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CASCADIC MULTIGRID METHOD FOR ISOPARAMETRIC FINITE ELEMENT WITH NUMERICAL INTEGRATION *1)

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Abstract

The purpose of this paper is to study the cascadic multigrid method for the second-order elliptic problems with curved boundary in two-dimension which are discretized by the isoparametric finite element method with numerical integration. We show that the CCG method is accurate with optimal complexity and traditional multigrid smoother (like symmetric Gauss-Seidel, SSOR or damped Jacobi iteration) is accurate with suboptimal complexity.

Key words: Cascadic multigrid method, Isoparametric element, Numerical integration.

1. Introduction

In this paper, we consider the second-order linear elliptic problems posed over a bounded domain $\Omega \subset R^2$ with curved boundary Γ. The problem can be described as

$$Lu = -\text{div}(a(x)\text{grad}u) + b(x)u = f, \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0, \quad \text{on} \quad \Gamma, \tag{1.2}$$

where $a(x)$ is a (sufficiently smooth) uniformly positive definite matrix in Ω, $b(x)$ is sufficiently smooth and $0 < b \leq b(x)$.

The weak form of the problem (1.1)–(1.2) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall v \in H_0^1(\Omega), \tag{1.3}$$

where

$$a(u,v) = \int_\Omega ((a(x)\nabla u) \cdot \nabla v + b(x)uv)\text{d}x,$$

$$(f,v) = \int_\Omega fv\text{d}x.$$

In this paper, we will need to assume the H^2-regularity on Problem (1.1)–(1.2). We formalize it into assumption (A.1).

A.1 For any $f \in L^2(\Omega)$, the corresponding solution u of Problem (1.1)-(1.2) is in the space $H^2(\Omega) \cap H_0^1(\Omega)$ and there exists a constant C independent of u and f such that

$$\|u\|_{2,\Omega} \leq C\|f\|_{0,\Omega}, \qquad \forall f \in L^2(\Omega). \tag{1.4}$$

For the second-order selfadjoint elliptic boundary value problems, taking into account numerical integration, Ciarlet [5], Ciarlet and Raviart [6] and Li [8] obtained the error estimates

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in $H^1(\Omega_h)$-norm and $H^1(\Omega)$-norm respectively. In general, Ω_h does not contain Ω and vice versa. Because the identical quadrature scheme is used in [5] [6] and [8], the finite element solution obtained in [8] is the same as that in [5] and [6] respectively. In [8], Li avoid to extend the partial differential equation and only need the H^2-regularity assumption on the differential equation (1.1)-(1.2) to obtain the error estimate in $H^1(\Omega)-$norm. While in [5] and [6], in order to make (4.4.68) in [5] hold and obtain the error estimate in $H^1(\Omega_h)-$norm, the higher regularity assumption on the differential equation will be needed.

On the other hand, Bornemann and Deuflhard [2] have recently proposed the so-called cascadic multigrid method. As a distinctive feature this method performs more iterations on coarser levels so as to obtain less iterations on finer level. A first candidate of such a cascadic multigrid method was the recently suggested cascadic conjugate gradient method in Deuflhard [7], in short CCG method, which used the CG method as a basic iteration method on each level. The first publication of this algorithm in Deuflhard [7] contained rather convincing numerical results, but no theoretical justification. For the second-order elliptic problem in 2D which is discretized by the P1 conforming element, Bornemann and Deuflhard [2] have proved that the CCG method is accurate with optimal computational complexity for the conjugate gradient method as a smoother and only nearly optimal complexity for other conventional iterative smoother, like symmetric Gauss-Seidel, SSOR or damped Jacobi method. Shi and Xu [11] establish the general framework to analyse the cascadic multigrid method. In [12], Shi and Xu develop the cascadic multigrid method for parabolic problems and obtain the optimal convergence accuracy and computational complexity. Shaidurov and Tobiska [10] study the convergence of the CCG method which is used to solve the elliptic problems in domain with re-entrant corners.

In this paper we use the cascadic multigrid method to solve the second-order elliptic problems with curved boundary discretized by the isoparametric finite element method taking into account numerical integration. We show that in this case the CCG method is accurate with optimal complexity and traditional multigrid smoother (like symmetric Gauss-Seidel, SSOR or damped Jacobi iteration) is accurate with suboptimal complexity.

Apart from the introduction, this paper comprises three sections. In Sect.2, we give a special discretization using the isoparametric finite element. In Sect.3, we take into account into numerical integration and derive the error estimate in L^2–norm over the domain Ω. Finally, we prove the accuracy and complexity of cascadic multigrid method for H^2–regular elliptic problems in Sect.4.

2. A special Discreziation

We start with a coarse approximate triagulation $\mathcal{T}_0$ of Ω for a sufficiently small h_0. The triangulation $\{\tau_k\}$ will be defined from $\{\tau_{k-1}\}$ as follows:

(1). If τ_{k-1} is a triangle with two vertices in Ω then τ_{k-1} is broken into four finer-grid triangles by the line connecting the midpoints of the edges of the triangle τ_{k-1}.

(2). If τ_{k-1} is a triangle with two vertices on $\partial\Omega$, we take a new boundary point to be the crosspoint of the boundary arc and the vertical bisector between the two boundary vertices of τ_{k-1} and form two curved triangles and two straight triangles by connecting the nodes of the element τ_{k-1}. From the construction of $\mathcal{T}_k$, we know that $h_k \approx \frac{1}{2}h_{k-1}$.

In this paper, as in Li [8], we give a special triangulation $\mathcal{T}_k$, that is

$$\overline{\Omega} = \cup_{K\in\mathcal{T}_k}\overline{K}, \quad h_{K,k} = \mathrm{diam}(K), \quad h_k = \max_{K\in\mathcal{T}_k} h_{K,k}, \tag{2.1}$$

where the interior finite element $(K, P_K, \sum_K)(K \in \mathcal{T}_k)$ is obtained from a reference finite element $(\widehat{K}, \widehat{P}, \widehat{\sum})$ through an affine mapping $F_K(\widehat{K})$ which is uniquely determined by the data of the nodes of the finite element K (see [5] and [6]). While the boundary finite element

$(K, P_K, \sum_K)(K \in \mathcal{T}_k)$ can be obtained from a modified reference finite element $(\widetilde{K}, \widetilde{P}, \widetilde{\sum})$ through an isoparametric mapping $F_K \in P_2(\widetilde{K})$ which is obtained by generalizing original mapping $F_K(\widehat{K})$, where $\widetilde{K}$ is expanded from $\widehat{K}$ and the mapping $F_K(\widetilde{K})$ satisfies $K \subset F_K(\widetilde{K})$, for details see [8]. We always assume the isoparametric family is regular (see Ciarlet [5]).

