Journal of Computational Mathematics, Vol.26, No.3, 2008, 410–436.

FINITE ELEMENT METHODS FOR THE NAVIER-STOKES EQUATIONS BY $H(\mathrm{div})$ ELEMENTS*

Junping Wang
Division of Mathematical Sciences, National Science Foundation, Arlington, VA 22230, USA
Email: jwang@nsf.gov
Xiaoshen Wang and Xiu Ye
Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AK 72204, USA
Email: xxwang@ualr.edu, xxye@ualr.edu

Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

We derived and analyzed a new numerical scheme for the Navier-Stokes equations by using $H(\mathrm{div})$ conforming finite elements. A great deal of effort was given to an establishment of some Sobolev-type inequalities for piecewise smooth functions. In particular, the newly derived Sobolev inequalities were employed to provide a mathematical theory for the $H(\mathrm{div})$ finite element scheme. For example, it was proved that the new finite element scheme has solutions which admit a certain boundedness in terms of the input data. A solution uniqueness was also possible when the input data satisfies a certain smallness condition. Optimal-order error estimates for the corresponding finite element solutions were established in various Sobolev norms. The finite element solutions from the new scheme feature a full satisfaction of the continuity equation which is highly demanded in scientific computing.

Mathematics subject classification: 65N15, 65N30, 75D07; Secondary 35B45, 35J50.
Key words: Finite element methods, Navier-Stokes equations, CFD.

1. Introduction

We are concerned with numerical solutions of the Navier-Stokes equations: find a pair of unknown functions $(\mathbf{u};p)$ satisfying

$$-\nu\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{f}\quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u}=0\quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u}=0\quad \text{on } \partial\Omega, \tag{1.3}$$

where ν denotes the fluid viscosity; Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators, respectively; $\Omega\subset\mathbb{R}^n$ is the region occupied by the fluid; $\mathbf{f}=\mathbf{f}(\mathbf{x})\in[L^2(\Omega)]^n$ is the unit external volumetric force acting on the fluid at $\mathbf{x}\in\Omega$.

The commonly used finite element methods for the Navier-Stokes problem (1.1)-(1.3) are based on a variational equation which is obtained by testing the momentum equation (1.1) by functions in $[H_0^1(\Omega)]^n$ and the continuity equation (1.2) by functions in $L^2(\Omega)$ (see Section 2 for their definition). The corresponding finite element method requires a pair of finite element spaces which are conforming in $H^1\times L^2$ and satisfy the *inf-sup* condition of Babuška [3] and

* Received January 31, 2008 / accepted March 14, 2008 /

Brezzi [4]. These constraints result in finite element approximations, denoted by $(\mathbf{u}_h; p_h)$, which hardly satisfy the continuity equation

$$\nabla \cdot \mathbf{u}_h(\mathbf{x}) = 0, \quad \forall\, \mathbf{x} \in \Omega. \tag{1.4}$$

Readers are referred to [19] and [9] for more details regarding the approximation methods and their properties.

The recent development in discontinuous Gelerkin methods [2, 5–7, 11, 12, 14] provides new means in solving the incompressible problems numerically. However, the corresponding finite element solutions are usually totally discontinuous and fail to satisfy the continuity equation (1.4) immediately [13, 23, 26, 30].

Eq. (1.4) requires that the numerical solution $\mathbf{u}_h$ be a member of the Sobolev space $H(\text{div}; \Omega)$. Therefore, the discontinuous Galerkin methods [13,23,26,30] may not be appropriate when (1.4) needs to be satisfied. On the other hand, the $H^1 \times L^2$ conforming finite element methods require the total continuity of $\mathbf{u}_h$, which is beyond what is required for a satisfaction of (1.4). Therefore, it appears that the $H(\text{div})$ elements of Raviart-Thomas type [27] might be appropriate for approximating the solution of the Navier-Stokes equations.

In [29], a finite element scheme for the Stokes equations was derived and analyzed by using existing $H(\text{div})$ finite elements of the Raviart-Thomas type. The numerical solutions of the finite element schemes developed in [29] satisfy the incompressibility constraint (1.2) exactly. The goal of this paper is to continue our investigation in $H(\text{div})$ finite element methods by extending the results of [29] to the Navier-Stokes equations. There are two main difficulties in this extension. The first one lies on a treatment of the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ in designing a numerical discretization scheme for (1.1)-(1.3). An up-winding approach shall be used to tackle this difficulty, yielding a numerical scheme that should be stable for small viscosities. The second difficult is associated with a mathematical analysis for the numerical scheme; namely, one has to deal with the difficulties caused by discontinuity of the finite elements and the corresponding integral forms over the element boundaries. Some Sobolev-type inequalities are established to address this challenge.

This paper is organized as follows. In Section 2, we introduce some preliminaries and notations for Sobolev spaces. A variational formula is presented in Section 3 for the Navier-Stokes equations. In Section 4, we present a $H(\text{div})$ finite element method for the Navier-Stokes equations, based on the variational formula developed in Section 3. In Section 5, we derive some Sobolev-type inequalities for piecewise smooth functions. Section 6 is devoted to a mathematical study of the finite element scheme. Here it was proved that the new finite element scheme has solutions and the solutions are unique when the input data is sufficiently small. In Section 7, we establish some optimal-order error estimates for the finite element approximations in a discrete H^1-norm for the velocity approximation and L^2-norms for the pressure.

2. Preliminaries and Notations

Let D be any domain in $\mathbb{R}^n, n = 2, 3$. For simplicity, we take the case $n = 2$ as a protocol in the presentation and analysis. Extension to problems in three space variables is possible for all the results to be presented in this manuscript.

We use standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot, \cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for $s \geq 0$. For example, for any integer $s \geq 0$, the seminorm $|\cdot|_{s,D}$ is given by

$$|v|_{s,D} = \left(\sum_{|\alpha|=s} \int_D |\partial^\alpha v|^2 dD \right)^{\frac{1}{2}}$$

with the usual notation

$$\alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2, \quad \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}.$$

The Sobolev norm $\|\cdot\|_{m,D}$ is given by

$$\|v\|_{m,D} = \left(\sum_{j=0}^{m} |v|_{j,D}^2 \right)^{\frac{1}{2}}.$$

The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and inner product notation. We also use $L_0^2(\Omega)$ to denote the subspace of $L^2(\Omega)$ consisting of functions with mean value zero.

The space $H(\mathrm{div};\Omega)$ is defined as the set of vector-valued functions on Ω which, together with their divergence, are square integrable; i.e.,

$$H(\mathrm{div};\Omega) = \left\{ \mathbf{v} : \ \mathbf{v} \in [L^2(\Omega)]^2, \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}.$$

The norm in $H(\mathrm{div};\Omega)$ is defined by

$$\|\mathbf{v}\|_{H(\mathrm{div};\Omega)} = \left(\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2 \right)^{\frac{1}{2}}.$$

Let $K \subset \Omega$ be a triangle or quadrilateral. For any smooth vector-valued functions $\mathbf{w}$ and $\mathbf{v}$, it follows from the divergence theorem that

$$\int_K (-\Delta \mathbf{w}) \cdot \mathbf{v} dK = (\nabla \mathbf{w}, \nabla \mathbf{v})_K - \int_{\partial K} \frac{\partial \mathbf{w}}{\partial \mathbf{n}_K} \cdot \mathbf{v} \ ds, \tag{2.1}$$

where ds represents the boundary element, $\mathbf{n}_K$ is the outward normal direction on ∂K, and

$$(\nabla \mathbf{w}, \nabla \mathbf{v})_K = \sum_{i,j=1}^{2} \int_K \frac{\partial w_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dK.$$

Let τ_K be the tangential direction to ∂K so that $\mathbf{n}_K$ and τ_k form a right-hand coordinate system. It follows from the representation

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{n}_K)\mathbf{n}_K + (\mathbf{v} \cdot \tau_K)\tau_K$$

that

$$\frac{\partial \mathbf{w}}{\partial \mathbf{n}_K} \cdot \mathbf{v} = \frac{\partial (\mathbf{w} \cdot \mathbf{n}_K)}{\partial \mathbf{n}_K} (\mathbf{v} \cdot \mathbf{n}_K) + \frac{\partial (\mathbf{w} \cdot \tau_K)}{\partial \mathbf{n}_K} (\mathbf{v} \cdot \tau_K). \tag{2.2}$$

3. A Variational Formula

Let $\mathcal{T}_h$ be a finite element partition of the domain Ω with mesh size h. Assume that the partition $\mathcal{T}_h$ is quasi-uniform; i.e., it is regular and satisfies the inverse assumption (see [10]). Define the finite element spaces V_h and W_h for the velocity and pressure respectively by

$$V_h = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \ \mathbf{v}|_K \in V_r(K), \quad \forall K \in \mathcal{T}_h, \quad \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$$

and

$$W_h = \{q \in L_0^2(\Omega) : \ q|_K \in W_m(K), \quad \forall K \in \mathcal{T}_h\},$$

where $\mathbf{n}$ is the outward normal direction on the boundary of Ω, $V_r(K)$ is a space of vector-valued polynomials on the element K with index $r \geq 1$, and $W_m(K)$ is a set of polynomials on the element K with index $m \geq 0$.

To derive a weak formulation for the Navier-Stokes equations, we shall test the Navier-Stokes system (1.1)-(1.2) by discontinuous finite element functions in V_h and W_h, respectively. The first obvious equation is given by testing Eq. (1.2) against any $q \in W_h$, yielding

$$(\nabla\cdot\mathbf{u}, q) = 0. \tag{3.1}$$

A second equation can be obtained by testing the momentum equation (1.1) against any $\mathbf{v} \in V_h$. The main body of this section is devoted to a discussion of the momentum equation, particularly the treatment of the nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$.

To this end, let us multiply Eq. (1.1) by any $\mathbf{v} \in V_h$ and use (2.1) to obtain

$$\sum_{K\in\mathcal{T}_h}\left(\nu(\nabla\mathbf{u},\nabla\mathbf{v})_K - \nu\int_{\partial K}\frac{\partial\mathbf{u}}{\partial\mathbf{n}_K}\cdot\mathbf{v}\ ds + (\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v})_K\right) - (p,\nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}),\mathbf{v}), \tag{3.2}$$

where we have also used the integration by parts to deduce

$$\int_\Omega \nabla p\cdot\mathbf{v}d\Omega = -(p,\nabla\cdot\mathbf{v}).$$

The fact that $\mathbf{v} \in V_h$ implies that $\mathbf{v}\cdot\mathbf{n}_K$ is continuous across each interior boundary. Thus, it follows from (2.2) that

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial\mathbf{u}}{\partial\mathbf{n}_K}\cdot\mathbf{v}\ ds = \sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial(\mathbf{u}\cdot\tau_K)}{\partial\mathbf{n}_K}\ \mathbf{v}\cdot\tau_K\ ds. \tag{3.3}$$

For convenience, we introduce a product space

$$\mathbf{X}_h = \prod_{K\in\mathcal{T}_h}[H^1(K)]^2$$

and the following notation:

$$(\nabla_h\mathbf{w},\nabla_h\mathbf{q}) = \sum_{K\in\mathcal{T}_h}(\nabla\mathbf{w},\nabla\mathbf{q})_K, \quad \forall\ \mathbf{w},\mathbf{q}\in\mathbf{X}_h.$$

By substituting (3.3) into (3.2) we obtain

$$\begin{aligned}\nu(\nabla_h\mathbf{u},\nabla_h\mathbf{v}) + \sum_{K\in\mathcal{T}_h}(\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v})_K - (p,\nabla\cdot\mathbf{v})\\ -\nu\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial(\mathbf{u}\cdot\tau_K)}{\partial\mathbf{n}_K}\ \mathbf{v}\cdot\tau_K ds = (\mathbf{f},\mathbf{v}).\end{aligned} \tag{3.4}$$

We now reformulate the boundary integrals in (3.4). Let e be an interior edge shared by two elements K_1 and K_2, and $\mathbf{n}_1$ and $\mathbf{n}_2$ be unit normal vectors on e pointing exterior to K_1 and K_2, respectively. Denote by τ_1 and τ_2 the two tangential directions which make the right-hand coordinate systems with $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. We define the average $\{\cdot\}$ and jump $[\![\cdot]\!]$ on e for vector-valued functions $\mathbf{w}$ as follows:

$$\begin{aligned}\{\varepsilon(\mathbf{w})\} &= \frac{1}{2}\left(\mathbf{n}_1\cdot\nabla(\mathbf{w}\cdot\tau_1)|_{\partial K_1}+\mathbf{n}_2\cdot\nabla(\mathbf{w}\cdot\tau_2)|_{\partial K_2}\right),\\ [\![\mathbf{w}]\!] &= \mathbf{w}|_{\partial K_1}\cdot\tau_1+\mathbf{w}|_{\partial K_2}\cdot\tau_2.\end{aligned}$$

For a boundary edge $e=\partial K_1\cap\partial\Omega$, the above two operations must be modified by

$$\{\varepsilon(\mathbf{w})\}=\mathbf{n}_1\cdot\nabla(\mathbf{w}\cdot\tau_1)|_{\partial K_1},\quad [\![\mathbf{w}]\!]=\mathbf{w}|_{\partial K_1}\cdot\tau_1.$$

