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A KIND OF MULTIVARIATE NURBS SURFACES *1)

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Abstract

The purpose of this paper is to construct a kind of multivariate NURBS surfaces by using the bivariate B-splines in the space $S_2^1(\triangle_{mn}^{(2)})$ and discuss some properties of this kind of NURBS surfaces with multiple knots on the type-2 triangulation.

Key words: Bivariate B-spline, Type-2 triangulation ($\triangle_{mn}^{(2)}$), NURBS surfaces.

1. Introduction

As we know that the NURBS surface was usually obtained by using the tensor product B-splines. For example, the bi-quadratic B-spline surface

$$\mathbf{S}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} N_{i,2}(u)N_{j,2}(v)\mathbf{P}_{i,j}, \tag{1}$$

in general, is a surface of degree 2 in the u or v direction. However, it should be a surface of degree 4 in other ways. As a result, there may be some inflection points on the surface. In some cases, we would prefer quadratic surfaces (or piecewise quadratic surfaces) to the tensor product surfaces. In order to resolve the problem, we can construct another kind of NURBS surfaces by using the local supported bivariate B-splines in the space $S_2^1(\triangle_{mn}^{(2)})$ because they span the whole space as proved in (Ren-Hong Wang, 1985; Ren-Hong Wang, 1994). Moreover, since the basis of B-splines satisfies the partition of unity, these surfaces have stronger convex hull property and transformation invariance. After presenting the definition of the bivariate NURBS surface, section 2 shows some properties of this kind of NURBS surfaces with multiple knots on the type-2 triangulation, and presents several examples to demonstrate these properties.

2. Construction of Surface

2.1. Parametric Representation

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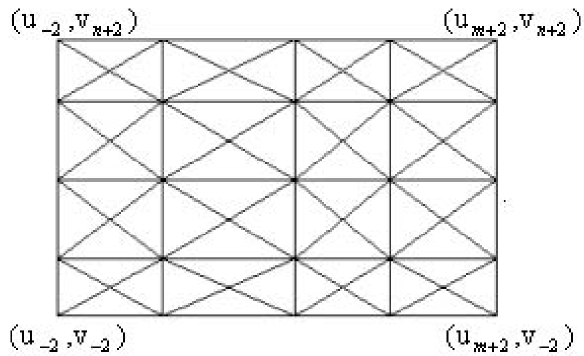

Figure 1. A type-2 triangulation of parametric domain.

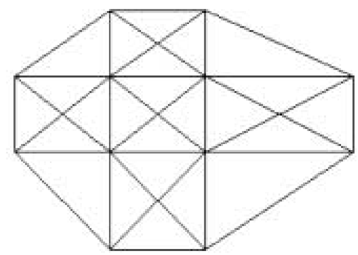

Figure 2. The support of B-spline.

Let $\mathbf{c}_{ij} \in R^3$ be control points, and $w_{ij} \in R$ be weights $(i = -1, 0, \cdots, m; j = -1, 0, \cdots, n)$. By using appropriate parameters u and v, we obtain a type-2 triangulation shown in Figure 1. In this paper we consider only control points corresponding to those intersection points of two diagonals in each subrectangle. The support of B-spline is shown in Figure 2, so the parametric knots should be:

$$\begin{cases} u_{-2} \leq u_{-1} \leq u_0 \leq \cdots \leq u_m \leq u_{m+1} \leq u_{m+2} \\ v_{-2} \leq v_{-1} \leq v_0 \leq \cdots \leq v_n \leq v_{n+1} \leq v_{n+2} \end{cases} .$$

Define non-uniform B-splines $B_{ij}(u,v) \in S_2^1(\triangle_{mn}^{(2)})$ which are shown in appendix, $(u,v) \in [u_{i-1}, u_{i+2}] \times [v_{j-1}, v_{j+2}], i = -1, 0, \cdots, m; j = -1, 0, \cdots, n$(Ren-Hong Wang, 1994). Then the bivariate NURBS surface is defined by:

$$\mathbf{S}(u,v) = \frac{\sum_{i=-1}^{m} \sum_{j=-1}^{n} \mathbf{c}_{ij} w_{ij} B_{ij}(u,v)}{\sum_{i=-1}^{m} \sum_{j=-1}^{n} w_{ij} B_{ij}(u,v)}, \quad (u,v) \in [u_0, u_m] \times [v_0, v_n]. \tag{2}$$

