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A TWO-GRID METHOD FOR THE STEADY PENALIZED NAVIER-STOKES EQUATIONS *1)

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Abstract

A two-grid method for the steady penalized incompressible Navier-Stokes equations is presented. Convergence results are proved. If $h = O(H^{3-s})$ and $\epsilon = O(H^{5-2s})$ ($s = 0$ ($n = 2$); $s = \frac{1}{2}$ ($n = 3$)) are chosen, the convergence order of this two-grid method is the same as that of the usual finite element method. Numerical results show that this method is efficient and can save a lot of computation time.

Key words: Penalized Navier-Stokes equations, Two-grid method, Error estimate, Numerical test.

Introduction

It is well known that numerically solving the incompressible Navier-Stokes equations has two difficulties: the nonlinear term and the incompressibility condition. Firstly, we use the penalized Navier-Stokes equations to conquer the second difficulty, and a two-grid method presented in [1-2] to save a lot of computation time. Secondly, we analyze the convergence of the numerical solution, and derive that if $h = O(H^{3-s})$ and $\epsilon = O(H^{5-2s})$ ($s = 0$ ($n = 2$); $s = \frac{1}{2}$ ($n = 3$)) are chosen, the convergence order of this method is the same as that of the usual finite element method. However, the computational attraction of this two-grid method is that it finds a solution for a small nonlinear problem on a coarse mesh finite element space X^H, and a solution for a linear problem on a fine mesh finite element space X^h ($h \ll H$), compared with the usual finite element method finding a solution for the same large nonlinear problem on X^h. Thus, this method can save a lot of computation time. Finally, numerical tests are given to support our theoretic results.

It is noticeable that this method is totally different from the Nonlinear Galerkin method presented in [3-4]. The Nonlinear Galerkin method is a numerical method for dissipative evolution partial differential equations where the spatial discretization relies on a nonlinear manifold instead of a linear space as in the classical Galerkin method. More precisely, one considers a finite dimensional space X^h which is split as $X^h = X^H + W^h$, where $H \gg h$ and W^h is a convenient supplementary of X^H in X^h. One looks for an approximate solution u^h lying in a manifold Σ= graphϕ of X^h; u^h takes the form $u^h = v^H + \phi(v^H)$ where v^H lies in X^H and ϕ is a mapping from X^H to W^h. The method reduces to an evolution equation for X^H, obtained by projecting the equations under consideration on the manifold Σ=graphϕ. In the usual finite element method, typically, we have $\phi = 0$. The two-grid method is based on a coarse grid finite element space X^H and a fine grid finite element space X^h ($X^H \subset X^h$, $H \gg h$). This method consists of finding a solution v^H for a nonlinear problem on X^H by the usual finite element method, a solution v^h for a linear problem on X^h by one-step Newton method, and a solution w^H for a linear correctness problem on X^H, where an approximate solution $u^h = w^H + v^h$ is defined as in the following step 1-step 4 in §2.

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1. Notations and Mathematical Preliminaries

Consider the incompressible Navier-Stokes problem

$$\begin{cases} -\lambda\Delta u + (u\cdot\nabla)u + \nabla p = f & in \quad \Omega, \\ \nabla\cdot u = 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \end{cases} \tag{1.1}$$

in a convex polygon or polyhedron domain Ω of R^n, with $n = 2$ or 3. Here, $\lambda = Re^{-1}$, Re is the Reynolds number, $u : \Omega \to R^n$ the velocity, $p : \Omega \to R$ the pressure and f the prescribed external force.

