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THE DERIVATIVE PATCH INTERPOLATING RECOVERY TECHNIQUE FOR FINITE ELEMENT APPROXIMATIONS *1)

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Abstract

A derivative patch interpolating recovery technique is analyzed for the finite element approximation to the second order elliptic boundary value problems in two dimensional case. It is shown that the convergence rate of the recovered gradient admits superc onvergence on the recovered subdomain, and is two order higher than the optimal global convergence rate (ultracovergence) at an internal node point when even order finite element spaces and local uniform meshes are used.

Key words: Finite element, Derivative recovery, Ultraconvergence.

1. Introduction

Finite element superconvergence property has long attracted considerable attentions since its practical importance in enhancing the accuracy of finite element calculation and in adaptive computing via a posteriori error estimate. In this field affluent research results have been achieved. For some complete literature on superconvergence, the reader is referred to Wahlbin's book [1], Chen and Huang's book [2], and a recent conference proceeding edited by Krizek et al. [3]. In article [4,5], Lin Qun et al. proposed a new type of interpolation operator into the finite element spaces, that is the interpolation operator of projection type, and remarked that it will approximate the finite element solutions much better than the usual Lagrange interpolation. Thus, the interpolation operator of projection type provides a new powerful means in the research of finite element superconvergence, and we will use it as main analysis means in this paper.

In a previous work[6], the ultraconvergence (i.e., two order higher than the optimal global convergence rate) of the derivative patch interpolating recovery technique was analyzed for a class of two-point boundary value problems. The current work is devoted to the superconvergence and ultraconvergence properties of the derivative interpolating recovery technique for finite element approximation to the elliptic equation $Au = f$ on a rectangular domain with the general partial differential operator of second order

$$A = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} a_{ij} \frac{\partial}{\partial x_i} + \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} + a_0 I$$

and $A = -\Delta + a_0 I$ when ultraconvergence is concerned. In this article, we will assume that the rectangular partition mesh is regular for general case, or quasi-regular when superconvergence is considered. Moreover, when we analysis the ultraconvergence at an interior nodal point p_0, we will also assume that the mesh is local uniform, that is the four elements which share

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the common interior nodal point p_0 are uniform. In general, the solution of the second order elliptic boundary value problems on a rectangular domain may have corner singularity, and consequently, the finite element approximation may suffer from the "pollution effect" which will result in the failure of the recovery procedure. There have been many techniques to treat the pollution effect caused by domain singularity, for example, the local mesh refinement. In order to concentrate on the local recovery method, in this paper, we assume that the solution is smooth enough on domain Ω for our purpose, otherwise some local analysis methods should be used[4,7].

Recently, many research works focus on the so-called $Z-Z$ derivative patch recovery technique[8-11], and this technique is considered to be one of the most effectiveness techniques in the research of asymptotically exact a posteriori error estimates[12]. This technique uses the least square method to fit the first order derivatives of finite element solution and results in superconvergence. The ultraconvergence property of $Z-Z$ technique has been analyzed by Zhang[13] for the Laplace equation in the two dimensional setting. Comparing with the $Z-Z$ technique, our recovery method is more simple and easier to implement, and possesses the explicit expression.

In this paper, we shall use notation H_0^1 and W_p^m to represent the usual Sobolev spaces on domain Ω with norm and seminorm $\|\cdot\|_{m.p}$ and $|\cdot|_{m.p}$ on W_p^m, respectively, and use letter C to denote a generic constant.

The plan of this paper is as follows: In Section 2 we introduce the interpolation operator of projection type and discuss its approximation properties. In Section 3 the derivative patch interpolating recovery operator is defined and its super-approximation and ultra-approximation properties are analyzed. Section 4 is devoted to the superconvergence and ultraconvergence properties for the finite element approximation to the second order elliptic boundary value problems.

2. Interpolation Operator of Projection Type and Its Super-approximation Properties

Let element $e = e_1 \times e_2 = (x_e - h_e, x_e + h_e) \times (y_e - \hbar_e, y_e + \hbar_e)$, $\{L_j(x)\}_{j=0}^{\infty}$ and $\{\widetilde{L}_j(y)\}_{j=0}^{\infty}$ be the normalized orthogonal *Legendre* polynomial systems on $L_2(e_1)$ and $L_2(e_2)$, respectively. Set

$$\omega_0(x) = \widetilde{\omega}_0(y) = 1, \quad \omega_{j+1}(x) = \int_{x_e-h_e}^{x} L_j(x)\,dx, \;\; \widetilde{\omega}_{j+1}(y) = \int_{y_e-\hbar_e}^{y} L_j(y)\,dy, \;\; j \geq 0$$