2.2. Uniform B-spline Surface

Let $\Delta u_i = u_i - u_{i-1} = a$, $\Delta v_j = v_j - v_{j-1} = b$, $i = -1, 0, \cdots, m+2$; $j = -1, 0, \cdots, n+2$, $a, b \in R$. Then $B_{ij}(u,v) = B_{00}(u - ia, v - jb)$, $i = -1, 0, \cdots, m$; $j = -1, 0, \cdots, n$, and a uniform B-spline surface is defined by:

$$\mathbf{S}(u,v) = \sum_{i=-1}^{m} \sum_{j=-1}^{n} \mathbf{c}_{ij} B_{ij}(u,v), \quad (u,v) \in [u_0, u_m] \times [v_0, v_n]. \tag{3}$$

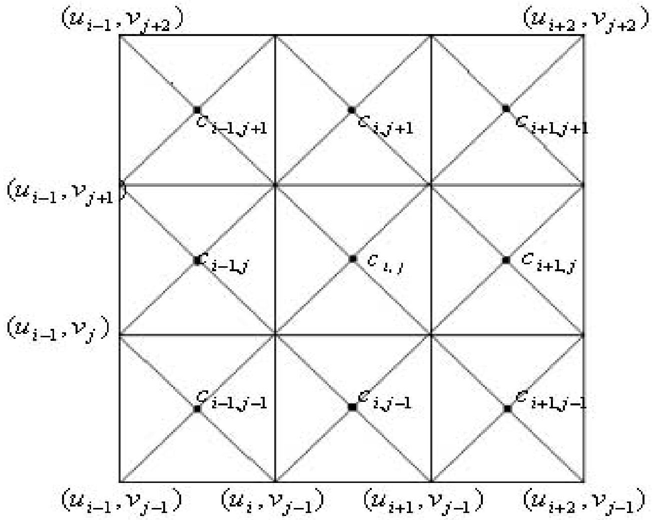

Figure 3. Control points and uniform parametric domain.

The surface is C^1 continuous on the domain (see Figure 3). The surface $\mathbf{S}(u,v)$ has the following properties:

$$\mathbf{S}(u_i,v_j) = \frac{1}{4}(\mathbf{c}_{i-1,j} + \mathbf{c}_{i-1,j-1} + \mathbf{c}_{i,j} + \mathbf{c}_{i,j-1});$$

$$\mathbf{S}'_u(u_i,v_j) = \frac{1}{2}[(\mathbf{c}_{i,j} - \mathbf{c}_{i-1,j}) + (\mathbf{c}_{i,j-1} - \mathbf{c}_{i-1,j-1})];$$

$$\mathbf{S}'_v(u_i,v_j) = \frac{1}{2}[(\mathbf{c}_{i-1,j} - \mathbf{c}_{i-1,j-1}) + (\mathbf{c}_{i,j} - \mathbf{c}_{i,j-1})];$$

$$\mathbf{S}'_u(\frac{u_i+u_{i+1}}{2}, \frac{v_j+v_{j+1}}{2}) = \frac{1}{2}(\mathbf{c}_{i+1,j} - \mathbf{c}_{i-1,j});$$

$$\mathbf{S}'_v(\frac{u_i+u_{i+1}}{2}, \frac{v_j+v_{j+1}}{2}) = \frac{1}{2}(\mathbf{c}_{i,j+1} - \mathbf{c}_{i,j-1}).$$

Moreover,

$$\mathbf{S}(u,v_j) = \frac{1}{4}(1-u)^2(\mathbf{c}_{i-1,j} + \mathbf{c}_{i-1,j-1}) + \frac{1}{4}(1+2u-2u^2)(\mathbf{c}_{i,j} + \mathbf{c}_{i,j-1}) + \frac{1}{4}u^2(\mathbf{c}_{i+1,j} + \mathbf{c}_{i+1,j-1}),$$