Hereafter, we will need the following functional spaces:

$$X = H_0^1(\Omega)^n \equiv \{u \in (H^1(\Omega))^n : \quad u = 0 \quad on \quad \partial\Omega\},$$
$$V \equiv \{u \in H_0^1(\Omega)^n : \nabla\cdot u = 0 \quad in \quad \Omega\}$$

with scalar product $(u, v)_1 := (\nabla u, \nabla v), \quad u, v \in H_0^1(\Omega)^n$, and

$$M = L_0^2(\Omega) \equiv \{p \in L^2(\Omega) : \quad \int_\Omega p dx = 0\}.$$

Let $H^{-1}(\Omega)^n$ be a dual space of $H_0^1(\Omega)^n$ with the corresponding norm:

$$\|f\|_{-1} \equiv \sup_{0\neq u\in H_0^1(\Omega)^n} \frac{\langle f, u\rangle}{|u|_1}, \quad f \in H^{-1}(\Omega)^n.$$

We will use the standard notations $L^2(\Omega)^n$, $H^k(\Omega)^n$ and $H_0^k(\Omega)^n$ to denote the usual Sobolev spaces over Ω. The norm and seminorm corresponding to $H^k(\Omega)^n$ will be denoted by $\|\cdot\|_k$ and $|\cdot|_k$, respectively. In particular, we will use $\|\cdot\|_0$ and $(\cdot,\cdot)$ to denote the norm and the scalar product in $L^2(\Omega)^n$, respectively.

With the above notations, the weak form of problem (1.1) reads: find $(u, p) \in (X, M)$, such that

$$\begin{cases} a(u, v) + N(u, u, v) - b(p, v) & = \langle f, v\rangle, \quad \forall v \in X, \\ b(q, u) & = 0, \quad \forall q \in M, \end{cases} \tag{1.2}$$

where

$$a(u, v) = (u, v)_1, \quad b(p, v) = (p, \nabla\cdot v),$$
$$\text{and} \quad N(u, v, w) = \tfrac{1}{2}[((u\cdot\nabla)v, w) - ((u\cdot\nabla)w, v)].$$

Because of pressure not being in the second equation of problem (1.2), the algebraic equations generated by a finite dimensional approximation are not positive definite, which results in difficulty in solving the numerical solution of problem (1.2). If the positive definite form connected with the pressure is obtained in problem (1.2), then this difficulty will be conquered. For brevity, $\epsilon(p, q)$ is introduced, where ϵ is a penalty parameter. Then, introduce the following penalized problem: find $u_\epsilon \in X$ satisfying

$$\begin{cases} a_\epsilon(u_\epsilon, v) + N(u_\epsilon, u_\epsilon, v) = \langle f, v\rangle, \quad \forall v \in X, \\ p_\epsilon = -\dfrac{1}{\epsilon}\nabla\cdot u_\epsilon, \end{cases} \tag{1.3}$$

where

$$a_\epsilon(u, v) \equiv a(u, v) + \epsilon^{-1}(\nabla\cdot u, \nabla\cdot v).$$

Furthermore, we shall use the following estimates.

Lemma 1.1[5−7]**.** *Given $u,\ v,\ w \in X$ and $f \in H^{-1}(\Omega)^n$, there exists a positive constant c such that*

$$\begin{aligned}
&|\langle f, v\rangle| \le \|f\|_{-1}|v|_1,\\
&a_\epsilon(u,u) = \lambda|u|_1^2 + \epsilon^{-1}\|\nabla\cdot u\|_0^2,\\
&a_\epsilon(u,v) \le \lambda|u|_1|v|_1 + \epsilon^{-1}\|\nabla\cdot u\|_0\|\nabla\cdot v\|_0,\\
&N(u,v,w) = -N(u,w,v), \quad |N(u,v,w)| \le c|u|_1|v|_1|w|_1,\\
&|N(u,v,w)| \le c\|u\|_0^{1-s}|u|_1^s|v|_1|w|_1,\\
&|N(u,v,w)| \le \frac{c}{2}\|u\|_0^{\frac12}|u|_1^{\frac12}|v|_1\|w\|_0^{\frac12}|w|_1^{\frac12} + \frac{c}{2}\|u\|_0^{\frac12}|u|_1^{\frac12}\|v\|_0^{\frac12}|v|_1^{\frac12}|w|_1, \quad (n=2),
\end{aligned}$$

where s is a sufficiently small positive constant for n=2 and $s=\frac12$ for n=3.