It is well known that polynomials $\omega_{k+1}(x)$ and $L_k(x)\,(k \geq 1)$ have $k+1$ and k zero points on $\overline{e}_1$ and e_1, respectively, and these zero points are symmetrically distributed with respect to the middle point x_e. Denote the two kinds of zero point set by $N_k^{(0)} = \{\, g_j^{(0)} \,\}$ and $N_k = \{\, g_j \,\}$, respectively, and we call $N_k^{(0)}$ the Lobatto point set and N_k the Gauss point set. Moreover, we know that these polynomials also possess the following symmetry and antisymmetry

$$\omega_{2j}(x_e + x) = \omega_{2j}(x_e - x), \quad \omega_{2j-1}(x_e + x) = -\omega_{2j-1}(x_e - x) \tag{1}$$

$$L_{2j}(x_e + x) = L_{2j}(x_e - x), \quad L_{2j-1}(x_e + x) = -L_{2j-1}(x_e - x) \tag{2}$$

The completely parallel conclusions hold for the polynomials $\widetilde{\omega}_{k+1}(y)$ and $\widetilde{L}_k(y)$ on element $e_2 = (y_e - \hbar_e, y_e + \hbar_e)$.

Below we denote the Lobatto and Gauss points on element $e = e_1 \times e_2$ by $\{\, G_{ij}^{(0)} = (g_i^{(0)}, \widetilde{g}_j^{(0)}) \,\}$ and $\{\, G_{ij} = (g_i, \widetilde{g}_j) \,\}$, respectively, and also denote the *Gauss* lines by $G_{x,j} = \{\, (x, \widetilde{g}_j);\; x \in e_1,\; \widetilde{g}_j \in \widetilde{N}_k \}$ and $G_{i,y} = \{\, (g_i, y)\,;\; g_i \in N_k,\; y \in e_2 \,\}$.

Now let $u \in H^2(e)$, then we have Fourier expansion [4]

$$u(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \beta_{ij}\,\omega_i(x)\,\tilde{\omega}_j\,(y)\,,\;(x,y)\in e \tag{3}$$

$$\beta_{00} = u(x_e - h_e, y_e - \hbar_e), \quad \beta_{ij} = \int_e u_{xy}\,L_{i-1}(x)\,\tilde{L}_{j-1}\,(y)\,dxdy \tag{4}$$

$$\beta_{i0} = \int_{e_1} u_x(x, y_e - \hbar_e)\,L_{i-1}(x)\,dx\,, \quad \beta_{0j} = \int_{e_2} u_y(x_e - h_e, y)\,\tilde{L}_{j-1}\,(y)\,dy\,,\; i,j\geq 1 \tag{5}$$

Introduce the k-order and bicomplete k-order polynomial spaces P_k and Q_k, i.e.

$$p(x,y) = \sum_{i+j=0}^{k} a_{ij}\,x^i y^j\,,\;\forall\, p\in P_k\,; \quad q(x,y) = \sum_{i=0}^{k}\sum_{j=0}^{k} a_{ij}\,x^i y^j\,,\;\forall\, q\in Q_k$$

Define the k-order interpolation operator of projection type by $\pi_k\,:\,H^2(e)\to Q_k(e)$ such that

$$\pi_k u(x,y) = \sum_{i=0}^{k}\sum_{j=0}^{k} \beta_{ij}\,\omega_i(x)\,\tilde{\omega}_j\,(y),\;\;(x,y)\in e \tag{6}$$

Then π_k is uniquely solvable with respect to $Q_k(e)$ and for $k\geq 1$ possesses properties[4]

$$\pi_k u(x_e \pm h_e, y_e \pm \hbar_e) = u(x_e \pm h_e, y_e \pm \hbar_e) \tag{7}$$

$$|u - \pi_k u|_{m.p.e} \leq Ch^{k+1-m}|u|_{k+1.p.e},\; 1\leq p\leq\infty,\; 0\leq m\leq k+1 \tag{8}$$

here $h = \sqrt{h_e^2 + \hbar_e^2}$. From (3) and (6) we have

$$u - \pi_k u = (\sum_{i=0}^{k}\sum_{j=k+1}^{\infty} + \sum_{i=k+1}^{\infty}\sum_{j=0}^{k} + \sum_{i=k+1}^{\infty}\sum_{j=k+1}^{\infty})\,\beta_{ij}\,\omega_i(x)\,\tilde{\omega}_j\,(y) \tag{9}$$

Lemma 1. *Let $u\in Q_k(e)\bigcup\{x^{k+1}, y^{k+1}\}$, $D_1 = \frac{\partial}{\partial x}$, $D_2 = \frac{\partial}{\partial y}$. Then*

$$u - \pi_k u = \beta_{k+1,0}\omega_{k+1}(x) + \beta_{0,k+1}\,\tilde{\omega}_{k+1}\,(y)$$
$$D_1(u - \pi_k u) = \beta_{k+1,0}L_k(x)\,, \quad D_2(u - \pi_k u) = \beta_{0,k+1}\,\tilde{L}_k\,(y)$$