$u \in [u_i, u_{i+1}]$, i.e., $\mathbf{S}(u,v_j)$ is a uniform quadratic B-spline curve with the control points $\frac{1}{2}(\mathbf{c}_{i-1,j} + \mathbf{c}_{i-1,j-1})$, $\frac{1}{2}(\mathbf{c}_{i,j} + \mathbf{c}_{i,j-1})$, and $\frac{1}{2}(\mathbf{c}_{i+1,j} + \mathbf{c}_{i+1,j-1})$. $\mathbf{S}(u_i,v)$ has a similar representation.

2.3. Non-uniform B-spline Surface

The non-uniform B-spline surface is a bivariate piecewise quadric polynomial with C^1 smoothness when the parameters u and v both have no multiple knots. When u and (or) v have (has) multiple knots, then the supports of B-splines will be changed, and the smoothness will be changed as well. In general, the surface is $(2-r)$ differentiable in somewhere, here r is the multiplicity of knot. However, the basis of B-splines satisfies also the partition of unity.

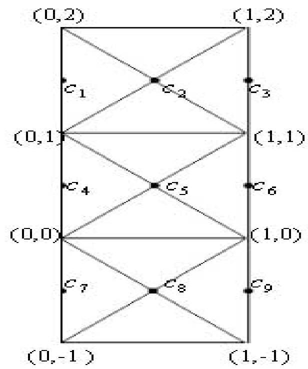

Figure 4. Control points and non-uniform parametric domain in Example 1.

Example 1. We consider a surface which has multiple knots along one parametric vector. Let knot vectors be (see Figure 4):

$$\begin{cases} U = \{u_{-2}, u_{-1}, u_0, u_1, u_2, u_3\} = \{0,0,0,1,1,1\} \\ V = \{v_{-2}, v_{-1}, v_0, v_1, v_2, v_3\} = \{-2,-1,0,1,2,3\} \end{cases},$$

$c_1, c_2, \cdots, c_8, c_9$ be the control points, the basis of B-splines be $B_1(u,v)$, $B_2(u,v)$, $\cdots$, $B_8(u,v)$, $B_9(u,v)$ accordingly. Then the surface is defined by:

$$\mathbf{S}(u,v) = \sum_{i=1}^{9} \mathbf{c}_i B_i(u,v), \quad (u,v) \in [0,1] \times [0,1]. \tag{4}$$

It has the following properties:

$$\mathbf{S}(0,0) = \frac{1}{2}(\mathbf{c}_4 + \mathbf{c}_7); \mathbf{S}(1,0) = \frac{1}{2}(\mathbf{c}_6 + \mathbf{c}_9);$$

$$\mathbf{S}(0,1) = \frac{1}{2}(\mathbf{c}_1 + \mathbf{c}_4); \mathbf{S}(1,1) = \frac{1}{2}(\mathbf{c}_3 + \mathbf{c}_6);$$

$$\mathbf{S}'_u(0,0) = (\mathbf{c}_5 - \mathbf{c}_4) + (\mathbf{c}_8 - \mathbf{c}_7); \mathbf{S}'_v(0,0) = \mathbf{c}_4 - \mathbf{c}_7;$$

$$\mathbf{S}'_u(1,0) = (\mathbf{c}_6 - \mathbf{c}_5) + (\mathbf{c}_9 - \mathbf{c}_8); \mathbf{S}'_v(1,0) = \mathbf{c}_6 - \mathbf{c}_9;$$

$$\mathbf{S}(u,0) = \frac{1}{2}(1-u)^2(\mathbf{c}_4 + \mathbf{c}_7) + u(1-u)(\mathbf{c}_5 + \mathbf{c}_8) + \frac{1}{2}u^2(\mathbf{c}_6 + \mathbf{c}_9);$$

$$\mathbf{S}(u,1) = \frac{1}{2}(1-u)^2(\mathbf{c}_1 + \mathbf{c}_4) + u(1-u)(\mathbf{c}_2 + \mathbf{c}_5) + \frac{1}{2}u^2(\mathbf{c}_3 + \mathbf{c}_6).$$