Thanks to [8-9], the solution to problem (1.3) has the following properties.

Theorem 1.1. *Assume that there are two constants $\alpha > 0$ and $\beta > 0$ satisfying $\lambda > \alpha$ and $\lambda - c\beta \ge \alpha > 0$, where c is defined in Lemma 1.1. Then when $\alpha^{-2}B\|f\|_{-1} < 1$ and $(\epsilon\alpha^{-1})^{\frac12}\|f\|_{-1} \le \beta$ are satisfied, there exists a unique solution to problem (1.3) in $\{u|u \in (H_0^1(\Omega)\cap H^2(\Omega))^n,\ \|\nabla\cdot u\|_0 \le \beta,\ |u|_1 \le \alpha^{-1}\|f\|_{-1}\}$. The following estimate also holds (see [8, Theorem 3.4, p309]):*

$$|u-u_\epsilon|_1 + \|p-p_\epsilon\|_0 \le c\epsilon, \tag{1.4}$$

where

$$B \equiv \sup_{u,v,w\in X}\frac{|N(u,v,w)|}{|u|_1|v|_1|w|_1},$$

(u,p) and (u_ϵ,p_ϵ) are solutions to problems (1.2) and (1.3), respectively, and $c>0$ is a constant independent of ϵ .

2. Two-grid Discretizations

Let $\tau_h = \{K\}$ be a uniformly regular subdivision of mesh size h, $0<h<1$, of the domain $\bar\Omega$ into closed subsets K, quadrilaterals in two dimensions and hexahedrons in three dimensions. Assume that X^h and M^h are finite element subspaces of X and M, respectively. Let I_h and J_h be the finite element interpolation operators associated with X^h and M^h, respectively.

Then the usual finite element approximations u_h and $u_{\epsilon h} \in X^h$ are calculated by solving the large nonlinear systems given by

$$\begin{cases} a(u_h,v) + N(u_h,u_h,v) - b(p_h,v) = \langle f,v\rangle, \quad \forall v\in X^h,\\ b(q,u_h) = 0, \quad \forall q \in M^h, \end{cases} \tag{2.1}$$

and

$$a_\epsilon(u_{\epsilon h},v) + N(u_{\epsilon h},u_{\epsilon h},v) = \langle f,v\rangle, \quad \forall v \in X^h, \tag{2.2}$$

respectively.

As usual, we make the following standard assumptions on the finite element subspaces X^h and M^h:

(H1) Approximation hypothesis: for any $(u,p) \in (X\times M)\cap(H^2(\Omega)^n \times H^1(\Omega))$,

$$\inf_{(v_h,q_h)\in X^h\times M^h}\{h|u-v_h|_1 + \|u-v_h\|_0 + h\|p-q_h\|_0\} \le ch^2\{\|u\|_2 + \|p\|_1\}.$$

(H2) Interpolation hypothesis: for any $(v,q) \in (X\times M)\cap(H^2(\Omega)^n\times H^1(\Omega))$,

$$|v-I_hv|_1 + \|q-J_hq\|_0 \le ch(\|v\|_2 + \|q\|_1).$$

(H3) Inverse hypothesis:

$$|v_h|_1 \le ch^{-1}\|v_h\|_0, \quad \forall v_h \in X^h.$$

(H4) Stability hypothesis:

$$\inf_{q\in M^h}\sup_{v\in X^h}\frac{(q,\nabla\cdot v)}{\|q\|_0|v|_1} \ge \beta > 0,$$

where β is a constant independent of h.

It is well known[8−9] that u_h and $u_{\epsilon h} \in X^h$ satisfy the following results from above hypotheses.

