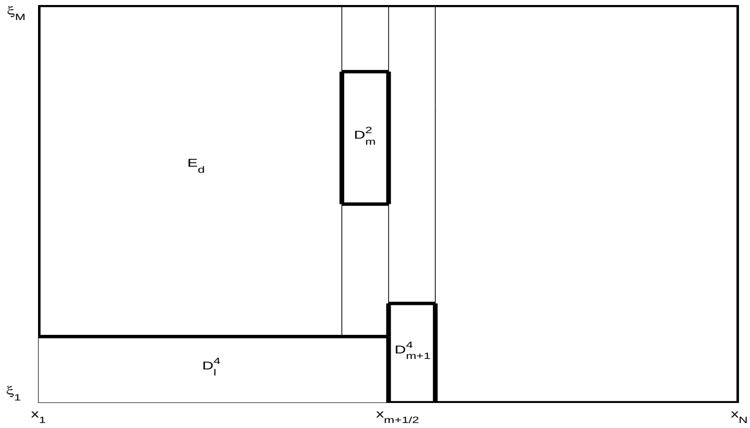

Figure 3.1 Sketch of the index sets D^2_m, D^4_{m+1}, D^4_l

If $N^m_j = 1$, denote the element in S^m_j to be j^1. Directly checking from formulas (3.4) and (3.6) one has

$$\alpha_{mj} \le 1 - \mu_j + \mu_{j^1} < 2. \tag{3.22}$$

Recall the notation $k^{0,+}$ defined in Remark 2.1 s.t. $\xi_{k^{0,+}} = \frac{1}{2}\Delta\xi$. Under the mesh size restriction (3.13), if $k^{0,+} \in S^2_m$ then $N^m_{k^{0,+}}$=1.

If $N^m_j = 2$, denote the elements in S^m_j to be j^1, j^2. Denote

$$\xi^1_l = \sqrt{(\xi_{j^1})^2 - 2D}, \quad \xi^2_l = \sqrt{(\xi_{j^2})^2 - 2D}.$$

Then from (3.4) and (3.6), Algorithm I, one gets

$$\alpha_{mj} = 1 - \mu_j + \mu_{j^1}\big(1 - \big|\xi^1_l - \xi_j\big|/\Delta\xi\big) + \mu_{j^2}\big(1 - \big|\xi^2_l - \xi_j\big|/\Delta\xi\big). \tag{3.23}$$

Since $\big|\xi^1_l - \xi^2_l\big| > \Delta\xi$, then $(\xi^1_l - \xi_j)(\xi^2_l - \xi_j) < 0$. From (3.23) one has

$$\begin{aligned}\alpha_{mj} &< 1 - \mu_j + \big(1 - \big|\xi^1_l - \xi_j\big|/\Delta\xi\big) + \big(1 - \big|\xi^2_l - \xi_j\big|/\Delta\xi\big)\\ &= 1 - \mu_j + \big(2 - \big|\xi^1_l - \xi^2_l\big|/\Delta\xi\big) < 2 - \mu_j < 2.\end{aligned} \tag{3.24}$$

Combining (3.22) and (3.24) one gets

$$\alpha_{mj} < 2, \quad \forall (m,j) \in D^2_m. \tag{3.25}$$

Therefore we have

$$M_1 < 2. \tag{3.26}$$

Next we study M_2. Define the set

$$S^{m+1}_j = \Big\{j' \Big| \xi_{j'} < 0, \big| - \sqrt{\xi^2_{j'} + 2D} - \xi_j\big| < \Delta\xi\Big\} \quad \text{for} \quad (m+1,j) \in D^4_{m+1}$$

and its subdivisions

$$\begin{aligned} S^{m+1,+}_j &= \Big\{j' \in S^{m+1}_j \Big| 0 \le -\sqrt{(\xi_{j'})^2 + 2D} - \xi_j < \Delta\xi\Big\},\\ S^{m+1,-}_j &= \Big\{j' \in S^{m+1}_j \Big| -\Delta\xi < -\sqrt{(\xi_{j'})^2 + 2D} - \xi_j < 0\Big\}.\end{aligned}$$

Denote j_D the index such that $\xi_{j_D-1}<-\sqrt{2D}$ and $\xi_{j_D}>-\sqrt{2D}$. Then D^4_{m+1} can also be defined as

$$D^4_{m+1}=\big\{(m+1,j)|\ 1\le j\le j_D\big\}.$$

Define function $T_r(x)=-\sqrt{x^2+2D}$. For $j'\in S^{m+1}_j$, define

$$w^{j'}_j=1-|T_r(\xi_{j'})-\xi_j|/\Delta\xi.$$

Let $k^{0,-}$ be the index such that $\xi_{k^{0,-}}=-\frac{1}{2}\Delta\xi$. For $l<k^{0,-}$ one has

$$\frac{T_r(\xi_{l+1})-T_r(\xi_l)}{\Delta\xi}=\frac{|\xi'|}{\sqrt{(\xi')^2+2D}}>\frac{|\xi_{l+1}|}{\sqrt{(\xi_{l+1})^2+2D}}=\left|\frac{\xi_{l+1}}{T_r(\xi_{l+1})}\right|. \tag{3.27}$$

According to (3.5) and (3.8), Algorithm I, definition of the computational domain in (3.1), for $(m+1,j)\in D^4_{m+1}$, $\alpha_{m+1,j}$ are given by

$$\alpha_{m+1,1}=1-\mu_1+\sum_{j'\in S^{m+1,+}_1}\mu_{j'}w^{j'}_1, \tag{3.28}$$

$$\alpha_{m+1,j}=1-\mu_j+\sum_{j'\in S^{m+1}_j}\mu_{j'}w^{j'}_j,\quad 1<j<j_D, \tag{3.29}$$

$$\alpha_{m+1,j_D}=1-\mu_{j_D}+\sum_{j'\in S^{m+1,-}_{j_D}}\mu_{j'}w^{j'}_{j_D}. \tag{3.30}$$

Let $N^{j,+},N^{j,-}$ be the number of elements in $S^{m+1,+}_j$ and $S^{m+1,-}_j$ respectively. We name the elements in $S^{m+1,+}_j$ as $k^+_i, i=1,2,\cdots,N^{j,+}$ such that $k^+_1<k^+_2<\cdots<k^+_{N^{j,+}}$, and the elements in $S^{m+1,-}_j$ as $k^-_i, i=1,2,\cdots,N^{j,-}$ such that $k^-_1<k^-_2<\cdots<k^-_{N^{j,-}}$. Define

$$\begin{aligned}
&\widehat{\alpha}_1=\frac{T_r(\xi_{k^+_1})-\xi_j}{\Delta\xi},\qquad \widehat{\alpha}_{N^{j,+}+1}=\frac{\xi_{j+1}-T_r(\xi_{k^+_{N^{j,+}}})}{\Delta\xi},\\
&\widehat{\alpha}_i=\frac{T_r(\xi_{k^+_i})-T_r(\xi_{k^+_{i-1}})}{\Delta\xi},\quad 2\le i\le N^{j,+},\\
&\widehat{\beta}_1=\frac{T_r(\xi_{k^-_1})-\xi_{j-1}}{\Delta\xi},\qquad \widehat{\beta}_{N^{j,-}+1}=\frac{\xi_j-T_r(\xi_{k^-_{N^{j,-}}})}{\Delta\xi},\\
&\widehat{\beta}_i=\frac{T_r(\xi_{k^-_i})-T_r(\xi_{k^-_{i-1}})}{\Delta\xi},\quad 2\le i\le N^{j,-}.
\end{aligned}$$