Therefore, $\mathbf{S}(u,0)$ and $\mathbf{S}(u,1)$ are quadratic Bézier curves with $\frac{1}{2}(\mathbf{c}_4+\mathbf{c}_7)$, $\frac{1}{2}(\mathbf{c}_5+\mathbf{c}_8)$, $\frac{1}{2}(\mathbf{c}_6+\mathbf{c}_9)$ and $\frac{1}{2}(\mathbf{c}_1 + \mathbf{c}_4)$, $\frac{1}{2}(\mathbf{c}_2 + \mathbf{c}_5)$, $\frac{1}{2}(\mathbf{c}_3 + \mathbf{c}_6)$ as their control points respectively. It is clear that the quadratic B-spline curve with a multiple knot $U = \{0,0,0,1,1,1\}$ can be regarded as a generalization of the quadratic Bézier representation.

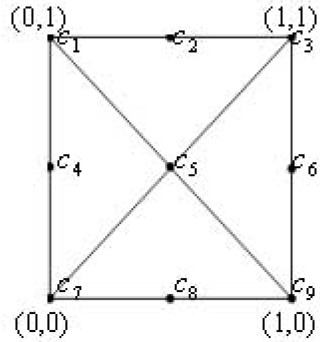

Figure 5. Control points and parametric domain in Example 2.

Example 2. We construct a surface which has multiple knots along two parametric vectors. Let knot vectors be $U = \{0,0,0,1,1,1\}; V = \{0,0,0,1,1,1\}$ (see Figure 5), $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_8, \mathbf{c}_9$ be the control points, the basis of B-splines be $B_1(u,v)$, $B_2(u,v)$, $\cdots$, $B_8(u,v)$, $B_9(u,v)$. Then the surface is defined by:

$$\mathbf{S}(u,v) = \sum_{i=1}^{9} \mathbf{c}_i B_i(u,v), \quad (u,v) \in [0,1] \times [0,1]. \tag{5}$$

Because the B-spline with knot vectors $U = \{a,a,b,b\}; V = \{c,c,d,d\}$ is zero, hence, $B_5(u,v) \equiv 0$. It means that the surface is only determined by all control points except $\mathbf{c}_5$ corresponding to $B_5(u,v)$. It is easy to see that

$$\mathbf{S}(0,0) = \mathbf{c}_7; \mathbf{S}(1,0) = \mathbf{c}_9; \mathbf{S}(0,1) = \mathbf{c}_1; \mathbf{S}(1,1) = \mathbf{c}_3;$$
$$\mathbf{S}'_u(0,0) = 2(\mathbf{c}_8 - \mathbf{c}_7); \mathbf{S}'_u(1,0) = 2(\mathbf{c}_9 - \mathbf{c}_8);$$
$$\mathbf{S}'_v(0,0) = 2(\mathbf{c}_4 - \mathbf{c}_7); \mathbf{S}'_v(0,1) = 2(\mathbf{c}_1 - \mathbf{c}_4);$$
$$\mathbf{S}(u,0) = (1-u)^2 \mathbf{c}_7 + 2u(1-u)\mathbf{c}_8 + u^2 \mathbf{c}_9;$$
$$\mathbf{S}(0,v) = (1-v)^2 \mathbf{c}_7 + 2v(1-v)\mathbf{c}_4 + v^2 \mathbf{c}_1.$$

So, the surface shown in (5) interpolates the four corner control points in Figure 5; the four boundaries of the surface are quadratic Bézier curves with the corresponding control points respectively.

2.4. Non-uniform Rational B-spline Surface

The above results show that we can construct the bivariate NURBS surface by using both methods of the B-spline curve and the Bézier curve on the adjusting of control points and weights (G.E.Farin,1993; L.Piegl,1997; Y.S.Zhou,1993).