Theorem 2.1. *Suppose $(u,p) \in (X\times M)\cap(H^2(\Omega)^n \times H^1(\Omega))$ is a nonsingular solution of (1.2) and the finite element subspaces X^h and M^h satisfy assumptions (H1)-(H4). Then there exists a unique solution (u_h,p_h) to problem (2.1) satisfying the following error estimate:*

$$h|u-u_h|_1 + \|u-u_h\|_0 + h\|p-p_h\|_0 \le ch^2(\|u\|_2 + \|p\|_1).$$

Theorem 2.2. *Solutions to problem (2.2) exist and satisfy $|u_{\epsilon h}|_1 \le \lambda^{-1}\|f\|_{-1}$. Suppose $\lambda^{-2}B\|f\|_{-1} < 1$. Then the solution $u_{\epsilon h}$ to (2.2) is unique. Moreover, if assumption (H4) is satisfied, then for all $\epsilon \le \epsilon_0$ small enough the following estimate holds*[9,Theorem9.22,p347]*:*

$$|u_h - u_{\epsilon h}|_1 + \|p_h - p_{\epsilon h}\|_0 \le c\epsilon,$$

where $c > 0$ is a constant independent of h and ϵ.

Proof. Set $v = u_{\epsilon h}$ in (2.2). This gives

$$\lambda|u_{\epsilon h}|_1^2 + \epsilon^{-1}\|\nabla\cdot u_{\epsilon h}\|_0^2 = \langle f, u_{\epsilon h}\rangle \le \|f\|_{-1}|u_{\epsilon h}|_1,$$

whence $|u_{\epsilon h}|_1 \le \lambda^{-1}\|f\|_{-1}$ and $\|\nabla\cdot u_{\epsilon h}\|_0 \le \sqrt{\epsilon}\|f\|_{-1}^{\frac{1}{2}}|u_{\epsilon h}|_1^{\frac{1}{2}} \le \lambda^{-\frac{1}{2}}\sqrt{\epsilon}\|f\|_{-1} \le \beta$. This a priori bound ensures that solutions to problem (2.2) exist (see [9, Theorem 9.21, p346] for details). For the uniqueness, let $u_{\epsilon h}^1$ and $u_{\epsilon h}^2$ be two solutions to problem (2.2) and $z_{\epsilon h} = u_{\epsilon h}^1 - u_{\epsilon h}^2$ their difference. Then

$$\begin{aligned}\lambda|z_{\epsilon h}|_1^2 + \frac{1}{\epsilon}\|\nabla\cdot z_{\epsilon h}\|_0^2 &= -N(z_{\epsilon h}, u_{\epsilon h}^2, z_{\epsilon h})\\ &\le B|z_{\epsilon h}|_1^2|u_{\epsilon h}|_1 \le B\lambda^{-1}\|f\|_{-1}|z_{\epsilon h}|_1^2.\end{aligned}$$

Thus $|z_{\epsilon h}|_1^2 = 0$ provided $(1 - B\lambda^{-2}\|f\|_{-1}) > 0$

Theorem 2.3. *Under the conditions of Theorem 2.1 and Theorem 2.2, if $u \in H^2(\Omega)^n \cap H_0^1(\Omega)^n$ is the solution to problem (1.2), then the following estimates hold:*

$$|u - u_{\epsilon h}|_1 \le c(\epsilon + h)\|u\|_2,$$
$$\|u - u_{\epsilon h}\|_0 \le c(h^2 + \epsilon)\|u\|_2.$$

Combining Theorem 2.1 with Theorem 2.2 and using the triangle inequality, we easily get Theorem 2.3.