Proof. Let $u\in Q_k(e)\bigcup\{x^{k+1}, y^{k+1}\}$ so that $u_{xy}\in Q_{k-1}(e)$, $u_x\in P_k(e_1)$ and $u_y\in P_k(e_2)$. Then, from $(4)-(5)$ and the orthogonality of *Legendre* polynomial system, we have

$$\beta_{ij} = 0,\;\; i\geq k+1, j\geq 1 \text{ or } i\geq 1,\; j\geq k+1, \quad \beta_{i0} = \beta_{0j} = 0,\;\; i\geq k+2,\; j\geq k+2$$

combining this with (9) to obtain the representation of $u-\pi_k u$, the other representations follow from taking partial derivatives for the representation formula of $u-\pi_k u$, the proof is completed.

Corollary. *Let $u\in W_{\infty}^{k+2}(e)$. Then, the interpolation operator π_k possesses the super-approximation properties*

$$|(u - \pi_k u)(G_{ij}^{(0)})| \leq Ch^{k+2}|u|_{k+2.\infty.e},\; k\geq 1 \tag{10}$$

$$|D_l(u - \pi_k u)(G_{ij})| \leq Ch^{k+1}|u|_{k+2.\infty.e},\; k\geq 1,\; l = 1,2 \tag{11}$$

Proof. We only prove the estimate (11), the proof of estimate (10) is similar. Introduce bilinear transformation $F\;:\hat{e}\to\; e$ by $(x,y) = F(\xi,\eta) = (x_e + \xi h_e, y_e + \eta\hbar_e)$, $(\xi,\eta)\in\hat{e}=$

$(-1,1)\times(-1,1)$. Denote $\hat{u}(\xi,\eta) = u(F(\xi,\eta))$, $\hat{D} = D_\xi$ (or D_η). For fixed point $G_{ij} = F(\xi_i,\eta_j)$ and smooth function u, define the linear functional

$$E(\hat{u}) = \hat{D}(\hat{u} - \hat{\pi}_k \hat{u})(\xi_i,\eta_j) = h_e D_1(u - \pi_k u)(G_{ij}) \tag{12}$$

From (8) we see that E is a linear bounded functional on $W_\infty^{k+2}(\hat{e})$, and it follows from Lemma 1 that

$$E(\hat{u}) = 0, \ \forall \ \hat{u} \in \ P_{k+1}(\hat{e})$$

Then, according to the Bramble-Hilbert Lemma[14], we have

$$|E(\hat{u})| \leq C|\ \hat{u}\ |_{k+2.\infty.\hat{e}} \leq Ch^{k+2}|u|_{k+2.\infty.e}$$

Combining this with (12), estimate (11) is proved.

Remark 1. From the proof above, it is easy to see that the super-approximation points G_{ij} in (11) can be replaced by super-approximation lines $G_{x,j}$ (or $G_{i,y}$).

Some delicate properties (see (15)-(17)) of the interpolation operator of projection type, that are not shared by the usual Lagrange interpolation, can be shown as follows.

From Lemma 1 we see that

$$D_1 D_2(u - \pi_k u)(x,y) = 0, \ \ (x,y) \in e, \ \ \forall u \in P_{k+1}(e) \tag{13}$$

Moreover, when $u \in P_{k+2}(e)$, from (4)-(5) and (9) we can obtain

$$D_1 D_2(u - \pi_k u) = \sum_{i=1}^{k} \beta_{i,k+1} L_{i-1}(x)\ \tilde{L}_k\ (y) + \sum_{j=1}^{k} \beta_{k+1,j} L_k(x)\ \tilde{L}_{j-1}\ (y) + \beta_{k+1,k+1} L_k(x)\ \tilde{L}_k\ (y)$$

hence

$$D_1 D_2(u - \pi_k u)(G_{ij}) = 0, \ \ \forall u \in P_{k+2}(e) \tag{14}$$

By (13), (14) and Bramble-Hilbert Lemma, we can prove similarly as in the Corollary that the following global super-approximation and ultra-approximation results hold

$$|D_1 D_2(u - \pi_k u)(x,y)\,| \leq Ch^k |u|_{k+2.\infty.e}, \ (x,y) \in e, \ k \geq 1 \tag{15}$$

$$|D_1 D_2(u - \pi_k u)(G_{ij})| \ \leq Ch^{k+1}|u|_{k+3.\infty.e}\,, \ k \geq 1 \tag{16}$$

From Lemma 1 and the orthogonality of Legender polynomial system, we also see that for $u \in Q_k(e) \bigcup \{x^{k+1},\, y^{k+1}\}$

$$\int_e \nabla(u - \pi_k u) \nabla v \, dxdy = 0, \ \forall v \in Q_k(e) \tag{17}$$

This shows that the interpolation approximation $\pi_k u$ can be considered as the finite element solution of the Laplace equation with exact solution $u \in Q_k(e) \bigcup \{x^{k+1},\, y^{k+1}\}$.