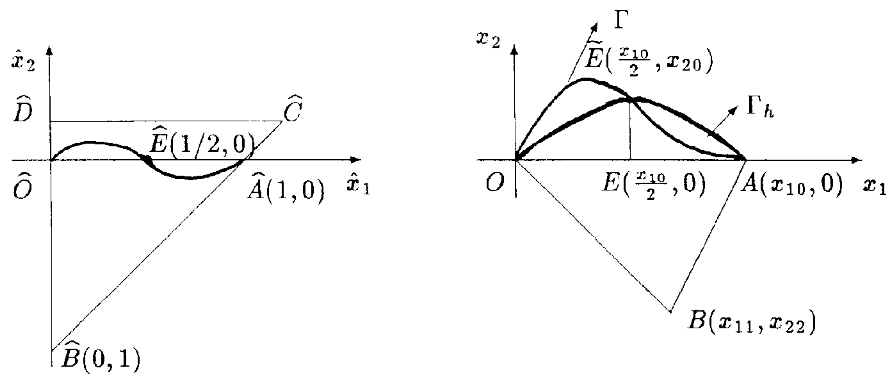

Fig.1.

In this paper, as usual, we still denote the modified reference finite element by $(\widehat{K}, \widehat{P}, \widehat{\sum})$ if there is no confusion.

For convenience, Let $E_{k,h}$ denote the set of nodes of the boundary finite elements corresponding to $\mathcal{T}_k$ which are on Γ.

Let M_k denote the isoparametric finite element space associated with the triangulation $\mathcal{T}_k$

$$M_k = \{v_k \in C^0(\overline{\Omega}) : v_k(x) = 0, \forall x \in E_{k,h}, v_k|_K = \hat{v}_k \cdot F_K^{-1}, \quad \hat{v}_k \in P_2(\widehat{K})\}.$$

Note that $M_{k-1} \not\subseteq M_k$. At each level k, the discrete problem of the problem (1.3) is: Find $u_k \in M_k$, such that

$$a(u_k, v_k) = (f, v_k), \quad \forall v_k \in M_k, \tag{2.2}$$

where

$$a(u_k, v_k) = \int_\Omega ((a(x)\nabla u_k) \cdot \nabla v_k + b(x) u_k v_k) \mathrm{d}x,$$

$$(f, v_k) = \int_\Omega f v_k \mathrm{d}x.$$

In this paper, the notation of Sobolev spaces and associated norms are the same as those in Ciarlet [5], and C denotes the positive constant independent of h_k and the number of the levels and may be different at different occurrence.

3. Numerical Integration and Error Estimate

In this section, we take into account numerical integration and derive the error estimate $||u - u_k||_{0,\Omega}$ under the H^2- regularity assumption which will serve as a crucial ingredient of the cascadic multigrid optimality analysis in section 4.

In paper [6], Ciarlet and Raviart obtained the error estimate $||\tilde{u} - u_h||_{L^2(\Omega_h)}$. Because the extension $\tilde{u}$ of the exact solution u to the set Ω_h must satisfy the original differential equation, the higher regularity assumption on the differential equation (1.1)-(1.2) will be needed.

Let $K = F_K(\widehat{K})$, $\hat{\varphi} : \widehat{K} \to R$, $\varphi : K \to R$, $\varphi = \hat{\varphi} \cdot F_K^{-1}$. Over the reference element $\widehat{K}$, we apply the same quadrature scheme as that used in [5] and [6],

$$\int_{\widehat{K}} \hat{\varphi}(\hat{x}) d\hat{x} \sim \sum_{l=1}^{L} \hat{\omega}_l \hat{\varphi}(\hat{b}_l), \quad \hat{\omega}_l > 0, \quad \hat{b}_l \in \widehat{K}, \quad 1 \le l \le L. \tag{3.1}$$

where $\hat{b}_l (1 \le l \le L)$ are the vertices and midpoints of the edges of the reference element $\widehat{K}$.

Given an isoparametric finite element $K = F_K(\widehat{K})$ corresponding to an invertible mapping $F : \widehat{K} \to K$, we may, and will, assume that

$$J_{F_K}(\hat{x}) = \det(DF_K(\hat{x})) > 0, \quad \forall \hat{x} \in \widehat{K}.$$

Using the standard formula for the change of variables in multiple integrals, we find that the quadrature scheme (3.1) over the reference element $\widehat{K}$ automatically induces the quadrature scheme over the finite element K, namely,

$$\int_K \varphi(x) dx \sim \sum_{l=1}^{6} \omega_{l,K} \varphi(b_{l,K}), \tag{3.2}$$

with weights $\omega_{l,K}$ and nodes $b_{l,K}$ defined by

$$\omega_{l,K} = \hat{\omega}_l J_{F_K}(\hat{b}_l) \quad \text{and} \quad b_{l,K} = F_K(\hat{b}_l), \quad 1 \le l \le 6,$$

where $b_{l,K} (1 \le l \le 6)$ are the nodes of the element $K \in \mathcal{T}_k$ (see Ciarlet [5] and Li [8]).

Accordingly, we introduce the quadrature error functionals

$$E_K(\varphi) = \int_K \varphi(x) dx - \sum_{l=1}^{6} \omega_{l,K} \varphi(b_{l,K}); \widehat{E}(\hat{\varphi}) = \int_{\widehat{K}} \hat{\varphi}(\hat{x}) d\hat{x} - \sum_{l=1}^{L} \hat{\omega}_l \hat{\varphi}(\hat{b}_l),$$

which are related by $E_K(\varphi) = \widehat{E}(\hat{\varphi} J_{F_K})$ and $\widehat{E}(\hat{\varphi}) = E(\varphi J_{F_K}^{-1})$.

