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CONVERGENCE OF AN IMMERSED INTERFACE UPWIND SCHEME FOR LINEAR ADVECTION EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS I: L^1-ERROR ESTIMATES*

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Abstract

We study the L^1-error estimates for the upwind scheme to the linear advection equations with a piecewise constant coefficients modeling linear waves crossing interfaces. Here the interface condition is immersed into the upwind scheme. We prove that, for initial data with a bounded variation, the numerical solution of the immersed interface upwind scheme converges in L^1-norm to the differential equation with the corresponding interface condition. We derive the one-halfth order L^1-error bounds with explicit coefficients following a technique used in [25]. We also use some inequalities on binomial coefficients proved in a consecutive paper [32].

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Key words: Linear advection equations, Immersed interface upwind scheme, Piecewise constant coefficients, Error estimate, Half order error bound.

1. Introduction

In this paper we study the L^1-error estimates for the upwind difference scheme to the linear advection equation

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

$$u|_{t=0} = u_0(x), \tag{1.2}$$

with piecewise constant (without loss of generality, a step function in this paper) wave speed

$$c(x) = \begin{cases} c^- & x < 0, \\ c^+ & x > 0. \end{cases} \tag{1.3}$$

Without loss of generality, we assume $c(x) > 0$, which is the local sound speed of the media. At the interface between two different media, c is discontinuous.

Eqs. (1.1)-(1.3) is the simplest case of a hyperbolic equation with singular coefficients. For hyperbolic conservation laws with Lipschitz continuous coefficients, there were numerous works

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on convergence rate estimates for numerical methods. Half-order optimal convergence rates for monotone type or viscosity type methods were established in [2, 24-27, 29]. In contrast, for hyperbolic equations with singular coefficients, or conservation laws with discontinuous flux functions, the convergence rate results for numerical methods are much less studied, although many authors have studied the convergence of the numerical methods. The convergence studies include the convergence of a front tracking method for conservation laws with discontinuous flux functions [4], the convergence of front tracking schemes [3, 5, 18, 19], the Lax-Friedrichs scheme [17] and convergence rate estimates for Godunov's and Glimm's methods [21, 28] for the resonant systems of conservation laws, the convergence of monotone schemes for synthetic aperture radar shape-from-shading equations with discontinuous intensities [23], the convergence of a class of finite difference schemes for the linear conservation equation and the transport equation with discontinuous coefficients [6], the convergence of a difference scheme, based on Godunov or Engquist-Osher flux, for scaler conservation laws with a discontinuous convex flux [30] and the extension to the nonconvex flux [31], the convergence of an upwind difference scheme of Engquist-Osher type for degenerate parabolic convection-diffusion equations with a discontinuous coefficient [16], the convergence of a relaxation scheme for conservation laws with a discontinuous coefficient [15], the convergence of Godunov-type methods for conservation laws with a flux function discontinuous in space [1], the convergence of upwind difference schemes of Godunov and Engquist-Osher type for a scalar conservation law with indefinite discontinuities in the flux function [22]. In the above cases, except for the resonant systems of conservation laws, convergence rates for numerical methods were not studied.

One approach to treat Eqs. (1.1)-(1.3) is to use the equation on domains $x < 0$ and $x > 0$ respectively. Then one needs to provide an interface condition at $x = 0$ to connect the solutions at the two sides of the interface. Once an appropriate interface condition is given, a unique solution of (1.1)-(1.3) can be determined using the method of characteristics. See [12] for the justification of the well-posedness of the Liouville equation with partial transmissions and reflections using this approach in the case of a piecewise constant wave speed with a vertical interface.

The physically relevant interface conditions for (1.1)-(1.3) are not necessarily unique [34]. For example, one can require that u is continuous across the interface,

$$u(0^-, t) = u(0^+, t). \tag{1.4}$$

On the other hand, one can also assume that the flux cu is continuous across the interface,

$$c^- u(0^-, t) = c^+ u(0^+, t). \tag{1.5}$$

Depending on applications, (1.4) and (1.5) are both physically relevant interface conditions being studied. See [34] for more detailed discussions.

A natural and successful approach for computing hyperbolic equations with singular coefficients is to build the interface condition into the numerical scheme. Many efficient numerical methods have been designed using this technique. For example, we mention the immersed interface methods by LeVeque and Li [20, 34].

For Eqs. (1.1)-(1.2) with a general $c(x)$ including indefinite sign changes, the convergence of a class of finite difference schemes to the duality solutions was proved in [6]. For Eqs. (1.1)-(1.3), the duality solution is the one corresponding to the interface condition (1.4). To our knowledge, no error bounds with explicit coefficients have been established for the upwind difference scheme or other monotone difference schemes to discontinuous solutions of Eqs. (1.1)-(1.3) with general interface conditions.

In this paper, we will prove that given a general interface condition, the upwind difference scheme with the immersed interface condition produces numerical solutions converging in L^1-norm to the solution of Eqs. (1.1)-(1.3) with the corresponding interface condition. We will derive the L^1-error bounds for the numerical solutions. Our approach makes use of the linearities of both Eq. (1.1) and the upwind difference scheme. In fact, due to the linearities of both the equation and the scheme, the error estimates for general BV initial data can be derived based on error estimates for some Riemann initial data. For these Riemann initial data, the numerical solutions and the exact solutions can both be expressed in terms of the initial data. Then the L^1-error (upper bound) for the numerical solutions can be explicitly expressed. We can then estimate the upper bounds for the L^1-error (upper bound) expressions to derive error estimates for the upwind difference scheme to Eqs. (1.1)-(1.3). This strategy is specifically suitable for linear schemes and linear equations and has been used in [25] to estimate lower error bounds for monotone difference schemes to the linear advection equation with a constant wave speed. This approach can not only derive the optimal convergence rate but also give explicit coefficients in the error bound estimates.

This is the first step toward establishing a convergence theory for the Hamiltonian preserving schemes for the Liouville equation with singular–both discontinuous and measure-valued–coefficients [10-12].Such schemes have important applications in computational high frequency waves through heterogeneous media [7-9, 13, 14, 33]But so far only stability results are available [33].

This paper is organized as follows. In Section 2 we present the main result of this paper. In Section 3 we focus on proving the error bounds for the upwind difference scheme to Eqs. (1.1)-(1.3) with a general interface condition and some Riemann initial data. We use some inequalities on binomial coefficients established in a consecutive paper [32]. In Section 4 we give the proof of the main theorem using the results derived in Section 3. We conclude the paper in Section 5.

2. Main Theorem

Firstly we introduce some notations. We employ a uniform mesh with grid points at $x_{i+\frac{1}{2}} = i\Delta x, i \in \mathbb{Z}$, where Δx is the mesh size. This means the wave speed interface $x = 0$ is located at a grid point. The cells are centered at $x_i = (i - \frac{1}{2})\Delta x, i \in \mathbb{Z}$. We also assume a uniform time step Δt and the discrete times are given by $t_n = n\Delta t, n \in \mathbb{N} \cup \{0\}$. We introduce the quantities $\lambda^- = c^- \frac{\Delta t}{\Delta x}, \lambda^+ = c^+ \frac{\Delta t}{\Delta x}$. The condition $0 < \lambda^-, \lambda^+ < 1$ is the CFL condition.

We consider a general class of interface conditions for (1.1)-(1.3),

$$u(0^+, t) = \rho u(0^-, t), \qquad \rho > 0, \tag{2.1}$$

where $\rho = 1$ corresponds to (1.4), and $\rho = c^-/c^+$ to (1.5).

The upwind difference scheme with forward Euler time discretization for Eqs. (1.1)-(1.3) by building the interface condition (2.1) into reads

$$v_i^{n+1} = (1 - \lambda^-)v_i^n + \lambda^- v_{i-1}^n, \quad \text{if } x_i < 0, \tag{2.2}$$

$$v_i^{n+1} = (1 - \lambda^+)v_i^n + \lambda^+ \rho v_{i-1}^n, \quad \text{if } x_i = \frac{\Delta x}{2}, \tag{2.3}$$

$$v_i^{n+1} = (1 - \lambda^+)v_i^n + \lambda^+ v_{i-1}^n, \quad \text{if } x_i > \Delta x, \tag{2.4}$$

where

$$v_i^0 = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_0(x)dx. \tag{2.5}$$

The exact solution of (1.1)-(1.3) with the interface condition (2.1) can be constructed following characteristics and is of the form

$$u(x,t;u_0) = \begin{cases} u_0(x-c^-t) & x<0, \\ \rho u_0\left(\frac{c^-}{c^+}x - c^-t\right) & 0<x<c^+t, \\ u_0(x-c^+t) & x>c^+t. \end{cases} \tag{2.6}$$

To compare the numerical solution computed from (2.2)-(2.5) with the exact solution (2.6), we introduce

$$v(x,t;u_0) = v_i^n, \quad \text{for } (x,t)\in[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}})\times[t_n,t_{n+1}). \tag{2.7}$$

The main theorem to be proved in this paper is as follows.