Let $\mathcal{E}_h$ denote the union of the boundaries of all elements K in $\mathcal{T}_h$. For sufficiently smooth $\mathbf{u}$ (e.g., $\mathbf{u}\in H^{\frac{3}{2}}(\Omega)$), by grouping terms associated with each edge $e\in\mathcal{E}_h$ it is not hard to see that

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial(\mathbf{u}\cdot\tau_K)}{\partial\mathbf{n}_K}\,\mathbf{v}\cdot\tau_K ds=\sum_{e\in\mathcal{E}_h}\int_e\{\varepsilon(\mathbf{u})\}[\![\mathbf{v}]\!]ds. \tag{3.5}$$

Next we present a treatment of the nonlinear term $\sum_{K\in\mathcal{T}_h}(\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v})_K$ by adding some stabilization terms. To avoid any possible confusion, we remark that $\mathbf{u}\cdot\nabla\mathbf{v}$ should be viewed as a row vector $\mathbf{u}$ times a matrix $\nabla\mathbf{v}$ from left with

$$\nabla\mathbf{v}=\left[\begin{array}{cc}\partial_{x_1}v_1 & \partial_{x_1}v_2\\ \partial_{x_2}v_1 & \partial_{x_2}v_2\end{array}\right].$$

Let us introduce a trilinear form on $\mathbf{X}_h\times\mathbf{X}_h\times\mathbf{X}_h$ as follows:

$$a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w}):=\frac{1}{2}\left(\sum_{K\in\mathcal{T}_h}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})_K-\sum_{K\in\mathcal{T}_h}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})_K\right). \tag{3.6}$$

This trilinear form is skew symmetric in the last two variables. Through a straight forward use of integration by parts, one arrives at the following identity (see, e.g. [17]):

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})_K &= a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w})-\frac{1}{2}\sum_{K\in\mathcal{T}_h}((\nabla\cdot\mathbf{u})\mathbf{v},\mathbf{w})_K\\ &\quad+\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{w})ds\end{aligned} \tag{3.7}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}_h$. If particular, if $\mathbf{u},\mathbf{v}\in\mathbf{X}_h\cap[H_0^1(\Omega)]^2$ and $\nabla\cdot\mathbf{u}=0$, then

$$\sum_{K\in\mathcal{T}_h}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})_K=a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w})+\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{w})ds. \tag{3.8}$$

Since $\mathbf{v}=0$ on the boundary of Ω, we have

$$\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{w})ds=\frac{1}{2}\sum_{e\in\mathcal{E}_h^0}\int_e\mathbf{u}\cdot\mathbf{n}(\mathbf{v}_L\cdot\mathbf{w}_L-\mathbf{v}_R\cdot\mathbf{w}_R)ds, \tag{3.9}$$

where $\mathcal{E}_h^0$ is the collection of all interior edges, $\mathbf{n}$ is an orientation of $e \in \mathcal{E}_h^0$, $\mathbf{w}_L$ is the trace of $\mathbf{w}$ on e as seen from the left, and $\mathbf{w}_R$ is the trace of $\mathbf{w}$ on e as seen from the right. More precisely, $\mathbf{w}_L$ and $\mathbf{w}_R$ are defined as follows:

$$\mathbf{w}_L(x) = \lim_{t\to 0^+} \mathbf{w}(x - t\mathbf{n}), \qquad \mathbf{w}_R(x) = \lim_{t\to 0^+} \mathbf{w}(x + t\mathbf{n}).$$

Note that $\mathbf{v} \in [H_0^1(\Omega)]^2$ implies $\mathbf{v}_L = \mathbf{v}_R$ on each interior edge e. It follows that

$$\mathbf{v}_L \cdot \mathbf{w}_L - \mathbf{v}_R \cdot \mathbf{w}_R = \mathbf{v}_R \cdot \mathbf{w}_L - \mathbf{v}_L \cdot \mathbf{w}_R.$$

Substituting the above into (3.9) yields

$$\frac{1}{2} \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{w})ds = \frac{1}{2} \sum_{e\in\mathcal{E}_h^0} \int_e \mathbf{u}\cdot\mathbf{n}(\mathbf{v}_R\cdot\mathbf{w}_L - \mathbf{v}_L\cdot\mathbf{w}_R)ds, \tag{3.10}$$

which, together with (3.8), implies

$$\sum_{K\in\mathcal{T}_h} (\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{w})_K = a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w}) + \frac{1}{2} \sum_{e\in\mathcal{E}_h^0} \int_e \mathbf{u}\cdot\mathbf{n}(\mathbf{v}_R\cdot\mathbf{w}_L - \mathbf{v}_L\cdot\mathbf{w}_R)ds \tag{3.11}$$

for any $\mathbf{w} \in \mathbf{X}_h$ and $\mathbf{u},\mathbf{v} \in \mathbf{X}_h \cap [H_0^1(\Omega)]^2$ with $\nabla\cdot\mathbf{u} = 0$. The right-hand side of (3.11) can be further stabilized as follows:

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} (\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{w})_K \\ &= a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w}) + \frac{1}{2} \sum_{e\in\mathcal{E}_h^0} \int_e \mathbf{u}\cdot\mathbf{n}(\mathbf{v}_R\cdot\mathbf{w}_L - \mathbf{v}_L\cdot\mathbf{w}_R)ds + \gamma \sum_{e\in\mathcal{E}_h} \int_e |\mathbf{u}\cdot\mathbf{n}| [\![\mathbf{v}]\!][\![\mathbf{w}]\!]ds,\end{aligned} \tag{3.12}$$

where $\gamma > 0$ is a stabilization parameter.

The right-hand side of (3.12) provides a suitable weak form for the nonlinear inertial term of the Navier-Stokes equations. For this purpose, we introduce a quasi-trilinear form as follows:

$$\begin{aligned}a_1(\mathbf{u},\mathbf{v},\mathbf{w}) := a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w}) &+ \frac{1}{2} \sum_{e\in\mathcal{E}_h^0} \int_e \mathbf{u}\cdot\mathbf{n}(\mathbf{v}_R\cdot\mathbf{w}_L - \mathbf{v}_L\cdot\mathbf{w}_R)ds \\ &+\gamma \sum_{e\in\mathcal{E}_h^0} \int_e |\mathbf{u}\cdot\mathbf{n}| [\![\mathbf{v}]\!][\![\mathbf{w}]\!]ds.\end{aligned} \tag{3.13}$$

In particular, if $\mathbf{u}$ is the weak solution of the Navier-Stokes equations (1.1)-(1.3) with sufficient regularity (e.g., $\mathbf{u} \in H^{\frac{3}{2}+\epsilon}(\Omega)$ with $\epsilon > 0$), then by substituting (3.5) and (3.13)/(3.12) into (3.4) we obtain

$$\nu(\nabla_h\mathbf{u}, \nabla_h\mathbf{v}) + a_1(\mathbf{u},\mathbf{u},\mathbf{v}) - (\nabla\cdot\mathbf{v}, p) - \nu \sum_{e\in\mathcal{E}_h} \int_e \{\varepsilon(\mathbf{u})\}[\![\mathbf{v}]\!]ds = (\mathbf{f},\mathbf{v}) \tag{3.14}$$

for all $\mathbf{v} \in V_h$. As in the usual discontinuous finite element method, we further stabilize Eq. (3.14) by adding the following term to its left-hand side:

$$S_\beta(\mathbf{u},\mathbf{v}) := \nu \sum_{e\in\mathcal{E}_h} \int_e \left(\alpha h_e^{-1} [\![\mathbf{v}]\!][\![\mathbf{u}]\!] - \beta\{\varepsilon(\mathbf{v})\}[\![\mathbf{u}]\!]\right) ds, \tag{3.15}$$

where $\alpha > 0$ is another stabilization parameter, $\beta = \pm 1$, and h_e is the length of the edge e. It is easy to see that $S_\beta(\mathbf{u}, \mathbf{v}) = 0$ for any $\mathbf{u} \in [H_0^1(\Omega)]^2$ and $\mathbf{v} \in V_h$. It follows that a variational form for the momentum equation in the Navier-Stokes system can be given as follows:

$$\nu(\nabla_h \mathbf{u}, \nabla_h \mathbf{v}) + S_\beta(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) - \nu \sum_{e \in \mathcal{E}_h} \int_e \{\varepsilon(\mathbf{u})\}[\![\mathbf{v}]\!] ds = (\mathbf{f}, \mathbf{v}).$$

For simplification of notation, we introduce a functional space

$$V(h, s) = V_h + \prod_{K \in \mathcal{T}_h} \left[H^s(K)^2 \cap H_0^1(\Omega)^2\right], \quad s > \frac{3}{2} \tag{3.16}$$

and two bilinear forms

$$d_\beta(\mathbf{u}, \mathbf{v}) = \nu(\nabla_h \mathbf{u}, \nabla_h \mathbf{v}) + S_\beta(\mathbf{u}, \mathbf{v}) - \nu \sum_{e \in \mathcal{E}_h} \int_e \{\varepsilon(\mathbf{u})\}[\![\mathbf{v}]\!] ds \tag{3.17}$$

and

$$b(\mathbf{v}, q) = (\nabla \cdot \mathbf{v}, q)$$

on $V(h, s) \times V(h, s)$ and $V(h, s) \times L_0^2(\Omega)$, respectively. Notice that $d_\beta(\cdot, \cdot)$ is symmetric for $\beta = 1$. To summarize, our variational form is given by seeking $\mathbf{u} \in V(h, s)$ and $p \in L_0^2(\Omega)$ such that

$$d_\beta(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \quad \forall\ \mathbf{v} \in V(h, s), \tag{3.18}$$

$$b(\mathbf{u}, q) = 0, \qquad \forall\ q \in L_0^2(\Omega). \tag{3.19}$$

With the conditions specified in this paper, it can be proved that the standard weak solution $(\mathbf{u}; p)$ of the Navier-Stokes problem belongs to $V(h, s)$ for some $s > \frac{3}{2}$; readers are referred to [15,16,21,25] for details on solution regularity. Therefore, the variational problem (3.18) and (3.19) has at least one solution and this solution also satisfies the Navier-Stokes equations. It should be possible to show that all solutions of (3.18) and (3.19) also satisfy the Navier-Stokes equations in a weak sense. Details are left to readers for verification.

4. Finite Element Approximations and Their Properties

The solution of (3.18) and (3.19), hence the solution of the Navier-Stokes equations (1.1)-(1.3), can be approximated by restricting $V(h, s)$ and $L_0^2(\Omega)$ to properly-defined subspaces such as the finite element spaces V_h and W_h associated with a prescribed finite element partition $\mathcal{T}_h$. The resulting approximation, denoted by $(\mathbf{u}_h; p_h) \in V_h \times W_h$, is given as solution of the following discrete equations:

$$d_\beta(\mathbf{u}_h, \mathbf{v}) + a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}), \quad \forall\ \mathbf{v} \in V_h \tag{4.1}$$

$$b(\mathbf{u}_h, q) = 0, \qquad \forall\ q \in W_h. \tag{4.2}$$

There are two main issues for the finite element scheme (4.1) and (4.2). The first one is on an efficient computation of $(\mathbf{u}_h; p_h)$, the second is about qualitative properties of the finite element approximation, such as solution existence, uniqueness, and convergence as the mesh size h tends to zero. While both issues are equally important, we would like to focus our attention on the second one in the rest of this paper.

Our mathematical analysis for the finite element scheme (4.1) and (4.2) needs two discrete norms, denoted by $|||\cdot|||_1$ and $|||\cdot|||$, in the linear space $V(h,s)$ which are defined as follows

$$|||\mathbf{v}|||_1^2 = |\mathbf{v}|_{1,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\mathbf{v}]\!]\|_e^2, \tag{4.3}$$

$$|||\mathbf{v}|||^2 = |||\mathbf{v}|||_1^2 + \sum_{e\in\mathcal{E}_h} h_e\|\{\varepsilon(\mathbf{v})\}\|_e^2, \tag{4.4}$$

where $|\mathbf{v}|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} |\mathbf{v}|_{1,K}^2$ and $\|\mathbf{v}\|_e = \left(\int_e |\mathbf{v}|^2 ds\right)^{\frac{1}{2}}$ is the standard norm in $L^2(e)$. It is clear that the norm $|||\cdot|||_1$ resembles the usual H^1-norm of the Sobolev space $H^1(\Omega)$ for piecewise H^1 functions.

Let K be an element with e as an edge and $p>1$ be any real number. It is well-known that there exists a constant $C=C(p)$ such that for any function $g\in H^1(K)$,

$$\|g\|_{L^p(e)}^p \le C\left(h_K^{-1}\|g\|_{L^p(K)}^2 + h_K^{p-1}\|\nabla g\|_{L^p(K)}^p\right), \tag{4.5}$$

where, and in what follows of this paper, h_K stands for the size of K. Observe that the quasi-uniformity of $\mathcal{T}_h$ implies that h_K is proportional to h_e for all the edges/faces $e\subset\partial K$. In particular, with $p=2$ one has for any $\mathbf{v}\in V_h$,

$$h_e\|\{\varepsilon(\mathbf{v})\}\|_e^2 \le C\left(\|\nabla\mathbf{v}\|_K^2 + h_K^2\|\nabla^2\mathbf{v}\|_K^2\right).$$

The standard inverse inequality can be employed to the last term of the above inequality, yielding

$$h_e\|\{\varepsilon(\mathbf{v})\}\|_e^2 \le C\|\nabla\mathbf{v}\|_K^2$$

for some constant C independent of the mesh size h. Consequently, there is a constant C independent of h such that

$$|||\mathbf{v}||| \le C_0|||\mathbf{v}|||_1, \qquad \forall\mathbf{v}\in V_h. \tag{4.6}$$

This shows that the two norms $|||\cdot|||_1$ and $|||\cdot|||$ are equivalent in the finite element space V_h.