In this paper, in order to obtain the error estimate $||u - u_k||_{0,\Omega}$, as in the paper Ciarlet and Raviart[6], the quadrature scheme over the reference element $\widehat{K}$ is exact for the space $P_4(\widehat{K})$, i.e.,

$$\widehat{E}(\hat{v}) = 0, \quad \forall \hat{v} \in P_4(\widehat{K}). \tag{3.3}$$

The discrete problem of the problem (1.3) with numerical integration is: Find $u_k \in M_k$ such that

$$a^*_{h,k}(u_k, v_k) = f^*_{h,k}(v_k), \quad \forall v_k \in M_k, \tag{3.4}$$

where

$$a^*_{h,k}(u_k, v_k) = \sum_{K \in \mathcal{T}_k} \sum_{l=1}^{6} \omega_{l,K}((a \nabla u_k) \cdot \nabla v_k + b u_k v_k)(b_{l,K}), \tag{3.5}$$

$$f^*_{h,k}(v_k) = \sum_{K \in \mathcal{T}_k} \sum_{l=1}^{6} \omega_{l,K}(f v_k)(b_{l,K}). \tag{3.6}$$

Lemma 3.1. (see [9]) *Suppose the quadrature scheme (3.1) is exact for the space $P_4(\widehat{K})$, i.e., (3.3) holds, then there exist $C_1, C_2 > 0$, $h_0' \in (0,1)$, such that*

$$C_1||v_k||_{1,\Omega}^2 \leq a_{h,k}^*(v_k,v_k) \leq C_2||v_k||_{1,\Omega}^2, \quad \forall h \in (0,h_0], \quad v_k \in M_k. \tag{3.7}$$

From Lemma 3.1, we know that the discrete problem (3.4) has a unique solution $u_k \in M_k$. The following estimate is proved in Li [8]:

$$||u - \Pi_k u||_{1,\Omega} \leq Ch_k^2||u||_{3,\Omega}, \quad \forall u \in H^3(\Omega) \cap H_0^1(\Omega), \tag{3.8}$$

where $\Pi_k u \in M_k$ is the interpolant of u.

Using the method in Li[8], we can obtain the following estimate:

$$||u - \Pi_k u||_{0,\Omega} \leq Ch_k^2||u||_{2,\Omega}, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3.9}$$

From Li [8], we have the following error estimate:

Lemma 3.2. *Suppose the union $\cup_{l=1}^L\{\hat{b}_l\}$ contains a $P_1(\widehat{K})$-unisolvent subset or the quadrature scheme (3.1) is exact for the space $P_2(\widehat{K})$, then*

$$||u - u_k||_{1,\Omega} \leq Ch_k^2(||u||_{3,\Omega} + ||f||_{2,q,\Omega}), \tag{3.10}$$

where $f \in W^{2,q}(\Omega)$, $q \geq 2$.

Lemma 3.3. *There exists a constant C independent of h_k such that*

$$||v||_{0,\Omega_h\backslash\Omega} \leq Ch_k^3|v|_{1,\Omega_h\backslash\Omega}, \quad ||v||_{0,\Omega\backslash\Omega_h} \leq Ch_k^3|v|_{1,\Omega\backslash\Omega_h}, \tag{3.11}$$

either for all functions $v \in H_0^1(\Omega_h)$ or all functions $v \in H^1(\widetilde{\Omega})$ such that $v = 0$ on Γ, where $\widetilde{\Omega} \subset R^2$ is a bounded open subset such that $\Omega \subset \widetilde{\Omega}$ and $\Omega_h \subset \widetilde{\Omega}$ for all h_k.

Proof. The first inequality in (3.11) is proved in Lemma 2 in Ciarlet and Raviart [6]. Using the method in the proof of Lemma 2 in [6], we can easily see that the second inequality in (3.11) is also valid.

Lemma 3.4. (Lemma 3 in [6]) *For any function $v \in H^1(\Omega_h)$, there exists a constant C independent of h_k such that*

$$||v||_{0,\Gamma_h} \leq C||v||_{1,\Omega_h}, \tag{3.12}$$

where $\Gamma_h = \partial\Omega_h$.

In order to estimate $||u - u_k||_{0,\Omega}$, we use the duality argument. We consider the following auxiliary problem:

$$L\phi = -\text{div}(a(x)\text{grad}\phi) + b(x)\phi \;=\; g, \quad \text{in} \quad \Omega, \tag{3.13}$$

$$\phi \;=\; 0, \quad \text{on} \quad \Gamma, \tag{3.14}$$

We assume the H^2–regularity on Problem (1.1)-(1.2). Obviously, Problem (3.12)–(3.14) also has H^2–regularity, for any $g \in L^2(\Omega)$, the corresponding solution ϕ is in the space $H^2(\Omega) \cap H_0^1(\Omega)$ and there exists a constant C independent of ϕ and g such that

$$||\phi||_{2,\Omega} \leq C||g||_{0,\Omega}, \qquad \forall g \in L^2(\Omega). \tag{3.15}$$

From the Tietze-Urysohn extension theorem, we know that there exists an extension operator $\mathcal{E}$: $H^2(\Omega) \to H^2(R^2)$ satisfying $||\mathcal{E}v||_{2,R^2} \leq C||v||_{2,\Omega}$. For different function spaces, the extension operator is different. To simplify notation, in this paper, we denote the extension operator by $\mathcal{E}$ if there is no confusion. We have the following extension problem:

$$L(\mathcal{E}\phi) = -\text{div}((\mathcal{E}a)\text{grad}(\mathcal{E}\phi)) + (\mathcal{E}b)(\mathcal{E}\phi) \;=\; \mathcal{E}g, \quad \text{in} \quad \Omega_h, \tag{3.16}$$

where the functions $\mathcal{E}a, \mathcal{E}\phi$ and $\mathcal{E}b$ are the extensions of the functions a, ϕ and b to the set Ω_h, when $\Omega_h \not\subset \Omega$. Fixed $\mathcal{E}a, \mathcal{E}\phi$ and $\mathcal{E}b$, $\mathcal{E}g$ will be determined by the left side of the equation (3.16).