Theorem 2.1. *For any* $\rho>0$ *in the interface condition* (2.1)*, the upwind difference scheme* (2.2)-(2.5)*, under the CFL condition* $0<\lambda^-,\lambda^+<1$*, has the following* L^1*-error bound to the exact solution* (2.6)*:*

$$\begin{aligned} &\|v(\cdot,t_n;u_0)-u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})} \\ \le\ & \max\{\rho,1\}\|u_0\|_{BV}\left[\gamma_m\frac{c^M}{c^m}\sqrt{\Delta x}+\left(1+\frac{c^M}{c^m}+\max\left\{\frac{c^+}{c^-},1\right\}\right)\Delta x\right] \\ &+|\rho-1|\max_{-\Delta x<x<0}|u_0(x)|\gamma_+\sqrt{\Delta x}, \end{aligned} \tag{2.8}$$

where

$$c^m=\min\{c^-,c^+\},\quad c^M=\max\{c^-,c^+\}, \tag{2.9}$$

$$\gamma_m=\sqrt{\frac{2}{e}c^m\left(1-c^m\frac{\Delta t}{\Delta x}\right)t_{n+1}}, \tag{2.10}$$

$$\gamma_+=\sqrt{\frac{2}{e}c^+\left(1-c^+\frac{\Delta t}{\Delta x}\right)t_{n+1}}, \tag{2.11}$$

and the definition of the BV norm is given by

$$\|u_0\|_{BV}=\sup_{|h|\neq 0}\frac{1}{|h|}\|u_0(\cdot+h)-u_0(\cdot)\|_{L^1(\mathbb{R})}. \tag{2.12}$$

Remark 2.1. The half-order convergence rate in (2.8) is the same as the optimal convergence rate for monotone difference schemes to conservation laws with a constant c [25]. When $\rho\neq 1$, in which cases the solution jumps across the interface, the error bounds depend not only on $\|u_0\|_{BV}$ but also on the initial data on the cell to the left side of the interface $\max_{-\Delta x<x<0}|u_0(x)|$.

3. Error Bounds for the Riemann Problem

The strategy for proving Theorem 2.1 is to focus on proving the error bounds for some Riemann initial data. The error bounds for general BV initial data can be derived by using the results for these Riemann initial data. We only consider the Riemann initial data whose jump is at a mesh point and whose value is zero on the cell to the left side of the interface. The results for these Riemann initial data are sufficient for proving Theorem 2.1. Assume x^* is a mesh point and denote

$$x^* = J\Delta x, \qquad \text{for } J \in \mathbb{Z}. \tag{3.1}$$

We classify these Riemann initial data into two cases:

$$u_0(x) = \begin{cases} 0 & x < x^* \\ Z & x > x^* \end{cases} \quad x^* \geq 0, \tag{3.2}$$

$$u_0(x) = \begin{cases} Z & x < x^* \\ 0 & x > x^* \end{cases} \quad x^* < 0. \tag{3.3}$$

From (3.1)-(3.3) the initial cell average values are given by

$$v_i^0 = \begin{cases} 0 & \frac{x_i}{\Delta x} < J, \\ Z & \frac{x_i}{\Delta x} > J, \end{cases} \tag{3.4}$$

for initial data (3.2) and

$$v_i^0 = \begin{cases} Z & \frac{x_i}{\Delta x} < J, \\ 0 & \frac{x_i}{\Delta x} > J, \end{cases} \tag{3.5}$$

for initial data (3.3).

In this section we aim at proving the following theorem.

Theorem 3.1. *Let $u_0(x)$ be the Riemann initial data falling into one of the two cases* (3.2)-(3.3). *Then $\forall \rho > 0$ in the interface condition* (2.1), *the upwind difference scheme* (2.2)-(2.5), *under the CFL condition $0 < \lambda^-, \lambda^+ < 1$, has the following L^1-error bounds:*

$$\|v(\cdot, t_n; u_0) - u(\cdot, t_n; u_0)\|_{L^1(\mathbb{R})} \leq \max\{\rho, 1\}|Z| \left[\gamma_m \frac{c^M}{c^m}\sqrt{\Delta x} + \left(1 + \frac{c^M}{c^m}\right)\Delta x\right], \tag{3.6}$$

where c^m, c^M, γ_m are defined in (2.9) *and* (2.10).

As stated in Section 1, our approach consists of firstly expressing the exact solution and the numerical solutions in terms of the initial data and then deriving the explicit L^1 error (upper bound) for the numerical solutions.

3.1. The numerical solutions

The next step is to express the numerical solutions at t^n in terms of the initial data. We have the following theorem.

Theorem 3.2. *By recursively using scheme* (2.2)-(2.4), *one can express v_i^n in terms of the initial data v_i^0 as*

$$v_i^n = \sum_{l=0}^{n} \Gamma_{n,l}^- v_{l-n+i}^0, \quad \text{if } x_i < 0, \tag{3.7}$$

$$v_i^n = \sum_{l=0}^{n} \Gamma_{n,l}^+ v_{l-n+i}^0, \quad \text{if } x_i > n\Delta x, \tag{3.8}$$

$$v_i^n = \rho \sum_{l=0}^{M_{n,i}} \sum_{j=M_{n,i}+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^n v_{l-n+i}^0 + \sum_{l=M_{n,i}+1}^{n} \Gamma_{n,l}^+ v_{l-n+i}^0, \quad \text{if } 0 < x_i < n\Delta x, \tag{3.9}$$

where we define

$$\Gamma^-_{n,l} = C^l_n(\lambda^-)^{n-l}(1-\lambda^-)^l, \tag{3.10}$$

$$\Gamma^+_{n,l} = C^l_n(\lambda^+)^{n-l}(1-\lambda^+)^l, \tag{3.11}$$

$$\Lambda^n_{j,k,l} = C^k_{j+k-1}C^{l-k}_{n-j-k}(\lambda^+)^{n-l-j+1}(1-\lambda^+)^{l-k}(\lambda^-)^{j-1}(1-\lambda^-)^k, \tag{3.12}$$

$$M_{n,i} = n - \left[\frac{x_i}{\Delta x}\right]^+, \tag{3.13}$$

with C^l_n *denoting the binomial coefficients, and* $[x]^+$ *the smallest integer not less than* x.

Proof. One can directly check that (3.7)-(3.9) hold for $n=1$.

Now suppose (3.7)-(3.9) hold for n. We will prove they are also true for $n+1$. For $x_i < 0$,

$$v^{n+1}_i = (1-\lambda^-)v^n_i + \lambda^- v^n_{i-1}. \tag{3.14}$$

Since $x_i < 0, x_{i-1} < 0$, the assumption that (3.7) holds for n gives

$$v^n_{i+m} = \sum_{l=0}^{n} \Gamma^-_{n,l} v^0_{l-n+i+q}, \quad m = 0, -1. \tag{3.15}$$

Consequently,

$$v^{n+1}_i = (1-\lambda^-)\sum_{l=1}^{n+1}\Gamma^-_{n,l-1}v^0_{l-n+i-1} + \lambda^-\sum_{l=0}^{n}\Gamma^-_{n,l}v^0_{l-n+i-1} = \sum_{l=0}^{n+1}\Gamma^-_{n+1,l}v^0_{l-(n+1)+i}. \tag{3.16}$$

So (3.7) holds for $n+1$. Similarly, for $x_i > (n+1)\Delta x$,

$$v^{n+1}_i = (1-\lambda^+)v^n_i + \lambda^+ v^n_{i-1}. \tag{3.17}$$

Since $x_i > (n+1)\Delta x, x_{i-1} > n\Delta x$, the assumption that (3.8) holds for n gives

$$v^n_{i+m} = \sum_{l=0}^{n} \Gamma^+_{n,l} v^0_{l-n+i+q}, \quad m = 0, -1. \tag{3.18}$$

Combining (3.17) and (3.18) one can prove that (3.8) holds for $n+1$. To prove (3.9) holds for $n+1$, we consider the following three cases.