In addition to the norms introduced in this section, our theoretical analysis for the finite element scheme (4.1) and (4.2) requires some Sobolev-type inequalities for functions which are piecewise smooth. Details are provided in the next section.

5. Sobolev-type Inequalities for Piecewise Smooth Functions

Let $D\subset\mathbb{R}^n$ be an open bounded domain in the n-dimensional space $\mathbb{R}^n, n\ge 2$, and $W^{1,p}(D)$ be the usual Sobolev space with $p\in(1,\infty)$. We recall the following trace inequality: for any $f\in W^{1,p}(D)$ and $p\in(1,n)$ one has

$$\|f\|_{L^{\tilde p}(\partial D)} \le C(D,p)\|f\|_{1,p,D}, \tag{5.1}$$

where $\tilde p = (n-1)p/(n-p)$ and $\|\cdot\|_{1,p,D}$ stands for the usual Sobolev norm in $W^{1,p}(D)$. The dependence of the constant $C(D,p)$ with respect to the size of the domain D can be explicitly estimated when applied to finite element partitions.

Lemma 5.1. *Let $\mathcal{T}_h$ be a quasi-regular finite element partition of an open bounded domain $\Omega\subset\mathbb{R}^n, n\ge 2$. Then there exists a constant $C(p)$ such that for any $K\in\mathcal{T}_h$ and $w\in W^{1,p}(K)$ with $p\in(1,n)$ and $\tilde p = \frac{(n-1)p}{n-p}$ one has*

$$\|w\|_{L^{\tilde p}(\partial K)} \le C(p)\left(h_K^{-1}\|w\|_{L^p(K)} + \|\nabla w\|_{L^p(K)}\right), \tag{5.2}$$

where h_K is the diameter of the element K.

Proof. To figure out the dependence, let $\hat{K}$ be a reference element with an affine mapping $\hat{F}$ that maps $\hat{K}$ to K:

$$\hat{F}: \ \hat{K} \to K.$$

For any function $w \in W^{1,p}(K)$, denote by $\hat{w}$ the composition of w and $\hat{F}$:

$$\hat{w}(\hat{x}) = w \circ \hat{F}(\hat{x}), \quad \hat{x} \in \hat{K}.$$

Let h_K be the diameter of K, then the usual scaling argument can be applied to yield

$$\|w\|_{L^{\tilde{p}}(\partial K)} \leq C h_K^{\frac{n-1}{\tilde{p}}} \|\hat{w}\|_{L^{\tilde{p}}(\partial \hat{K})}.$$

Applying the trace inequality (5.1) to the right-hand side of the above inequality we arrive at

$$\|w\|_{L^{\tilde{p}}(\partial K)} \leq C(\hat{K}, p) h_K^{\frac{n-1}{\tilde{p}}} \|\hat{w}\|_{1,p,\hat{K}}. \tag{5.3}$$

Now we go back to the original element K through the affine map $\hat{F}$:

$$\begin{aligned} \|\hat{w}\|_{1,p,\hat{K}}^p &= \int_{\hat{K}} |\hat{w}|^p d\hat{K} + \int_{\hat{K}} |\hat{\nabla} \hat{w}|^p d\hat{K} \\ &\leq C h_K^{-n} \int_K |w|^p dK + C h_K^{-n+p} \int_K |\nabla w|^p dK. \end{aligned} \tag{5.4}$$

Substituting (5.4) into (5.3) yields

$$\begin{aligned} \|w\|_{L^{\tilde{p}}(\partial K)} &\leq C(\hat{K}, p) h_K^{\frac{n-1}{\tilde{p}}} \left(h_K^{-n} \int_K |w|^p dK + h_K^{-n+p} \int_K |\nabla w|^p dK \right)^{1/p} \\ &= C(\hat{K}, p) \left(h_K^{-p} \int_K |w|^p dK + \int_K |\nabla w|^p dK \right)^{1/p}, \end{aligned}$$

which is the desired estimate (5.2). □

Lemma 5.2. *Let $q \in (1, \infty)$ be any real number and $\hat{q} = \frac{nq}{n+q-1}$. There exists a constant C such that for any $w \in \prod_{K \in \mathcal{T}_h} W^{1,\hat{q}}(K)$ we have*

$$\left(\sum_{e \in \mathcal{E}_h} h_e \|w\|_{L^q(e)}^q \right)^{\frac{1}{q}} \leq C \|w\|_{L^q(\Omega)} + C h^{\frac{1}{q}} \left(\sum_{K \in \mathcal{T}_h} \|\nabla w\|_{L^{\hat{q}}(K)}^{\hat{q}} \right)^{\frac{1}{\hat{q}}}. \tag{5.5}$$

Proof. Since $\hat{q} \equiv \frac{nq}{n+q-1}$, it is not hard to verify that for $n > 1$ one has

$$1 < \hat{q} < n, \quad q = \frac{(n-1)\hat{q}}{n - \hat{q}}.$$

Thus, it follows from (5.2) (with $\tilde{p} = q$ and $p = \hat{q}$) that

$$\|w\|_{L^q(e)} \leq C \left(h_K^{-1} \|w\|_{L^{\hat{q}}(K)} + \|\nabla w\|_{L^{\hat{q}}(K)} \right). \tag{5.6}$$

Next by observing $\hat{q} \leq q$ and $n/\hat{q} - n/q = (q-1)/q$ we have from the Hölder inequality

$$\begin{aligned} \|w\|_{L^{\hat{q}}(K)} &\leq C h_K^{\frac{n}{\hat{q}} - \frac{n}{q}} \|w\|_{L^q(K)} \\ &= C h_K^{\frac{q-1}{q}} \|w\|_{L^q(K)}. \end{aligned}$$

Substituting the above into (5.6) yields

$$\|w\|_{L^q(e)} \le C\left(h_K^{-\frac{1}{q}}\|w\|_{L^q(K)} + \|\nabla w\|_{L^{\hat{q}}(K)}\right).$$

It follows that

$$h_e\,\|w\|_{L^q(e)}^q \le C\left(\|w\|_{L^q(K)}^q + h_K\|\nabla w\|_{L^{\hat{q}}(K)}^q\right).$$

Summing over all the edges leads to

$$\sum_{e\in\mathcal{E}_h} h_e\,\|w\|_{L^q(e)}^q \le C\|w\|_{L^q(\Omega)}^q + C\sum_{K\in\mathcal{T}_h} h_K\|\nabla w\|_{L^{\hat{q}}(K)}^q. \tag{5.7}$$

Note that the following inequality holds true

$$\sum_{j=1}^m |a_j|^\lambda \le \left(\sum_{j=1}^m |a_j|\right)^\lambda$$

for any $\lambda > 1$. Thus,

$$\sum_{K\in\mathcal{T}_h} h_K\|\nabla w\|_{L^{\hat{q}}(K)}^q \le \left(\sum_{K\in\mathcal{T}_h} h_K^{\frac{\hat{q}}{q}}\|\nabla w\|_{L^{\hat{q}}(K)}^{\hat{q}}\right)^{\frac{q}{\hat{q}}}.$$

Substituting the last inequality into (5.7) gives

$$\sum_{e\in\mathcal{E}_h} h_e\,\|w\|_{L^q(e)}^q \le C\|w\|_{L^q(\Omega)}^q + C\left(\sum_{K\in\mathcal{T}_h} h_K^{\frac{\hat{q}}{q}}\|\nabla w\|_{L^{\hat{q}}(K)}^{\hat{q}}\right)^{\frac{q}{\hat{q}}}, \tag{5.8}$$

which implies the desired inequality (5.5). □

Let $p > 1$ and $s > 1$ be two real numbers, q and t be the conjugate of p and s respectively (i.e., $1/p + 1/q = 1$ and $1/s + 1/t = 1$). Introduce the following space:

$$J(s,p;\Omega) := L^s(\Omega) \cap \left(\prod_{K\in\mathcal{T}_h} W^{1,p}(K)\right).$$

The following lemma provides an estimate for the L^s-norm of functions in $J(s,p;\Omega)$:

Lemma 5.3. *Let $p > 1$ and $s > 1$ be two real numbers, q and t be the conjugate of p and s respectively. Assume that $n/(n-1) < s \le np/(n-p)$. Then for any function $w \in J(s,p;\Omega)$, the following estimate holds true*

$$\begin{aligned}\|w\|_{L^s(\Omega)} \le C(\epsilon)&\left(\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^p(K)}^p + \sum_{e\in\mathcal{E}_h} h_e^{1-p}\int_e |[\![w]\!]|^p de\right)^{\frac{1}{p}}\\ &+\epsilon h^{\frac{1}{q(s-1)}}\,\|w\|_{L^{\hat{q}(s-1)}(\Omega)},\end{aligned} \tag{5.9}$$

where $\epsilon > 0$ is an arbitrary, but positive real number and $\hat{q} = nq/(n+q-1)$. In particular, if s additionally satisfies $s \le np/(n-1)$, then one has

$$\|w\|_{L^s(\Omega)} \le C\left(\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^p(K)}^p + \sum_{e\in\mathcal{E}_h} h_e^{1-p}\int_e |[\![w]\!]|^p de\right)^{\frac{1}{p}}. \tag{5.10}$$

Proof. For any $w \in J(s,p;\Omega)$, we consider the following auxiliary problem: Find $g \in W^{2,t}(\Omega)$ such that $g = 0$ on the boundary $\partial\Omega$ of Ω and

$$-\Delta g = \mathrm{sgn}(w)|w|^{s-1} \qquad \text{in} \;\; \Omega, \tag{5.11}$$

where $\mathrm{sgn}(\cdot)$ is the sign function with values $1, 0, -1$ when the argument is positive, vanishing, or negative, as appropriate. Without loss of generality (if necessary, one may consider the same problem on a domain covering Ω with w being extended by zero outside of Ω), we may assume that Ω has a very smooth boundary so that the problem (5.11) has a unique solution with the following a priori estimates:

$$\|g\|_{2,t} \le C\|w\|_{L^s(\Omega)}^{s-1}, \tag{5.12}$$

and

$$\|g\|_{2,\hat{q}} \le C\|w\|_{L^{\hat{q}(s-1)}(\Omega)}^{s-1}, \tag{5.13}$$

where $\|w\|_{L^{\hat{q}(s-1)}(\Omega)}$ formally stands for the "norm" of w even though $\hat{q}(s-1) \ge 1$ may fail to be true.

Now, multiplying (5.11) by w and then integrating over the domain Ω gives

$$\begin{aligned}
\|w\|_{L^s(\Omega)}^s &= \int_\Omega (-\Delta g) w d\Omega = \sum_{K\in\mathcal{T}_h} \int_K (-\Delta g) w dK \\
&= \sum_{K\in\mathcal{T}_h} \left(\int_K \nabla g \cdot \nabla w dK - \int_{\partial K} \frac{\partial g}{\partial n} w \right) \\
&\le \left| \sum_{K\in\mathcal{T}_h} \int_K \nabla g \cdot \nabla w dK \right| + \left| \sum_{e\in\mathcal{E}_h} \int_e \frac{\partial g}{\partial n} [\![w]\!] de \right|,
\end{aligned} \tag{5.14}$$

where we have used the Green's formula in the second line. The first term on the right-hand side of (5.14) can be estimated by using the Hölder's inequality:

$$\begin{aligned}
\left| \sum_{K\in\mathcal{T}_h} \int_K \nabla g \cdot \nabla w dK \right| &\le \sum_{K\in\mathcal{T}_h} \|\nabla g\|_{L^q(K)} \|\nabla w\|_{L^p(K)} \\
&\le \|\nabla g\|_{L^q(\Omega)} \left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{L^p(K)}^p \right)^{\frac{1}{p}}.
\end{aligned} \tag{5.15}$$

To deal with the second term of (5.14), we use the Hölder's inequality to arrive at

$$\begin{aligned}
\left| \sum_{e\in\mathcal{E}_h} \int_e \frac{\partial g}{\partial n} [\![w]\!] de \right| &\le \sum_{e\in\mathcal{E}_h} \left\| \frac{\partial g}{\partial n} \right\|_{L^q(e)} \|[\![w]\!]\|_{L^p(e)} \\
&\le \left(\sum_{e\in\mathcal{E}_h} h_e \left\| \frac{\partial g}{\partial n} \right\|_{L^q(e)}^q \right)^{\frac{1}{q}} \left(\sum_{e\in\mathcal{E}_h} h_e^{1-p} \|[\![w]\!]\|_{L^p(e)}^p \right)^{\frac{1}{p}}.
\end{aligned} \tag{5.16}$$

Applying the estimate (5.5) to the first factor of (5.16) we obtain

$$\left| \sum_{e\in\mathcal{E}_h} \int_e \frac{\partial g}{\partial n} [\![w]\!] de \right| \le C \left[\|\nabla g\|_{L^q(\Omega)} + h^{\frac{1}{q}} \|\nabla^2 g\|_{L^{\hat{q}}(\Omega)} \right] \left[\sum_{e\in\mathcal{E}_h} h_e^{1-p} \|[\![w]\!]\|_{L^p(e)}^p \right]^{\frac{1}{p}}. \tag{5.17}$$