The solution to nonlinear problem (2.2) can still be quite computationally intensive. Therefore, this article considers an attractive two-grid method as follows:

Step 1: Find $u_\epsilon^H \in X^H$, such that

$$a_\epsilon(u_\epsilon^H, v) + N(u_\epsilon^H, u_\epsilon^H, v) = \langle f, v\rangle, \quad \forall v \in X^H. \tag{2.3}$$

Step 2: Find $u_\epsilon^h \in X^h$, such that

$$a_\epsilon(u_\epsilon^h, v) + N(u_\epsilon^h, u_\epsilon^H, v) + N(u_\epsilon^H, u_\epsilon^h, v) - N(u_\epsilon^H, u_\epsilon^H, v) = \langle f, v\rangle, \quad \forall v \in X^h. \tag{2.4}$$

Step 3: Find $u_\epsilon^H \in X^H$, such that

$$a_\epsilon(e_\epsilon^H, v) + N(u_\epsilon^H, e_\epsilon^H, v) + N(e_\epsilon^H, u_\epsilon^H, v) = -N(u_\epsilon^h - u_\epsilon^H, u_\epsilon^h - u_\epsilon^H, v), \quad \forall v \in X^H. \tag{2.5}$$

Step 4: Set $u^* = u_\epsilon^h + e_\epsilon^H$.

Owing to [1], we can similarly get the following results.

Lemma 2.1. *Given a solution u_ϵ^H to problem (2.3), if $\lambda^{-2}B\|f\|_{-1} < 1$, ϵ is small enough and there is a constant γ such that $\epsilon\lambda^{-2}\|f\|_{-1}\beta_1(1 + B\beta_1 + B\lambda^{-1}\|f\|_{-1}) < \gamma^2$, then the solution to problem (2.4) exists uniquely in $\{u|u \in X^h, |u|_1 \le \beta_1, \|\nabla \cdot u\|_0 \le \gamma\}$, where $\beta_1 = (\lambda - \lambda^{-1}B\|f\|_{-1})^{-1}(1 + B\lambda^{-2}\|f\|_{-1})\|f\|_{-1}$.*

Lemma 2.2. *Given solutions u_ϵ^H and u_ϵ^h to problems (2.3) and (2.4), respectively, if $\lambda^{-2}B\|f\|_{-1} < 1$, then problem (2.5) has a unique solution.*

3. Error Estimates

As well known [8,§3.1,p297] that u is a nonsingular solution of the Navier-Stokes equations if and only if the following inf-sup conditions hold:

$$\begin{aligned} &\inf_{w\in X}\sup_{v\in X}\frac{a(w,v) + N(u,w,v) + N(w,u,v)}{|w|_1|v|_1} \ge \gamma(u) > 0,\\ &\inf_{v\in X}\sup_{w\in X}\frac{a(w,v) + N(u,w,v) + N(w,u,v)}{|w|_1|v|_1} \ge \gamma(u) > 0. \end{aligned} \tag{3.1}$$

Thus, for $|u - u_\epsilon^H|_1$ small enough (which holds for all H and ϵ with $0 < H \le H_0$ and $0 < \epsilon \le \epsilon_0$, where H_0 and ϵ_0 are two positive constants)

$$\begin{aligned} &\inf_{w\in X^h}\sup_{v\in X^h}\frac{a(w,v) + N(u_\epsilon^H,w,v) + N(w,u_\epsilon^H,v)}{|w|_1|v|_1} \ge \frac{\gamma(u)}{2} > 0,\\ &\inf_{v\in X^h}\sup_{w\in X^h}\frac{a(w,v) + N(u_\epsilon^H,w,v) + N(w,u_\epsilon^H,v)}{|w|_1|v|_1} \ge \frac{\gamma(u)}{2} > 0. \end{aligned} \tag{3.2}$$

(3.2) also holds for w and v in X^H.