3. Derivative Patch Interpolating Recovery Technique

In this section we shall introduce the derivative patch interpolating recovery operator and discuss its super-approximation and ultra-approximation properties. Let $e^{(s)}$ $(s = 1, 2, 3, 4)$ be four elements which share a common interior node point (x_0, y_0). Corresponding to the point (x_0, y_0), we define the derivative patch recovery domain (see figure 1)

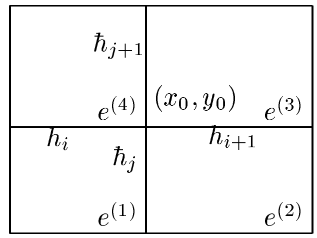

Figure 1

$$D_0 = \bigcup_{s=1}^{4} e^{(s)} = (x_0 - h_i, x_0 + h_{i+1}) \times (y_0 - \hbar_j, y_0 + \hbar_{j+1})$$

Let $G_{ij} = (g_i, \tilde{g}_j)$, $i, j = \pm 1, \cdots, \pm k$ are the $4k^2$ *Gauss* points in D_0, i.e.

$$x_0 - h_i < g_{-k} < \cdots < g_{-1} < x_0 < g_1 < \cdots < g_k < x_0 + h_{i+1}$$
$$y_0 - \hbar_j < \tilde{g}_{-k} < \cdots < \tilde{g}_{-1} < y_0 < \tilde{g}_1 < \cdots < \tilde{g}_k < y_0 + \hbar_{j+1}$$

Because of $4k^2 = (2k)^2$, then a polynomial in Q_{2k-1} can be uniquely determined by its values on the $4k^2$ Gauss points $\{G_{ij}\}$.

Corresponding to the point g_i ($\tilde{g}_j$), introduce the $2k-1$ order Lagrange interpolation basis function $\varphi_i(x) \in P_{2k-1}(x_0 - h_i, x_0 + h_{i+1})$ ($\tilde{\varphi}_i$ $(x) \in P_{2k-1}(y_0 - \hbar_j, y_0 + \hbar_{j+1})$) by

$$\varphi_i(x) = \prod_{l=\pm 1, l \neq i}^{\pm k} \frac{(x - g_l)}{(g_i - g_l)}, \quad \tilde{\varphi}_j\ (x) = \prod_{l=\pm 1, l \neq j}^{\pm k} \frac{(y - \tilde{g}_l)}{(\tilde{g}_j - \tilde{g}_l)}$$

Then, $\{\, \varphi_{ij}(x, y) = \varphi_i(x)\, \tilde{\varphi}_j\ (y) \,\}$ form the basis function system of space $Q_{2k-1}(D_0)$.

Now, we define the derivative patch interpolation recovery operator $R : W^1_\infty(D_0) \to Q_{2k-1}$ (D_0) such that

$$RD_l w(x, y) = \sum_{i=\pm 1}^{\pm k} \sum_{j=\pm 1}^{\pm k} D_l w(G_{ij}) \varphi_{ij}(x, y), \ l = 1, 2, \ \ (x, y) \in D_0 \tag{18}$$

Note that $D_l u_h(x, y))$ may be discontinuous across element edges when u_h is a piecewise polynomial on D_0, but the recovery derivative $RD_l u_h(x, y)$ is a smooth polynomial on D_0.

Lemma 2. *The derivative recovery operator R possesses the following properties*

$$R\nabla u_{k+1}(x, y) = \nabla u_{k+1}(x, y), \ \forall\, u_{k+1} \in Q_k(D_0) \bigcup \{x^{k+1}, y^{k+1}\} \tag{19}$$

$$R\nabla \pi_k u_{k+1}(x, y) = \nabla u_{k+1}(x, y), \ \forall\, u_{k+1} \in Q_k(D_0) \bigcup \{x^{k+1}, y^{k+1}\} \tag{20}$$

$$\|R\nabla u\|_{0.\infty.D_0} \le \|\nabla u\|_{0.\infty.D_0}, \ \forall\, u \in W^1_\infty(D_0) \tag{21}$$