Now substituting (5.17) and (5.15) into (5.14) yields

$$\begin{aligned}
\|w\|_{L^s(\Omega)}^s \le\ & \|\nabla g\|_{L^q(\Omega)} \left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{L^p(K)}^p\right)^{\frac{1}{p}} \\
&+C\left(\|\nabla g\|_{L^q(\Omega)} + h^{\frac{1}{q}}\left\|\nabla^2 g\right\|_{L^{\hat{q}}(\Omega)}\right)\left[\sum_{e\in\mathcal{E}_h} h_e^{1-p}\int_e |[\![w]\!]|^p de\right]^{\frac{1}{p}} \\
\le\ & C\left\|\nabla g\right\|_{L^q(\Omega)}\left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{L^p(K)}^p + \sum_{e\in\mathcal{E}_h} h_e^{1-p}\int_e |[\![w]\!]|^p de\right)^{\frac{1}{p}} \\
&+Ch^{\frac{1}{q}}\left\|\nabla^2 g\right\|_{L^{\hat{q}}(\Omega)}\left[\sum_{e\in\mathcal{E}_h} h_e^{1-p}\int_e |[\![w]\!]|^p de\right]^{\frac{1}{p}}.
\end{aligned} \tag{5.18}$$

The condition $\frac{n}{n-1} < s \le \frac{np}{n-p}$ implies the following:

$$t = \frac{s}{s-1} < n, \quad q \le \frac{nt}{n-t}.$$

Thus, we have from the usual Sobolev embedding theorem that

$$\|\nabla g\|_{L^q(\Omega)} \le C\|\nabla g\|_{W^{1,t}(\Omega)},$$

which, together with (5.12), yields

$$\|\nabla g\|_{L^q(\Omega)} \le C\|w\|_{L^s(\Omega)}^{s-1}.$$

Substituting the above estimate and (5.13) into (5.18) gives

$$\begin{aligned}
\|w\|_{L^s(\Omega)}^s \le\ & C\left\|w\right\|_{L^s(\Omega)}^{s-1}\left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{L^p(K)}^p + \sum_{e\in\mathcal{E}_h} h_e^{1-p}\int_e |[\![w]\!]|^p de\right)^{\frac{1}{p}} \\
&+Ch^{\frac{1}{q}}\left\|w\right\|_{L^{\hat{q}(s-1)}(\Omega)}^{s-1}\left[\sum_{e\in\mathcal{E}_h} h_e^{1-p}\int_e |[\![w]\!]|^p de\right]^{\frac{1}{p}}.
\end{aligned} \tag{5.19}$$

The estimate (5.19) further leads to

$$\begin{aligned}
\|w\|_{L^s(\Omega)}^s \le\ & C\left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{L^p(K)}^p + \sum_{e\in\mathcal{E}_h} h_e^{1-p}\int_e |[\![w]\!]|^p de\right)^{\frac{s}{p}} \\
&+Ch^{\frac{1}{q}}\left\|w\right\|_{L^{\hat{q}(s-1)}(\Omega)}^{s-1}\left[\sum_{e\in\mathcal{E}_h} h_e^{1-p}\int_e |[\![w]\!]|^p de\right]^{\frac{1}{p}}.
\end{aligned} \tag{5.20}$$

Note that for any $a, b \ge 0$ and positive m and n with $m^{-1} + n^{-1} = 1$ one has

$$ab \le a^m + b^n.$$

Applying the last inequality to the last term of (5.20) with $a = \epsilon^{s-1} h^{\frac{1}{q}} \|w\|_{L^{\hat{q}(s-1)}(\Omega)}^{s-1}$ and $m = t \equiv s/(s-1)$ yields

$$\|w\|_{L^s(\Omega)}^s \leq C \left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{L^p(K)}^p + \sum_{e\in\mathcal{E}_h} h_e^{1-p} \int_e |[\![w]\!]|^p de \right)^{\frac{s}{p}}$$
$$+\epsilon^s h^{\frac{t}{q}} \|w\|_{L^{\hat{q}(s-1)}(\Omega)}^s + C\epsilon^{s(1-s)} \left[\sum_{e\in\mathcal{E}_h} h_e^{1-p} \int_e |[\![w]\!]|^p de \right]^{\frac{s}{p}}, \qquad (5.21)$$

where $\epsilon > 0$ is an arbitrary, but positive real number. The inequality (5.21) implies the desired estimate (5.9).

As to the inequality (5.10), we observe that for $s \leq \frac{np}{n-1}$ one has

$$\hat{q}(s-1) \leq s.$$

It follows that

$$\|w\|_{L^{\hat{q}(s-1)}(\Omega)} \leq C \|w\|_{L^s(\Omega)}.$$

Substituting the last inequality into (5.9) yields

$$\left(1 - C\epsilon h^{\frac{1}{q(s-1)}}\right) \|w\|_{L^s(\Omega)}$$
$$\leq C \left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{L^p(K)}^p + \sum_{e\in\mathcal{E}_h} h_e^{1-p} \int_e |[\![w]\!]|^p de \right)^{\frac{1}{p}} + C\epsilon^{(1-s)} \left[\sum_{e\in\mathcal{E}_h} h_e^{1-p} \int_e |[\![w]\!]|^p de \right]^{\frac{1}{p}}.$$

By choosing ϵ sufficiently small such that $\left(1 - C\epsilon h^{t/s}\right) \geq \frac{1}{2}$ one obtains the desired estimate (5.10). $\square$

The next two corollaries emphasize the case $p = 2$ for the estimates established in Lemmas 5.3 and 5.2.

Corollary 5.1. *Let n be the dimension of the domain Ω and $\hat{q} = 2n/(n+1)$. Then the following results hold true.*

1. *For any real number s satisfying $\frac{n}{n-1} < s \leq \frac{2n}{n-1}$, there is a constant C such that for any function $w \in J(s, 2; \Omega)$ one has*

$$\|w\|_{L^s(\Omega)} \leq C(\epsilon) \left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1} \int_e |[\![w]\!]|^2 de \right)^{\frac{1}{2}}$$
$$+\epsilon h^{\frac{1}{2(s-1)}} \|w\|_{L^{\hat{q}(s-1)}(\Omega)}, \qquad (5.22)$$

   where $\epsilon > 0$ is an arbitrary, but positive real number.

2. *For any real number s satisfying $\frac{n}{n-1} < s \leq \frac{2n}{n-1}$, there is a constant C such that for any function $w \in J(s, 2; \Omega)$ one has*

$$\|w\|_{L^s(\Omega)} \leq C \left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1} \int_e |[\![w]\!]|^2 de \right)^{\frac{1}{2}}. \qquad (5.23)$$

Proof. The estimate (5.22) is a direct application of (5.9) when $p = 2$. Similarly, the estimate (5.23) stems from (5.10) since the condition $s \le np/(n-1)$ is satisfied when $p = 2$. □

Corollary 5.2. *Let n be the dimension of the domain Ω, $\hat{q} = 2n/(n+1)$, and s be a real number such that*

$$\frac{n}{n-1} < s \le \frac{2n}{n-2}. \tag{5.24}$$

Let $\hat{s} = ns/(n+s-1)$. Then the following results hold true.

1. *There is a constant C such that for any function $w \in J(s,2;\Omega)$:*

$$\begin{aligned}\left(\sum_{e\in\mathcal{E}_h} h_e \|w\|_{L^s(e)}^s\right)^{\frac{1}{s}} \le C(\epsilon)\left(\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e |[\![w]\!]|^2 de\right)^{\frac{1}{2}} \\ +\epsilon h^{\frac{1}{2(s-1)}}\|w\|_{L^{\hat{q}(s-1)}(\Omega)} + Ch^{\frac{1}{s}}\left(\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^{\hat{s}}(K)}^{\hat{s}}\right)^{\frac{1}{\hat{s}}},\end{aligned} \tag{5.25}$$

   where $\epsilon > 0$ is an arbitrary, but positive real number.

2. *If, in addition, s satisfies $s \le 2n/(n-1)$, then there exists a constant C such that*

$$\left[\sum_{e\in\mathcal{E}_h} h_e \|w\|_{L^s(e)}^s\right]^{\frac{1}{s}} \le C\left[\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e |[\![w]\!]|^2 de\right]^{\frac{1}{2}}. \tag{5.26}$$

Proof. With $q = s$ and $\hat{q} = \hat{s}$ we have from (5.5) of Lemma 5.2 that

$$\left(\sum_{e\in\mathcal{E}_h} h_e \|w\|_{L^s(e)}^s\right)^{\frac{1}{s}} \le C\|w\|_{L^s(\Omega)} + Ch^{\frac{1}{s}}\left(\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^{\hat{s}}(K)}^{\hat{s}}\right)^{\frac{1}{\hat{s}}}. \tag{5.27}$$

The first term on the right-hand side of (5.27) can be estimated by using (5.22). The combined inequality is exactly the desired estimate (5.25).

To derive (5.26), we use (5.23) to bound the first term of the right-hand side of (5.27) (note that the condition for this application is satisfied!), yielding

$$\begin{aligned}\left(\sum_{e\in\mathcal{E}_h} h_e \|w\|_{L^s(e)}^s\right)^{\frac{1}{s}} \le C\left(\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e |[\![w]\!]|^2 de\right)^{\frac{1}{2}} \\ +Ch^{\frac{1}{s}}\left(\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^{\hat{s}}(K)}^{\hat{s}}\right)^{\frac{1}{\hat{s}}}.\end{aligned} \tag{5.28}$$

The additional condition of $s \le 2n/(n-1)$ implies that $\hat{s} \le 2$. Thus, we have from the Hölder's inequality that

$$\left(\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^{\hat{s}}(K)}^{\hat{s}}\right)^{\frac{1}{\hat{s}}} \le C\left(\sum_{K\in\mathcal{T}_h}\|\nabla w\|_{L^2(K)}^2\right)^{\frac{1}{2}}.$$

Substituting the above inequality into (5.28) gives the desired inequality (5.26). □

We end this section by establishing some estimates for functions in the finite element space V_h.

Corollary 5.3. *Let n be the dimension of the domain Ω and s be an arbitrary real number satisfying $\frac{n}{n-1} < s \leq \frac{2n}{n-2}$. Then there exists a constant C such that for any finite element function $w \in V_h$ one has*

$$\|w\|_{L^s(\Omega)} \leq C\left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e |[\![w]\!]|^2 de\right)^{\frac{1}{2}}, \tag{5.29}$$

$$\left(\sum_{e\in\mathcal{E}_h} h_e \|w\|^s_{L^s(e)}\right)^{\frac{1}{s}} \leq C\left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e |[\![w]\!]|^2 de\right)^{\frac{1}{2}}. \tag{5.30}$$

Note that the right-hand sides of (5.29) *and* (5.30) *are the discrete H^1 norm $|\!|\!|w|\!|\!|_1$ as defined in* (4.3).

Proof. Corollary 5.3 should be viewed as an application of the Corollaries 5.1 and 5.2 to finite element functions. For simplicity, we adopt the notations used in Corollaries 5.1 and 5.2 without redefining them.

The estimate (5.22) holds true when $w \in V_h$ is a finite element function. Without loss of generality, we consider only the case $\hat{q}(s-1) \geq s$. Note that the condition $s \leq 2n/(n-2)$ implies $2s + 2n - ns \geq 0$. Using the standard inverse inequality one obtains

$$\begin{aligned} h^{\frac{1}{2(s-1)}}\|w\|_{L^{\hat{q}(s-1)}(\Omega)} &\leq Ch^{\left(\frac{1}{2(s-1)}+\frac{n}{\hat{q}(s-1)}-\frac{n}{s}\right)}\|w\|_{L^s(\Omega)} \\ &= Ch^{\frac{2s+2n-ns}{2s(s-1)}}\|w\|_{L^s(\Omega)} \leq C\|w\|_{L^s(\Omega)}. \end{aligned} \tag{5.31}$$

Now substituting the above into (5.22) yields

$$\|w\|_{L^s(\Omega)} \leq C(\epsilon)\left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e |[\![w]\!]|^2 de\right)^{\frac{1}{2}} + \epsilon C\,\|w\|_{L^s(\Omega)}.$$

It follows that (5.29) holds true for appropriately chosen value of ϵ.