I) For $\Delta x < x_i < n\Delta x$,

$$v^{n+1}_i = (1-\lambda^+)v^n_i + \lambda^+ v^n_{i-1}. \tag{3.19}$$

Since $0 < x_{i-1}, x_i < n\Delta x$, the assumption that (3.9) holds for n gives

$$\begin{aligned} v^n_{i+m} = \rho &\sum_{l=0}^{M_{n,i}-m} \sum_{j=M_{n,i}+1-m-l} \sum_{k=0}^{l} \Lambda^n_{j,k,l} v^0_{l-n+i+m} \\ &+ \sum_{l=M_{n,i}+1-m}^{n} \Gamma^+_{n,l} v^0_{l-n+i+m}, \quad m = 0, -1. \end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20) gives

$$\begin{aligned} v^{n+1}_i = &(1-\lambda^+)\left[\rho\sum_{l=1}^{M_{n,i}+1}\sum_{j=M_{n,i}+2-l}\sum_{k=0}^{l-1}\Lambda^n_{j,k,l-1}v^0_{l-n+i-1} + \sum_{l=M_{n,i}+2}^{n+1}\Gamma^+_{n,l-1}v^0_{l-n+i-1}\right] \\ &+\lambda^+\left[\rho\sum_{l=0}^{M_{n,i}+1}\sum_{j=M_{n,i}+2-l}\sum_{k=0}^{l}\Lambda^n_{j,k,l}v^0_{l-n+i-1} + \sum_{l=M_{n,i}+2}^{n}\Gamma^+_{n,l}v^0_{l-n+i-1}\right] \\ = &\rho\sum_{l=0}^{M_{n,i}+1}\sum_{j=M_{n,i}+2-l}\sum_{k=0}^{l}\Lambda^{n+1}_{j,k,l}v^0_{l-(n+1)+i} + \sum_{l=M_{n+1,i}+1}^{n+1}\Gamma^+_{n+1,l}v^0_{l-(n+1)+i}. \end{aligned} \tag{3.21}$$

II) For $0 < x_i < \Delta x$,

$$v_i^{n+1} = (1-\lambda^+)v_i^n + \lambda^+ \rho v_{i-1}^n. \tag{3.22}$$

Since $-\Delta x < x_{i-1} < 0 < x_i < \Delta x$, the assumption that (3.7), (3.9) hold for n gives

$$v_i^n = \rho \sum_{l=0}^{n-1} \sum_{k=0}^{l} Q_{k,l}^n v_{l-n+i}^0 + (1-\lambda^+)^n v_i^0, \tag{3.23}$$

$$v_{i-1}^n = \sum_{l=0}^{n} \Gamma_{n,l}^- v_{l-n+i-1}^0, \tag{3.24}$$

where $Q_{k,l}^n$ in (3.23) is defined by

$$Q_{k,l}^n = C_{n-1-l+k}^k \lambda^+ (1-\lambda^+)^{l-k} (\lambda^-)^{n-l-1} (1-\lambda^-)^k. \tag{3.25}$$

Combining (3.22), (3.23) and (3.24) one has

$$\begin{aligned} v_i^{n+1} &= \rho \sum_{l=1}^{n} \sum_{k=0}^{l-1} Q_{k,l}^{n+1} v_{l-n+i-1}^0 + (1-\lambda^+)^{n+1} v_i^0 + \rho \sum_{l=0}^{n} \lambda^+ \Gamma_{n,l}^- v_{l-n+i-1}^0 \\ &= \rho \sum_{l=0}^{(n+1)-1} \sum_{k=0}^{l} Q_{k,l}^{n+1} v_{l-(n+1)+i}^0 + (1-\lambda^+)^{n+1} v_i^0. \end{aligned} \tag{3.26}$$

III) For $n\Delta x < x_i < (n+1)\Delta x$, the scheme is given by (3.19), and $0 < x_{i-1} < n\Delta x$, the assumption that (3.8), (3.9) hold for n gives

$$v_i^n = \sum_{l=0}^{n} \Gamma_{n,l}^+ v_{l-n+i}^0, \tag{3.27}$$

$$v_{i-1}^n = \rho (\lambda^+)^n v_{-n+i-1}^0 + \sum_{l=1}^{n} \Gamma_{n,l}^+ v_{l-n+i-1}^0. \tag{3.28}$$

Combining (3.19), (3.27) and (3.28) one obtains

$$v_i^{n+1} = \rho (\lambda^+)^{n+1} v_{-(n+1)+i}^0 + \sum_{l=1}^{n+1} \Gamma_{n+1,l}^+ v_{l-(n+1)+i}^0. \tag{3.29}$$

Together with (3.21), (3.26) and (3.29) one has (3.9) holds for $n+1$. □

The expression (3.9) and the fact that the scheme (2.2)-(2.4) preserves the constant solution when $\rho = 1$ can be summarized into the following lemma.

Lemma 3.1.

$$\sum_{l=p+1}^{n} \Gamma_{n,l}^+ + \sum_{l=0}^{p} \sum_{j=p+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^n = 1, \tag{3.30}$$

for $n \in \mathbb{N}, 0 \le p \le n-1, 0 < \lambda^-, \lambda^+ < 1$.

This lemma will be used later in this paper.

3.2. The exact solutions

For the two type of initial data (3.2), (3.3), the exact solution (2.6) at time t_n can be written as follows.

- Case (3.2):

$$u(x,t_n;u_0) = \begin{cases} 0 & x < x^* + c^+ t_n, \\ Z & x > x^* + c^+ t_n; \end{cases} \tag{3.31}$$

- Case (3.3), if $x^* + c^- t_n \le 0$:

$$u(x,t_n;u_0) = \begin{cases} Z & x < x^* + c^- t_n, \\ 0 & x > x^* + c^- t_n; \end{cases} \tag{3.32}$$

- Case (3.3), if $x^* + c^- t_n > 0$:

$$u(x,t_n;u_0) = \begin{cases} Z & x < 0, \\ \rho Z & 0 < x < \frac{c^+}{c^-}(x^* + c^- t_n), \\ 0 & x > \frac{c^+}{c^-}(x^* + c^- t_n). \end{cases} \tag{3.33}$$

3.3. The upper bound of the L^1-error

Now we have both expressions in terms of the initial data for the exact solutions in (3.31)-(3.33) and for the numerical solutions in (3.7)-(3.9) with the initial cell average values given by (3.4), (3.5). In this subsection we derive the L^1-error (upper bound) expressions for the numerical solutions with respect to the initial data (3.2), (3.3).

3.3.1. Case (3.2)

The exact solution is given by (3.31). The initial cell average values v_i^0 are given by (3.4) with $J \ge 0$. One has $v_i^0 = 0$ when $x_i < 0$.

From (3.7), for $x_i < 0$, one has

$$v_i^n = \sum_{l=0}^{n} \Gamma_{n,l}^- v_{l-n+i}^0 = 0 = \sum_{l=0}^{n} \Gamma_{n,l}^+ v_{l-n+i}^0. \tag{3.34}$$

From (3.9), for $0 < x_i < n\Delta x$, noticing $x_{l-n+i} < 0$ for $l \le M_{n,i}$, one has

$$\begin{aligned} v_i^n &= \rho \sum_{l=0}^{M_{n,i}} \sum_{j=M_{n,i}+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^n v_{l-n+i}^0 + \sum_{l=M_{n,i}+1}^{n} \Gamma_{n,l}^+ v_{l-n+i}^0 \\ &= \sum_{l=M_{n,i}+1}^{n} \Gamma_{n,l}^+ v_{l-n+i}^0 = \sum_{l=0}^{n} \Gamma_{n,l}^+ v_{l-n+i}^0. \end{aligned} \tag{3.35}$$

Combining (3.34), (3.35) and (3.8), the numerical solution v_i^n always takes the following form whenever $x_i < 0, 0 < x_i < n\Delta x$ and $x_i > n\Delta x$:

$$v_i^n = \sum_{l=0}^{n} \Gamma_{n,l}^+ v_{l-n+i}^0. \tag{3.36}$$

The L^1-error between the numerical solution and the exact solution is

$$\|v(\cdot,t_n;u_0)-u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}=E_1^I+E_2^I, \tag{3.37}$$

where

$$E_1^I=\int_{x<x^*+c^+t_n}|v(x,t_n;u_0)|\,dx, \tag{3.38}$$
$$E_2^I=\int_{x>x^*+c^+t_n}|v(x,t_n;u_0)-Z|\,dx. \tag{3.39}$$

We introduce the sets

$$J_1=\left\{i\middle|\frac{x_i}{\Delta x}<[J+n\lambda^+]^-\right\}, \tag{3.40}$$
$$J_2=\left\{i\middle|[J+n\lambda^+]^-<\frac{x_i}{\Delta x}<[J+n\lambda^+]^+\right\}, \tag{3.41}$$
$$J_3=\left\{i\middle|\frac{x_i}{\Delta x}>[J+n\lambda^+]^+\right\}. \tag{3.42}$$

The two terms (3.38), (3.39) can be deduced as follows. For the first term,

$$\begin{aligned}\frac{E_1^I}{\Delta x}&=\sum_{i\in J_1}|v_i^n|+\sum_{i\in J_2}|v_i^n|\left(n\lambda^+-[n\lambda^+]^-\right)\\&=\sum_{i\in J_1}\left|\sum_{l=0}^n\Gamma_{n,l}^+v_{l-n+i}^0\right|+\sum_{i\in J_2}\left|\sum_{l=0}^n\Gamma_{n,l}^+v_{l-n+i}^0\right|\left(n\lambda^+-[n\lambda^+]^-\right)\\&=|Z|\sum_{i\in J_1}\sum_{l=M_{n,i}+J+1}^n\Gamma_{n,l}^++|Z|\sum_{i\in J_2}\sum_{l=M_{n,i}+J+1}^n\Gamma_{n,l}^+\left(n\lambda^+-[n\lambda^+]^-\right)\\&=|Z|\sum_{l=n-[n\lambda^+]^-+1}^n\Gamma_{n,l}^+(l-n+[n\lambda^+]^-)+|Z|\sum_{l=n-[n\lambda^+]^++1}^n\Gamma_{n,l}^+\left(n\lambda^+-[n\lambda^+]^-\right)\\&=|Z|\sum_{l=n-[n\lambda^+]^++1}^n\Gamma_{n,l}^+(l-n+n\lambda^+).\end{aligned} \tag{3.43}$$