From (3.29) one has for $1<j<j_D$

$$\begin{aligned}
\alpha_{m+1,j}&=1-\mu_j+\sum_{j'\in S^{m+1,+}_j}\mu_{j'}w^{j'}_j+\sum_{j'\in S^{m+1,-}_j}\mu_{j'}w^{j'}_j\\
&=1-\mu_j+J_1+J_2,
\end{aligned}\tag{3.31}$$

where

$$J_1=\sum_{j'\in S^{m+1,+}_j}\mu_{j'}w^{j'}_j=\sum_{i=1}^{N^{j,+}}\mu_{k^+_i}\Big(\xi_{j+1}-T_r(\xi_{k^+_i})\Big)/\Delta\xi, \tag{3.32}$$

$$J_2=\sum_{j'\in S^{m+1,-}_j}\mu_{j'}w^{j'}_j=\sum_{i=1}^{N^{j,-}}\mu_{k^-_i}\Big(T_r(\xi_{k^-_i})-\xi_{j-1}\Big)/\Delta\xi. \tag{3.33}$$

It can be checked that

$$\Big(\xi_{j+1} - T_r(\xi_{k_i^+})\Big)/\Delta\xi = \sum_{k=i+1}^{N^{j,+}+1} \widehat{\alpha}_k, \quad 1 \le i \le N^{j,+}, \tag{3.34}$$

$$\Big(T_r(\xi_{k_i^-}) - \xi_{j-1}\Big)/\Delta\xi = \sum_{k=1}^{i} \widehat{\beta}_k, \quad 1 \le i \le N^{j,-}, \tag{3.35}$$

$$k_{i-1}^+ = k_i^+ - 1, \quad 2 \le i \le N^{j,+}, \tag{3.36}$$

$$k_{i-1}^- = k_i^- - 1, \quad 2 \le i \le N^{j,-}, \tag{3.37}$$

$$k_1^+ - 1 = k_{N^{j,-}}^-. \tag{3.38}$$

Using (3.27), (3.34), (3.36) and (3.38), J_1 in (3.32) can be estimated by

$$\begin{aligned}
J_1 &= \sum_{i=1}^{N^{j,+}} \left(\lambda_x^t \left|\xi_{k_i^+}\right| \sum_{k=i+1}^{N^{j,+}+1} \widehat{\alpha}_k\right) \\
&< \sum_{i=1}^{N^{j,+}} \left(\lambda_x^t \left|T_r(\xi_{k_i^+})\right| \left|\big(T_r(\xi_{k_i^+}) - T_r(\xi_{k_i^+-1})\big)/\Delta\xi\right| \sum_{k=i+1}^{N^{j,+}+1} \widehat{\alpha}_k\right) \\
&< \sum_{i=2}^{N^{j,+}} \left(\lambda_x^t |\xi_j| \widehat{\alpha}_i \sum_{k=i+1}^{N^{j,+}+1} \widehat{\alpha}_k\right) + \lambda_x^t |\xi_j| \big(\widehat{\alpha}_1 + \widehat{\beta}_{N^{j,-}+1}\big) \sum_{k=2}^{N^{j,+}+1} \widehat{\alpha}_k \\
&< \mu_j \sum_{i=1}^{N^{j,+}} \left(\widehat{\alpha}_i \sum_{k=i+1}^{N^{j,+}+1} \widehat{\alpha}_k\right) + \mu_j \widehat{\beta}_{N^{j,-}+1} \\
&< \mu_j \frac{1}{2} \left(\sum_{i=1}^{N^{j,+}+1} \widehat{\alpha}_i\right)^2 + \mu_j \widehat{\beta}_{N^{j,-}+1} \\
&= \frac{1}{2}\mu_j + \mu_j \widehat{\beta}_{N^{j,-}+1}.
\end{aligned} \tag{3.39}$$

Using (3.35) and (3.37), J_2 in (3.33) can be estimated by

$$\begin{aligned}
J_2 &= \sum_{i=1}^{N^{j,-}} \left(\mu_{k_i^-} \sum_{k=1}^{i} \widehat{\beta}_k\right) < \sum_{i=2}^{N^{j,-}} \mu_{k_i^-} + \lambda_x^t \left|\xi_{j-1}\right| \widehat{\beta}_1 \\
&< \mu_{j-1}\widehat{\beta}_1 + \lambda_x^t \sum_{i=2}^{N^{j,-}} \left|T_r(\xi_{k_i^-})\right| \left|\big(T_r(\xi_{k_i^-}) - T_r(\xi_{k_i^--1})\big)/\Delta\xi\right| < \mu_{j-1} \sum_{i=1}^{N^{j,-}} \widehat{\beta}_i.
\end{aligned} \tag{3.40}$$

Together with (3.31), (3.39) and (3.40) one gets

$$\begin{aligned}
\alpha_{m+1,j} &= 1 - \mu_j + J_1 + J_2 < 1 - \mu_j + \frac{1}{2}\mu_j + \mu_j \widehat{\beta}_{N^{j,-}+1} + \mu_{j-1} \sum_{i=1}^{N^{j,-}} \widehat{\beta}_i \\
&< 1 - \frac{1}{2}\mu_j + \mu_{j-1} = 1 + \frac{1}{2}\mu_{j-1} + \frac{1}{2}\lambda_x^t \Delta\xi \\
&\le \frac{3}{2} + \frac{1}{2}\lambda_x^t \Delta\xi, \quad 1 < j < j_D.
\end{aligned} \tag{3.41}$$

Similarly one can deduce

$$\alpha_{m+1,j} < \frac{3}{2} + \frac{1}{2}\lambda_x^t \Delta\xi, \quad j = 1, j_D. \tag{3.42}$$

Combining (3.41) and (3.42) one has

$$\alpha_{m+1,j} < \frac{3}{2} + \frac{1}{2}\lambda_x^t \Delta\xi, \quad \forall (m+1,j) \in D_{m+1}^4. \tag{3.43}$$

Therefore we have

$$M_2 < \frac{3}{2} + \frac{1}{2}\lambda_x^t \Delta\xi. \tag{3.44}$$

Combining (3.19)-(3.21), (3.26) and (3.44) one obtains

$$|f^{n+1}|_1 \le |f^n|_1 + \frac{1}{N_d} \sum_{(i,j)\in D_m^2} |f_{ij}^n| + \frac{\frac{1}{2} + \frac{1}{2}\lambda_x^t \Delta\xi}{N_d} \sum_{(i,j)\in D_{m+1}^4} |f_{ij}^n|. \tag{3.45}$$

Repeatedly using (3.45) gives proof for Lemma 3.1. □

We define some notations

$$S_m^{2,1} = \left\{ j \in S_m^2 \middle| \xi_j \ge \xi_z \right\}, \quad S_m^{2,2} = \left\{ j \in S_m^2 \middle| \xi_j < \xi_z \right\}, \tag{3.46}$$

where ξ_z is the constant in Assumption 3.1. Dividing the set D_m^2 into two parts:

$$D_m^{2,1} = \big\{(i,j) \in D_m^2 | j \in S_m^{2,1}\big\}, \quad D_m^{2,2} = \big\{(i,j) \in D_m^2 | j \in S_m^{2,2}\big\}. \tag{3.47}$$