Proof. When $u_{k+1} \in Q_k(D_0) \bigcup \{x^{k+1}, y^{k+1}\}$, we have $D_l u_{k+1} \in Q_k(D_0) \subset Q_{2k-1}(D_0)$, then equality (19) comes from the uniqueness of interpolation polynomial. By Lemma 1 we see that $D_l u_{k+1}(G_{ij}) = D_l \pi_k u_{k+1}(G_{ij})$, $i, j = \pm 1, \cdots, \pm k$, $l = 1, 2$ for $u_{k+1} \in Q_k(D_0) \bigcup \{x^{k+1}, y^{k+1}\}$, then from (18) we have $R\nabla \pi_k u_{k+1}(x, y) = R\nabla u_{k+1}(x, y)$. Now the equality (20) follows from (19). Estimate (21) can be dire ctly verified by mapping D_0 into the standard element $\hat{D} = (-1, 1) \times (-1, 1)$.

Lemma 3. *Let $u \in W_\infty^{k+2}(D_0)$. Then, the operator R possesses the following global super-approximation property*

$$\|\nabla u - R\nabla \pi_k u\|_{0.\infty.D_0} \le Ch^{k+1}|u|_{k+2.\infty.D_0},\ k \ge 1 \tag{22}$$

Proof. The estimate (22) can be obtained by using (20) and Bramble-Hilbert Lemma.

Now we investigate the ultra-approximation property of the derivative recovery operator at an interior nodal point (x_0, y_0). Let D_0 be composed of four rectangular elements, that is $D_0 = (x_0 - h_1, x_0 + h_1) \times (y_0 - h_2, y_0 + h_2)$. Because the Gauss points of element are symmetrically distributed with respect to the middle point of element, so we have

$$x_0 - g_{-l} = g_l - x_0, \quad y_0 - \tilde{g}_{-l} = \tilde{g}_l - y_0,\ \ l = 1, 2, \cdots, k \tag{23}$$

This results in $\varphi_i(x_0) = \varphi_{-i}(x_0)$, $\tilde{\varphi}_j(y_0) = \tilde{\varphi}_{-j}(y_0)$, $i, j = 1, 2, \cdots, k$. Note that the basis functions $\varphi_{ij} = \varphi_i(x)\tilde{\varphi}_j(y)$, then, at the point (x_0, y_0), the definition of operator R (see (18)) can be simplified as

$$RD_l w(x_0, y_0) = \sum_{i=1}^{k}\sum_{j=1}^{k}[D_l w(G_{i,j}) + D_l w(G_{i,-j}) + D_l w(G_{-i,j}) + D_l w(G_{-i,-j})]\varphi_{ij}(x_0, y_0) \tag{24}$$

Lemma 4. *Let $u \in W_\infty^{k+3}(D_0)$, $k \ge 2$ be even, (x_0, y_0) be an interior nodal point, $D_0 = (x_0 - h_1, x_0 + h_1) \times (y_0 - h_2, y_0 + h_2)$. Then, at point (x_0, y_0), the derivative recovery operator R possesses the following ultra-approximation property*

$$|D_l u(x_0, y_0) - RD_l \pi_k u(x_0, y_0)| \le Ch^{k+2}|u|_{k+3.\infty.D_0},\ l = 1, 2,\ k \ge 2$$

Proof. For simplification, without loss of generality, we assume that D_0 is centered at the origin $(x_0, y_0) = (0, 0)$. Then we have

$$RD_l w(0,0) = \sum_{i=1}^{k}\sum_{j=1}^{k}[D_l w(G_{i,j}) + D_l w(G_{i,-j}) + D_l w(G_{-i,j}) + D_l w(G_{-i,-j})]\varphi_{ij}(0,0) \tag{25}$$

If we can prove that for $k \ge 2$ even, hold

$$D_l u(0,0) = RD_l \pi_k u(0,0),\ l = 1, 2,\ \ \forall u \in P_{k+2}(D_0) \tag{26}$$

then, the conclusion of Lemma 4 can be derived from Bramble-Hilbert Lemma. Since $P_{k+2}(D_0) \subset Q_k(D_0) \bigcup \{x^{k+1}, y^{k+1}, x^{k+2}, y^{k+2}, xy^{k+1}, x^{k+1}y\}$, according to (20) in Lemma 2, we only need to prove (26) for $u \in \{x^{k+2}, y^{k+2}, xy^{k+1}, x^{k+1}y\}$. Below denote by $\{l_j(t)\}$ the normalized orthogonal Legendre polynomial system on the standard element $\hat{e} = (-1, 1)$. Then, the Legendre polynomial system on the elements $(-h_1, 0)$ and $(0, h_1)$ can be expressed as, respectively,

$$L_j^-(x) = l_j(t),\ x = -\frac{h_1}{2} + \frac{h_1}{2}t,\ t \in (-1, 1)$$
$$L_j^+(x) = l_j(t),\ x = \frac{h_1}{2} + \frac{h_1}{2}t,\ t \in (-1, 1)$$