To derive (5.30), we observe that the estimate (5.25) is valid for $w \in V_h$. It suffices to treat the last two terms on the right-hand side of (5.25). The $L^{\hat{q}(s-1)}$-norm of w can be estimated by using (5.31). The other term, which is

$$h^{\frac{1}{s}}\|\nabla_h w\|_{L^{\hat{s}}(\Omega)} := h^{\frac{1}{s}}\left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|^{\hat{s}}_{L^{\hat{s}}(K)}\right)^{\frac{1}{\hat{s}}},$$

can be estimated by using the standard inverse inequality as follows:

$$\begin{aligned} h^{\frac{1}{s}}\|\nabla_h w\|_{L^{\hat{s}}(\Omega)} &\leq Ch^{\frac{1}{s}+\frac{n}{\hat{s}}-\frac{n}{2}}\|\nabla_h w\|_{L^2(\Omega)} \\ &= Ch^{\frac{2s+2n-ns}{2s}}\|\nabla_h w\|_{L^2(\Omega)} \leq C\|\nabla_h w\|_{L^2(\Omega)}. \end{aligned}$$

Substituting the above and the estimate (5.31) into (5.25) yields

$$\begin{aligned} \left(\sum_{e\in\mathcal{E}_h} h_e \|w\|^s_{L^s(e)}\right)^{\frac{1}{s}} &\leq C(\epsilon)\left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e |[\![w]\!]|^2 de\right)^{\frac{1}{2}} \\ &\quad + \epsilon C\,\|w\|_{L^s(\Omega)} + C\left(\sum_{K\in\mathcal{T}_h} \|\nabla w\|^2_{L^2(K)}\right)^{\frac{1}{2}}, \end{aligned}$$

which, together with (5.29), verifies the desired inequality (5.30). □

6. Theory for the Finite Element Method

6.1. Coercivity and boundedness

The goal of this subsection is to establish some properties for the bilinear form $d_\beta(\cdot,\cdot)$ and the quasi-trilinear form $a_1(\cdot,\cdot,\cdot)$ which were used in the discrete equation (4.1). Notice that the quasi-trilinear form $a_1(\cdot,\cdot,\cdot)$ disappears for Stokes problems, and the properties for the bilinear form $d_\beta(\cdot,\cdot)$ has been well studied in [29]. Thus, our attention shall be focused on the quasi-trilinear form $a_1(\cdot,\cdot,\cdot)$.

Recall that the bilinear form $d_\beta(\cdot,\cdot)$ was defined by (3.17) and (3.15), and the following ellipticity has been established in [29].

Lemma 6.1. *Let $d_\beta(\cdot,\cdot)$ be defined as in* (3.17)*. Then the following results hold true, regardless of the dimension $n > 1$ for the domain Ω.*

1. *For the symmetric scheme $\beta = 1$, there exists a constant α_0 independent of h such that for any $\mathbf{v} \in V_h$*
$$d_\beta(\mathbf{v},\mathbf{v}) \geq \nu\alpha_0 |\!|\!|\mathbf{v}|\!|\!|^2, \tag{6.1}$$
provided that the stabilization parameter α in $S_\beta(\cdot,\cdot)$ is sufficiently large.

2. *For the non-symmetric case $\beta = -1$, it is easy to see that for any finite element function $\mathbf{v} \in V_h$ one has*
$$\begin{aligned} d_\beta(\mathbf{v},\mathbf{v}) &= \nu(\nabla_h \mathbf{v}, \nabla_h \mathbf{v}) + \nu\alpha \sum_{e\in\mathcal{E}_h} h_e^{-1} \int_e [\![\mathbf{v}]\!]^2 ds \\ &\geq \nu \ \min(1,\alpha) |\!|\!|\mathbf{v}|\!|\!|_1^2 \geq \nu \ \min(1,\alpha)/C_0 |\!|\!|\mathbf{v}|\!|\!|^2, \end{aligned}$$
where the relation (4.6) *has been employed in the last inequality. Thus, the coercivity* (6.1) *holds true for the bilinear form $d_\beta(\cdot,\cdot)$ with any positive value of α when $\beta = -1$.*

Since the symmetric case is conditionally coercive, we shall assume that the parameter α is chosen appropriately so that the ellipticity (6.1) holds true. The advantage for the symmetric scheme is that the resulting matrix from the bilinear form $d_\beta(\cdot,\cdot)$ is symmetric and positive definite, and hence there are more tools available in solution techniques than non-symmetric forms.

As to the quasi-trilinear form $a_1(\mathbf{u},\mathbf{v},\mathbf{w})$, we recall that it is defined by (3.13) and (3.6) as follows:

$$\begin{aligned} a_1(\mathbf{u},\mathbf{v},\mathbf{w}) := a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w}) &+ \frac{1}{2}\sum_{e\in\mathcal{E}_h^0}\int_e \mathbf{u}\cdot\mathbf{n}(\mathbf{v}_R\cdot\mathbf{w}_L - \mathbf{v}_L\cdot\mathbf{w}_R)ds \\ &+\gamma\sum_{e\in\mathcal{E}_h^0}\int_e |\mathbf{u}\cdot\mathbf{n}|[\![\mathbf{v}]\!][\![\mathbf{w}]\!]ds, \end{aligned} \tag{6.2}$$

where

$$a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}\left(\sum_{K\in\mathcal{T}_h}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})_K - \sum_{K\in\mathcal{T}_h}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})_K\right). \tag{6.3}$$

For simplicity, we introduce the following notation

$$\begin{aligned} b_1(\mathbf{u},\mathbf{v},\mathbf{w}) &:= \frac{1}{2}\sum_{e\in\mathcal{E}_h^0}\int_e \mathbf{u}\cdot\mathbf{n}(\mathbf{v}_R\cdot\mathbf{w}_L-\mathbf{v}_L\cdot\mathbf{w}_R)ds,\\ b_2(\mathbf{u},\mathbf{v},\mathbf{w}) &:= \gamma\sum_{e\in\mathcal{E}_h^0}\int_e |\mathbf{u}\cdot\mathbf{n}|[\![\mathbf{v}]\!][\![\mathbf{w}]\!]ds. \end{aligned} \tag{6.4}$$

It is clear from (6.2) that

$$a_1(\mathbf{u},\mathbf{v},\mathbf{w})) = a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w}) + b_1(\mathbf{u},\mathbf{v},\mathbf{w}) + b_2(\mathbf{u},\mathbf{v},\mathbf{w}). \tag{6.5}$$

It is also obvious that the trilinear form b_1 can be rewritten as follows

$$b_1(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}\sum_{e\in\mathcal{E}_h^0}\int_e \mathbf{u}\cdot\mathbf{n}(\mathbf{v}_R\cdot[\![\mathbf{w}]\!]-[\![\mathbf{v}]\!]\cdot\mathbf{w}_R)ds. \tag{6.6}$$

Lemma 6.2. *Let $a_1(\cdot,\cdot,\cdot)$ be defined by* (6.5), (6.6), *and* (6.4). *Regardless of the dimension $n>1$ for the domain Ω, for any $\mathbf{u}$ and $\mathbf{v}\in V_h$ we have*

$$a_1(\mathbf{u},\mathbf{v},\mathbf{v}) = \gamma\sum_{e\in\mathcal{E}_h^0}\int_e |\mathbf{u}\cdot\mathbf{n}|[\![\mathbf{v}]\!]^2 ds. \tag{6.7}$$

The rest of this subsection is devoted to a discussion of boundedness for the bilinear form $d_\beta(\cdot,\cdot)$ and the quasi-trilinear form $a_1(\cdot,\cdot,\cdot)$ in the finite element spaces under consideration. First of all, we recall the following boundedness result [29] for the bilinear form $d_\beta(\cdot,\cdot)$ in the linear space $V(h,s)$ as defined in 3.16).

Lemma 6.3. *Regardless of the dimension $n>1$ for the domain Ω, there exists a constant C independent of h such that*

$$|d_\beta(\mathbf{w},\mathbf{v})| \le C\nu|\!|\!|\mathbf{w}|\!|\!||\!|\!|\mathbf{v}|\!|\!|, \quad \forall \mathbf{w},\mathbf{v}\in V(h,s). \tag{6.8}$$

As to the boundedness of the quasi-trilinear form $a_1(\cdot,\cdot,\cdot)$, it suffices to establish some results for each component in the decomposition (6.5). Our first result along this line concerns the trilinear form $a_{sk}(\cdot,\cdot,\cdot)$.

Lemma 6.4. *Let the trilinear form $a_{sk}(\cdot,\cdot,\cdot)$ be given as in* (6.3). *Assume that the dimension n for the domain Ω is no more than 4 (i.e., $n\le 4$), then there exists a constant C such that for any $\mathbf{u},\mathbf{v},\mathbf{w}\in V_h$ we have*

$$|a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w})| \le C|\!|\!|\mathbf{u}|\!|\!|_1\ |\!|\!|\mathbf{v}|\!|\!|_1\ |\!|\!|\mathbf{w}|\!|\!|_1. \tag{6.9}$$

In particular, the same estimate (6.9) *holds true for any $\mathbf{u},\mathbf{v},\mathbf{w}\in V(h,s)$ when $n=2$.*

Proof. It follows from (6.3) that

$$\begin{aligned} |a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w})| &\le \frac{1}{2}\sum_{K\in\mathcal{T}_h}\|\mathbf{u}\|_{L^4(K)}\left(\|\mathbf{w}\|_{L^4(K)}\|\nabla\mathbf{v}\|_{L^2(K)}+\|\mathbf{v}\|_{L^4(K)}\|\nabla\mathbf{w}\|_{L^2(K)}\right)\\ &\le \frac{1}{2}\|\mathbf{u}\|_{L^4(\Omega)}\left(\|\mathbf{w}\|_{L^4(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)}+\|\mathbf{v}\|_{L^4(\Omega)}\|\nabla\mathbf{w}\|_{L^2(\Omega)}\right). \end{aligned}$$

Using the inequality (5.29) in Corollary 5.3 with $s = 4$ (note that all the conditions are satisfied with $s = 4$) we obtain

$$\|\mathbf{w}\|_{L^4(\Omega)} \le C|\!|\!|\mathbf{w}|\!|\!|_1 \tag{6.10}$$

for any $\mathbf{w} \in V_h$. Thus, there exists a constant C such that

$$|a_{sk}(\mathbf{u},\mathbf{v},\mathbf{w})| \le C|\!|\!|\mathbf{u}|\!|\!|_1 |\!|\!|\mathbf{v}|\!|\!|_1 |\!|\!|\mathbf{w}|\!|\!|_1$$

for any $\mathbf{u},\mathbf{v},\mathbf{w} \in V_h$. This completes the proof of the lemma.

In the special case of $n = 2$, one may use the inequality (5.23) to arrive at the estimate (6.10) for any $\mathbf{w} \in V(h,s)$. This shows that (6.9) holds true when the functional arguments vary in $V(h,s)$. $\square$

Lemma 6.5. *Let the trilinear form $b_1(\cdot,\cdot,\cdot)$ be given as in (6.6). Assume that the dimension n for the domain Ω is no more than 4 (i.e., $n \le 4$), then there exists a constant C such that for any $\mathbf{u},\mathbf{v},\mathbf{w} \in V_h$ we have*

$$|b_1(\mathbf{u},\mathbf{v},\mathbf{w})| \le C|\!|\!|\mathbf{u}|\!|\!|_1\ |\!|\!|\mathbf{v}|\!|\!|_1\ |\!|\!|\mathbf{w}|\!|\!|_1. \tag{6.11}$$

In particular, the same estimate (6.11) holds true for any $\mathbf{u},\mathbf{v},\mathbf{w} \in V(h,s)$ when $n = 2$.

Proof. The trilinear form $b_1(\cdot,\cdot,\cdot)$ contains two parts in its definition, but it suffices to deal with the first one as the second one can be handled similarly. To this end, we observe that

$$\begin{aligned}\left|\sum_{e\in\mathcal{E}_h^0}\int_e \mathbf{u}\cdot\mathbf{n}(\mathbf{v}_R\cdot[\![\mathbf{w}]\!])de\right| &\le \sum_{e\in\mathcal{E}_h^0}\|\mathbf{u}_R\|_{L^4(e)}\|\mathbf{v}_R\|_{L^4(e)}\|[\![\mathbf{w}]\!]\|_{L^2(e)}\\ &\le \left(\sum_{e\in\mathcal{E}_h^0} h_e\|\mathbf{u}_R\|^4_{L^4(e)}\right)^{\frac14}\left(\sum_{e\in\mathcal{E}_h^0} h_e\|\mathbf{v}_R\|^4_{L^4(e)}\right)^{\frac14}\left(\sum_{e\in\mathcal{E}_h^0} h_e^{-1}[\![\mathbf{w}]\!]^2_{L^2(e)}\right)^{\frac12},\end{aligned} \tag{6.12}$$

which, together with the estimate (5.30) when applied to both $\mathbf{u}$ and $\mathbf{v}$ with $s = 4$, leads to

$$\left|\sum_{e\in\mathcal{E}_h^0}\int_e \mathbf{u}\cdot\mathbf{n}(\mathbf{v}_R\cdot[\![\mathbf{w}]\!])de\right| \le C|\!|\!|\mathbf{u}|\!|\!|_1\ |\!|\!|\mathbf{v}|\!|\!|_1\ |\!|\!|\mathbf{w}|\!|\!|_1.$$

The inequality (5.26) can be used to deal with the case of $n = 2$ when the functional arguments are no longer finite element functions. This completes the proof of the lemma. $\square$

The same argument as in the proof of Lemma 6.5 can be applied to estimate the quasi-trilinear form $b_2(\cdot,\cdot,\cdot)$, yielding a boundedness result stated as follows.

Lemma 6.6. *Let the quasi-trilinear form $b_2(\cdot,\cdot,\cdot)$ be given as in (6.4). Assume that the dimension n for the domain Ω is no more than 4 (i.e., $n \le 4$), then there exists a constant C such that for any $\mathbf{u},\mathbf{v},\mathbf{w} \in V_h$ we have*

$$|b_2(\mathbf{u},\mathbf{v},\mathbf{w})| \le C|\!|\!|\mathbf{u}|\!|\!|_1\ |\!|\!|\mathbf{v}|\!|\!|_1\ |\!|\!|\mathbf{w}|\!|\!|_1. \tag{6.13}$$

In particular, the same estimate (6.13) holds true for any $\mathbf{u},\mathbf{v},\mathbf{w} \in V(h,s)$ when $n = 2$.