Example 3. Let knot vectors be $U = \{0,0,0,1,1,1\}; V = \{0,0,0,1,1,1\}$, the control points be

$$\mathbf{c}_1 = (-1,1,0); \mathbf{c}_2 = (0,1,1); \mathbf{c}_3 = (1,1,0); \mathbf{c}_4 = (-1,0,1);$$
$$\mathbf{c}_6 = (1,0,1); \mathbf{c}_7(-1,-1,0); \mathbf{c}_8 = (0,-1,1); \mathbf{c}_9 = (1,-1,0),$$

the weights be $w_1 = w_3 = w_7 = w_9 = 1; w_2 = w_4 = w_6 = w_8 = \frac{\sqrt{2}}{2}$, and

$$\mathbf{S}(u,v) = \frac{\sum_{i=1,i\neq 5}^{9} \mathbf{c}_i w_i B_i(u,v)}{\sum_{i=1,i\neq 5}^{9} w_i B_i(u,v)}, \quad (u,v) \in [0,1] \times [0,1]. \tag{6}$$

According to the properties of the quadratic rational Bézier curve, the four boundaries of the surface (6) are quarter circles (see Figure 6).

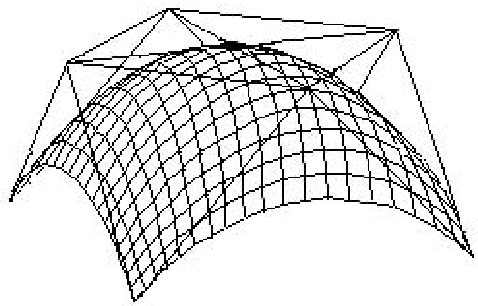

Figure 6. The four boundaries of the surface (6) in Example 3 are quarter circles.

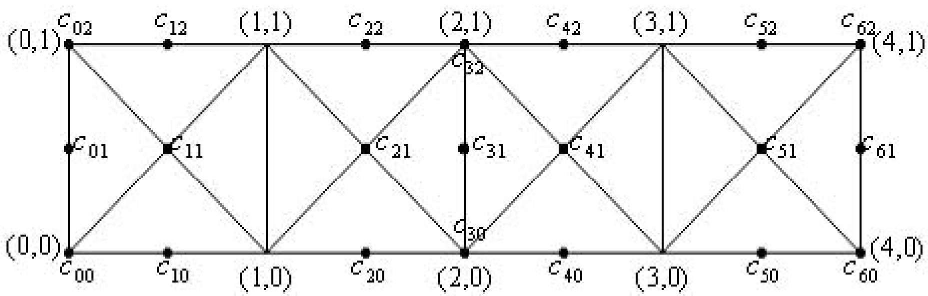

Figure 7. Control points and parametric domain in Example 4.

Example 4. Let knot vectors be $U = \{0,0,0,1,2,2,3,4,4,4\}; V = \{0,0,0,1,1,1\}$ (see Figure 7), the control points be

$$\mathbf{c}_{00} = (0,0,1); \mathbf{c}_{10} = (1,0,1); \mathbf{c}_{20} = (1,0,-1); \mathbf{c}_{30} = (0,0,-1);$$

$$\mathbf{c}_{40} = (-1,0,-1); \mathbf{c}_{50} = (-1,0,1); \mathbf{c}_{60} = \mathbf{c}_{00}; \mathbf{c}_{01} = (0,1,1);$$

$$\mathbf{c}_{11} = (1,1,1); \mathbf{c}_{21} = (1,1,-1); \mathbf{c}_{31} = (0,1,-1); \mathbf{c}_{41} = (-1,1,-1);$$

$$\mathbf{c}_{51} = (-1,1,1); \mathbf{c}_{61} = \mathbf{c}_{01}; \mathbf{c}_{02} = (0,2,1); \mathbf{c}_{12} = (1,2,1); \mathbf{c}_{22} = (1,2,-1);$$