By the symmetry and antisymmetry of $l_j(t)$, we have

$$L_{2j}^-(-\tau) = L_{2j}^+(\tau),\ L_{2j-1}^-(-\tau) = -L_{2j-1}^+(\tau),\ 0 \le \tau \le h_1 \tag{27}$$
$$\omega_{2j}^-(-\tau) = \omega_{2j}^+(\tau),\ \omega_{2j-1}^-(-\tau) = -\omega_{2j-1}^+(\tau),\ 0 \le \tau \le h_1 \tag{28}$$

Similar results hold for $\tilde{L}_j(y)$ and $\tilde{\omega}_j(y)$. Now we will prove (26) for $u = x^{k+2}$, y^{k+2}, xy^{k+1}, yx^{k+1} separately.

(1) $u = x^{k+2}$. From (4)-(5) and the orthogonality of Legendre polynomial, we obtain $\beta_{ij} = 0,\ i \geq 0, j \geq 1,\ \beta_{i0} = 0,\ i \geq k+3$. Then, from (9) we have

$$D_1(u - \pi_k u)(x,y) = \begin{cases} \beta^-_{k+1,0} L^-_k(x) + \beta^-_{k+2,0} L^-_{k+1}(x),\ x \in (-h_1, 0) \\ \beta^+_{k+1,0} L^+_k(x) + \beta^+_{k+2,0} L^+_{k+1}(x),\ x \in (0, h_1) \end{cases} \tag{29}$$

$$\beta^-_{j+1,0} = \int_{-h_1}^{0} (k+2) x^{k+1} L^-_j(x)\,dx,\ \beta^+_{j+1,0} = \int_0^{h_1} (k+2) x^{k+1} L^+_j(x)\,dx$$

Note that k is even, then by (27) and integration by substitution, we have $\beta^-_{k+1,0} = -\beta^+_{k+1,0}$, $\beta^-_{k+2,0} = \beta^+_{k+2,0}$. Thus, from (27) and (29), we obtain

$$\begin{aligned} &D_1(u-\pi_k u)(-\tau, y) + D_1(u - \pi_k u)(\tau, y) \\ &= (\beta^-_{k+1,0} + \beta^+_{k+1,0}) L^+_k(\tau) + (\beta^+_{k+2,0} - \beta^-_{k+2,0}) L^+_k(\tau) = 0,\ 0 \leq \tau \leq h_1 \end{aligned}$$

taking $\tau = -g_{-i} = g_i,\ i = 1, 2, \cdots, k$, it implies from (25) that

$$RD_1(u - \pi_k u)(0,0) = 0, \quad \mathit{or} \quad RD_1 \pi_k u(0,0) = RD_1 u(0,0)$$

Since $D_1 u(x,y) = (k+2)x^{k+1},\ D_1 u(g_{-i}, y) = -D_1 u(g_i, y)$, then, from (25) we also obtain $RD_1 u(0,0) = 0$, hence $RD_1 \pi_k u(0,0) = 0 = D_1 u(0,0)$.

(2) $u = y^{k+2}$. From (4)-(5) and the orthogonality of Legendre polynomial, we have $\beta_{ij} = 0,\ i \geq 1, j \geq 0$. Then from (6) we see $D_1 \pi_k u(x,y) = 0$. Therefore $RD_1 \pi_k u(0,0) = 0 = D_1 u(0,0)$.

(3) $u = xy^{k+1}$. From (4)-(5) and the orthogonality of Legendre polynomial, we obtain $\beta_{ij} = 0,\ i \geq 2, j \geq 0$ *or* $i \geq 0, j \geq k+2$. Then, from (9) we have

$$D_1(u - \pi_k u)(x,y) = \begin{cases} \frac{1}{\sqrt{h_1}} \beta^-_{1,k+1}\ \tilde{\omega}^-_{k+1}(y),\ y \in (-h_2, 0) \\ \frac{1}{\sqrt{h_1}} \beta^+_{1,k+1}\ \tilde{\omega}^+_{k+1}(y),\ y \in (0, h_2) \end{cases} \tag{30}$$

$$\beta^-_{1,k+1} = \int_{-h_2}^{0} (k+1) y^k\ \tilde{L}^-_k(y)\,dy,\ \ \beta^+_{1,k+1} = \int_0^{h_2} (k+1) y^k\ \tilde{L}^+_k(y)\,dy$$