From equation (3.16), $||\mathcal{E}v||_{2,R^2} \leq C||v||_{2,\Omega}$ and H^2–regularity, we have

$$||\mathcal{E}g||_{0,\Omega_h} \leq C||g||_{0,\Omega}.$$

In this paper, for $v_k \in M_k$, $\mathcal{E}v_k$ is the natural extension of v_k and for simplicity we still denote the extension $\mathcal{E}v_k$ by v_k.

Theorem 3.5. *Suppose the quadrature scheme (3.1) is exact for the space $P_4(\widehat{K})$, i.e., (3.3) holds, then for any sufficiently small h_k, we have*

$$||u - u_k||_{0,\Omega} \leq Ch_k^3(||u||_{3,\Omega} + ||f||_{3,\Omega}), \tag{3.17}$$

where $f \in H^3(\Omega)$.

Proof. By the definition of $||\cdot||_{0,\Omega}$, we have

$$||u - u_k||_{0,\Omega} = \sup_{g \in L^2(\Omega)\backslash\{0\}} \frac{(u - u_k, g)_{0,\Omega}}{||g||_{0,\Omega}}. \tag{3.18}$$

From (3.16), we may write $(u - u_k, g)_{0,\Omega}$ as follows:

$$\begin{aligned} (u - u_k, g)_{0,\Omega} &= (\mathcal{E}u - u_k, \mathcal{E}g)_{0,\Omega_h} - (\mathcal{E}u - u_k, \mathcal{E}g)_{0,\Omega_h\backslash\Omega} \\ &\quad + (u - u_k, g)_{0,\Omega\backslash\Omega_h} \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.19}$$

From the Cauchy–Schwarz inequality, the property of the extension operator and Lemma 3.3, we obtain

$$\begin{aligned} |I_2| &\leq ||\mathcal{E}u - u_k||_{0,\Omega_h\backslash\Omega} ||\mathcal{E}g||_{0,\Omega_h\backslash\Omega} \\ &\leq C(||\mathcal{E}u||_{0,\Omega_h\backslash\Omega} + ||u_k||_{0,\Omega_h\backslash\Omega})||g||_{0,\Omega} \\ &\leq C(h_k^3||u||_{3,\Omega} + ||u_k||_{0,\Omega_h\backslash\Omega})||g||_{0,\Omega}. \end{aligned} \tag{3.20}$$

For the term $||u_k||_{0,\Omega_h\backslash\Omega}$, from (3.11), we have

$$||u_k||_{0,\Omega_h\backslash\Omega} \leq Ch_k^3||u_k||_{1,\Omega_h\backslash\Omega} \leq Ch_k^3(||u - u_k||_{1,\Omega} + ||u||_{1,\Omega}). \tag{3.21}$$

Combining (3.20) with (3.21) yields

$$|I_2| \leq Ch_k^3(||u - u_k||_{1,\Omega} + ||u||_{3,\Omega})||g||_{0,\Omega}. \tag{3.22}$$

Similar as above, using (3.11), we get

$$|I_3| \leq Ch_k^3(||u - u_k||_{1,\Omega} + ||u||_{3,\Omega})||g||_{0,\Omega}. \tag{3.23}$$

Using the Green's formula and $u_k = 0$ on Γ_h, we obtain

$$\begin{aligned} I_1 &= a_{\Omega_h}(\mathcal{E}u - u_k, \mathcal{E}\phi) + \int_{\Gamma_h} (\mathcal{E}a)\mathrm{grad}(\mathcal{E}\phi) \cdot \mathbf{n}(\mathcal{E}u)\mathrm{d}s \\ &= R_1 + R_2. \end{aligned} \tag{3.24}$$

where the bilinear form $a_{\Omega_h}(\cdot,\cdot)$ is defined over the domain Ω_h and $\mathbf{n} = (n_1, n_2)$ is the unit outer normal along Γ_h.
By the Cauchy–Schwarz inequality, we have

$$|R_2| \leq C||\mathcal{E}u||_{0,\Gamma_h}||\mathrm{grad}(\mathcal{E}\phi)||_{0,\Gamma_h} \tag{3.25}$$

By Lemma 3.4, we get

$$||\mathrm{grad}(\mathcal{E}\phi)||_{0,\Gamma_h} \leq C||\mathcal{E}\phi||_{2,\Omega_h} \leq C||\phi||_{2,\Omega}. \tag{3.26}$$

It follows from (5.13) in Ciarlet and Raviart [6] that

$$||\mathcal{E}u||_{0,\Gamma_h} \leq Ch_k^3||\mathcal{E}u||_{1,\infty,\Omega_h} \leq Ch_k^3||u||_{3,\Omega}. \tag{3.27}$$

From (3.25)–(3.27), we get

$$|R_2| \leq Ch_k^3||u||_{3,\Omega}||g||_{0,\Omega}, \tag{3.28}$$

where the regularity assumption $||\phi||_{2,\Omega} \leq C||g||_{0,\Omega}$ is used.

To estimate $R_1 = a_{\Omega_h}(\mathcal{E}u - u_k, \mathcal{E}\phi)$, we may write

$$\begin{aligned} R_1 = \; & a_{\Omega_h}(\mathcal{E}u - u_k, \mathcal{E}\phi - \Pi_k\phi) - a_{\Omega_h}(u_k, \Pi_k\phi) \\ & + a^*_{\Omega_h}(u_k, \Pi_k\phi) + a_{\Omega_h}(\mathcal{E}u, \Pi_k\phi) - f^*_{h,k}(\Pi_k\phi). \end{aligned} \tag{3.29}$$

where the representations of the numerical integration terms $a^*_{\Omega_h}(u_k, \Pi_k\phi)$ and $f^*_{h,k}(\Pi_k\phi)$ are given by (3.5) and (3.6) over the domain Ω_h, respectively, and $\Pi_k\phi \in M_k$ is the interpolant of ϕ.