Define

$$S_{11} = \sum_{n=0}^{L-1} \bigg\{ \sum_{(i,j)\in D_m^{2,1}} |f_{ij}^n| \bigg\}, \quad S_{12} = \sum_{n=0}^{L-1} \bigg\{ \sum_{(i,j)\in D_m^{2,2}} |f_{ij}^n| \bigg\}. \tag{3.48}$$

Define the sets

$$S_r = \big\{(i,j)| \ \ x_i > x_{m+\frac{1}{2}}, (m+1,j) \in D_{m+1}^4\big\}, \tag{3.49}$$

$$S_l = \Big\{(i,j)| \ \ x_i < x_{m+\frac{1}{2}}, (m,j) \in D_m^2\Big\}, \tag{3.50}$$

$$S_l^2 = \{(i,j) \in S_l | j \in S_m^{2,2}\}. \tag{3.51}$$

With the zero incoming boundary condition, repeatedly using the schemes (3.5) and (3.4) yields

$$f_{ij}^n = \sum_{(p,q)\in S_r} \beta_{pq}^{ijn0} f_{pq}^0, \quad (i,j) \in S_r, \tag{3.52}$$

$$f_{ij}^n = \sum_{(p,q)\in S_l} \gamma_{pq}^{ijn0} f_{pq}^0, \quad (i,j) \in S_l. \tag{3.53}$$

Under the hyperbolic CFL condition (2.7), $\beta_{pq}^{ijn0}, \gamma_{pq}^{ijn0} \ge 0$. $\gamma_{pq}^{ijn0} \ne 0$ only when $p \le i$ and $q = j$, and $\beta_{pq}^{ijn0} \ne 0$ only when $p \ge i$ and $q = j$ due to the upwind flux and the constant potential. Define

$$F(p,q) = \sum_{n=0}^{L-1} \beta_{pq}^{m+1,qn0}, \quad (p,q) \in S_r, \tag{3.54}$$

$$G(p,q) = \sum_{n=0}^{L-1} \gamma_{pq}^{mqn0}, \quad (p,q) \in S_l. \tag{3.55}$$

We further give some lemmas before presenting the l^1-stability theorem for Scheme I.

Lemma 3.2. *Under the hyperbolic CFL condition* (2.7)*, $F(p,q), G(p,q)$ defined in* (3.54)*,* (3.55) *satisfy*

$$\begin{aligned} F(p-1,q) &\geq F(p,q), && \text{for } p > m+1, && (3.56)\\ F(p,q) &< \frac{1}{\mu_q}, && \text{for } (p,q) \in S_r, && (3.57)\\ G(p+1,q) &\geq G(p,q), && \text{for } p < m, && (3.58)\\ G(p,q) &< \frac{1}{\mu_q}, && \text{for } (p,q) \in S_l. && (3.59) \end{aligned}$$

Proof. We give proof for (3.56) and (3.57). The other two estimates (3.58) and (3.59) can be proved similarly. One can calculate

$$\begin{aligned} &\beta_{pq}^{iqn0} = C_n^{p-i} (1-\mu_q)^{n-p+i} \mu_q{}^{p-i}, \quad i \leq p \leq i+n,\\ &\beta_{pq}^{iqn0} = 0, \quad p < i \text{ or } p > i+n. \end{aligned} \tag{3.60}$$

From scheme (3.5), for $p > m+1$ one has

$$\begin{aligned} \beta_{pq}^{m+1,q,n+1,0} &= (1-\mu_q)\beta_{pq}^{m+1,qn0} + \mu_q \beta_{pq}^{m+2,qn0}\\ &= (1-\mu_q)\beta_{pq}^{m+1,qn0} + \mu_q \beta_{p-1,q}^{m+1,qn0}. \end{aligned} \tag{3.61}$$

Adding (3.61) from $n = 0$ to $L-1$ leads to

$$\begin{aligned} &\beta_{pq}^{m+1,qL0} + \sum_{n=0}^{L-1} \beta_{pq}^{m+1,qn0} = (1-\mu_q) \sum_{n=0}^{L-1} \beta_{pq}^{m+1,qn0} + \mu_q \sum_{n=0}^{L-1} \beta_{p-1,q}^{m+1,qn0}\\ \Rightarrow \quad &\beta_{pq}^{m+1,qL0} + \mu_q F(p,q) = \mu_q F(p-1,q), \end{aligned}$$

which gives (3.56).

Using (3.56), (3.54) and (3.60) one has

$$F(p,q) \leq F(m+1,q) = \sum_{n=0}^{L-1} \beta_{m+1,q}^{m+1,qn0} = \sum_{n=0}^{L-1} (1-\mu_q)^n < \frac{1}{\mu_q}, \quad (p,q) \in S_r. \tag{3.62}$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Under the hyperbolic CFL condition* (2.7)*, the mesh size restriction* (3.13) *and the zero incoming boundary condition, S_2 and S_{11} defined in* (3.15) *and* (3.48) *satisfy*

$$S_2 < \frac{N_d}{\lambda_x^t \left(\sqrt{2D} - \Delta\xi\right)} |f^0|_1, \quad S_{11} < \frac{N_d}{\lambda_x^t \xi_z} |f^0|_1. \tag{3.63}$$

Proof. We give proof for the estimate for S_2. The estimate for S_{11} can be similarly proved. Notice $D_{m+1}^4 \subset S_r$, substituting (3.52) into the expression of S_2 in (3.48) gives

$$S_2 \leq \sum_{(p,q)\in S_r} \left(\sum_{n=0}^{L-1} \sum_{(i,j)\in D_{m+1}^4} \beta_{pq}^{ijn0} \right) |f_{pq}^0| = \sum_{(p,q)\in S_r} F(p,q) |f_{pq}^0|. \tag{3.64}$$

From definition of D^4_{m+1}, one has

$$\mu_q > \lambda^t_x \left(\sqrt{2D} - \Delta\xi\right), \quad \forall (m+1, q) \in D^4_{m+1}.$$

Applying (3.57) in Lemma 3.2 one has

$$F(p,q) < \frac{1}{\mu_q} < \frac{1}{\lambda^t_x \left(\sqrt{2D} - \Delta\xi\right)}, \quad (p,q) \in S_r. \tag{3.65}$$

Combining (3.64), (3.65) and the definition (3.2) gives the estimate for S_2 in (3.63). $\square$

With the above preparation, we now establish the l^1-stability theorem for Scheme I:

Theorem 3.1. *Under Assumption* 3.1*, Scheme I given by* (3.4)-(3.8) *is* l^1*-stable*

$$|f^L|_1 < C|f^0|_1 \tag{3.66}$$

under the hyperbolic CFL condition (2.7)*, the mesh size restriction* (3.13) *and the zero incoming boundary condition, where the constant* C *in* (3.66) *is given by*

$$C = 1 + \frac{1}{\lambda^t_x \xi_z} + \frac{\frac{1}{2} + \frac{1}{2}\lambda^t_x \Delta\xi}{\lambda^t_x \left(\sqrt{2D} - \Delta\xi\right)} + \frac{C_1}{\lambda^t_x \left(\xi_M + \sqrt{\xi_1^2 - 2D}\right)} \left(\frac{8}{3} + \frac{3}{(2D)^{\frac{1}{4}}}\sqrt{\xi_z}\right), \tag{3.67}$$

which is independent of the computational time $T = L\Delta t$.