By (27) and integration by substitution we see that $\beta^-_{1,k+1} = \beta^+_{1,k+1}$. Then from (28) and (30) we obtain

$$D_1(u - \pi_k u)(x, -\tau) + D_1(u - \pi_k u)(x, \tau) = 0,\ 0 \leq \tau \leq h_2$$

taking $\tau = -\tilde{g}_{-j} = \tilde{g}_j,\ j = 1, 2, \cdots, k$, it implies from (25) that $RD_1(u - \pi_k u)(0,0) = 0$. Since $D_1 u(x,y) = y^{k+1},\ D_1 u(x, \tilde{g}_{-j}) = -D_1 u(x, \tilde{g}_j)$, then from (25) we also obtain $RD_1 u(0,0) = 0$. Hence $RD_1 \pi_k u(0,0) = 0 = D_1 u(0,0)$.

(4) $u = x^{k+1} y$. By (4)-(5) and the orthogonality of Legendre polynomial, we have $\beta_{ij} = 0,\ i \geq 0, j \geq 2$ or $i \geq k+2,\ j \geq 0$. Then, from (9) we obtain

$$D_1(u - \pi_k u)(x,y) = \beta_{k+1,0} L_k(x) + \beta_{k+1,1} L_k(x)\ \tilde{\omega}_1\ (y)$$

hence $D_1(u - \pi_k u)(G_{ij}) = 0,\ i,j = \pm 1, \cdots, \pm k$, this results in $RD_1(u - \pi_k u)(0,0) = 0$. Note that $D_1 u = (k+1) x^k y,\ D_1 u(x, g_{-j}) = -D_1 u(x, g_j)$, we also obtain $RD_1 u(0,0) = 0$. Hence $RD_1 \pi_k u(0,0) = 0 = D_1 u(0,0)$.

Completely similarly we can show that

$$RD_2 \pi_k u(0,0) = D_2 u(0,0),\ \forall\ u \in \{x^{k+2}, y^{k+2}, xy^{k+1}, x^{k+1} y\}$$

Thus, equality (26) is verified and the proof of Lemma 4 is completed.

4. Superconvergence and Ultraconvergence

Let $\Omega \subset R^2$ be a rectangular domain, $J_h = \{e\}$ be a sequence of subdivisions of $\overline{\Omega}$ parameterized by mesh size h so that $\overline{\Omega} = \bigcup_{e \in J_h} \overline{e}$. We assume that the partition is regular and all elements $\{e\}$ are rectangles with sides parallel to the coordinate axes, respectively. Define the k order tensor product finite element space $S_h \subset H_0^1(\Omega)$ as usual. On each element $e \in J_h$, we define the k-order interpolation operator of projection type π_k as in Section 2 so that π_k is defined on J_h. By Lemma 1 and the orthogonality of Legendre polynomial system, it is easy to see that for $k \geq 2$

$$\int_e (u - \pi_k u)\, q\, dxdy = 0\,,\ \forall\, q \in Q_{k-2}(e),\, e \in J_h \tag{31}$$

$$\int_l (u - \pi_k u)\, p\, ds = 0\,,\ \forall\, p \in P_{k-2}(l)\,,\ \text{line segment } l \in \partial e \tag{32}$$

Thus, properties (7) and (32) make $\pi_k :\ H^2(\Omega) \to S_h$, $k \geq 1$. Introduce the bilinear form

$$A(u,v) = \sum_{i,j=1}^{2} (a_{ij}\, D_i u, D_j v) + \sum_{i=1}^{2} (a_i\, D_i u, v) + (a_0\, u, v) \tag{33}$$

where (,) represents the inner product on $L_2(\Omega)$, $a_{ij}(x,y)$, $a_i(x,y)$ and $a_0(x,y)$ are properly smooth functions. It is well known that elementary estimate of interpolation operator (also called as interpolation weak estimate) play an important role in the research of superconvergence. Many detai led discussions have been given for the interpolation operator of Lagrange type [4,7]. For the interpolation operator of projection type, by properties (31) and (32) and using the Bilinear Lemma[14], we can prove that (a detailed proof can be found in article [15,Theorem 7.5-7.6])

Theorem 1. *Let bilinear form $A(u,v)$ be defined by (33) (not necessary positive definite and symmetric), $u \in H_0^1(\Omega) \bigcap W_p^{k+2}(\Omega)$. Then, the interpolation operator π_k satisfies the following super-approximation elementary estimate*

$$|A(u - \pi_k u, v_h)| \leq C h^{k+1} \|u\|_{k+2.p} \|v_h\|_{1.q}\,,\ k \geq 1 \tag{34}$$

For the special case $A(u,v) = (\nabla u, \nabla v) + (a_0 u, v)$, we have the ultra-approximation elementary estimate

$$|A(u - \pi_k u, v_h)| \leq C h^{k+2} \|u\|_{k+3.p} \|v_h\|_{1.q}\,,\ k \geq 2 \tag{35}$$

where $v_h \in S_h$, $2 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Now further assume that $A(u,v)$ is a continuous and uniformly elliptic on $H_0^1(\Omega) \times H_0^1(\Omega)$ (not necessary symmetry). For function $u \in H_0^1(\Omega)$, define its finite element approximation by $u_h \in S_h$ such that

$$A(u - u_h, v_h) = 0,\ \forall\ v_h \in S_h \tag{36}$$

Afterwards we assume that the partition J_h is quasi-regular. By means of the elementary estimates (34)-(35) and the Green function methods[7], we can prove the following results.