We may rewrite $a_{\Omega_h}(\mathcal{E}u, \Pi_k\phi) - f^*_{h,k}(\Pi_k\phi)$ as follows.

$$\begin{aligned} & a_{\Omega_h}(\mathcal{E}u, \Pi_k\phi) - f^*_{h,k}(\Pi_k\phi) \\ & \quad = (L(\mathcal{E}u), \Pi_k\phi)_{0,\Omega_h} - f^*_{h,k}(\Pi_k\phi) \\ & \quad = (f, \Pi_k\phi)_{0,\Omega_h\cap\Omega} - f^*_{h,k}(\Pi_k\phi) + (L(\mathcal{E}u), \Pi_k\phi)_{0,\Omega_h\setminus\Omega} \\ & \quad = (\mathcal{E}f, \Pi_k\phi)_{0,\Omega_h} - f^*_{h,k}(\Pi_k\phi) - (\mathcal{E}f, \Pi_k\phi)_{0,\Omega_h\setminus\Omega} \\ & \qquad + (L(\mathcal{E}u), \Pi_k\phi)_{0,\Omega_h\setminus\Omega}, \end{aligned} \tag{3.30}$$

where the extension $\mathcal{E}f$ of the function f is different from the extension $\mathcal{E}g$ of the function g. The extension $\mathcal{E}f$ of the function f is obtained by the extension theorem and doesn't require $L(\mathcal{E}u) = \mathcal{E}f$ over Ω_h.

From the Cauchy–Schwarz inequality, (3.11) and the regularity assumption (3.15), we obtain

$$\begin{aligned} |(\mathcal{E}f, \Pi_k\phi)_{0,\Omega_h\setminus\Omega}| & \leq ||\mathcal{E}f||_{0,\Omega_h\setminus\Omega}||\Pi_k\phi||_{0,\Omega_h\setminus\Omega} \\ & \leq Ch_k^3||f||_{1,\Omega}||\Pi_k\phi||_{1,\Omega_h} \\ & \leq Ch_k^3||f||_{1,\Omega}(||\mathcal{E}\phi - \Pi_k\phi||_{1,\Omega_h} + ||\mathcal{E}\phi||_{1,\Omega_h}) \\ & \leq Ch_k^3||f||_{1,\Omega}||g||_{0,\Omega}. \end{aligned} \tag{3.31}$$

$$|(L(\mathcal{E}u), \Pi_k\phi)_{0,\Omega_h\setminus\Omega}| \leq Ch_k^3||u||_{3,\Omega}||g||_{0,\Omega}. \tag{3.32}$$

From Theorem 5 in Ciarlet and Raviart [6], we have

$$\begin{aligned} |(\mathcal{E}f, \Pi_k\phi)_{0,\Omega_h} - f^*_{h,k}(\Pi_k\phi)| & \leq Ch_k^3||\mathcal{E}f||_{3,\Omega_h}||\Pi_k\phi||_{2,\Omega_h,h} \\ & \leq Ch_k^3||f||_{3,\Omega}||\phi||_{2,\Omega} \\ & \leq Ch_k^3||f||_{3,\Omega}||g||_{0,\Omega}. \end{aligned} \tag{3.33}$$

where $||\Pi_k\phi||_{2,\Omega_h,h} = (\sum\limits_{K\in\mathcal{T}_k} ||\Pi_k\phi||^2_{2,K})^{1/2}$ and we have used (5.16) in [6], i.e., $||\Pi_k\phi||_{2,\Omega_h,h} \le ||\phi||_{2,\Omega}$.

From (3.30)–(3.33), we have

$$|a_{\Omega_h}(\mathcal{E}u,\Pi_k\phi) - f^*_{h,k}(\Pi_k\phi)| \le Ch_k^3(||f||_{3,\Omega} + ||u||_{3,\Omega})||g||_{0,\Omega}. \tag{3.34}$$

From Lemma 3.2 and (3.9), we have

$$|a_{\Omega_h}(\mathcal{E}u - u_k, \mathcal{E}\phi - \Pi_k\phi)| \le Ch_k^3(||f||_{2,q,\Omega} + ||u||_{3,\Omega})||g||_{0,\Omega}. \tag{3.35}$$

From (5.20) in [6], we know that

$$\begin{aligned} &|a_{\Omega_h}(u_k,\Pi_k\phi) - a^*_{\Omega_h}(u_k,\Pi_k\phi)| \\ &\quad\le C(h_k||u_k - \Pi_k u||_{1,\Omega_h} + h_k^3||\mathcal{E}u||_{3,\Omega_h})||g||_{0,\Omega} \\ &\quad\le C(h_k||u_k - \Pi_k u||_{1,\Omega} + h_k^3||u||_{3,\Omega})||g||_{0,\Omega} \\ &\quad\le Ch_k^3(||u||_{3,\Omega} + ||f||_{2,q,\Omega})||g||_{0,\Omega}, \end{aligned} \tag{3.36}$$

where the triangle inequality, Lemma 3.2 and (3.8) have been used.

Combining (3.29), (3.34), (3.35) with (3.36), we obtain

$$|R_1| = |a_{\Omega_h}(\mathcal{E}u - u_k, \mathcal{E}\phi)| \le Ch_k^3(||u||_{3,\Omega} + ||f||_{3,\Omega})||g||_{0,\Omega}. \tag{3.37}$$

From (3.18), (3.19), (3.22), (3.23), (3.24), (3.28) and (3.37), we get the desired result (3.17).

4. Cascadic Multigrid Method

We now construct a cascadic multigrid algorithm for the isoparametric finite element method. Since $M_{k-1} \not\subseteq M_k$, we introduce coarse-to-fine intergrid operator $I_k : M_{k-1} \to M_k$.

For $v \in M_{k-1}$, if p is a vertex of a element in $\mathcal{T}_k$ and $\bar{p}$ is the interior midpoint of an edge of a element in $\mathcal{T}_k$, $\bar{q}$ is the crosspoint of the arc and vertical bisector between two vertices (which are on Γ) of the curved boundary element in $\mathcal{T}_k$, then $I_k v \in M_k$ is determined by

$$(I_k)(p) = \begin{cases} 0, & \text{if} \quad p \in \Gamma, \\ v(p), & \text{if} \quad p \notin \Gamma, \end{cases}$$

$$(I_k v)(\bar{p}) = v(\bar{p}), \quad (I_k v)(\bar{q}) = 0.$$

Using the special triangulation given in section 2 and the standard scaling argument [5], we can show that the following lemma holds for the isoparametric finite element space M_k.