Let $C_H(w,v) = a(w,v) + N(u_\epsilon^H, w, v) + N(w, u_\epsilon^H, v)$, and Q_H be a projection operator from X to X^H satisfying

$$C_H(w, Q_H v) = C_H(w,v), \qquad \forall w \in X^H, v \in X. \tag{3.3}$$

Lemma 3.1. *With H small enough, Q_H is well defined and satisfies*

$$|v - Q_H v|_1 \le \inf_{v^H\in X^H}|v - v^H|_1, \tag{3.4}$$

and

$$\|v - Q_H v\|_0 \le cH|v - Q_H v|_1. \tag{3.5}$$

Proof. For all $v \in X$, by Lemma 1.1, Theorem 2.2 with h replaced by H and (3.2)-(3.3), it follows that

$$\begin{aligned} |v^H - Q_H v|_1 &\le c \sup_{\phi^H \in X^H} \frac{C_H(\phi^H, Q_H v - v^H)}{|\phi^H|_1} = c \sup_{\phi^H \in X^H} \frac{C_H(\phi^H, v - v^H)}{|\phi^H|_1} \\ &= c \sup_{\phi^H \in X^H} \frac{a(\phi^H, v - v^H) + N(u_\epsilon^H, \phi^H, v - v^H) + N(\phi^H, u_\epsilon^H, v - v^H)}{|\phi^H|_1} \\ &\le c|v - v^H|_1. \end{aligned}$$

Then the triangle inequality gives

$$|v - Q_H v|_1 \le c \inf_{v^H \in X^H} |v - v^H|_1.$$

To estimate L^2-error, we consider a dual problem: for given $g \in L^2(\Omega)^n$, find $\xi \in X$ such that

$$C_H(\xi, \phi) = (\phi, g), \quad \forall \phi \in X. \tag{3.6}$$

We shall assume that the solution ξ to problem (3.6) is H^2-regular, i.e.,

$$\|\xi\|_2 \le c\|g\|_0. \tag{3.7}$$

It follows from problems (3.6), (3.7) and (3.3) that

$$\begin{aligned} \|v - Q_H v\|_0 &= \sup_{g \in L^2(\Omega)^n} \frac{\displaystyle\int_\Omega (v - Q_H v) g dx}{\|g\|_0} = \sup_{g \in L^2(\Omega)^n} \frac{C_H(\xi, v - Q_H v)}{\|g\|_0} \\ &= \sup_{g \in L^2(\Omega)^n} \frac{C_H(\xi - Q_H \xi, v - Q_H v)}{\|g\|_0} \le cH|v - Q_H v|_1. \end{aligned}$$

Lemma 3.2. *(a) If the global uniqueness condition $\lambda^{-2} B\|f\|_{-1} < 1$ holds, then both u_ϵ^H and u_ϵ^h exist uniquely. Further, the error $|u_{\epsilon h} - u_\epsilon^h|_1$ satisfies*

$$|u_{\epsilon h} - u_\epsilon^h|_1 \le c(\epsilon^2 + H^{3-s} + \epsilon H) \quad (\ s \text{ is small enough}(n=2); \quad s = \frac{1}{2}(n=3)).$$

(b) If the uniqueness condition fails, suppose u is the nonsingular solution of the Navier-Stokes equations. Then for H small enough, the following estimate holds:

$$|u_{\epsilon h} - u_\epsilon^h|_1 \le c(\epsilon^2 + H^{3-s} + \epsilon H) \quad (\ s \text{ is small enough}(n=2); \quad s = \frac{1}{2}(n=3)).$$

Proof. For brevity, we only prove part (b).
A straightforward calculation shows that

$$N(u, u, v) = N(u, u_\epsilon^H, v) + N(u_\epsilon^H, u, v) - N(u_\epsilon^H, u_\epsilon^H, v) + N(u - u_\epsilon^H, u - u_\epsilon^H, v). \tag{3.8}$$

Subtracting (2.4) from (2.2), setting $e = u_{\epsilon h} - u_\epsilon^h$ and using (3.8), we have

$$a_\epsilon(e, v) + N(e, u_\epsilon^H, v) + N(u_\epsilon^H, e, v) = -N(u_{\epsilon h} - u_\epsilon^H, u_{\epsilon h} - u_\epsilon^H, v). \tag{3.9}$$