Theorem 2. *Let u and u_h satisfy equation (36), $u \in H_0^1(\Omega) \bigcap W_\infty^{k+2}(\Omega)$. Then, the following superapproximation estimate holds*

$$\|\pi_k u - u_h\|_{1.\infty} \leq C h^{k+1} |\, ln\, h|\, \|u\|_{k+2.\infty},\ k \geq 1 \tag{37}$$

For the special case $A(u,v) = (\nabla u, \nabla v) + (a_0 u, v)$ and $u \in H_0^1(\Omega) \bigcap W_\infty^{k+3}(\Omega)$, the ultra-approximation estimate holds

$$\|\pi_k u - u_h\|_{1.\infty} \leq C h^{k+2} |\, ln\, h|\, \|u\|_{k+3.\infty},\ k \geq 2 \tag{38}$$

Now we can prove the main results of our paper.

Theorem 3. *Let u and u_h satisfy equation (36), $u \in H_0^1(\Omega)\bigcap W_\infty^{k+2}(\Omega)$, $D_0 = (x_0 - h_i, x_0 + h_{i+1}) \times (y_0 - \hbar_j, y_0 + \hbar_{j+1})$, (x_0, y_0) be an interior nodal point, R be the derivative patch recovery operator defined by (18). Then we have the following global superconvergence estimate on recovery subdomain D_0*

$$|\nabla u(x,y) - R\nabla u_h(x,y)| \le Ch^{k+1}|ln\, h|\, \|u\|_{k+2.\infty},\ (x,y) \in D_0,\ k \ge 1 \tag{39}$$

Furthermore, for the special case $A(u,v) = (\nabla u, \nabla v) + (a_0 u, v)$, $u \in H_0^1(\Omega)\bigcap W_\infty^{k+3}(\Omega)$, and $h_i = h_{i+1}$, $\hbar_j = \hbar_{j+1}$, we have the following ultraconvergence result with $k \ge 2$ even

$$|\nabla u(x_0,y_0) - R\nabla u_h(x_0,y_0)| \le Ch^{k+2}|ln\, h|\, \|u\|_{k+3.\infty},\ k \ge 2 \tag{40}$$

Proof. From Lemma 2 we know that R is a linear bounded operator, then, using Lemma 3 and Theorem 2, we obtain

$$\begin{aligned}|\nabla u(x,y) - R\nabla u_h(x,y)| &\le |\nabla u(x,y) - R\nabla \pi_k u(x,y)| + \|R\|\, |\pi_k u(x,y) - u_h(x,y)\|_{1.\infty.D_0} \\ &\le Ch^{k+1}|u|_{k+2.\infty.D_0} + Ch^{k+1}|\, ln\, h|\, \|u\|_{k+2.\infty},\ (x,y) \in D_0\end{aligned}$$

Thus, estimate (39) is derived. Similarly, estimate (40) can be obtained by using Lemma 4 and Theorem 2, the proof is completed.

By the results in Theorem 3, it is easy to see that when D_0 is not a superconvergence subdomain of error $\nabla(u - u_h)$, we have

$$\|\nabla(u-u_h)\|_{0.\infty.D_0} \,/\, \|\nabla(Ru_h - u_h)\|_{0.\infty.D_0} \to 0,\ h \to 0 \tag{41}$$

and when the interior nodal point (x_0, y_0) is not a ultraconvergence point of error $\nabla(u - u_h)$, with even-order finite element space and $A = -\Delta + a_0 I$, we have

$$|\nabla(u-u_h)(x_0,y_0)| \,/\, |\nabla(Ru_h - u_h)(x_0,y_0)| \to 0,\ h \to 0 \tag{42}$$

Hence quantity $\|\nabla(Ru_h - u_h)\|_{0.\infty.D_0}$ ($|\nabla(Ru_h - u_h)(x_0,y_0)|$) also provides an asymptotically exact a posteriori error estimator for error $\|\nabla(u-u_h)\|_{0.\infty.D_0}$ ($|\nabla(u-u_h)(x_0,y_0)|$).

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