To summarize, we have proved the following boundedness result for the quasi-trilinear form $a_1(\cdot,\cdot,\cdot)$.

Lemma 6.7. *Let the quasi-trilinear form $a_1(\cdot,\cdot,\cdot)$ be given as in (6.2). Assume that the dimension n for the domain Ω is no more than 4 (i.e., $n \le 4$). Then, there exists a constant C independent of h such that*

$$|a_1(\mathbf{u},\mathbf{v},\mathbf{w})| \le C|\!|\!|\mathbf{u}|\!|\!|_1\ |\!|\!|\mathbf{v}|\!|\!|_1\ |\!|\!|\mathbf{w}|\!|\!|_1 \tag{6.14}$$

for all $\mathbf{u},\mathbf{w},\mathbf{v} \in V_h$. In particular, the same estimate (6.14) holds true for any $\mathbf{u},\mathbf{v},\mathbf{w} \in V(h,s)$ when $n = 2$.

Readers are referred to [20, 24] for boundedness results for related trilinear forms. We emphasize that, due to the absolute value in the form $b_2(\cdot,\cdot,\cdot)$, the form $a_1(\cdot,\cdot,\cdot)$ is not a trilinear form. However, the definition of $b_2(\cdot,\cdot,\cdot)$, together with the general inequality $||a|-|b|| \le |a-b|$ for any real numbers a and b, implies the following inequality:

$$|b_2(\mathbf{u},\mathbf{v},\mathbf{w}) - b_2(\bar{\mathbf{u}},\mathbf{v},\mathbf{w})| \le \gamma \sum_{e\in\mathcal{E}_h^0} \int_e |(\mathbf{u}-\bar{\mathbf{u}})\cdot\mathbf{n}||[\![\mathbf{v}]\!]||[\![\mathbf{w}]\!]|ds.$$

It follows that there is a constant $\tilde{\mathcal{N}}$ independent of h such that

$$|a_1(\mathbf{u},\mathbf{v},\mathbf{w}) - a_1(\bar{\mathbf{u}},\mathbf{v},\mathbf{w})| \le \tilde{\mathcal{N}}|\!|\!|\mathbf{u}-\bar{\mathbf{u}}|\!|\!|_1\ |\!|\!|\mathbf{v}|\!|\!|_1\ |\!|\!|\mathbf{w}|\!|\!|_1 \tag{6.15}$$

for all $\mathbf{u},\bar{\mathbf{u}},\mathbf{v},\mathbf{w} \in V_h$. The estimate (6.15) holds true for any $\mathbf{u},\mathbf{v},\mathbf{w} \in V(h,s)$ when $n = 2$.

6.2. Existence of finite element solutions

The Leray-Schauder fixed point theorem can be employed to justify solution existence for the finite element scheme given by (4.1) and (4.2). To this end, we recall that the Leray-Schauder fixed point theorem states that if a compact map F defined on the closure of an open convex subset U of a normed linear space X containing the origin has the property that $F(x) \ne \lambda x$ for all $\lambda > 1$ and all x on the boundary of U, then F must have a fixed point in the closure of U.

In applying the Leray-Schauder fixed point theorem to the finite element scheme (4.1) and (4.2), we introduce a divergent free subspace D_h of V_h as follows:

$$D_h = \{\mathbf{v} \in V_h :\ \nabla\cdot\mathbf{v}_h = 0\}.$$

It is easy to see that the discrete problem (4.1) and (4.2) can be reformulated as seeking $\mathbf{u}_h \in D_h$ satisfying

$$d_\beta(\mathbf{u}_h,\mathbf{v}) + a_1(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\ \mathbf{v} \in D_h. \tag{6.16}$$

Let $F : D_h \to D_h$ be a nonlinear map so that for each $\mathbf{w}_h \in D_h$, $\tilde{\mathbf{u}}_h := F(\mathbf{w}_h)$ is given as the solution of the following linear problem:

$$d_\beta(\tilde{\mathbf{u}}_h,\mathbf{v}) + a_1(\mathbf{w}_h,\tilde{\mathbf{u}}_h,\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\ \mathbf{v} \in D_h. \tag{6.17}$$

The map F is clearly continuous and therefore is compact in the finite dimensional space D_h. If $\lambda > 0$ and $\mathbf{w}_h$ satisfies $\tilde{\mathbf{u}}_h = F(\mathbf{w}_h) = \lambda\mathbf{w}_h$, then we have from (6.17) that

$$\lambda d_\beta(\mathbf{w}_h,\mathbf{v}) + \lambda a_1(\mathbf{w}_h,\mathbf{w}_h,\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\ \mathbf{v} \in D_h. \tag{6.18}$$

By choosing in (6.18) $\mathbf{v} = \mathbf{w}_h$, we come up with

$$\lambda \left(d_\beta(\mathbf{w}_h, \mathbf{w}_h) + a_1(\mathbf{w}_h, \mathbf{w}_h, \mathbf{w}_h) \right) = (\mathbf{f}, \mathbf{w}_h), \quad \forall\, \mathbf{v} \in D_h. \tag{6.19}$$

It now follows from (6.1) and (6.7) that

$$\lambda \left(\nu\alpha_0 |\!|\!| \mathbf{w}_h |\!|\!|^2 + \gamma \sum_{e \in \mathcal{E}_h^0} \int_e |\mathbf{w}_h \cdot \mathbf{n}| [\![\mathbf{w}_h]\!]^2 ds \right) \le |(\mathbf{f}, \mathbf{w}_h)|. \tag{6.20}$$

By introducing a mesh-dependent norm

$$\|\mathbf{f}\|_{*,h} = \sup_{\mathbf{v} \in D_h} \frac{(\mathbf{f}, \mathbf{v})}{|\!|\!| \mathbf{v} |\!|\!|}, \tag{6.21}$$

we have from (6.20) and (6.21) that

$$\lambda \left(\nu\alpha_0 |\!|\!| \mathbf{w}_h |\!|\!|^2 + \gamma \sum_{e \in \mathcal{E}_h^0} \int_e |\mathbf{w}_h \cdot \mathbf{n}| [\![\mathbf{w}_h]\!]^2 ds \right) \le \|\mathbf{f}\|_{*,h} |\!|\!| \mathbf{w}_h |\!|\!|.$$

It follows that

$$\lambda \le \frac{\|\mathbf{f}\|_{*,h}}{\alpha_0 \nu |\!|\!| \mathbf{w}_h |\!|\!|}.$$

Thus, $\lambda < 1$ holds true for any $\mathbf{w}_h$ being on the boundary of the ball in D_h centered at the origin with radius $\rho > \|\mathbf{f}\|_{*,h}/(\alpha_0\nu)$. Consequently, the Leray-Schauder fixed point theorem implies that the nonlinear map F defined by (6.17) has a fixed point $\mathbf{u}_h$:

$$F(\mathbf{u}_h) = \mathbf{u}_h$$

in any ball centered at the origin with radius $\rho > \|\mathbf{f}\|_{*,h}/(\alpha_0\nu)$. This fixed point $\mathbf{u}_h$ is clearly a solution of the finite element scheme (6.16), which in turn provides a solution of the original numerical scheme (4.1) and (4.2). The results can be summarized as follows.

Theorem 6.1. *The finite element discretization scheme* (6.16) *has at least one solution* $\mathbf{u}_h$ *in the divergence-free subspace* D_h. *Moreover, all the solutions of* (6.16) *satisfy the following estimates:*

$$|\!|\!| \mathbf{u}_h |\!|\!| \le \frac{\|\mathbf{f}\|_{*,h}}{\alpha_0 \nu} \tag{6.22}$$

and

$$\gamma \sum_{e \in \mathcal{E}_h^0} \int_e |\mathbf{u}_h \cdot \mathbf{n}| [\![\mathbf{u}_h]\!]^2 ds \le \frac{\|\mathbf{f}\|_{*,h}^2}{2\alpha_0 \nu}. \tag{6.23}$$

Proof. Note that $\mathbf{u}_h \in D_h$ is a solution of (6.16) if and only if it is a fixed-point of the nonlinear map F. Since F has at least one fixed point in the ball of D_h centered at the origin with radius $\rho = \|\mathbf{f}\|_{*,h}/(\alpha_0\nu)$, then the finite element scheme (6.16) must have a solution and all the solutions must satisfy the estimate (6.22).

It remains to establish the estimate (6.23). To this end, let $\mathbf{u}_h$ be a solution of (6.16). By choosing $\mathbf{v} = \mathbf{u}_h$ in (6.16) one arrives at

$$d_\beta(\mathbf{u}_h, \mathbf{u}_h) + a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_h) = (\mathbf{f}, \mathbf{u}_h). \tag{6.24}$$

Again, using (6.1) and (6.7) one has

$$\nu\alpha_0 |\!|\!| \mathbf{u}_h |\!|\!|^2 + \gamma \sum_{e\in\mathcal{E}_h^0} \int_e |\mathbf{u}_h \cdot \mathbf{n}| [\![\mathbf{u}_h]\!]^2 ds \le \|\mathbf{f}\|_{*,h} |\!|\!| \mathbf{u}_h |\!|\!|. \tag{6.25}$$

The right-hand side of (6.25) can be estimated as follows:

$$\|\mathbf{f}\|_{*,h} |\!|\!| \mathbf{u}_h |\!|\!| \le \frac{\alpha_0\nu}{2} |\!|\!| \mathbf{u}_h |\!|\!|^2 + \frac{1}{2\alpha_0\nu} \|\mathbf{f}\|_{*,h}^2.$$

Substituting the above into (6.25) yields

$$\frac{\nu\alpha_0}{2} |\!|\!| \mathbf{u}_h |\!|\!|^2 + \gamma \sum_{e\in\mathcal{E}_h^0} \int_e |\mathbf{u}_h \cdot \mathbf{n}| [\![\mathbf{u}_h]\!]^2 ds \le \frac{\|\mathbf{f}\|_{*,h}^2}{2\alpha_0\nu}, \tag{6.26}$$

which implies the desired estimate (6.23). □

6.3. A uniqueness result

The analysis here follows the idea presented in Girault and Raviart [19] on solution uniqueness for the Navier-Stokes equations. Let $\mathbf{u}_h$ and $\bar{\mathbf{u}}_h \in D_h$ be two solutions of the finite element scheme (6.16). Since both $\mathbf{u}_h$ and $\bar{\mathbf{u}}_h$ satisfy the nonlinear equation (6.16), then one has

$$d_\beta(\mathbf{u}_h, \mathbf{v}) + a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) = d_\beta(\bar{\mathbf{u}}_h, \mathbf{v}) + a_1(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{v})$$

for all $\mathbf{v} \in D_h$. By introducing $\mathbf{e}_h = \mathbf{u}_h - \bar{\mathbf{u}}_h$, the above equation can be rewritten as

$$d_\beta(\mathbf{e}_h, \mathbf{v}) + a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - a_1(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{v}) = 0.$$

Observe that

$$a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - a_1(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{v}) = a_1(\mathbf{u}_h, \mathbf{e}_h, \mathbf{v}) + a_1(\mathbf{u}_h, \bar{\mathbf{u}}_h, \mathbf{v}) - a_1(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{v}).$$

Thus, for any $\mathbf{v} \in D_h$

$$d_\beta(\mathbf{e}_h, \mathbf{v}) + a_1(\mathbf{u}_h, \mathbf{e}_h, \mathbf{v}) = a_1(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{v}) - a_1(\mathbf{u}_h, \bar{\mathbf{u}}_h, \mathbf{v}).$$

In particular, by letting $\mathbf{v} = \mathbf{e}_h$, we have from (6.1) and (6.7) that

$$\nu\alpha_0 |\!|\!| \mathbf{e}_h |\!|\!|^2 + \gamma \sum_{e\in\mathcal{E}_h^0} \int_e |\mathbf{u}_h \cdot \mathbf{n}| [\![\mathbf{e}_h]\!]^2 ds \le \left| a_1(\mathbf{u}_h, \bar{\mathbf{u}}_h, \mathbf{e}_h) - a_1(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{e}_h) \right|.$$

Moreover, using (6.15) we arrive at the following estimate

$$\nu\alpha_0 |\!|\!| \mathbf{e}_h |\!|\!|^2 + \gamma \sum_{e\in\mathcal{E}_h^0} \int_e |\mathbf{u}_h \cdot \mathbf{n}| [\![\mathbf{e}_h]\!]^2 ds \le \tilde{\mathcal{N}} \; |\!|\!| \bar{\mathbf{u}}_h |\!|\!| \; |\!|\!| \mathbf{e}_h |\!|\!|^2. \tag{6.27}$$

Since $\bar{\mathbf{u}}_h$ is a solution of (6.16), then the estimate (6.22) is applicable, i.e., $|\!|\!| \bar{\mathbf{u}}_u |\!|\!| \le \|\mathbf{f}\|_{*,h}/(\alpha_0\nu)$. Substituting the above into the right-hand side of (6.27) yields

$$\nu\alpha_0 |\!|\!| \mathbf{e}_h |\!|\!|^2 + \gamma \sum_{e\in\mathcal{E}_h^0} \int_e |\mathbf{u}_h \cdot \mathbf{n}| [\![\mathbf{e}_h]\!]^2 ds \le \frac{\tilde{\mathcal{N}} \|\mathbf{f}\|_{*,h}}{\alpha_0\nu} |\!|\!| \mathbf{e}_h |\!|\!|^2. \tag{6.28}$$

The estimate (6.28) implies obvious uniqueness under certain conditions. We summarize the result as follows.