When $n=2$, by (3.9), (3.2) and Lemma 1.1, noting that $u_{\epsilon h}-u_\epsilon^H=(u_{\epsilon h}-u_h)+(u_h-u)+(u-u_H)+(u_H-u_\epsilon^H)$, with $h<H$ we have

$$\begin{aligned}\frac{\gamma(u)}{2}|u_{\epsilon h}-u_\epsilon^h|_1 &\le \sup_{v\in X^h}\frac{N(u_{\epsilon h}-u_\epsilon^H,u_{\epsilon h}-u_\epsilon^H,v)}{|v|_1}\\ &\le c\{\|u_H-u_\epsilon^H\|_0^{1-s}|u_H-u_\epsilon^H|_1^{1+s}+\|u-u_H\|_0^{1-s}|u-u_H|_1^{1+s}\\ &\quad+\|u-u_H\|_0^{1-s}|u-u_H|_1^s|u_H-u_\epsilon^H|_1+\|u_H-u_\epsilon^H\|_0^{1-s}|u_H-u_\epsilon^H|_1^s|u-u_H|_1\}.\end{aligned}$$

Thus, Theorem 2.1 and Theorem 2.2 give

$$\frac{\gamma(u)}{2}|u_{\epsilon h}-u_\epsilon^h|_1\le c(\epsilon^2+H^{3-s}+H^{2-s}\epsilon+\epsilon H)\le c(\epsilon^2+H^{3-s}+\epsilon H).$$

When $n=3$, we can similarly get

$$\begin{aligned}\frac{\gamma(u)}{2}|u_{\epsilon h}-u_\epsilon^h|_1 &\le \sup_{v\in X^h}\frac{N(u_{\epsilon h}-u_\epsilon^H,u_{\epsilon h}-u_\epsilon^H,v)}{|v|_1}\\ &\le c\{\|u_H-u_\epsilon^H\|_0^{\frac12}|u_H-u_\epsilon^H|_1^{\frac32}+\|u-u_H\|_0^{\frac12}|u-u_H|_1^{\frac32}\\ &\quad+\|u-u_H\|_0^{\frac12}|u-u_H|_1^{\frac12}|u_H-u_\epsilon^H|_1+\|u_H-u_\epsilon^H\|_0^{\frac12}|u_H-u_\epsilon^H|_1^{\frac12}|u-u_H|_1.\}\\ &\le c(H^{\frac52}+\epsilon^2+H^{\frac32}\epsilon+\epsilon H)\le c(H^{\frac52}+\epsilon^2+\epsilon H).\end{aligned}$$

Thus

$$|u_{\epsilon h}-u_\epsilon^h|_1\le c(\epsilon^2+H^{3-s}+\epsilon H),(\text{ s is small enough}(n=2);\quad s=\frac12(n=3)).$$

Theorem 3.1. *If the assumptions in Lemma 3.2 are satisfied and ϵ is small enough, then the following estimate is valid:*

$$|u_{\epsilon h}-u^*|_1\le c(\epsilon^2+H^{3-s}+\epsilon^{\frac12}H^{\frac12})\qquad(s=0(n=2),\quad s=\frac12(n=3)).$$

Proof. Thanks to the definition of projection operator Q_H, (2.5) can be written in the following form:

$$a_\epsilon(e_\epsilon^H,v)+N(u_\epsilon^H,e_\epsilon^H,v)+N(e_\epsilon^H,u_\epsilon^H,v)=-N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_\epsilon^H,Q_Hv).\tag{3.10}$$

Combining (2.4) with (3.10) yields

$$\begin{aligned}&a_\epsilon(u^*,v)+N(u_\epsilon^H,u^*,v)+N(u^*,u_\epsilon^H,v)-N(u_\epsilon^H,u_\epsilon^H,v)\\ &=\langle f,v\rangle-N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_\epsilon^H,Q_Hv-v)-N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_\epsilon^H,v).\end{aligned}\tag{3.11}$$