$$\mathbf{c}_{32} = (0,2,-1); \mathbf{c}_{42} = (-1,2,-1); \mathbf{c}_{52} = (-1,2,1); \mathbf{c}_{62} = \mathbf{c}_{02},$$

the weights be $w_{0j} = w_{3j} = w_{6j} = 1; w_{1j} = w_{2j} = w_{4j} = w_{5j} = \frac{1}{2}, j = 0,1,2$, and

$$\mathbf{S}(u,v) = \frac{\sum_{i=0}^{6}\sum_{j=0}^{2} \mathbf{c}_{ij} w_{ij} B_{ij}(u,v)}{\sum_{i=0}^{6}\sum_{j=0}^{2} w_{ij} B_{ij}(u,v)}, \quad (u,v) \in [0,4] \times [0,1]. \tag{7}$$

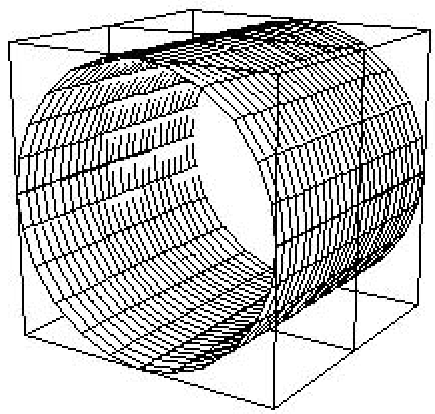

Figure 8. The surface (7) is a circular cylinder in Example 4.

According to the properties of the quadratic rational B-spline curve, the surface (7) is a circular cylinder (see Figure 8).

Appendix.

Referring to (Ren-Hong Wang, 1985; Ren-Hong Wang, 1994) let parametric knots be

$$\begin{cases} u_{-2} \le u_{-1} \le u_0 \le \cdots \le u_m \le u_{m+1} \le u_{m+2} \\ v_{-2} \le v_{-1} \le v_0 \le \cdots \le v_n \le v_{n+1} \le v_{n+2} \end{cases}, h_i = u_i - u_{i-1}, k_j = v_j - v_{j-1},$$

$$A_i = \frac{h_i}{h_i + h_{i+1}}, A_i' = \frac{h_{i+1}}{h_i + h_{i+1}}, B_j = \frac{k_j}{k_j + k_{j+1}}, B_j' = \frac{k_{j+1}}{k_j + k_{j+1}}.$$

A bivariate quadric polynomial defined on a triangle is determined by its values at three vertexes and midpoints of 3 edges of this triangle. So, for all $-1 \le i \le m+1$ and $-1 \le j \le n+1$, every non-uniform B-spline $B_{ij}(u,v) \in S_2^1(\triangle_{mn}^{(2)}), (u,v) \in [u_{i-1}, u_{i+2}] \times [v_{j-1}, v_{j+2}]$ can be represented as follows (see Figure 9).