Proof. Applying Lemma 3.1 one has

$$|f^L|_1 \le |f^0|_1 + \frac{1}{N_d}(S_{11} + S_{12}) + \frac{\frac{1}{2} + \frac{1}{2}\lambda^t_x \Delta\xi}{N_d} S_2, \tag{3.68}$$

where S_{11}, S_{12}, S_2 are defined in (3.48) and (3.15). The estimates for S_{11}, S_2 are provided in Lemma 3.3. In the following we estimate S_{12}.

Substituting (3.53) into the expression of S_{12} in (3.48) gives

$$S_{12} \le \sum_{(p,q)\in S^2_l} \left(\sum_{n=0}^{L-1} \sum_{(i,j)\in D^{2,2}_m} \gamma^{ijn0}_{pq} \right) |f^0_{pq}| = \sum_{(p,q)\in S^2_l} G(p,q) |f^0_{pq}|, \tag{3.69}$$

where S^2_l and $G(p,q)$ are defined in (3.51) and (3.55).

Let N_m be the number of elements in $S^{2,2}_m$. We name the elements in $S^{2,2}_m$ as $k^m_i, i = 1, 2, \cdots, N_m$ such that $k^m_1 < k^m_2 < \cdots < k^m_{N_m}$. Consequently $\mu_{k^m_1} < \mu_{k^m_2} < \cdots < \mu_{k^m_{N_m}}$. Since S^2_l is a subset of S_z defined in (3.9), applying Assumption 3.1, (3.69) and (3.59) one has

$$\begin{aligned} S_{12} &\le C_1 |f^0|_1 \sum_{(p,q)\in S^2_l} G(p,q) = C_1 |f^0|_1 \sum_{p=1}^{m} \sum_{q\in S^{2,2}_m} G(p,q) \\ &< C_1 |f^0|_1 m \sum_{q\in S^{2,2}_m} \frac{1}{\mu_q} = C_1 |f^0|_1 m \sum_{i=1}^{N_m} \frac{1}{\mu_{k^m_i}} = \frac{C_1 |f^0|_1 m}{\lambda^t_x \Delta\xi} I_B, \end{aligned} \tag{3.70}$$

where

$$I_B = \Delta\xi \sum_{i=1}^{N_m} \frac{1}{\xi_{k^m_i}}. \tag{3.71}$$

We then estimate the term I_B. Denote $N_{m,2} = [(N_m - 2)/2]^+$, where $[x]^+$ denotes the smallest integer no less than x. Define the set

$$S_R^B = \left\{ k \middle| \sqrt{2D} + \Delta\xi < \xi_k < \sqrt{2D} + (N_{m,2}+1)\Delta\xi \right\}. \tag{3.72}$$

Then the number of elements in S_R^B is $N_{m,2}$. We name the elements in S_R^B as $k_i^B, i = 1, 2, \cdots, N_{m,2}$ such that $k_1^B < k_2^B < \cdots < k_{N_{m,2}}^B$. Define the maps

$$\begin{aligned}
\widetilde{T}_1(k) = j \quad &\text{s.t.} \quad 0 \le \xi_j - \sqrt{(\xi_k)^2 - 2D} < \Delta\xi, \qquad \text{for } k \in S_R^B, \\
\widetilde{T}_2(k) = j \quad &\text{s.t.} \quad -\Delta\xi < \xi_j - \sqrt{(\xi_k)^2 - 2D} \le 0, \quad \text{for } k \in S_R^B.
\end{aligned}$$

Denote

$$T_i^1 = \widetilde{T}_1(k_i^B), \quad T_i^2 = \widetilde{T}_2(k_i^B), \quad i = 1, 2, \cdots, N_{m,2}.$$

Denote the index $k^{r,+}$ such that $\sqrt{2D} < \xi_{k^{r,+}} < \sqrt{2D} + \Delta\xi$. By definition of the set $S_m^{2,2}$,

$$\begin{aligned}
&\left| \xi_{k_1^m} - \sqrt{(\xi_{k^{r,+}})^2 - 2D} \right| < \Delta\xi \\
&\Rightarrow \xi_{k_3^m} \ge \xi_{k_1^m} + 2\Delta\xi > \sqrt{(\xi_{k^{r,+}})^2 - 2D} + \Delta\xi \\
&\Rightarrow \xi_{k_3^m} > \sqrt{\left(\xi_{k_1^B}\right)^2 - 2D} - \Delta\xi \quad \Rightarrow \xi_{T_1^2} \le \xi_{k_3^m}.
\end{aligned} \tag{3.73}$$

By definition of $\widetilde{T}_2$, $T_1^2 \in S_m^{2,2}$. Denote the index i_T such that $T_1^2 = k_{i_T}^m$, then $i_T \le 3$.

Since $\xi_{k_1^m} \ge \Delta\xi/2$ and $\xi_{k_2^m} \ge 3\Delta\xi/2$, the term (3.71) can be estimated by

$$\begin{aligned}
I_B &= \Delta\xi \sum_{i=1}^{N_m} \frac{1}{\xi_{k_i^m}} \;\le\; \frac{8}{3} + \Delta\xi \sum_{i=3}^{N_m} \frac{1}{\xi_{k_i^m}} \\
&\le \frac{8}{3} + \Delta\xi \sum_{i=i_T}^{N_m-3+i_T} \frac{1}{\xi_{k_i^m}} \\
&\le \frac{8}{3} + \Delta\xi \sum_{i=1}^{N_{m,2}} \frac{1}{\xi_{T_i^1}} + \Delta\xi \sum_{i=1}^{N_{m,2}} \frac{1}{\xi_{T_i^2}} \\
&< \frac{8}{3} + \Delta\xi \sum_{i=1}^{N_{m,2}} \left(\sqrt{(\xi_{k_i^B})^2 - 2D} \right)^{-1} + \Delta\xi \sum_{i=1}^{N_{m,2}} \left(\sqrt{(\xi_{k_i^B})^2 - 2D} - \Delta\xi \right)^{-1}.
\end{aligned} \tag{3.74}$$

Under the mesh size restriction (3.13), $\Delta\xi < \frac{1}{2}\sqrt{(\xi_{k_1^B})^2 - 2D}$. Then from (3.74) one has

$$\begin{aligned}
I_B &< \frac{8}{3} + 3\Delta\xi \sum_{i=1}^{N_{m,2}} \left(\sqrt{(\xi_{k_i^B})^2 - 2D} \right)^{-1} < \frac{8}{3} + \frac{3}{(8D)^{\frac{1}{4}}} \sum_{i=1}^{N_{m,2}} \frac{\Delta\xi}{\sqrt{i\Delta\xi}} \\
&< \frac{8}{3} + \frac{3}{(8D)^{\frac{1}{4}}} \int_0^{N_{m,2}\Delta\xi} \frac{1}{\sqrt{y}}\, dy = \frac{8}{3} + \frac{6}{(8D)^{\frac{1}{4}}} \sqrt{N_{m,2}\Delta\xi}.
\end{aligned} \tag{3.75}$$

From the definition of $S_m^{2,2}$ one has

$$(N_m - 1)\Delta\xi < \xi_z \Rightarrow N_{m,2}\Delta\xi \le \frac{N_m - 1}{2}\Delta\xi < \frac{\xi_z}{2}. \tag{3.76}$$

Combining (3.75) and (3.76) yields

$$I_B < \frac{8}{3} + \frac{3}{(2D)^{\frac{1}{4}}}\sqrt{\xi_z}. \tag{3.77}$$

Using the fact

$$N_d > \frac{\xi_M + \sqrt{\xi_1^2 - 2D}}{\Delta\xi} m \tag{3.78}$$

together with (3.70) and (3.77) obtains

$$\frac{S_{12}}{N_d} < \frac{C_1}{\lambda_x^t\left(\xi_M + \sqrt{\xi_1^2 - 2D}\right)}\left(\frac{8}{3} + \frac{3}{(2D)^{\frac{1}{4}}}\sqrt{\xi_z}\right)|f^0|_1. \tag{3.79}$$

Now applying the estimates (3.63) and (3.79) in (3.68) completes the proof for Theorem 3.1. □

Remark 3.2. In (3.67) the constant C depends on the reciprocal of λ_x^t. Since in practice, under the CFL condition, λ_x^t is chosen as large as possible, the estimate (3.66) gives practically useful information.