Lemma 4.1. *For any $v_k \in M_k, K \in \mathcal{T}_k$,*

$$c||v_k||^2_{0,K} \le h_k^2 \sum_{i=1}^{6} v_k^2(a_i) \le C||v_k||^2_{0,K}. \tag{4.1}$$

where $a_i (1 \le i \le 6)$ are the nodes of the element K.

Lemma 4.2. *For the operator I_k we have*

$$||I_k\Pi_{k-1}u - \Pi_k u||_{0,\Omega} \le Ch_k^3(|u|_{2,\Omega} + |u|_{3,\Omega}), \tag{4.2}$$

where $\Pi_k u \in M_k$ is the interpolant of u.

Proof. If the element $K \in \mathcal{T}_{k-1}$ is an interior element or a boundary element with two vertices in Ω then

$$I_k\Pi_{k-1}u - \Pi_k u = 0.$$

If the element $K_1 \in \mathcal{T}_{k-1}$ is a boundary element with two vertices on $\partial\Omega$, it is broken into two curved triangles and two straight triangles $e_i \in \mathcal{T}_k$ $(1 \le i \le 4)$. For the boundary element $e_1 \in \mathcal{T}_k$, we have from Lemma 4.1

$$||I_k\Pi_{k-1}u - \Pi_k u||^2_{0,e_1} \le Ch_k^2 \sum_{i=1}^{6} \phi_k^2(a_i), \tag{4.3}$$

where $\phi_k := (I_k\Pi_{k-1}u - \Pi_k u)|_{e_1}$, a_1, a_3 and a_5 are the vertices of the element e_1 and a_2, a_4 are the midpoints of the edges and a_6 is the node of the arc a_5a_3 on Γ of the element e_1 respectively.

By the definitions of I_k, Π_k and Π_{k-1},

$$\phi_k(a_1) = \phi_k(a_3) = \phi_k(a_5) = 0, \tag{4.4}$$

and

$$\begin{aligned} |\phi(a_2)| = |(I_k\Pi_{k-1}u - \Pi_k u)(a_2)| &= |\Pi_{k-1}u(a_2) - u(a_2)| \\ &\le |\Pi_{k-1}u - u|_{0,\infty,K_1} \end{aligned} \tag{4.5}$$

Similarly, we have

$$|\phi(a_4)| \le |\Pi_{k-1}u - u|_{0,\infty,K_1}, \quad |\phi(a_6)| \le |\Pi_{k-1}u - u|_{0,\infty,K_1}. \tag{4.6}$$

Arguing as in the case of Theorem 4.3.4 in Ciarlet [5], using the method given in Li [8], we know

$$||\Pi_{k-1}u - u||_{0,\infty,K_1} \le Ch_k^2(|u|_{2,K_1} + |u|_{3,K_1}). \tag{4.7}$$

From (4.3)–(4.7), we have

$$||I_k\Pi_{k-1}u - \Pi_k u||^2_{0,e_1} \le Ch_k^6(|u|^2_{2,K_1} + |u|^2_{3,K_1}). \tag{4.8}$$

Similar estimates can be obtained for the elements $e_i, i = 2, 3, 4$, i.e.,

$$||I_k\Pi_{k-1}u - \Pi_k u||^2_{0,e_i} \le Ch_k^6(|u|^2_{2,K_1} + |u|^2_{3,K_1}), \quad i = 2, 3, 4. \tag{4.9}$$

Summing all elements $K_1 \in \mathcal{T}_{k-1}$, we obtain the desired estimate (4.2).

Lemma 4.3. *For the operator I_k we have*

$$||v - I_k v||_{0,\Omega} \le Ch_k|v|_{1,\Omega}, \quad \forall v \in M_{k-1}. \tag{4.10}$$

Proof. Let a boundary element $K_1 \in \mathcal{T}_{k-1}$ be broken into four elements $e_i \in \mathcal{T}_k$ $(1 \le i \le 4)$. For the interior element $e_3 \in \mathcal{T}_k$, we have

$$||v - I_k v||_{0,e_3} \le Ch_k|v|_{1,K_1}, \tag{4.11}$$

where $I_k v|_{e_3}$ is the polynomial interpolant of v.

For the curved boundary element $e_1 \in \mathcal{T}_k$, from the definition of I_k, we have

$$\begin{aligned} ||v - I_k v||_{0,e_1} &\le ||v - \Pi_k v||_{1,e_1} + ||\Pi_k v - I_k v||_{0,e_1} \\ &\le Ch_k||v||_{1,e_1} + ||\sum_{i=1}^{6}(v - I_k v)(a_i)\phi_{a_i}||_{0,e_1} \\ &\le Ch_k||v||_{1,e_1} + ||(v - I_k v)(a_6)\phi_{a_6}||_{0,e_1} \\ &\le Ch_k||v||_{1,e_1} + |(v - I_k v)(a_6)|||\phi_{a_6}||_{0,e_1} \\ &\le Ch_k||v||_{1,e_1} + |v(a_6)|||\phi_{a_6}||_{0,e_1}. \end{aligned} \tag{4.12}$$

where $a_i (1 \le i \le 6)$ are the nodes of e_1, a_6 is the node on arc a_5a_3 along Γ and ϕ_{a_i} is the corresponding nodal basis.

Let a_6' be the point on the arc a_5a_3 along Γ_h satisfying $|a_6a_6'| \le Ch_k^3$. Obviously, $v(a_6') = 0$. From the Taylor expansion, we deduce

$$|v(a_6)| = |v(a_6) - v(a_6')| \le Ch_k^3 |v|_{1,\infty,e_1} \le Ch_k^2 |v|_{1,K_1}. \tag{4.13}$$

Combining (4.12), (4.13) with $||\phi_{a_6}||_{0,e_1} \le Ch_k$ yields

$$||v - I_k v||_{0,e_1} \le Ch_k^3 |v|_{1,K_1}. \tag{4.14}$$

Similarly, we get

$$||v - I_k v||_{0,e_2} \le Ch_k^3 |v|_{1,K_1}, \quad ||v - I_k v||_{0,e_4} \le Ch_k |v|_{1,K_1}. \tag{4.15}$$

Squaring both sides of the inequalities (4.11), (4.14), (4.15) and summing up all $K_1 \in \mathcal{T}_{k-1}$, we obtain the desired estimate (4.10).