For the second term, we have

$$\begin{aligned}\frac{E_2^I}{\Delta x}&=\sum_{i\in J_3}|v_i^n-Z|+\sum_{i\in J_2}|v_i^n-Z|\left([n\lambda^+]^+-n\lambda^+\right)\\&=\sum_{i\in J_3}\left|\sum_{l=0}^n\Gamma_{n,l}^+\left(v_{l-n+i}^0-Z\right)\right|+\sum_{i\in J_2}\left|\sum_{l=0}^n\Gamma_{n,l}^+\left(v_{l-n+i}^0-Z\right)\right|\left([n\lambda^+]^+-n\lambda^+\right)\\&=|Z|\sum_{i\in J_3}\sum_{l=0}^{M_{n,i}+J}\Gamma_{n,l}^++|Z|\sum_{i\in J_2}\sum_{l=0}^{M_{n,i}+J}\Gamma_{n,l}^+\left([n\lambda^+]^+-n\lambda^+\right)\\&=|Z|\sum_{l=0}^{n-[n\lambda^+]^+-1}\Gamma_{n,l}^+(n-[n\lambda^+]^+-l)+|Z|\sum_{l=0}^{n-[n\lambda^+]^+}\Gamma_{n,l}^+\left([n\lambda^+]^+-n\lambda^+\right)\\&=|Z|\sum_{l=0}^{n-[n\lambda^+]^+}\Gamma_{n,l}^+(n-n\lambda^+-l).\end{aligned} \tag{3.44}$$

By combining (3.37), (3.43) and (3.44), the error expression is given by

$$\|v(\cdot,t_n;u_0)-u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}=|Z|\Delta x\sum_{l=0}^{n}\Gamma^+_{n,l}|n-n\lambda^+-l|. \tag{3.45}$$

3.3.2. Case (3.3), if $x^*+c^-t_n\le 0$

The exact solution is given by (3.32). The initial cell average values v_i^0 are given by (3.5) with $J+n\lambda^-\le 0$. The L^1-error between the numerical solution and the exact solution is

$$\|v(\cdot,t_n;u_0)-u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}=\sum_{k=1}^{3}E_k^{II}, \tag{3.46}$$

where

$$E_1^{II}=\int_{x<x^*+c^-t_n}|v(x,t_n;u_0)-Z|\,dx, \tag{3.47}$$

$$E_2^{II}=\int_{x^*+c^-t_n<x<0}|v(x,t_n;u_0)|\,dx, \tag{3.48}$$

$$E_3^{II}=\int_{x>0}|v(x,t_n;u_0)|\,dx. \tag{3.49}$$

We introduce the sets

$$J_4=\left\{i\middle|\frac{x_i}{\Delta x}<[J+n\lambda^-]^-\right\}, \tag{3.50}$$

$$J_5=\left\{i\middle|[J+n\lambda^-]^-<\frac{x_i}{\Delta x}<[J+n\lambda^-]^+\right\}, \tag{3.51}$$

$$J_6=\left\{i\middle|[J+n\lambda^-]^+<\frac{x_i}{\Delta x}<0\right\}. \tag{3.52}$$

The terms (3.47)-(3.49) can be deduced as follows. For the first term,

$$\begin{aligned}\frac{E_1^{II}}{\Delta x}&=\sum_{i\in J_4}|v_i^n-Z|+\sum_{i\in J_5}|v_i^n-Z|\left(n\lambda^--[n\lambda^-]^-\right)\\&=\sum_{i\in J_4}\left|\sum_{l=0}^{n}\Gamma^-_{n,l}\left(v^0_{l-n+i}-Z\right)\right|+\sum_{i\in J_5}\left|\sum_{l=0}^{n}\Gamma^-_{n,l}\left(v^0_{l-n+i}-Z\right)\right|\left(n\lambda^--[n\lambda^-]^-\right)\\&=|Z|\sum_{i\in J_4}\sum_{l=M_{n,i}+J+1}^{n}\Gamma^-_{n,l}+|Z|\sum_{i\in J_5}\sum_{l=M_{n,i}+J+1}^{n}\Gamma^-_{n,l}\left(n\lambda^--[n\lambda^-]^-\right)\\&=|Z|\sum_{l=n-[n\lambda^-]^-+1}^{n}\Gamma^-_{n,l}\left(l-n+[n\lambda^-]^-\right)+|Z|\sum_{l=n-[n\lambda^-]^++1}^{n}\Gamma^-_{n,l}\left(n\lambda^--[n\lambda^-]^-\right).\end{aligned} \tag{3.53}$$

For the second term, we have

$$\begin{aligned}\frac{E_2^{II}}{\Delta x}&=\sum_{i\in J_6}|v_i^n|+\sum_{i\in J_5}|v_i^n|\left([n\lambda^-]^+-n\lambda^-\right)\\&=\sum_{i\in J_6}\left|\sum_{l=0}^{n}\Gamma^-_{n,l}v^0_{l-n+i}\right|+\sum_{i\in J_5}\left|\sum_{l=0}^{n}\Gamma^-_{n,l}v^0_{l-n+i}\right|\left([n\lambda^-]^+-n\lambda^-\right)\\&=|Z|\sum_{i\in J_6}\sum_{l=0}^{M_{n,i}+J}\Gamma^-_{n,l}+|Z|\sum_{i\in J_5}\sum_{l=0}^{M_{n,i}+J}\Gamma^-_{n,l}\left([n\lambda^-]^+-n\lambda^-\right)\\&=|Z|\sum_{l=0}^{n-[n\lambda^-]^+}\Gamma^-_{n,l}\left([n\lambda^-]^+-n\lambda^-\right)+E_4^{II},\end{aligned} \tag{3.54}$$

where

$$E_4^{II} = |Z| \sum_{l=0}^{n-[n\lambda^-]^+-1} \Gamma_{n,l}^- \left(n - [n\lambda^-]^+ - l\right), \quad \text{if } J \le -n, \tag{3.55}$$

$$E_4^{II} = |Z| \left\{ \sum_{l=0}^{n+J-1} \Gamma_{n,l}^- \left(-J - [n\lambda^-]^+\right) + \sum_{l=n+J}^{n-[n\lambda^-]^+-1} \Gamma_{n,l}^- \left(n - [n\lambda^-]^+ - l\right) \right\}, \quad \text{if } J > -n. \tag{3.56}$$

Finally, for the third term we have

$$\begin{aligned} \frac{E_3^{II}}{\Delta x} &= \sum_{\frac{x_i}{\Delta x}>0} |v_i^n| = \sum_{\frac{x_i}{\Delta x}>0} \left| \rho \sum_{l=0}^{M_{n,i}} \sum_{j=M_{n,i}+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^n v_{l-n+i}^0 \right| \\ &= \rho |Z| \sum_{0<\frac{x_i}{\Delta x}<n+J} \sum_{l=0}^{M_{n,i}+J} \sum_{j=M_{n,i}+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^n. \end{aligned} \tag{3.57}$$

Consequently,

$$\frac{E_3^{II}}{\rho |Z| \Delta x} = 0, \quad \text{if } J \le -n, \tag{3.58}$$

$$\frac{E_3^{II}}{\rho |Z| \Delta x} = \sum_{l=0}^{n+J-1} \sum_{j=1-J}^{n-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^n, \quad \text{if } J > -n. \tag{3.59}$$

We now simplify the expression (3.59). We use the following lemma.