4. A Counter Example for Instability

In this section we show that Assumption 3.1 on initial data is necessary for the l^1-stability of Scheme I. We show that l^1-instability examples can be constructed for Scheme I when Assumption 3.1 is violated by the initial data.

Here we impose the assumption:

Assumption 4.1. *There exists a positive constant ξ_z such that $\forall (i,j) \in S_z$ in (3.9), it holds that*

$$|f_{ij}^0| \le \frac{C_1|f^0|_1}{\Delta x^r}, \quad r > 0 \tag{4.1}$$

with C_1 independent of the mesh size.

Remark 4.1. Assumption 4.1 reduces to Assumption 3.1 in the case $r = 0$.

We will construct l^1-instability examples for Scheme I under Assumption 4.1. We first introduce some notations. Define the sets

$$S'_m = \left\{k|\sqrt{2D} + \Delta\xi \le \xi_k \le \frac{1}{3}\sqrt{20D} - \Delta\xi\right\},$$
$$S_m = \left\{k|\exists j \in S'_m, \ \text{ s.t. } \ -\frac{1}{2}\Delta\xi < \xi_k - \sqrt{{\xi_j}^2 - 2D} \le \frac{1}{2}\Delta\xi\right\}.$$

Let N_s be the number of elements in S_m. We name the elements in S_m as $k_i, i = 1, 2, \cdots, N_s$ such that $k_1 < k_2 < \cdots < k_{N_s}$.

Define a one-to-one map from S_m to S'_m as

$$T_s(k) = j \ \text{ s.t. } \ j \in S'_m, \quad \left|\xi_k - \sqrt{{\xi_j}^2 - 2D}\right| \le \frac{1}{2}\Delta\xi, \quad k \in S_m. \tag{4.2}$$

Clearly $\xi_{T_s(k_i)} \geq \sqrt{2D} + i\Delta\xi, \quad i = 1, 2, \cdots, N_s$.

Let $r' = \min(\frac{r}{2}, \frac{1}{2})$. We choose T such that

$$T\lambda_t^x < x_{m+\frac{1}{2}} - x_{\frac{1}{2}}, \quad T\lambda_t^x < x_{N+\frac{1}{2}} - x_{m+\frac{1}{2}},$$

thus $L < m$ and $L < N - m - 1$. Let

$$L_0 = [L^{1-r'}]^- L, \tag{4.3}$$

where $[x]^-$ denotes the largest integer no more than x. Define

$$G(k) = [L\mu_k]^-, \quad k \in S_m.$$

Clearly, $G(k_1) \leq G(k_2) \leq \cdots \leq G(k_{N_s})$.

Since $L < m$, so $G(k) < m$ for $k \in S_m$. Define the following set of indices

$$H = \big\{(i,j) | j \in S_m, m - G(j) < i \leq m\big\}.$$

Let N_h be the number of elements in H. We have the following Lemma.

Lemma 4.1. $N_h > L_0$ *for sufficiently fine mesh.*

Proof. According to the definitions,

$$\begin{aligned}
N_h &= \sum_{k\in S_m} G(k) = \sum_{k\in S_m} \Big[L\xi_k\lambda_x^t\Big]^- > \Big(L\lambda_x^t \sum_{k\in S_m} \xi_k\Big) - N_s \\
&> L\lambda_x^t \sum_{k\in S_m'} \Big(\sqrt{\xi_k^2 - 2D} - \Delta\xi\Big) - \frac{1}{\Delta\xi}\frac{(\sqrt{10}-3)\sqrt{2D}}{3} \\
&> L\lambda_x^t \sum_{i=1}^{N_s} \Big(\sqrt{(\sqrt{2D} + i\Delta\xi)^2 - 2D}\Big) - \gamma \\
&> \frac{3L}{\sqrt{20D}}\lambda_x^t \sum_{i=1}^{N_s} \Big[\big(\sqrt{2D} + i\Delta\xi\big)\sqrt{\big(\sqrt{2D} + i\Delta\xi\big)^2 - 2D}\Big] - \gamma \\
&= \frac{L\lambda_x^t}{\sqrt{20D}\Delta\xi} \sum_{i=1}^{N_s} \Big[3\big(\sqrt{2D} + i\Delta\xi\big)\sqrt{\big(\sqrt{2D} + i\Delta\xi\big)^2 - 2D}\Delta\xi\Big] - \gamma \\
&> \frac{L\lambda_x^t}{\sqrt{20D}\Delta\xi} \int_{\sqrt{2D}}^{\frac{\sqrt{20D}}{3} - 4\Delta\xi} 3x\sqrt{x^2 - 2D}dx - \gamma \\
&> \frac{L\lambda_x^t}{\sqrt{20D}\Delta\xi}\Big(\sqrt{2D}/3 - \sqrt{(8\sqrt{20D}\Delta\xi)/3}\Big) - \gamma
\end{aligned} \tag{4.4}$$

where

$$\gamma := \frac{13}{L}\big((\lambda_x^t + \lambda_\xi^t)/T\big)\big(\sqrt{10} - 3\big)\sqrt{2D}.$$

If we impose the following mesh size restrictions

$$\sqrt{\frac{8\sqrt{20D}\Delta\xi}{3}} < \frac{\sqrt{2D}}{6}, \quad \left(1 + \frac{\lambda_\xi^t}{T\lambda_x^t}\right)\frac{(\sqrt{10}-3)\sqrt{2D}}{3}\Delta\xi < \frac{1}{12\sqrt{10}}, \tag{4.5}$$

then from (4.4),

$$N_h > \frac{L\lambda_x^t}{12\sqrt{10}\Delta\xi} = \frac{L\lambda_x^t\lambda_\xi^t}{12\sqrt{10}\Delta t}.$$

According to (4.3), $L_0 < LL^{1-r'} = LT^{1-r'}/(\Delta t)^{1-r'}$. Therefore $N_h > L_0$ holds under the mesh size restriction

$$\Delta t < \left(\frac{\lambda_x^t\lambda_\xi^t}{12\sqrt{10}T^{1-r'}}\right)^{\frac{1}{r'}}. \tag{4.6}$$

Thus $N_h > L_0$ holds under the mesh size restrictions (4.5) and (4.6). $\square$

We now prove the following l^1-instability theorem for Scheme I:

Theorem 4.1. *$\forall r > 0$ in Assumption 4.1, $\forall h_0 > 0$, $\exists \Delta x < h_0$, $T > 0$, $\forall B > 0$, $\exists f_{ij}^0, (i,j) \in E_d$ satisfying Assumption 4.1, such that*

$$|f^L|_1 > B|f^0|_1\,, \tag{4.7}$$

where f^L are yielded by Scheme I under the hyperbolic CFL condition (2.7) and the zero incoming boundary condition.