Lemma 4.4. *Assume that the hypotheses of Theorem 3.5 hold and u_k are the solution of (3.4), then, for any sufficiently small h_k, we have*

$$||u_k - I_k u_{k-1}||_{0,\Omega} \le Ch_k^3 (||u||_{3,\Omega} + ||f||_{3,\Omega}). \tag{4.16}$$

Proof. From Theorem 3.5 and (3.8), we have

$$\begin{aligned} ||u_k - I_k u_{k-1}||_{0,\Omega} &\le ||u - u_k||_{0,\Omega} + ||u - \Pi_k u||_{0,\Omega} + ||\Pi_k u - I_k u_{k-1}||_{0,\Omega} \\ &\le Ch_k^3 (||u||_{3,\Omega} + ||f||_{3,\Omega}) + ||\Pi_k u - I_k u_{k-1}||_{0,\Omega}. \end{aligned} \tag{4.17}$$

From Lemma 4.2 and Lemma 4.3, we get

$$\begin{aligned} ||\Pi_k u - I_k u_{k-1}||_{0,\Omega} &\le ||\Pi_k u - I_k \Pi_{k-1} u||_{0,\Omega} + ||I_k(\Pi_{k-1} u - u_{k-1})||_{0,\Omega} \\ &\le Ch_k^3 ||u||_{3,\Omega} + ||\Pi_{k-1} u - u_{k-1}||_{0,\Omega} \\ &\quad + ||(\Pi_{k-1} u - u_{k-1}) - I_k(\Pi_{k-1} u - u_{k-1})||_{0,\Omega} \\ &\le Ch_k^3 ||u||_{3,\Omega} + ||\Pi_{k-1} u - u||_{0,\Omega} + ||u - u_{k-1}||_{0,\Omega} \\ &\quad + Ch_k ||\Pi_{k-1} u - u_{k-1}||_{1,\Omega} \\ &\le Ch_k^3 (||u||_{3,\Omega} + ||f||_{3,\Omega}) + Ch_k ||\Pi_{k-1} u - u||_{1,\Omega} \\ &\quad + Ch_k ||u - u_{k-1}||_{1,\Omega} \\ &\le Ch_k^3 (||u||_{3,\Omega} + ||f||_{3,\Omega}). \end{aligned} \tag{4.18}$$

Combining (4.17) with (4.18) completes the proof of Lemma 4.4.

Lemma 4.5. (see [8]) *There exists an $h_0' \in (0,1)$ such that*

$$||v_k||_{0,\Gamma} \le Ch_k^{\frac{5}{2}} |v_k|_{1,\cup K'}, \quad \forall v_k \in M_k, \quad h_k \in (0, h_0'], \tag{4.19}$$

where K' is the boundary finite element.

Following [3] and [11], we introduce a projection operator $P_k : M_{k-1} + M_k \to M_k$ defined by

$$a(P_k u, v) = a(u, v), \quad \forall v \in M_k. \tag{4.20}$$

We note that P_k is the elliptic projection operator defined by (4.20) without numerical integration. From the definition of P_k, we know that

$$|||P_k v|||_{1,k} \le |||v|||_{1,k-1}, \quad \forall v \in M_{k-1}, \tag{4.21}$$

where the norm $|||v|||_{1,k-1}$ of $v \in M_{k-1}$ is defined by $|||v|||_{1,k-1} = \sqrt{a(v,v)}$.

It is not difficult to see that the norm induced by $\sqrt{a(\cdot,\cdot)}$ is equivalent to the standard $H^1(\Omega)$ norm:

$$C_1|||v|||_{1,k} \leq ||v||_{1,\Omega} \leq C_2|||v|||_{1,k}, \quad \forall v \in M_k. \tag{4.22}$$

Lemma 4.6. *For the operator P_k, we have*

$$||v - P_k v||_{0,\Omega} \leq Ch_k||v||_{1,\Omega}, \quad \forall v \in M_{k-1}. \tag{4.23}$$

Proof. We use the duality argument (see [3] and [11]) to complete the proof. Consider the following problem: for a given $v \in M_{k-1}$, find $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$L\zeta = -\mathrm{div}(a(x)\mathrm{grad}\zeta) + b(x)\zeta = v - P_k v, \quad \text{in} \quad \Omega, \tag{4.24}$$

$$\zeta = 0, \quad \text{on} \quad \Gamma, \tag{4.25}$$

Using the Green formula, we obtain

$$\begin{aligned} ||v - P_k v||_0^2 &= a(\zeta, v - P_k v) - \int_\Gamma (a\mathrm{grad}\zeta)\cdot \mathbf{n} v \mathrm{d}s \\ &\quad + \int_\Gamma (a\mathrm{grad}\zeta)\cdot \mathbf{n} P_k v \mathrm{d}s \\ &= T_1 + T_2 + T_3, \end{aligned} \tag{4.26}$$

where n is the unit outerward normal vector along Γ.

Let ζ_k be any function in M_k. It follows from the definition of the operator P_k, the norm equivalence (4.22) and the regularity assumption $||\zeta||_{2,\Omega} \leq C||v - P_k v||_{0,\Omega}$ that

$$\begin{aligned} |T_1| &= |a(\zeta, v - P_k v)| = |a(\zeta - \zeta_k, v - P_k v)| \\ &\leq C \inf_{\zeta_k \in M_k} ||\zeta - \zeta_k||_{1,\Omega}||v - P_k v||_{1,\Omega} \\ &\leq Ch_k||\zeta||_{2,\Omega}||v - P_k v||_{1,\Omega} \\ &\leq Ch_k||v - P_k v||_{0,\Omega}||v||_{1,\Omega}. \end{aligned} \tag{4.27}$$

Noting that $|\zeta|_{1,\Gamma} \leq C||\zeta||_{2,\Omega}$, we have, by Lemma 4.5 and the regularity assumption $||\zeta||_{2,\Omega} \leq C||v - P_k v||_{0,\Omega}$,

$$|T_2| \leq Ch_k^{\frac{5}{2}}||v - P_k||_{0,\Omega}||v||_{1,\Omega}.$$

Using the norm equivalence (4.22), we know that similar estimation to $|T_2|$ is also valid for $|T_3|$. Combining above inequalities completes the proof.