Lemma 3.2. *For $m, n \in \mathbb{N}$, $m \le n$, $0 < \lambda^-$, $\lambda^+ < 1$,*

$$\sum_{l=0}^{m-1} \sum_{j=n-m+1}^{n-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^n = \sum_{l=0}^{m-1} \Gamma_{n,l}^- (m-l) \frac{\lambda^+}{\lambda^-}. \tag{3.60}$$

The proof of this lemma will be given in the Appendix. □

Applying Lemma 3.2 to (3.59), one has if $J > -n$, then

$$\frac{E_3^{II}}{\rho |Z| \Delta x} = \sum_{l=0}^{n+J-1} \Gamma_{n,l}^- (n+J-l) \frac{\lambda^+}{\lambda^-}. \tag{3.61}$$

Combining (3.46), (3.53)-(3.55) and (3.58), for $J \le -n$, the error expression is given by

$$\begin{aligned} &\|v(\cdot, t_n; u_0) - u(\cdot, t_n; u_0)\|_{L^1(\mathbb{R})} \\ &= |Z| \Delta x \left[\sum_{l=n-[n\lambda^-]^-+1}^{n} \Gamma_{n,l}^- \left(l - n + [n\lambda^-]^-\right) + \sum_{l=n-[n\lambda^-]^++1}^{n} \Gamma_{n,l}^- \left(n\lambda^- - [n\lambda^-]^-\right) \right. \\ &\quad \left. + \sum_{l=0}^{n-[n\lambda^-]^+} \Gamma_{n,l}^- \left([n\lambda^-]^+ - n\lambda^-\right) + \sum_{l=0}^{n-[n\lambda^-]^+-1} \Gamma_{n,l}^- \left(n - [n\lambda^-]^+ - l\right) \right] \\ &= |Z| \Delta x \sum_{l=0}^{n} \Gamma_{n,l}^- |l - n + n\lambda^-|. \end{aligned} \tag{3.62}$$

Combining (3.46), (3.53), (3.54), (3.56) and (3.61), for $J > -n$, the error upper bound expression is given by

$$\begin{aligned}
&\|v(\cdot,t_n;u_0)-u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}\\
=&|Z|\Delta x\Bigg[\sum_{l=n-[n\lambda^-]^-+1}^{n}\Gamma^-_{n,l}\left(l-n+[n\lambda^-]^-\right)+\sum_{l=n-[n\lambda^-]^++1}^{n}\Gamma^-_{n,l}\left(n\lambda^- - [n\lambda^-]^-\right)\\
&+\sum_{l=0}^{n-[n\lambda^-]^+}\Gamma^-_{n,l}\left([n\lambda^-]^+-n\lambda^-\right)+\sum_{l=0}^{n+J-1}\Gamma^-_{n,l}\left(-J-[n\lambda^-]^+\right)\\
&+\sum_{l=n+J}^{n-[n\lambda^-]^+-1}\Gamma^-_{n,l}\left(n-[n\lambda^-]^+-l\right)+\rho\sum_{l=0}^{n+J-1}\Gamma^-_{n,l}(n+J-l)\frac{\lambda^+}{\lambda^-}\Bigg]\\
\le&|Z|\Delta x\max\left\{\rho\frac{\lambda^+}{\lambda^-},1\right\}\sum_{l=0}^{n}\Gamma^-_{n,l}|l-n+n\lambda^-|. \qquad (3.63)
\end{aligned}$$

Together with (3.62) and (3.63), in both cases $J \le -n$ and $J > -n$, the error upper bound expression is given by

$$\|v(\cdot,t_n;u_0)-u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}\le|Z|\Delta x\max\left\{\rho\frac{\lambda^+}{\lambda^-},1\right\}\sum_{l=0}^{n}\Gamma^-_{n,l}|l-n+n\lambda^-|.$$

3.3.3. Case (3.3), if $x^*+c^-t_n>0$

The exact solution is given by (3.33). The initial cell average values v_i^0 are given by (3.5) with $J<0, J+n\lambda^->0$. The L^1-error between the numerical solution and the exact solution is

$$\|v(\cdot,t_n;u_0)-u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}=\sum_{k=1}^{5}E_k^{III}, \qquad (3.64)$$

where

$$E_1^{III}=\int_{\frac{x}{\Delta x}<0}|v(x,t_n;u_0)-Z|\,dx, \qquad (3.65)$$

$$E_2^{III}=\int_{0<\frac{x}{\Delta x}<[K]^-}|v(x,t_n;u_0)-\rho Z|\,dx, \qquad (3.66)$$

$$E_3^{III}=\int_{[K]^-<\frac{x}{\Delta x}<K}|v(x,t_n;u_0)-\rho Z|\,dx, \qquad (3.67)$$

$$E_4^{III}=\int_{K<\frac{x}{\Delta x}<[K]^+}|v(x,t_n;u_0)|\,dx, \qquad (3.68)$$

$$E_5^{III}=\int_{\frac{x}{\Delta x}>[K]^+}|v(x,t_n;u_0)|\,dx, \qquad (3.69)$$

with $K=\frac{\lambda^+}{\lambda^-}(J+n\lambda^-)$.

The terms (3.65)-(3.69) can be deduced as follows.

$$\frac{E_1^{III}}{\Delta x} = \sum_{J<\frac{x_i}{\Delta x}<0} \left| \sum_{l=0}^{n} \Gamma_{n,l}^{-} \left(v_{l-n+i}^{0} - Z \right) \right| = |Z| \sum_{J<\frac{x_i}{\Delta x}<0} \sum_{l=M_{n,i}+J+1}^{n} \Gamma_{n,l}^{-}$$
$$= |Z| \sum_{l=n+J+1}^{n} \Gamma_{n,l}^{-}(l-n-J). \tag{3.70}$$

If $[K]^- = 0$, then

$$E_2^{III} = 0. \tag{3.71}$$

If $[K]^- > 0$, utilizing Lemma 3.1 one has

$$\frac{E_2^{III}}{\Delta x}$$
$$= \sum_{0<\frac{x_i}{\Delta x}<[K]^-} \left| \rho \sum_{l=0}^{M_{n,i}} \sum_{j=M_{n,i}+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^{n} \left(v_{l-n+i}^{0} - Z \right) - \sum_{l=M_{n,i}+1}^{n} \Gamma_{n,l}^{+} \rho Z \right|$$
$$= \rho |Z| \sum_{0<\frac{x_i}{\Delta x}<[K]^-} \left[\sum_{l=M_{n,i}+J+1}^{M_{n,i}} \sum_{j=M_{n,i}+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^{n} + \sum_{l=M_{n,i}+1}^{n} \Gamma_{n,l}^{+} \right]$$
$$= \rho |Z| \left\{ \sum_{l=n+1-[K]^-}^{n} \Gamma_{n,l}^{+}(l-n+[K]^-) + \sum_{l=n+J+1-[K]^-}^{n-1} \sum_{j=\max(n+1-l-[K]^-,1)}^{\min(n-l,-J)} \sum_{k=0}^{l} \Lambda_{j,k,l}^{n} \right\}. \tag{3.72}$$

Combining (3.70)-(3.72) one has

$$\frac{E_1^{III} + E_2^{III}}{\Delta x} \le \max\{\rho, 1\} |Z| T_1, \tag{3.73}$$

where

$$T_1 = \sum_{l=n+J+1}^{n} \Gamma_{n,l}^{-}(l-n-J), \qquad \text{if } [K]^- = 0, \tag{3.74}$$
$$T_1 = \sum_{l=n+J+1}^{n} \Gamma_{n,l}^{-}(l-n-J) + \sum_{l=n+J+1-[K]^-}^{n-1} \sum_{j=\max(n+1-l-[K]^-,1)}^{\min(n-l,-J)} \sum_{k=0}^{l} \Lambda_{j,k,l}^{n}$$
$$+ \sum_{l=n+1-[K]^-}^{n} \Gamma_{n,l}^{+}(l-n+[K]^-), \qquad \text{if } [K]^- > 0, \tag{3.75}$$

with $K = \frac{\lambda^+}{\lambda^-}(J + n\lambda^-)$. Furthermore,

$$E_3^{III} + E_4^{III} \le \Delta x \rho |Z|, \tag{3.76}$$
$$\frac{E_5^{III}}{\Delta x} = \sum_{\frac{x_i}{\Delta x}>[K]^+} |v_i^n| = \sum_{\frac{x_i}{\Delta x}>[K]^+} \left| \rho \sum_{l=0}^{M_{n,i}} \sum_{j=M_{n,i}+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^{n} v_{l-n+i}^{0} \right|$$
$$= \rho |Z| \sum_{[K]^+<\frac{x_i}{\Delta x}<n+J} \sum_{l=0}^{M_{n,i}+J} \sum_{j=M_{n,i}+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^{n} = \rho |Z| T_2, \tag{3.77}$$

where

$$T_2 = \sum_{l=0}^{n+J-[K]^+-1} \sum_{j=1-J}^{n-l-[K]^+} \sum_{k=0}^{l} \Lambda_{j,k,l}^n, \tag{3.78}$$

with $K = \frac{\lambda^+}{\lambda^-}(J + n\lambda^-)$. Together with (3.64), (3.73), (3.76) and (3.77), the error upper bound expression is given by

$$\begin{aligned} \|v(\cdot, t_n; u_0) - u(\cdot, t_n; u_0)\|_{L^1(\mathbb{R})} &\leq \max\{\rho, 1\}|Z|\Delta x T_1 + \rho|Z|\Delta x + \rho|Z|\Delta x T_2 \\ &\leq \rho|Z|\Delta x + \max\{\rho, 1\}|Z|\Delta x \left(T_1 + T_2\right), \end{aligned} \tag{3.79}$$

where T_1, T_2 are given in (3.74), (3.75) and (3.78) respectively.

3.4. Proof of Theorem 3.1

By using the L^1-error (upper bound) expressions derived in the last subsection, we can give proof of Theorem 3.1 by using the following binomial coefficients inequalities.