Proof. We define a function F_H in H to be

$$F_H(i,j) = m - G(j) + 1 + \sum_{l=1}^{s-1} G(k_l) \quad \text{if} \quad j = k_s, \quad (i,j) \in H.$$

F_H is a one-to-one map from H to $(1, 2, \cdots, N_h)$. Define the set

$$H_L = \big\{(i,j)|(i,j) \in H, F_H(i,j) \le L_0\big\}.$$

Since we are considering fine enough mesh, $N_h > L_0$ holds by Lemma 4.1. Thus the number of elements in H_L is L_0.

We now introduce the initial value f_{ij}^0 satisfying the condition of Theorem 4.1:

$$f_{ij}^0 = c_0, \quad (i,j) \in H_L, \tag{4.8}$$
$$f_{ij}^0 = 0, \quad (i,j) \in E_d \setminus H_L, \tag{4.9}$$

where $c_0 > 0$ is a constant. We first check that these initial values satisfy Assumption 4.1. Since

$$\frac{|f_{ij}^0|}{|f^0|_1} = \frac{N_d}{L_0} < \frac{2MN}{L^{2-r'}} = \frac{2(x_{N+\frac12} - x_{\frac12})(\xi_{M+\frac12} - \xi_{\frac12}){\lambda_x^t}^{2-r'}}{\lambda_x^\xi T^{2-r'}\Delta x^{r'}},$$

Assumption 4.1 is satisfied if

$$\frac{2(x_{N+\frac12} - x_{\frac12})(\xi_{M+\frac12} - \xi_{\frac12}){\lambda_x^t}^{2-r'}}{\lambda_x^\xi T^{2-r'}\Delta x^{r'}} < \frac{C_1}{\Delta x^r}. \tag{4.10}$$

Condition (4.10) is satisfied for fine enough mesh since by definition $r' < r$.

Next we analyze the relation between $|f^L|_1$ and $|f^0|_1$. Since $L < N - m - 1$, the solution at the boundary cells remains zero for all the time steps. Define the sets

$$S_m^m = \big\{(i,j)|i = m, j \in S_m\big\},$$
$$S_m^l = \big\{(i,j)|x_i < x_{m+\frac12}, j \in S_m\big\},$$
$$S_m^r = \big\{(i,j)|x_i > x_{m+\frac12}, j \in S'_m\big\}.$$

At each time step, only solutions at cells belonging to S_m^l or S_m^r are possibly nonzero. Namely

$$f_{ij}^n = 0 \quad \text{for} \quad (i,j) \in E_d \setminus \{S_m^l \cup S_m^r\}. \tag{4.11}$$

Since Scheme I is positive preserving, and the initial values (4.8) and (4.9) are nonnegative, the numerical solutions at each time step are always nonnegative. Similar to the proof of (3.19), at each time step

$$\begin{aligned}|f^{n+1}|_1 = &{N_d}^{-1} \sum_{(i,j)\in E_d} f_{ij}^{n+1} = {N_d}^{-1} \sum_{(i,j)\in E_d} \alpha_{ij} f_{ij}^n \\ =&{N_d}^{-1} \sum_{(i,j)\in S_m^l} \alpha_{ij} f_{ij}^n + {N_d}^{-1} \sum_{(i,j)\in S_m^r} \alpha_{ij} f_{ij}^n + {N_d}^{-1} \sum_{(i,j)\in E_d\setminus\{S_m^l\cup S_m^r\}} f_{ij}^n.\end{aligned} \tag{4.12}$$

The last term in (4.12) is zero by (4.11).

From schemes (3.4), (3.6) one sees that among those α_{ij} with $(i,j) \in S_m^l \cup S_m^r$, $\alpha_{ij} \neq 1$ only when $(i,j) \in S_m^m$, so from (4.12) one has

$$|f^{n+1}|_1 = {N_d}^{-1} \sum_{(m,j)\in S_m^m} \alpha_{mj} f_{mj}^n + {N_d}^{-1} \sum_{(i,j)\in E_d\setminus S_m^m} f_{ij}^n. \tag{4.13}$$

From schemes (3.4) and (3.6), for $(m,j) \in S_m^m$, by setting $j' = T_s(j)$, where T_s is defined in (4.2), one has

$$\alpha_{mj} = 1 - \mu_j + \mu_{j'} c_{j'j}, \tag{4.14}$$

where $c_{j'j}$ are the coefficients in (3.6).

According to the definitions of S_m and S_m', $c_{j'j} \geq 1/2$, $\xi_j < \sqrt{2D}/3$, $\xi_{j'} > \sqrt{2D}$. So from (4.14) one has

$$\alpha_{mj} > 1 + \frac{\sqrt{2D}}{6} \lambda_x^t. \tag{4.15}$$

Combining (4.15) and (4.13) gives

$$\begin{aligned}|f^{n+1}|_1 >& \frac{\sqrt{2D}}{6} \lambda_x^t {N_d}^{-1} \sum_{(m,j)\in S_m^m} f_{mj}^n + {N_d}^{-1} \sum_{(i,j)\in E_d} f_{ij}^n \\ =& \frac{\sqrt{2D}}{6} \lambda_x^t {N_d}^{-1} \sum_{(m,j)\in S_m^m} f_{mj}^n + |f^n|_1.\end{aligned} \tag{4.16}$$

Summing up (4.16) from $n = 0$ to $L - 1$ yields

$$|f^L|_1 > |f^0|_1 + \frac{\sqrt{2D}}{6} \lambda_x^t {N_d}^{-1} \sum_{n=0}^{L-1} \sum_{(m,j)\in S_m^m} f_{mj}^n. \tag{4.17}$$

Let

$$f_{ij}^n = \sum_{(p,q)\in S_m^l} \eta_{pq}^{ijn0} f_{pq}^0, \quad (i,j) \in S_m^l. \tag{4.18}$$

Since $S_m^m \in S_m^l$, substituting (4.18) into (4.17) gives

$$\begin{aligned}
|f^L|_1 > & |f^0|_1 + \frac{1}{6}\sqrt{2D}\lambda_x^t {N_d}^{-1} \sum_{(p,q)\in S_m^l} \Big(\sum_{n=0}^{L-1} \sum_{(m,j)\in S_m^m} \eta_{pq}^{mjn0} \Big) f_{pq}^0 \\
= & |f^0|_1 + \frac{1}{6}\sqrt{2D}\lambda_x^t {N_d}^{-1} \sum_{(p,q)\in H_L} \Big(\sum_{n=0}^{L-1} \sum_{(m,j)\in S_m^m} \eta_{pq}^{mjn0} \Big) c_0 \\
= & |f^0|_1 + \frac{1}{6}\sqrt{2D}\lambda_x^t \frac{c_0}{N_d} \sum_{(m,j)\in S_m^m} \Big(\sum_{(p,j)\in H_L} \sum_{n=0}^{L-1} \eta_{pj}^{mjn0} \Big) \\
\geq & |f^0|_1 + \frac{1}{6}\sqrt{2D}\lambda_x^t \frac{c_0}{N_d} \sum_{l=1}^{s-1} \Big(\sum_{(p,k_l)\in H} \sum_{n=0}^{L-1} \eta_{pk_l}^{mk_ln0} \Big),
\end{aligned} \tag{4.19}$$

where $k_s \in S_m$ is the quantity such that $\exists\ i$ satisfying $m - G(k_s) < i \leq m$ and $F_H(i,k_s) = L_0$.