Theorem 6.2. *Let $a_1(\cdot,\cdot,\cdot)$ be given in* (6.2) *and define*

$$\tilde{\mathcal{N}}_h = \sup_{\mathbf{u},\bar{\mathbf{u}},\mathbf{v},\mathbf{w}\in V_h} \frac{|a_1(\mathbf{u};\mathbf{v},\mathbf{w}) - a_1(\bar{\mathbf{u}};\mathbf{v},\mathbf{w})|}{\|\!|\mathbf{u}-\bar{\mathbf{u}}|\!\|\,\|\!|\mathbf{v}|\!\|\,\|\!|\mathbf{w}|\!\|}. \tag{6.29}$$

Assume that $\rho \equiv \tilde{\mathcal{N}}_h\|\mathbf{f}\|_{*,h}/(\alpha_0\nu)^2 < 1$ holds true, where α_0 is the ellipticity constant in (6.1) *and $\|\mathbf{f}\|_{*,h}$ is given by* (6.21). *Then the finite element discretization scheme* (6.16) *has at most one solution in the divergence-free subspace D_h.*

7. Error Estimates

In this section we shall establish the error estimates for the finite element schemes (4.1)-(4.2). Our main objective is to derive an optimal-order error estimate for the pressure in $L^2(\Omega)$ and the velocity in the discrete H^1-norm given by (4.4). For simplicity, we consider only the case of two space variables (i.e., $n=2$); problems in higher dimensions such as $n=3$ and $n=4$ can be handled by using the Sobolev-type inequalities presented in Section 5.

Assumption A1: *There exists an operator $\Pi_h : (H^1(\Omega))^2 \to V_h$ such that*

$$b(\mathbf{v}-\Pi_h\mathbf{v}, q) = 0, \qquad \forall q \in W_h. \tag{7.1}$$

In addition, the operator Π_h is assumed to satisfy the following:

$$|\mathbf{v}-\Pi_h\mathbf{v}|_{s,K} \le Ch^{t-s}|\mathbf{v}|_{t,K}, \quad \forall K \in \mathcal{T}_h,\ s=0,1, \tag{7.2}$$

where the constant C depends only on the shape of K and $1 \le t \le r+1$.

Inequalities (4.5) and (7.2) imply

$$\|\!|\mathbf{v}-\Pi_h\mathbf{v}|\!\|_1 \le C\|\mathbf{v}\|_1.$$

Since $\|\!|\mathbf{v}|\!\|_1 = |\mathbf{v}|_1 \le \|\mathbf{v}\|_1$ for $\mathbf{v} \in (H_0^1(\Omega))^2$, it follows by the above and triangle inequalities

$$\|\!|\Pi_h\mathbf{v}|\!\|_1 \le C\|\mathbf{v}\|_1. \tag{7.3}$$

For our finite element formulations, the *inf-sup* condition given in Brezzi's framework would read as follows: there exists a positive constant β, independent of h, such that

$$\sup_{\mathbf{v}\in V_h} \frac{b(\mathbf{v},q)}{\|\!|\mathbf{v}|\!\|} \ge \beta\|q\|, \quad \forall q \in W_h. \tag{7.4}$$

To verify (7.4), we first use the operator Π_h to obtain

$$\sup_{\mathbf{v}\in V_h} \frac{b(\mathbf{v},q)}{\|\!|\mathbf{v}|\!\|} \ge \sup_{\mathbf{v}\in(H_0^1(\Omega))^2} \frac{b(\Pi_h\mathbf{v},q)}{\|\!|\Pi_h\mathbf{v}|\!\|} = \sup_{\mathbf{v}\in(H_0^1(\Omega))^2} \frac{b(\mathbf{v},q)}{\|\!|\Pi_h\mathbf{v}|\!\|}. \tag{7.5}$$

Observe that by using (7.3), and (4.6), we have for all $\mathbf{v} \in (H_0^1(\Omega))^2$

$$\|\!|\Pi_h\mathbf{v}|\!\| \le C\|\!|\Pi_h\mathbf{v}|\!\|_1 \le C\|\mathbf{v}\|_1. \tag{7.6}$$

Thus, substituting (7.6) into the inequality (7.5) gives

$$\sup_{\mathbf{v}\in V_h} \frac{b(\mathbf{v},q)}{\|\!|\mathbf{v}|\!\|} \ge C^{-1} \sup_{\mathbf{v}\in(H_0^1(\Omega))^2} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1} \ge \beta\|q\|,$$

where we have used the *inf-sup* condition for the continuous case [9, 19].

Our error analysis requires a use of the L^2 projection from $L_0^2(\Omega)$ to the finite element space W_h, which is denoted by Q_h. In addition, we need the following error equation: for all $\mathbf{v} \in V_h$ and $q \in W_h$ one has

$$d_\beta(\mathbf{u}-\mathbf{u}_h, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) - a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p - p_h) = 0, \tag{7.7}$$

$$b(\mathbf{u}-\mathbf{u}_h, q) = 0. \tag{7.8}$$

The above error equations can be obtained from subtracting (4.1)-(4.2) from (3.18)-(3.19). We recall that the exact solution of (1.1)-(1.3) satisfies the following boundedness

$$\|\mathbf{u}\|_1 \le \mu^{-1}\|\mathbf{f}\|_{-1}, \tag{7.9}$$

where, as usual

$$\|\mathbf{f}\|_{-1} = \sup_{\mathbf{v}\in[H_0^1(\Omega)]^n} \frac{(\mathbf{f}, \mathbf{v})}{\|\mathbf{v}\|_1}.$$

The following is an error estimate for the velocity approximation in the discrete H^1 norm.

Theorem 7.1. *Let* $(\mathbf{u}; p)$ *be the solution of* (1.1)-(1.3) *and* $(\mathbf{u}_h; p_h) \in V_h \times W_h$ *be obtained from* (4.1)-(4.2). *Assume that the Assumption* A1 *holds true. Let*

$$\rho = \frac{\tilde{\mathcal{N}}_h \|\mathbf{f}\|_{*,h}}{\alpha_0^2 \nu^2},$$

where α_0 *is the ellipticity constant in* (6.1) *and* $\|\mathbf{f}\|_{*,h}$ *is given by* (6.21). *Assume that* $\rho < 1$ *so that the finite element scheme* (4.1)-(4.2) *has a unique solution. Then, there exists a constant* C *independent of* h *such that*

$$|||\Pi_h\mathbf{u} - \mathbf{u}_h||| \le \frac{C}{(1-\rho)\alpha_0\nu}\Big(\mathcal{M}|||\mathbf{u}-\Pi_h\mathbf{u}||| + \|p - Q_h p\|_0\Big) \tag{7.10}$$

and

$$\gamma \sum_{e\in\mathcal{E}_h} \int_e [\![\Pi_h\mathbf{u} - \mathbf{u}_h]\!]^2 ds \le \frac{C}{(1-\rho)\alpha_0\nu}\Big(\mathcal{M}|||\mathbf{u}-\Pi_h\mathbf{u}||| + \|p - Q_h p\|_0\Big)^2, \tag{7.11}$$

where

$$\mathcal{M} = \nu + \frac{\|\mathbf{f}\|_{-1}}{\nu} + \frac{\tilde{\mathcal{N}}\|\mathbf{f}\|_{*,h}}{\alpha_0\nu}. \tag{7.12}$$

Proof. Let

$$\xi_h = \mathbf{u}_h - \Pi_h\mathbf{u}, \qquad \eta_h = p_h - Q_h p \tag{7.13}$$

be the error between the finite element solution $(\mathbf{u}_h; p_h)$ and the projection $(\Pi_h\mathbf{u}; Q_h p)$ of the exact solution. Denote by

$$\xi = \mathbf{u} - \Pi_h\mathbf{u}, \qquad \eta = p - Q_h p \tag{7.14}$$

the error between the exact solution $(\mathbf{u}; p)$ and it projection. It follows from the error equations (7.7) and (7.8) that

$$d_\beta(\xi_h, \mathbf{v}) - b(\mathbf{v}, \eta_h) = d_\beta(\xi, \mathbf{v}) - b(\mathbf{v}, \eta) + a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) - a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}), \tag{7.15}$$

$$b(\xi_h, q) = b(\xi, q) = 0 \tag{7.16}$$

for any $\mathbf{v} \in V_h$ and $q \in W_h$. By letting $\mathbf{v} = \xi_h$ in (7.15) and $q = \eta_h$ in (7.16), the sum of (7.15) and (7.16) gives

$$d_\beta(\xi_h, \xi_h) = d_\beta(\xi, \xi_h) - b(\xi_h, \eta) + a_1(\mathbf{u}, \mathbf{u}, \xi_h) - a_1(\mathbf{u}_h, \mathbf{u}_h, \xi_h). \tag{7.17}$$

It is easy to see that

$$\begin{aligned} & a_1(\mathbf{u}, \mathbf{u}, \xi_h) - a_1(\mathbf{u}_h, \mathbf{u}_h, \xi_h) \\ &= a_1(\mathbf{u}, \mathbf{u}, \xi_h) - a_1(\mathbf{u}, \mathbf{u}_h, \xi_h) + a_1(\mathbf{u}, \mathbf{u}_h, \xi_h) - a_1(\mathbf{u}_h, \mathbf{u}_h, \xi_h) \\ &= a_1(\mathbf{u}, \xi, \xi_h) - a_1(\mathbf{u}, \xi_h, \xi_h) + a_1(\mathbf{u}, \mathbf{u}_h, \xi_h) - a_1(\mathbf{u}_h, \mathbf{u}_h, \xi_h). \end{aligned}$$

Substituting the above equation into (7.17) yields

$$\begin{aligned} & d_\beta(\xi_h, \xi_h) + a_1(\mathbf{u}, \xi_h, \xi_h) \\ &= d_\beta(\xi, \xi_h) - b(\xi_h, \eta) + a_1(\mathbf{u}, \xi, \xi_h) + a_1(\mathbf{u}, \mathbf{u}_h, \xi_h) - a_1(\mathbf{u}_h, \mathbf{u}_h, \xi_h). \end{aligned} \tag{7.18}$$

To estimate each term on the right-hand side of (7.18), we use the boundedness result (6.8) to deduce

$$|d_\beta(\xi, \xi_h)| \le C\nu |\!|\!| \xi |\!|\!| \, |\!|\!| \xi_h |\!|\!|, \tag{7.19}$$

and it is trivial to see the following

$$|b(\xi_h, \eta)| \le |\!|\!| \xi_h |\!|\!|_1 \|\eta\|_0. \tag{7.20}$$

As to the third term, we have from (6.14) that

$$\begin{aligned} |a_1(\mathbf{u}, \xi, \xi_h)| &\le C |\!|\!| \mathbf{u} |\!|\!|_1 |\!|\!| \xi |\!|\!|_1 |\!|\!| \xi_h |\!|\!|_1 \\ &= C\|\mathbf{u}\|_1 |\!|\!| \xi |\!|\!|_1 |\!|\!| \xi_h |\!|\!|_1 \le C\nu^{-1}\|\mathbf{f}\|_{-1} |\!|\!| \xi |\!|\!|_1 |\!|\!| \xi_h |\!|\!|_1, \end{aligned} \tag{7.21}$$

where we have used the estimate (7.9) in the last inequality. The last two terms on the right-hand side of (7.18) can be handled by using (6.15) and (6.29) as follows:

$$\begin{aligned} & |a_1(\mathbf{u}, \mathbf{u}_h, \xi_h) - a_1(\mathbf{u}_h, \mathbf{u}_h, \xi_h)| \\ &\le |a_1(\mathbf{u}, \mathbf{u}_h, \xi_h) - a_1(\Pi_h\mathbf{u}, \mathbf{u}_h, \xi_h)| + |a_1(\mathbf{u}_h, \mathbf{u}_h, \xi_h) - a_1(\Pi_h\mathbf{u}, \mathbf{u}_h, \xi_h)| \\ &\le \tilde{\mathcal{N}} |\!|\!| \xi |\!|\!|_1 |\!|\!| \mathbf{u}_h |\!|\!|_1 |\!|\!| \xi_h |\!|\!|_1 + \tilde{\mathcal{N}}_h |\!|\!| \mathbf{u}_h |\!|\!|_1 |\!|\!| \xi_h |\!|\!|^2. \end{aligned}$$

Furthermore, using the boundedness estimate (6.22) one obtains

$$|a_1(\mathbf{u}, \mathbf{u}_h, \xi_h) - a_1(\mathbf{u}_h, \mathbf{u}_h, \xi_h)| \le \frac{\tilde{\mathcal{N}}\|\mathbf{f}\|_{*,h}}{\alpha_0\nu} |\!|\!| \xi |\!|\!|_1 |\!|\!| \xi_h |\!|\!|_1 + \frac{\tilde{\mathcal{N}}_h\|\mathbf{f}\|_{*,h}}{\alpha_0\nu} |\!|\!| \xi_h |\!|\!|_1^2. \tag{7.22}$$