Lemma 3.3. *For any* $0 < \lambda < 1$, $n \in \mathbb{N}$,

$$\sum_{l=0}^{n} C_n^l \lambda^{n-l}(1-\lambda)^l |n - n\lambda - l| \leq \sqrt{\frac{2}{e}}\sqrt{\lambda(1-\lambda)(n+1)}. \tag{3.80}$$

Lemma 3.4. T_1 *given by* (3.74)-(3.75) *satisfies*

$$\begin{aligned} T_1 \leq &\sum_{l=n-[n\lambda^m]^+ +1}^{n} C_n^l (\lambda^m)^{n-l}(1-\lambda^m)^l (l - n + [n\lambda^m]^+)\frac{\lambda^M}{\lambda^m}, \\ &\forall\, 0 < \lambda^-,\ \lambda^+ < 1,\ n \in \mathbb{N},\ J \in \mathbb{Z},\ -n\lambda^- < J < 0, \end{aligned} \tag{3.81}$$

where $\lambda^m = \min\{\lambda^-, \lambda^+\}, \lambda^M = \max\{\lambda^-, \lambda^+\}$.

Lemma 3.5. T_2 *given by* (3.78) *satisfies*

$$\begin{aligned} T_2 \leq &\sum_{l=0}^{n-[n\lambda^m]^- -1} C_n^l (\lambda^m)^{n-l}(1-\lambda^m)^l (n - [n\lambda^m]^- - l)\frac{\lambda^M}{\lambda^m}, \\ &\forall\, 0 < \lambda^-,\ \lambda^+ < 1,\ n \in \mathbb{N},\ J \in \mathbb{Z},\ -n\lambda^- < J < 0, \end{aligned} \tag{3.82}$$

where $\lambda^m = \min\{\lambda^-, \lambda^+\}, \lambda^M = \max\{\lambda^-, \lambda^+\}$.

We leave the proofs of Lemmas 3.3, 3.4 and 3.5 for a consecutive paper [32].

Proof of Theorem 3.1

Proof. For the two cases (3.2), (3.3), we have

- Case (3.2)

  In this case the error expression is given by (3.45). Applying Lemma 3.3 one has

$$\begin{aligned} \|v(\cdot, t_n; u_0) - u(\cdot, t_n; u_0)\|_{L^1(\mathbb{R})} &= |Z|\Delta x \sum_{l=0}^{n} \Gamma_{n,l}^+ |n - n\lambda^+ - l| \\ &\leq |Z|\Delta x \sqrt{\frac{2}{e}}\sqrt{\lambda^+(1-\lambda^+)(n+1)}. \end{aligned} \tag{3.83}$$

- Case (3.3), if $x^* + c^- t_n \leq 0$

  In this case the error expression is given by (3.64). Applying Lemma 3.3 one has

$$\|v(\cdot,t_n;u_0) - u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})} \leq |Z|\Delta x \max\left\{\rho\frac{\lambda^+}{\lambda^-},1\right\}\sum_{l=0}^{n}\Gamma^-_{n,l}|l-n+n\lambda^-|$$

$$\leq |Z|\Delta x \max\left\{\rho\frac{\lambda^+}{\lambda^-},1\right\}\sqrt{\frac{2}{e}}\sqrt{\lambda^-(1-\lambda^-)(n+1)}. \tag{3.84}$$

- Case (3.3), if $x^* + c^- t_n > 0$

  In this case the upper bound expression is given by (3.79). Applying Lemmas 3.4 and 3.5 one has

$$\begin{aligned}
&\|v(\cdot,t_n;u_0) - u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}\\
&\leq \rho|Z|\Delta x + \max\{\rho,1\}|Z|\Delta x\,(T_1+T_2)\\
&\leq \rho|Z|\Delta x + \max\{\rho,1\}|Z|\Delta x\Bigg(\sum_{l=n-[n\lambda^m]^++1}^{n} C_n^l(\lambda^m)^{n-l}(1-\lambda^m)^l(l-n+n\lambda^m+1)\frac{\lambda^M}{\lambda^m}\\
&\quad+\sum_{l=0}^{n-[n\lambda^m]^--1} C_n^l(\lambda^m)^{n-l}(1-\lambda^m)^l(n-n\lambda^m+1-l)\frac{\lambda^M}{\lambda^m}\Bigg)\\
&\leq \max\{\rho,1\}|Z|\Delta x\left(1+\sum_{l=0}^{n}C_n^l(\lambda^m)^{n-l}(1-\lambda^m)^l|n-n\lambda^m-l|\frac{\lambda^M}{\lambda^m}+\frac{\lambda^M}{\lambda^m}\right),
\end{aligned} \tag{3.85}$$

  where $\lambda^m = \min\{\lambda^-,\lambda^+\}, \lambda^M = \max\{\lambda^-,\lambda^+\}$.

Applying Lemma 3.3 to (3.85) gives

$$\begin{aligned}
&\|v(\cdot,t_n;u_0) - u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}\\
&\leq \max\{\rho,1\}|Z|\Delta x\left(\sqrt{\frac{2}{e}}\sqrt{\lambda^m(1-\lambda^m)(n+1)}\frac{\lambda^M}{\lambda^m}+1+\frac{\lambda^M}{\lambda^m}\right).
\end{aligned} \tag{3.86}$$

One can check that

$$\max\{\rho,1\}\sqrt{\lambda^m(1-\lambda^m)}\frac{\lambda^M}{\lambda^m} \geq \sqrt{\lambda^+(1-\lambda^+)}, \tag{3.87}$$

$$\max\{\rho,1\}\sqrt{\lambda^m(1-\lambda^m)}\frac{\lambda^M}{\lambda^m} \geq \max\left\{\rho\frac{\lambda^+}{\lambda^-},1\right\}\sqrt{\lambda^-(1-\lambda^-)}. \tag{3.88}$$

Combining (3.83), (3.84), (3.86), (3.87) and (3.88), for any initial data $u_0(x)$ belonging to (3.2), (3.3), the L^1-error bounds for the upwind difference scheme are given by

$$\begin{aligned}
&\|v(\cdot,t_n;u_0) - u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}\\
&\leq \max\{\rho,1\}|Z|\Delta x\left(\sqrt{\frac{2}{e}}\sqrt{\lambda^m(1-\lambda^m)(n+1)}\frac{\lambda^M}{\lambda^m}+1+\frac{\lambda^M}{\lambda^m}\right)\\
&= \max\{\rho,1\}|Z|\left[\gamma_m\frac{c^M}{c^m}\sqrt{\Delta x}+\left(1+\frac{c^M}{c^m}\right)\Delta x\right],
\end{aligned} \tag{3.89}$$

where c^m, c^M, γ_m are defined by (2.9) and (2.10).

4. Proof of Theorem 2.1

With Theorem 3.1, we can now give proof of the main theorem of this paper.

Proof of Theorem 2.1

Proof. For any BV initial data $u_0(x)$, the error between u_0 and its cell average approximation $v(x,0;u_0)$ clearly satisfies

$$\|u_0(\cdot) - v(\cdot,0;u_0)\|_{L^1(\mathbb{R})} \le \|u_0\|_{BV}\Delta x. \tag{4.1}$$

Using the L^1-boundedness of the solution of the linear equation (1.1)-(1.3) with the interface condition (2.1), one has

$$\begin{aligned}&\|u(\cdot,t_n;u_0(x)) - u(\cdot,t_n;v(x,0;u_0))\|_{L^1(\mathbb{R})} = \|u(\cdot,t_n;u_0(x) - v(x,0;u_0))\|_{L^1(\mathbb{R})}\\ \le{}& \max\left\{\rho\frac{c^+}{c^-},1\right\}\|u_0(\cdot)-v(\cdot,0;u_0)\|_{L^1(\mathbb{R})} \le \max\left\{\rho\frac{c^+}{c^-},1\right\}\|u_0\|_{BV}\Delta x.\end{aligned} \tag{4.2}$$

Recall that the cell center positions x_0 and x_1 are located at two sides of the interface. We define step functions

$$w_0(x) = \begin{cases} v_0^0 & x<0,\\ v_1^0 & x\ge 0,\end{cases} \tag{4.3}$$

$$w_i(x) = \begin{cases} v_i^0 - v_{i+1}^0 & x< i\Delta x,\\ 0 & x\ge i\Delta x,\end{cases} \qquad i\in\mathbb{Z}^-, \tag{4.4}$$

$$w_i(x) = \begin{cases} 0 & x< i\Delta x,\\ v_{i+1}^0 - v_i^0 & x\ge i\Delta x,\end{cases} \qquad i\in\mathbb{Z}^+. \tag{4.5}$$

Then

$$\sum_{i=-\infty}^{\infty} w_i(x) = v(x,0;u_0) \qquad \text{on } \mathbb{R}. \tag{4.6}$$

One has

$$\begin{aligned}&\|v(\cdot,t_n;u_0(x)) - u(\cdot,t_n;v(x,0;u_0))\|_{L^1(\mathbb{R})}\\ ={}& \|v(\cdot,t_n;v(x,0;u_0)) - u(\cdot,t_n;v(x,0;u_0))\|_{L^1(\mathbb{R})}\\ ={}& \int_{-\infty}^{\infty}\left|\sum_{i=-\infty}^{\infty}[v(x,t_n;w_i) - u(x,t_n;w_i)]\right|dx\\ \le{}& \int_{-\infty}^{\infty}\sum_{i=-\infty}^{\infty}|v(x,t_n;w_i) - u(x,t_n;w_i)|\,dx.\end{aligned} \tag{4.7}$$