We then estimate $\sum_{(p,j)\in H} \sum_{n=0}^{L-1} \eta_{pj}^{mjn0}$ for $(m,j) \in S_m^m$. Using (3.4) one can check the following relation

$$\sum_{n=0}^{L-1} \eta_{pj}^{mjn0} = \sum_{n=0}^{L-1} \eta_{mj}^{mjn0} - \frac{1}{\mu_j} \sum_{l=p}^{m-1} \eta_{lj}^{mjL0}. \tag{4.20}$$

For a fixed $j \in S_m$, adding (4.20) for p such that $(p,j) \in H$ gives

$$\sum_{(p,j)\in H} \sum_{n=0}^{L-1} \eta_{pj}^{mjn0} = G(j) \sum_{n=0}^{L-1} \eta_{mj}^{mjn0} - {\mu_j}^{-1} \sum_{l=m-G(j)+1}^{m-1} \big(l - m + G(j)\big) \eta_{lj}^{mjL0}. \tag{4.21}$$

According to the definition of S_m and S_m', when $j \in S_m$, $\xi_j < \sqrt{2D}/3$. The CFL condition (2.7) implies that $\sqrt{2D}\lambda_x^t < 1$, so $\mu_j < 1/3$ when $j \in S_m$. Using

$$\eta_{lj}^{mjL0} = (1-\mu_j)^{L+l-m} \mu_j^{m-l} C_L^{m-l}$$

gives

$$\begin{aligned}
\sum_{l=m-G(j)+1}^{m-1} \eta_{lj}^{mjL0} = & \sum_{l=m-G(j)+1}^{m-1} (1-\mu_j)^{L+l-m} \mu_j^{m-l} C_L^{m-l} \\
= \sum_{l=1}^{G(j)-1} (1-\mu_j)^{L-l} \mu_j^l C_L^l = & \sum_{l=1}^{[\mu_j L]^- -1} (1-\mu_j)^{L-l} \mu_j^l C_L^l < \frac{1}{2}.
\end{aligned} \tag{4.22}$$

The proof of the last inequality is given in the Appendix A. Substituting (4.22) into (4.21) gives

$$\begin{aligned}
\sum_{(p,j)\in H} \sum_{n=0}^{L-1} \eta_{pj}^{mjn0} > & G(j) \sum_{n=0}^{L-1} \eta_{mj}^{mjn0} - \frac{G(j)}{2} {\mu_j}^{-1} \\
= & G(j) \frac{1-(1-\mu_j)^L}{\mu_j} - \frac{G(j)}{2} {\mu_j}^{-1}.
\end{aligned} \tag{4.23}$$

By the definitions of S_m and S_m', for $j \in S_m$,

$$\xi_j > \sqrt{(\sqrt{2D}+\Delta\xi)^2 - 2D} - \Delta\xi > \sqrt{2\sqrt{2D}\Delta\xi} - \Delta\xi. \tag{4.24}$$

From (4.24), one can check that for fine enough mesh it holds $\mu_j = \xi_j \lambda_x^t > L/2 = (2\lambda_\xi^t/T)\Delta\xi,\ j \in S_m$. Thus

$$(1-\mu_j)^L < (1-L^{-1})^L < \frac{1}{e} < \frac{1}{2.5}\,. \tag{4.25}$$

Combining (4.23) and (4.25) obtains

$$\sum_{(p,j)\in H} \sum_{n=0}^{L-1} \eta_{pj}^{mjn0} > \frac{G(j)}{10\mu_j} > \frac{L-\frac{1}{\mu_j}}{10} > \frac{L}{20}, \tag{4.26}$$

where in the last inequality we used the condition $\mu_j > 2/L$ for fine enough mesh.

Next we estimate s appearing in (4.19) as the superscript of the summation. From the definition of s in (4.19),

$$\sum_{l=1}^{s} G(k_l) \geq L_0\,. \tag{4.27}$$

On the other hand, for $1 \leq s' \leq N_s$,

$$\begin{aligned}
\sum_{l=1}^{s'} G(k_l) &< L\lambda_x^t \sum_{l=1}^{s'} \xi_{k_l} + s' < L\lambda_x^t \sum_{l=1}^{s'} \sqrt{\xi_{k_l'}^2 - 2D} + s'\lambda_x^t L\Delta\xi + s' \\
&< \frac{L\lambda_x^t}{3\sqrt{2D}} \sum_{l=1}^{s'} 3\xi_{k_l'}\sqrt{\xi_{k_l'}^2 - 2D} + \frac{s'\lambda_x^t T}{\lambda_\xi^t} + s' \\
&< \frac{L\lambda_x^t}{3\sqrt{2D}\Delta\xi} \int_{\sqrt{2D}}^{\sqrt{2D}+(s'+1)\Delta\xi} 3\xi\sqrt{\xi^2-2D}\,d\xi + \frac{s'\lambda_x^t T}{\lambda_\xi^t} + s' \\
&< \frac{L\lambda_x^t}{3\sqrt{2D}\Delta\xi} \left[2\sqrt{2D}\,(s'+1)\,\Delta\xi\right]^{\frac{3}{2}} + \frac{s'\lambda_x^t T}{\lambda_\xi^t} + s'.
\end{aligned} \tag{4.28}$$

By choosing $s' = \left[{N_s}^{1-\frac{5}{6}r'}\right]^+$, for fine mesh ${N_s}^{1-\frac{5}{6}r'} > 2 \Rightarrow s'+1 < 2{N_s}^{1-\frac{5}{6}r'}$. Then (4.28) gives

$$\begin{aligned}
\sum_{l=1}^{s'} G(k_l) &< \frac{8\lambda_x^t (2D)^{\frac{1}{4}} L\sqrt{\Delta\xi}}{3} {N_s}^{\frac{3}{2}-\frac{5}{4}r'} + \frac{N_s\lambda_x^t T}{\lambda_\xi^t} + N_s \\
&< \frac{8\lambda_x^t (2D)^{\frac{1}{4}} \sqrt{T} L}{3\sqrt{\lambda_\xi^t}\sqrt{L}} \gamma^{\frac{3}{2}-\frac{5}{4}r'} + \frac{\gamma\lambda_x^t T}{\lambda_\xi^t} + \gamma \\
&= \frac{8\lambda_x^t (2D)^{\frac{1}{4}} \sqrt{T}\left(\lambda_\xi^t \gamma\Delta\xi/T\right)^{\frac{3}{2}-\frac{5}{4}r'}}{3\sqrt{\lambda_\xi^t}} L^{2-\frac{5}{4}r'} + \gamma\Delta\xi\lambda_x^t L + \frac{\gamma\Delta\xi\lambda_\xi^t L}{T}.
\end{aligned} \tag{4.29}$$

where $\gamma := \left(\sqrt{20D}/3 - \sqrt{2D}\right)/\Delta\xi$. From (4.29), one has for fine enough mesh and $s' = \left[{N_s}^{1-\frac{5}{6}r'}\right]^+$,

$$\sum_{l=1}^{s'} G(k_l) < \frac{3}{4} L^{2-r'} < L_0. \tag{4.30}$$

Comparing (4.30) with (4.27) gives for fine enough mesh

$$s \geq {N_s}^{1-\frac{5}{6}r'} + 1 > \left(\frac{\frac{1}{2}\left(\frac{\sqrt{20D}}{3} - \sqrt{2D}\right)\lambda_\xi^t}{T}\right)^{1-\frac{5}{6}r'} L^{1-\frac{5}{6}r'} + 1. \tag{4.31}$$