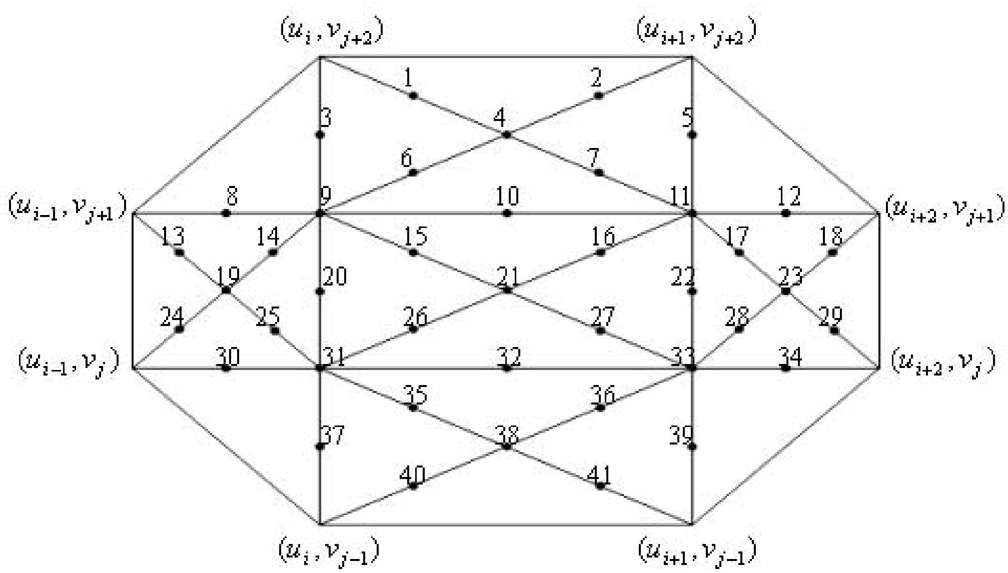

Figure 9. The support of B-spline.

$$1.\ \frac{1}{16}B'_{j+1};\ \ 2.\ \frac{1}{16}B'_{j+1};\ \ 3.\ \frac{1}{4}A_iB'_{j+1};\ \ 4.\ \frac{1}{4}B'_{j+1};\ \ 5.\ \frac{1}{4}A'_{i+1}B'_{j+1};$$

$$6.\ \frac{1}{16}(5+4A_i)B'_{j+1};\ \ 7.\ \frac{1}{16}(5+4A'_{i+1})B'_{j+1};\ \ 8.\ \frac{1}{4}A_iB'_{j+1};\ \ 9.\ A_iB'_{j+1};$$

$$10.\ \frac{1}{4}(2+A_i+A'_{i+1})B'_{j+1};\ \ 11.\ A'_{i+1}B'_{j+1};\ \ 12.\ \frac{1}{4}A'_{i+1}B'_{j+1};\ \ 13.\ \frac{1}{16}A_i;$$

$$14.\ \frac{1}{16}(5+4B'_{j+1})A_i;\ \ 15.\ \frac{1}{16}(A'_{i+1}+B_j+5A_i+5B'_{j+1}+4A_iB'_{j+1});$$

$$16.\ \frac{1}{16}(A_i+B_j+5A'_{i+1}+5B'_{j+1}+4A'_{i+1}B'_{j+1});\ \ 17.\ \frac{1}{16}(5+4B'_{j+1})A'_{i+1};$$

$$18.\ \frac{1}{16}A'_{i+1};\ \ 19.\ \frac{1}{4}A_i;\ \ 20.\ \frac{1}{4}(2+B_j+B'_{j+1})A_i;\ \ 21.\ \frac{1}{4}(A_i+A'_{i+1}+B_j+B'_{j+1});$$

$$22.\ \frac{1}{4}(2+B_j+B'_{j+1})A'_{i+1};\ \ 23.\ \frac{1}{4}A'_{i+1};\ \ 24.\ \frac{1}{16}A_i;\ \ 25.\ \frac{1}{16}(5+4B_j)A_i;$$

$$26.\ \frac{1}{16}(A'_{i+1}+B'_{j+1}+5A_i+5B_j+4A_iB_j);\ \ 27.\ \frac{1}{16}(A_i+B'_{j+1}+5A'_{i+1}+5B_j+4A_{i+1}B_j);$$

$$28.\ \frac{1}{16}(5+4B_j)A'_{i+1};\ \ 29.\ \frac{1}{16}A'_{i+1};\ \ 30.\ \frac{1}{4}A_iB_j;\ \ 31.\ A_iB_j;\ \ 32.\ \frac{1}{4}(2+A_i+A'_{i+1})B_j;$$

$$33.\ A'_{i+1}B_j;\ \ 34.\ \frac{1}{4}A'_{i+1}B_j;\ \ 35.\ \frac{1}{16}(5+4A_i)B_j;\ \ 36.\ \frac{1}{16}(5+4A'_{i+1})B_j;\ \ 37.\ \frac{1}{4}A_iB_j;$$

$$38.\ \frac{1}{4}B_j;\ \ 39.\ \frac{1}{4}A'_{i+1}B_j;\ \ 40.\ \frac{1}{16}B_j;\ \ 41.\ \frac{1}{16}B_j.$$

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