The conditions (4.4), (4.5) satisfy (3.2)-(3.3). Applying Theorem 3.1 one has,

$$\begin{aligned}&\|v(\cdot,t_n;w_i) - u(\cdot,t_n;w_i)\|_{L^1(\mathbb{R})}\\ \le{}& \max\{\rho,1\}\left|v_i^0 - v_{i+1}^0\right|\left[\gamma_m\frac{c^M}{c^m}\sqrt{\Delta x} + \left(1+\frac{c^M}{c^m}\right)\Delta x\right], \quad i\in\mathbb{Z}^-\cup\mathbb{Z}^+.\end{aligned} \tag{4.8}$$

Introduce two step functions

$$w_0^1(x) = \begin{cases} v_0^0 & x<0,\\ \rho v_0^0 & x\ge 0,\end{cases} \qquad w_0^2(x) = \begin{cases} 0 & x<0,\\ v_1^0 - \rho v_0^0 & x\ge 0.\end{cases} \tag{4.9}$$

Then $w_0 = w_0^1 + w_0^2$ and w_0^2 belongs to (3.2).

Using the fact that $v(x,t_n;w_0^1)=u(x,t_n;w_0^1)$ and applying (3.83) one has

$$\begin{aligned}&\|v(\cdot,t_n;w_0)-u(\cdot,t_n;w_0)\|_{L^1(\mathbb{R})}=\|v(\cdot,t_n;w_0^2)-u(\cdot,t_n;w_0^2)\|_{L^1(\mathbb{R})}\\ \le&\left|v_1^0-\rho v_0^0\right|\gamma_+\sqrt{\Delta x}\le\left|v_1^0-v_0^0\right|\gamma_+\sqrt{\Delta x}+|\rho-1|\left|v_0^0\right|\gamma_+\sqrt{\Delta x},\end{aligned}\tag{4.10}$$

where γ_+ is defined in (2.11).

Combining (4.8), (4.10) and utilizing the BV property of u_0 one has

$$\begin{aligned}&\sum_{i=-\infty}^{\infty}\int_{-\infty}^{\infty}|v(x,t_n;w_i)-u(x,t_n;w_i)|\,dx=\sum_{i=-\infty}^{\infty}\|v(x,t_n;w_i)-u(x,t_n;w_i)\|_{L^1(\mathbb{R})}\\ \le&\max\{\rho,1\}\sum_{i=-\infty}^{\infty}\left|v_i^0-v_{i+1}^0\right|\left[\gamma_m\frac{c^M}{c^m}\sqrt{\Delta x}+\left(1+\frac{c^M}{c^m}\right)\Delta x\right]+|\rho-1|\left|v_0^0\right|\gamma_+\sqrt{\Delta x}\\ \le&\max\{\rho,1\}\|u_0\|_{BV}\left[\gamma_m\frac{c^M}{c^m}\sqrt{\Delta x}+\left(1+\frac{c^M}{c^m}\right)\Delta x\right]\\ &+|\rho-1|\max_{-\Delta x<x<0}|u_0(x)|\gamma_+\sqrt{\Delta x}.\end{aligned}\tag{4.11}$$

Therefore,

$$\sum_{i=-\infty}^{\infty}|v(x,t_n;w_i)-u(x,t_n;w_i)|\in L^1(\mathbb{R})$$

and

$$\int_{-\infty}^{\infty}\sum_{i=-\infty}^{\infty}|v(x,t_n;w_i)-u(x,t_n;w_i)|\,dx=\sum_{i=-\infty}^{\infty}\int_{-\infty}^{\infty}|v(x,t_n;w_i)-u(x,t_n;w_i)|\,dx.\tag{4.12}$$

Combining (4.2), (4.7), (4.12) and (4.11) completes the proof of Theorem 2.1:

$$\begin{aligned}&\|v(\cdot,t_n;u_0)-u(\cdot,t_n;u_0)\|_{L^1(\mathbb{R})}\\ \le&\|v(\cdot,t_n;u_0(x))-u(\cdot,t_n;v(x,0;u_0))\|_{L^1(\mathbb{R})}+\|u(\cdot,t_n;u_0(x))-u(\cdot,t_n;v(x,0;u_0))\|_{L^1(\mathbb{R})}\\ \le&\max\{\rho,1\}\|u_0\|_{BV}\left[\gamma_m\frac{c^M}{c^m}\sqrt{\Delta x}+\left(1+\frac{c^M}{c^m}\right)\Delta x\right]\\ &+|\rho-1|\max_{-\Delta x<x<0}|u_0(x)|\gamma_+\sqrt{\Delta x}+\max\left\{\rho\frac{c^+}{c^-},1\right\}\|u_0\|_{BV}\Delta x\\ \le&\max\{\rho,1\}\|u_0\|_{BV}\left[\gamma_m\frac{c^M}{c^m}\sqrt{\Delta x}+\left(1+\frac{c^M}{c^m}+\max\left\{\frac{c^+}{c^-},1\right\}\right)\Delta x\right]\\ &+|\rho-1|\max_{-\Delta x<x<0}|u_0(x)|\gamma_+\sqrt{\Delta x}.\end{aligned}\tag{4.13}$$

5. Conclusion

In this paper we established the L^1-error estimates for the upwind difference scheme to the linear advection equation with piecewise constant wave speeds and a general interface condition. A natural and successful approach for computing such equations is to incorporate the interface conditions into the numerical schemes, in the spirit of the immersed interface method. We proved that, for initial data with a bounded variation, the numerical solution by the upwind scheme with an immersed interface condition converges in L^1-norm to the solution of the linear advection equation with the corresponding interface condition. We derived the half-order L^1-error bounds with explicit coefficients.

Our approach makes use of the linearities of both the equation (1.1) and the upwind difference scheme. We focused on error estimates for some Riemann initial data. Based on these results the error estimates for general BV initial data were derived. For these Riemann initial data, the numerical solutions and the exact solutions can both be expressed in terms of the initial data. Then the L^1-error (upper bound) for the numerical solutions were explicitly expressed in terms of some binomial coefficient. Some relevant inequalities on the binomial coefficients are proved in [32].

This paper deals with the upper bound error estimates. Similar techniques can be used to investigate the lower bound error estimates of the scheme. In that case one needs to estimate the lower bound of the L^1-error expressions. The half-order lower bound of the monotone schemes was proved in [25] for the linear advection equation with constant coefficients. Naturally, one expects that the upwind scheme for the same equation with piecewise constant coefficients can not achieve better accuracy, thus one should also have a half-order optimal convergence rate. Another interesting issue is to investigate whether the technique used in this paper can be applied to analyze more general monotone schemes for the linear advection equation with piecewise constant coefficients.

A related issue to the study in this paper is the error estimates for Hamiltonian-preserving schemes [10-12] to the Liouville equations with discontinuous Hamiltonians. The Liouville equations with discontinuous Hamiltonians arise in the high frequency limit of linear waves. They are linear hyperbolic equations with discontinuous and measure-valued coefficients. The Hamiltonian-preserving schemes are designed by incorporating interface conditions—Hamiltonian preservation, with transmission and reflection coefficients—into the numerical fluxes. In the future we will try to extend the same approach to investigate the L^1-error estimates for the Hamiltonian-preserving schemes to the Liouville equations with piecewise constant Hamiltonians.

Appendix

In this Appendix we prove Lemma 3.2. We first give some Lemmas A.1-A.3. Then we give the proof of Lemma 3.2.