Applying (4.26) and (4.31) in (4.19) gives

$$
\begin{aligned}
|f^L|_1 &> |f^0|_1 + \frac{\sqrt{2D}}{120}\lambda_x^t \frac{c_0}{N_d}\left(\frac{\frac{1}{2}\left(\frac{\sqrt{20D}}{3}-\sqrt{2D}\right)\lambda_\xi^t}{T}\right)^{1-\frac{5}{6}r'} L^{2-\frac{5}{6}r'} \\
&\geq |f^0|_1 + \frac{\sqrt{2D}}{120}\lambda_x^t \frac{L_0 c_0}{N_d}\left(\frac{\frac{1}{2}\left(\frac{\sqrt{20D}}{3}-\sqrt{2D}\right)\lambda_\xi^t}{T}\right)^{1-\frac{5}{6}r'} L^{\frac{1}{6}r'} \\
&= \left\{1 + \frac{\sqrt{2D}}{120}\lambda_x^t \left[\frac{\frac{1}{2}\left(\frac{\sqrt{20D}}{3}-\sqrt{2D}\right)\lambda_\xi^t}{T}\right]^{1-\frac{5}{6}r'} L^{\frac{1}{6}r'}\right\}|f^0|_1.
\end{aligned}
$$

So $\forall B > 0$, one can choose fine enough mesh size such that

$$
1 + \frac{\sqrt{2D}}{120}\lambda_x^t \left(\frac{\frac{1}{2}\left(\frac{\sqrt{20D}}{3}-\sqrt{2D}\right)\lambda_\xi^t}{T}\right)^{1-\frac{5}{6}r'} L^{\frac{1}{6}r'} > B
$$

under which the desired result (4.7) is obtained. □

Remark 4.2. A related issue to the study in this paper and the l^1-error estimate for Scheme I conducted in [19] is the l^1-error estimate for Scheme I with inexact initial data. Due to the linearity of Scheme I, this estimate can be obtained by applying the error estimate for Scheme I with exact initial data given in [19] and the l^1-norm estimate for the perturbation solution. Since the initial perturbation error may not satisfy Assumption 3.1, the perturbation solution can be l^1-unstable according to Theorem 4.1. Thus the l^1-norm estimate for the perturbation solution can not be achieved by directly applying the stability analysis performed in this paper. In the recent work [17], by extending the stability analysis in this paper we further proved that even if the solution of Scheme I can be l^1-unstable, the l^1-norm of the solution can still be estimated from the l^∞ and l^1-upper bounds of the initial data. Consequently we established in [17] the l^1-convergence of Scheme I given with a wide class of initial perturbation errors.

5. Conclusion

In this paper we studied the l^1-stability of a Hamiltonian-preserving scheme designed in [7] for the Liouville equation with a step function potential. The Hamiltonian-preserving scheme is designed by incorporating the particle behavior–transmission and reflection– at the potential barrier into the numerical fluxes. The l^1-stability of the scheme studied in this paper called Scheme I is more sophisticated among the two schemes designed in [7]. We proved that, with the zero incoming boundary condition and certain initial data condition, Scheme I is l^1-stable under the hyperbolic CFL condition. The stability constant is shown to be independent of the computational time. We also presented counter examples showing that Scheme I can be l^1-unstable if the initial data condition is violated. We observe that the initial data condition is satisfied when applying the decomposition technique proposed in [4] for solving the Liouville equation with measure-valued initial data arisen from the semiclassical limit of the linear Schrödinger equation. Recently we established the l^1-convergence of the same scheme with Dirichlet incoming boundary condition under certain initial data condition [19]. The initial data condition in this paper is more general than that considered in [19]. Thus the l^1-stability of Scheme I established in this paper is also valid for the initial data condition considered in [19].

This is reasonable since a convergent scheme for the Liouville equation with the zero incoming boundary condition should be l^1-stable.

Appendix

Lemma A.1. *Assume* $0<\mu<\frac{1}{2}$, $N\in\mathbb{N}$. *Then*

$$\sum_{l=0}^{[\mu N]^- -1}(1-\mu)^{N-l}\mu^l C_N^l<\frac{1}{2}. \tag{A.1}$$

Proof. Notice that

$$\sum_{l=0}^{N}(1-\mu)^{N-l}\mu^l C_N^l=1,$$

so proof of (A.1) is equivalent to prove

$$\begin{aligned}&\sum_{l=0}^{[\mu N]^- -1}(1-\mu)^{N-l}\mu^l C_N^l<\sum_{l=[\mu N]^-}^{N}(1-\mu)^{N-l}\mu^l C_N^l\\ \Leftrightarrow\quad&\sum_{l=0}^{[\mu N]^- -1}\left(\frac{\mu}{1-\mu}\right)^l C_N^l<\sum_{l=[\mu N]^-}^{N}\left(\frac{\mu}{1-\mu}\right)^l C_N^l.\end{aligned} \tag{A.2}$$

Denote $k=[\mu N]^-$, then $2k\le 2\mu N<N\Rightarrow k<N+1-k$. Denote $\Upsilon_l=(\frac{\mu}{1-\mu})^l C_N^l, l=0,1,\cdots,N$, we first compare the two terms Υ_{k-1} and Υ_k:

$$\frac{\Upsilon_k}{\Upsilon_{k-1}}=\frac{N+1-k}{k}\frac{\mu}{1-\mu}=\frac{N+1-k}{k}\frac{\mu N}{N-\mu N}\ge\frac{N+1-k}{k}\frac{k}{N-k}>1.$$

By comparing Υ_{k-2} and Υ_{k+1}, one has

$$\begin{aligned}\frac{\Upsilon_{k+1}}{\Upsilon_{k-2}}&=\frac{\Upsilon_{k+1}}{\Upsilon_k}\frac{\Upsilon_k}{\Upsilon_{k-1}}\frac{\Upsilon_{k-1}}{\Upsilon_{k-2}}>\frac{\Upsilon_{k+1}}{\Upsilon_k}\frac{\Upsilon_{k-1}}{\Upsilon_{k-2}}\\&=\frac{N+1-(k+1)}{k+1}\frac{N+1-(k-1)}{k-1}\left(\frac{\mu}{1-\mu}\right)^2\\&\ge\frac{(N+1-k)^2-1}{k^2-1}\left(\frac{k}{N-k}\right)^2>1.\end{aligned} \tag{A.3}$$

By induction, one can generally prove the following results,

$$\frac{\Upsilon_{k+l-1}}{\Upsilon_{k-l}}>1,\quad 1\le l\le k\Rightarrow\Upsilon_l<\Upsilon_{2k-1-l},\quad 0\le l\le k-1.$$

Thus the inequality (A.2) is proved. $\square$

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