Now substituting the estimates (7.19)-(7.22) into (7.18) we obtain

$$\begin{aligned} d_\beta(\xi_h, \xi_h) + a_1(\mathbf{u}, \xi_h, \xi_h) &\le C\nu |\!|\!| \xi |\!|\!| \, |\!|\!| \xi_h |\!|\!| + C |\!|\!| \xi_h |\!|\!|_1 \|\eta\|_0 + C\nu^{-1}\|\mathbf{f}\|_{-1} |\!|\!| \xi |\!|\!|_1 |\!|\!| \xi_h |\!|\!|_1 \\ &\quad + \frac{\tilde{\mathcal{N}}\|\mathbf{f}\|_{*,h}}{\alpha_0\nu} |\!|\!| \xi |\!|\!|_1 |\!|\!| \xi_h |\!|\!|_1 + \frac{\tilde{\mathcal{N}}_h\|\mathbf{f}\|_{*,h}}{\alpha_0\nu} |\!|\!| \xi_h |\!|\!|_1^2 \\ &\le C\left(\mathcal{M} |\!|\!| \xi |\!|\!| + \|\eta\|_0\right) |\!|\!| \xi_h |\!|\!| + \frac{\tilde{\mathcal{N}}_h\|\mathbf{f}\|_{*,h}}{\alpha_0\nu} |\!|\!| \xi_h |\!|\!|_1^2. \end{aligned} \tag{7.23}$$

where $\mathcal{M}$ is given by (7.12). Thus, it follows from the coercivity (6.1) and the above estimate that

$$\left(\nu\alpha_0 - \frac{\tilde{\mathcal{N}}_h\|\mathbf{f}\|_{*,h}}{\nu\alpha_0}\right) |\!|\!| \xi_h |\!|\!|^2 + a_1(\mathbf{u}, \xi_h, \xi_h) \le C(\mathcal{M} |\!|\!| \xi |\!|\!| + \|\eta\|_0) |\!|\!| \xi_h |\!|\!|. \tag{7.24}$$

Using the notation and the condition of Theorem 6.2 we arrive at

$$(1-\rho)\alpha_0\nu|\!|\!|\xi_h|\!|\!| + a_1(\mathbf{u},\xi_h,\xi_h) \le C\left(\mathcal{M}|\!|\!|\xi|\!|\!| + \|\eta\|_0\right),$$

which, together with (6.7), implies

$$(1-\rho)\alpha_0\nu|\!|\!|\xi_h|\!|\!|^2 + \gamma\sum_{e\in\mathcal{E}_h^0}\int_e |\mathbf{u}\cdot\mathbf{n}|[\![\xi_h]\!]^2 ds \le C\left(\mathcal{M}|\!|\!|\xi|\!|\!| + \|\eta\|_0\right)|\!|\!|\xi_h|\!|\!|.$$

Using the inequality $2ab \le \epsilon a^2 + \epsilon^{-1}b^2$ with $\epsilon = (1-\rho)\alpha_0\nu$ and $a = |\!|\!|\xi_h|\!|\!|$ we obtain

$$(1-\rho)\alpha_0\nu|\!|\!|\xi_h|\!|\!|^2 + 2\gamma\sum_{e\in\mathcal{E}_h^0}\int_e |\mathbf{u}\cdot\mathbf{n}|[\![\xi_h]\!]^2 ds \le \frac{C}{(1-\rho)\alpha_0\nu}\left(\mathcal{M}|\!|\!|\xi|\!|\!| + \|\eta\|_0\right)^2.$$

This leads to the error estimates (7.10) and (7.11). □

The following is a result on the pressure approximation.

Theorem 7.2. *Let* $(\mathbf{u};p)$ *be the solution of* (1.1)-(1.3) *and* $(\mathbf{u}_h;p_h)\in V_h\times W_h$ *be obtained from* (4.1)-(4.2). *Under the assumptions of Theorem 7.1, there exists a constant* C *independent of the mesh size* h *such that*

$$\|Q_hp - p_h\|_0 \le C(\nu+\mathcal{P})|\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!| + C\|p-Q_hp\|_0, \tag{7.25}$$

where

$$\mathcal{P} = \frac{C\|\mathbf{f}\|_{-1}}{\nu} + \frac{\tilde{\mathcal{N}}\|\mathbf{f}\|_{*,h}}{\alpha_0\nu}. \tag{7.26}$$

An error estimate for the pressure approximation is easily given by combining (7.25) *with* (7.10).

Proof. To establish (7.25), we use the discrete inf-sup condition (7.4) and the error equation (7.7) to obtain

$$\begin{aligned}\|p_h - Q_hp\|_0 &\le \sup_{\mathbf{v}\in V_h}\frac{b(\mathbf{v},p_h-Q_hp)}{|\!|\!|\mathbf{v}|\!|\!|}\\ &= \sup_{\mathbf{v}\in V_h}\frac{b(\mathbf{v},p_h-p)+b(\mathbf{v},p-Q_hp)}{|\!|\!|\mathbf{v}|\!|\!|}\\ &= \sup_{\mathbf{v}\in V_h}\frac{d_\beta(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) + a_1(\mathbf{u},\mathbf{u},\mathbf{v}) - a_1(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}) + b(\mathbf{v},p-Q_hp)}{|\!|\!|\mathbf{v}|\!|\!|}.\end{aligned} \tag{7.27}$$

Since

$$|b(\mathbf{v},p-Q_hp)| \le \|p-Q_hp\|_0|\!|\!|\mathbf{v}|\!|\!|,$$

we have from the boundedness (6.8) and the estimate (7.28) that

$$\begin{aligned}&|d_\beta(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) + a_1(\mathbf{u},\mathbf{u},\mathbf{v}) - a_1(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}) + b(\mathbf{v},p-Q_hp)|\\ &\le \left(C\nu|\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!| + \mathcal{P}|\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!| + C\|p-Q_hp\|_0\right)|\!|\!|\mathbf{v}|\!|\!|.\end{aligned}$$

Substituting the above into (7.27) yields

$$\|p_h - Q_hp\|_0 \le C(\nu+\mathcal{P})|\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!| + C\|p-Q_hp\|_0. \qquad \square$$

The following Lemma was used in the proof of Theorem 7.2.

Lemma 7.1. *Let* $(\mathbf{u}; p)$ *be the solution of* (1.1)-(1.3) *and* $(\mathbf{u}_h; p_h) \in V_h \times W_h$ *be obtained from* (4.1)-(4.2). *Under the assumptions of Theorem* 7.1, *for any* $\mathbf{w} \in V_h$ *we have*

$$|a_1(\mathbf{u}, \mathbf{u}, \mathbf{w}) - a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{w})| \leq \mathcal{P} |\!|\!| \mathbf{u} - \mathbf{u}_h |\!|\!|_1 |\!|\!| \mathbf{w} |\!|\!|_1, \tag{7.28}$$

where $\mathcal{P}$ *is given by* (7.26).

Proof. Note that

$$a_1(\mathbf{u}, \mathbf{u}, \mathbf{w}) - a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{w}) = a_1(\mathbf{u}, \mathbf{u} - \mathbf{u}_h, \mathbf{w}) + a_1(\mathbf{u}, \mathbf{u}_h, \mathbf{w}) - a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{w}).$$

Thus, it follows from (6.14) and (6.15) that

$$|a_1(\mathbf{u}, \mathbf{u}, \mathbf{w}) - a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{w})| \leq C \|\mathbf{u}\|_1 |\!|\!| \mathbf{u} - \mathbf{u}_h |\!|\!|_1 |\!|\!| \mathbf{w} |\!|\!|_1 + \tilde{\mathcal{N}} |\!|\!| \mathbf{u} - \mathbf{u}_h |\!|\!|_1 |\!|\!| \mathbf{u}_h |\!|\!|_1 |\!|\!| \mathbf{w} |\!|\!|_1.$$

Now we use the boundedness estimates (7.9) and (6.22) to obtain

$$|a_1(\mathbf{u}, \mathbf{u}, \mathbf{w}) - a_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{w})| \leq \left(\frac{C\|\mathbf{f}\|_{-1}}{\nu} + \frac{\tilde{\mathcal{N}} \|\mathbf{f}\|_{*,h}}{\alpha_0 \nu} \right) |\!|\!| \mathbf{u} - \mathbf{u}_h |\!|\!|_1 |\!|\!| \mathbf{w} |\!|\!|_1,$$

which completes the proof. □

Acknowledgments. The research of Wang was supported by the NSF IR/D program, while working at the National Science Foundation. However, any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation. The research of Ye was supported in part by National Science Foundation Grant DMS-0612435.

References

[1] Feng Kang, Differential vs. integral equations and finite vs. infinite elements, *Math. Numer. Sinica,* **2**:*1* (1980), 100-105.

[2] D. Arnold, F. Brezzi, B. Cockburn and D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.*, **39** (2002), 1749-1779.

[3] I. Babuška, The finite element method with Lagrangian multiplier, *Numer. Math.*, **20** (1973), 179–192.

[4] F. Brezzi, On the existence, uniqueness, and approximation of saddle point problems arising from Lagrangian multipliers, *RAIRO, Anal. Numér.*, **2** (1974), 129-151.

[5] I. Babuska and M. Zlamal, Nonconforming elements in the finite element method with penalty, *SIAM J. Numer. Anal.,* **10** (1973), 863-875.

[6] F. Bassi and S. Rebay, A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations, *J. Comput. Phys.*, **131** (1997), 267-279.

[7] C.E. Baumann and J.T. Oden, A discontinuous *hp* finite element method for convection- diffusion problems, *Comput. Method. Appl.*, **175** (1999), 311-341.

[8] F. Brezzi, J. Douglas and L. Marini, Two families of mixed finite elements for second order elliptic problems, *Numer. Math.*, **47** (1985), 217-235.

[9] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Elements, Spring-Verlag, New York, 1991.

[10] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, New York, 1978.

[11] B. Cockburn, S. Hou, and C.-W. Shu, TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: The multidimensional case, *Math. Comput.*, **54** (1990), 545-581.

[12] B. Cockburn, G. Kanschat and D. Schotzau, A locally conservative LDG method for the incompressible Navier-Stokes equations, *Math. Comput.*, **74** (2005), 1067-1095.

[13] B. Cockburn, G. Kanschat, D. Schoyzau and C. Schwab, Local discontinuous Galerkin methods for the Stokes system, *SIAM J. Numer. Anal.*, **40** (2002), 319-343.

[14] B. Cockburn and C.W. Shu, The local discontinuous Galerkin finite element method for convection-diffusion systems, *SIAM J. Numer. Anal.*, **35** (1998), 2440-2463.

[15] M. Costabel and M. Dauge, Crack singularities for general elliptic systems, *Math. Nachr.*, **235** (2002), 29-49.

[16] M. Dauge, Elliptic Boundary Value Problems in Corner Domains - Smoothness and Asymptotics of Solutions, Lecture Notes in Mathematics, Vol. 1341, Springer-Verlag, Berlin, 1988.

[17] J. Douglas, Z. Cai and X. Ye, A Stable Quadrilateral nonconforming Element for the Navier-Stokes Equations, *Calcolo*, **36** (1999) 215-232.

[18] J. Douglas and J. Wang, A new family of spaces in mixed finite element methods for rectangular elements, *Comput. Appl. Math.*, **12** (1993), 183-197.

[19] V. Girault and P.A. Raviart, Finite Element Methods for the Navier-Stokes Equations: Theory and Algorithms, Springer, Berlin, 1986.

[20] V. Girault, B. Rivière, and M.F. Wheeler, A discontinuous Gelerkin method with nonconforming domain decomposition for Stokes and Navier-Stokes problems, *Math. Comput.*, **74** (249) (2004), 53–84.

[21] P. Grisvard, Boundary Value Problems in Non-Smooth Domains, Pitman, London, 1985.

[22] M.D. Gunzburger, Finite Element Methods for Viscous Incompressible Flows, A Guide to Theory, Practice and Algorithms, Academic Press, San Diego, 1989.

[23] P. Hansbo and M. Larson, Discontinuous Galerkin methods for nearly incompressible elasticity by Nitsche's method, *Methods Appl. Mech. Engrg.*, **191** (2002), 1895-1908.

[24] O. A. Karakashian and W. N. Jureidini, A nonconforming finite element method for the stationary Navier-Stokes equations, *SIAM J. Numer. Anal.*, **35** (1998), 93-120.

[25] Jae Ryong Kweon and B. Kellogg, Regularity of Solutions to the Navier-Stokes equations for Compressible Barotropic Flows on a Polygon, *Arch. Rational Mech. Anal.*, **163** (2002).

[26] J. Liu and C. Shu, A high order discontinuous Galerkin method for 2D incompressible flows, *J. Comput. Phys.*, **160** (2000), 577-596.

[27] P-A. Raviart and J. Thomas, A mixed finite element method for second order elliptic problems, Mathematical Aspects of the Finite Element Method (I. Galligani, E. Magenes, eds.), *Lectures Notes in Math.* 606, Springer-Verlag, New York, 1977.

[28] J. Wang and T. Mathew, Mixed finite element method over quadrilaterals, In the Proceedings of the Third International Conference on Advances in Numerical Methods and Applications, Eds: I. T. Dimov, Bl. Sendov, and P. Vassilevski, World Scientific, 1994, 203-214.

[29] J. Wang and X. Ye, New finite element methods in computational fluid dynamics by $H(\mathrm{div})$ elements, *SIAM J. Numer. Anal.,* to appear.

[30] X. Ye, Discontinuous stable elements for incompressible flow, *Adv. Comput. Math.*, **20** (2004) 333-345.