Lemma A.1. *Define*

$$\Gamma_{n,l}(z) = C_n^l z^{n-l}(1-z)^l. \tag{A.1}$$

Then for $m \in \mathbb{N} \cup \{0\}$, $0 < z, \lambda^- < 1$,

$$\sum_{l=0}^{m} \sum_{k=0}^{l} \Gamma_{m-k,l-k}(z)(1-\lambda^-)^k \left[\frac{m-l}{z} - \frac{l-k}{1-z}\right] = 0. \tag{A.2}$$

Proof. It can be checked that (A.2) holds for $m = 0$. Now suppose (A.2) holds for $m \geq 0$, we will prove it also holds for $m+1$. One has

$$\begin{aligned}
&\sum_{l=0}^{m+1} \sum_{k=0}^{l} \Gamma_{m+1-k,l-k}(z)(1-\lambda^-)^k \left[\frac{m+1-l}{z} - \frac{l-k}{1-z}\right] \\
=& \sum_{l=0}^{m+1} \sum_{k=0} \Gamma_{m+1-k,l-k}(z)(1-\lambda^-)^k \left[\frac{m+1-l}{z} - \frac{l-k}{1-z}\right] \\
&+ \sum_{l=1}^{m+1} \sum_{k=1}^{l} \Gamma_{m+1-k,l-k}(z)(1-\lambda^-)^k \left[\frac{m+1-l}{z} - \frac{l-k}{1-z}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{m+1} \Gamma_{m+1,l}(z) \left[\frac{m+1-l}{z} - \frac{l}{1-z}\right] \\
&\quad + \sum_{l-1=0}^{m} \sum_{k-1=0}^{l-1} \Gamma_{m-(k-1),(l-1)-(k-1)}(z)(1-\lambda^-)^k \left[\frac{m-(l-1)}{z} - \frac{(l-1)-(k-1)}{1-z}\right] \\
&= d\left(\sum_{l=0}^{m+1} \Gamma_{m+1,l}(z)\right) \Big/ dz + (1-\lambda^-) \sum_{l=0}^{m} \sum_{k=0}^{l} \Gamma_{m-k,l-k}(z)(1-\lambda^-)^k \left[\frac{m-l}{z} - \frac{l-k}{1-z}\right] \\
&= (1-\lambda^-) \sum_{l=0}^{m} \sum_{k=0}^{l} \Gamma_{m-k,l-k}(z)(1-\lambda^-)^k \left[\frac{m-l}{z} - \frac{l-k}{1-z}\right]. \qquad \text{(A.3)}
\end{aligned}
$$

From (A.3), applying the assumption that (A.2) holds for m implies that (A.2) also holds for $m+1$. This completes the proof of Lemma A.1. □

Lemma A.2. *Define*

$$
\Lambda^{n,-}_{j,k,l}(z) = C^k_{j+k-1} C^{l-k}_{n-j-k} z^{n-l-j} (1-z)^{l-k} (\lambda^-)^{j-1} (1-\lambda^-)^k, \qquad \text{(A.4)}
$$

$$
\phi(n,m,z) = \sum_{l=0}^{m} \sum_{j=n-m} \sum_{k=0}^{l} \Lambda^{n,-}_{j,k,l}(z) = \sum_{l=0}^{m} \sum_{k=0}^{l} \Lambda^{n,-}_{n-m,k,l}(z),
$$

$$
n \in \mathbb{N},\ 0 \le m \le n-1,\ 0 < z,\ \lambda^- < 1. \qquad \text{(A.5)}
$$

Then

$$
\phi(n,m,z_1) = \phi(n,m,z_2), \quad \forall\, 0 < z_1,\ z_2 < 1,\ 0 \le m \le n-1,\ n \in \mathbb{N}. \qquad \text{(A.6)}
$$

Proof. (A.6) is equivalent to, for $n \in \mathbb{N}, 0 \le m \le n-1, 0 < z, \lambda^- < 1$,

$$
\partial\phi(n,m,z)/\partial z = \sum_{l=0}^{m} \sum_{k=0}^{l} \Lambda^{n,-}_{n-m,k,l}(z) \left[\frac{m-l}{z} - \frac{l-k}{1-z}\right] \equiv \widehat{\phi}(n,m,z) = 0. \qquad \text{(A.7)}
$$

Firstly we show that

$$
\widehat{\phi}(n,0,z) = 0, \quad \widehat{\phi}(n,n-1,z) = 0, \quad \forall n \in \mathbb{N},\ 0 < z < 1. \qquad \text{(A.8)}
$$

The first equality in (A.8) can be directly checked. For the second equality, one has

$$
\widehat{\phi}(n,n-1,z) = \sum_{l=0}^{n-1} \sum_{k=0}^{l} C^{l-k}_{n-1-k} z^{n-1-l} (1-z)^{l-k} (1-\lambda^-)^k \left[\frac{n-1-l}{z} - \frac{l-k}{1-z}\right]. \qquad \text{(A.9)}
$$

Applying Lemma A.1, the second equality in (A.8) is proved.

(A.8) implies that (A.7) holds for $n = 1, 2$. Now suppose (A.7) holds for $n \ge 2$, we will prove it is also true for $n+1$. For $1 \le m \le n-1, n \ge 2$, one has

$$
\begin{aligned}
\widehat{\phi}(n+1,m,z) &= \sum_{l=0}^{m} \sum_{k=0}^{l} \Lambda^{n+1,-}_{n+1-m,k,l}(z) \left[\frac{m-l}{z} - \frac{l-k}{1-z}\right] \\
&= \lambda^- \left\{ \sum_{l=0}^{m} \sum_{k=0}^{l} \Lambda^{n,-}_{n-m,k,l}(z) \left[\frac{m-l}{z} - \frac{l-k}{1-z}\right] \right. \\
&\qquad \left. + \frac{1-\lambda^-}{\lambda^-} \sum_{l=1}^{m} \sum_{k=1}^{l} \Lambda^{n,-}_{n-(m-1),k-1,l-1}(z) \left[\frac{m-l}{z} - \frac{l-k}{1-z}\right] \right\} \\
&= \lambda^- \left(\widehat{\phi}(n,m,z) + \frac{1-\lambda^-}{\lambda^-} \widehat{\phi}(n,m-1,z) \right). \qquad \text{(A.10)}
\end{aligned}
$$

From (A.10), applying the assumption that (A.7) holds for n, one has

$$\widehat{\phi}(n+1,m,z)=0, \quad \text{for } 1\le m\le n-1, 0<z<1. \tag{A.11}$$

(A.11) together with (A.8) implies that (A.7) holds for $n+1$. $\square$

Lemma A.3.

$$\sum_{k=0}^{l} C_{j+k-1}^{k} C_{n-j-k}^{l-k} = C_n^l, \quad \text{for } 1\le j\le n-l,\ 0\le l\le n-1, n\in\mathbb{N}. \tag{A.12}$$

Proof. Denote

$$\omega(n,l,j)=C_n^l-\sum_{k=0}^{l} C_{j+k-1}^{k} C_{n-j-k}^{l-k}. \tag{A.13}$$

Firstly we show that $\forall n\in\mathbb{N}, 1\le j\le n,\ 0\le l\le n-1$,

$$\omega(n,l,n-l)=0, \quad \omega(n,0,j)=0, \quad \omega(n,n-1,1)=0. \tag{A.14}$$

The second and third equalities in (A.14) can be checked directly. For the first equality, one can check that $\omega(n,0,n)=0, \forall n\in\mathbb{N}$, and for $0<l\le n-1, n\ge 2$ one has

$$\begin{aligned}\omega(n,l,n-l) &= C_n^l-\sum_{k=0}^{l} C_{n-l+k-1}^{k} = C_{n-1}^l + C_{n-1}^{l-1} - C_{n-1}^l - \sum_{k=0}^{l-1} C_{n-l+k-1}^{k}\\ &= \omega(n-1,l-1,n-l)=\cdots=\omega(n-l,0,n-l)=0. \end{aligned}\tag{A.15}$$

Thus the first equality in (A.14) is proved.

(A.14) implies that (A.12) holds for $n=1,2$. Now suppose (A.12) holds for $n\ge 2$, we will prove it is also true for $n+1$. For $1\le j\le n-l,\ 1\le l\le n-1$, one has

$$\begin{aligned}\omega(n+1,l,j) &= C_{n+1}^l-\sum_{k=0}^{l} C_{j+k-1}^{k} C_{n+1-j-k}^{l-k}\\ &= \left(C_n^l-\sum_{k=0}^{l} C_{j+k-1}^{k} C_{n-j-k}^{l-k}\right)+\left(C_n^{l-1}-\sum_{k=0}^{l-1} C_{j+k-1}^{k} C_{n-j-k}^{l-k-1}\right)\\ &= \omega(n,l,j)+\omega(n,l-1,j). \end{aligned}\tag{A.16}$$

From (A.16), applying the assumption that (A.12) holds for n, one has

$$\omega(n+1,l,j)=0, \quad \text{for } 1\le j\le n-l,\ 1\le l\le n-1. \tag{A.17}$$

(A.17) together with (A.14) implies that (A.12) holds for $n+1$. $\square$

Proof of Lemma 3.2

Proof.

$$P_1\equiv\sum_{l=0}^{m-1}\sum_{j=n-m+1}^{n-l}\sum_{k=0}^{l}\Lambda_{j,k,l}^n=\sum_{s=1}^{m}\sum_{l=0}^{m-s}\sum_{j=n-m+s}\sum_{k=0}^{l}\Lambda_{j,k,l}^n. \tag{A.18}$$

Applying Lemma A.2, one has from (A.18)

$$\begin{aligned} P_1 &= \sum_{s=1}^{m} \lambda^+ \phi(n, m-s, \lambda^+) = \sum_{s=1}^{m} \lambda^+ \phi(n, m-s, \lambda^-) \\ &= \frac{\lambda^+}{\lambda^-} \sum_{l=0}^{m-1} \sum_{j=n-m+1}^{n-l} \left(\sum_{k=0}^{l} C_{j+k-1}^{k} C_{n-j-k}^{l-k} \right) (\lambda^-)^{n-l} (1-\lambda^-)^l. \end{aligned} \tag{A.19}$$

From (A.19), (3.60) can be proved by applying Lemma A.3.

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