We use the operator $C_k : M_k \to M_k$ to denote the basic iterative procedure on the level k. Denoting m_k steps of the basic iteration on the level k by C_{k,m_k}, the cascadic multigrid method can be written as:

Cascadic multigrid method

(1). Set $u_0^0 = u_0^* \hat{=} u_0$, where u_0 is the solution of (3.3) on coarse initial triangulation $\mathcal{T}_0$. Let

$$u_k^0 = I_k u_{k-1}^*$$

(2). For $k = 1, \cdots, J$

$$u_k^{m_k} = C_{k,m_k} u_k^0$$

(3). Set $u_J^* \hat{=} u_J^{m_J}$.

Following [2], we call a cascadic multigrid method *optimal* for level J, if we obtain accuracy

$$||u_J - u_J^*||_{1,\Omega} \approx ||u - u_J||_{1,\Omega}$$

which means that the iteration error is comparable to the approximation error, and *multigrid complexity*

$$\text{amount of work} = O(n_J)$$

where $m_J = \dim M_J$.

We consider the following type of basic iterations for the finite element problem on the level k started with $u_k^0 \in M_k$:

$$u_k - C_{k,m_k} u_k^0 = T_k^{m_k}(u_k - u_k^0), \tag{4.28}$$

with a linear mapping $T_k : M_k \to M_k$ for the error propagation. We call the basic iteration an H^1-norm reducing smoother, if it obeys the smoothing properties

$$\text{(i).} \quad |||T_k^{m_k} v_k|||_{1,k} \leq C \frac{h_k^{-1}}{m_k^\gamma} ||v_k||_{0,\Omega}, \quad \forall v_k \in M_k, \tag{4.29}$$

$$\text{(ii).} \quad |||T_k^{m_k} v_k|||_{1,k} \leq |||v_k|||_{1,\Omega}, \quad \forall v_k \in M_k, \tag{4.30}$$

with a parameter $0 < \gamma \leq 1$.

For $k = 0, 1, \cdots, J$, define the operator $A_k : M_k \to M_k$ by

$$(A_k v, \omega) = a_{h,k}^*(v, \omega), \quad \forall v, \omega \in M_k.$$

It has been shown in Bank and Dupont [1] that the Richardson iterative operator $T_k = I - \frac{1}{\lambda_k} A_k$ satisfies (4.29) and (4.30) with $\gamma = 1/2$, where λ_k denotes the largest eigenvalue of A_k. Using the same method of constructing smoothers in Bramble [4], we can show that the operator $T_k = I - R_k A_k$, where R_k is the symmetric Gauss-Seidel, SSOR or Jacobi iteration on the level k, also satisfies (4.29) and (4.30) with $\gamma = 1/2$. Using the similar argument of Theorem 2.2 in [2], we can show that the conjugate gradient iteration is a smoother in the sense of (4.29) and (4.30) with a parameter $\gamma = 1$.

As in our example, by construction, h_{k+1} is approximately equal to $\frac{1}{2} h_k$. Thus, we have $2^{J-k} h_J / c \leq h_k \leq c 2^{J-k} h_J$. We consider sequences $m_1, \cdots, m_J$ of the kind

$$m_k = [\beta^{J-k} m_J] \tag{4.31}$$

for some fixed $\beta > 0$, where $[\cdot]$ denotes the chosen integral function.

Next, By means of the framework established in [11], from Theorem 3.5, Lemma 4.2, Lemma 4.3, Lemma 4.4 and Lemma 4.6, we obtain the following Theorem 4.7.

Theorem 4.7. *Assume that the hypotheses of Theorem 3.5 and (4.28)–(4.30) hold and the symmetric Gauss-Seidel, SSOR or Jacobi iteration is used as a smoother, then for any sufficiently small* h_k, *the error of the cascadic multigrid method can be estimated by*

$$||u_J - u_J^*||_{1,\Omega} \leq C \sum_{k=1}^{J} \frac{h_k^2}{m_k^\gamma} (||u||_{3,\Omega} + ||f||_{3,\Omega}). \tag{4.32}$$

The similar argument of Lemma 1.3 and Lemma 1.4 in [2] leads to the following theorems.

Theorem 4.8. *Let the number m_k of iterations on level k be given by (4.31). Then the accuracy of the cascadic multigrid method for the isoparametric element taking into account numerical integration is*

$$||u_J - u_J^*||_{1,\Omega} \leq \begin{cases} C \cdot \frac{1}{1-\frac{2}{\beta^\gamma}} \frac{h_J^2}{m_J^\gamma}(||u||_{3,\Omega} + ||f||_{3,\Omega}), & \text{for} \quad \beta > 2^{1/\gamma}, \\ C \cdot J \frac{h_J^2}{m_J^\gamma}(||u||_{3,\Omega} + ||f||_{3,\Omega}) & \text{for} \quad \beta = 2^{1/\gamma}. \end{cases} \tag{4.33}$$

Theorem 4.9. *Let the number m_k of iterations on level k be given by (4.31). Then the computational cost of the cascadic multigrid method for the isoparametric element is proportional to*

$$\sum_{k=1}^{J} m_k n_k \leq \begin{cases} C \frac{1}{1-\frac{\beta}{4}} m_J n_J, & \text{for} \quad \beta < 4, \\ C J m_J n_J, & \text{for} \quad \beta = 4. \end{cases} \tag{4.34}$$

If we choose to fix accuracy, we obtain as an immediate consequence of our results the following Theorem 4.10.

Theorem 4.10. *If the Gauss-Seidel or Jacobi iteration is used as a smoother and the number of iteration on level J is*

$$m_J = [m_* J^2].$$

The error of the cascadic multigrid method for the isoparametric element is

$$||u_J - u_J^*||_{1,\Omega} \leq C \frac{h_J^2}{m_*^{1/2}}(||u||_{3,\Omega} + ||f||_{3,\Omega}) \tag{4.35}$$

and the complexity of computation is

$$\sum_{k=1}^{J} m_k n_k \leq C m_* n_J (1 + \log n_J)^3. \tag{4.36}$$

Remark. If the conjugate gradient method is used as a smoother, Theorem 4.7 still holds. The reader may refer to [2] for details.

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