Subtracting (3.11) from (2.2), setting $u_{\epsilon h}-u^*=E$ and using (3.8), we obtain

$$\begin{aligned}&a_\epsilon(E,v)+N(u_\epsilon^H,E,v)+N(E,u_\epsilon^H,v)=N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_\epsilon^H,v)\\ &\quad-N(u_{\epsilon h}-u_\epsilon^H,u_{\epsilon h}-u_\epsilon^H,v)+N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_\epsilon^H,Q_Hv-v)\\ &=N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_{\epsilon h},v)+N(u_\epsilon^h-u_{\epsilon h},u_{\epsilon h}-u_\epsilon^H,v)\\ &\quad+N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_\epsilon^H,Q_Hv-v).\end{aligned}\tag{3.12}$$

From (3.2), it follows that

$$\frac{\gamma(u)}{2}|E|_1 \leq \sup_{v\in X^h} \frac{a_\epsilon(E,v)+N(u_\epsilon^H,E,v)+N(E,u_\epsilon^H,v)}{|v|_1}. \tag{3.13}$$

Next, we will estimate the right-hand side terms of (3.12).
From Lemma 1.1 and Lemma 3.1 with $n=2$, we have

$$\begin{aligned}
N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_{\epsilon h},v) &\leq c|u_\epsilon^h-u_\epsilon^H|_1|u_\epsilon^h-u_{\epsilon h}|_1|v|_1\\
&\leq c(|u_\epsilon^h-u_{\epsilon h}|_1+|u_{\epsilon h}-u_\epsilon^H|_1)|u_\epsilon^h-u_{\epsilon h}|_1|v|_1\\
&\leq c(\epsilon+H+\epsilon H)(\epsilon^2+H^{3-s}+\epsilon H)|v|_1\\
&\leq c(H^{4-s}+\epsilon^3+\epsilon H)|v|_1,\\
N(u_\epsilon^h-u_{\epsilon h},u_{\epsilon h}-u_\epsilon^H,v) &\leq c(H^{4-s}+\epsilon^3+\epsilon H)|v|_1,
\end{aligned}$$

$$\begin{aligned}
N(u_\epsilon^h-u_\epsilon^H,&u_\epsilon^h-u_\epsilon^H,Q_Hv-v) \leq \frac{c}{2}\|u_\epsilon^h-u_\epsilon^H\|_0|u_\epsilon^h-u_\epsilon^H|_1|Q_Hv-v|_1\\
&+\frac{c}{2}\|u_\epsilon^h-u_\epsilon^H\|_0^{\frac12}|u_\epsilon^h-u_\epsilon^H|_1^{\frac12}|u_\epsilon^h-u_\epsilon^H|_1|Q_Hv-v|_1^{\frac12}\|Q_Hv-v\|_0^{\frac12}\\
&\leq \frac{c}{2}(|u_\epsilon^h-u_{\epsilon h}|_1^{\frac12}+|u_{\epsilon h}-u_\epsilon^H|_1^{\frac12})\|u_\epsilon^h-u_\epsilon^H\|_0^{\frac12}|u_\epsilon^h-u_\epsilon^H|_1|Q_Hv-v|_1^{\frac12}\|Q_Hv-v\|_0^{\frac12}\\
&+\frac{c}{2}\|u_\epsilon^h-u_\epsilon^H\|_0|u_\epsilon^h-u_\epsilon^H|_1|Q_Hv-v|_1\\
&\leq c(\epsilon^{\frac12}H^{\frac12}+H^3)|v|_1+c(\epsilon^2+\epsilon H+H^3)|v|_1 \leq c(\epsilon^2+\epsilon^{\frac12}H^{\frac12}+H^3)|v|_1.
\end{aligned}$$

Similarly, for $n=3$ we have

$$\begin{aligned}
N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_\epsilon^H,Q_Hv-v) &\leq c\|u_\epsilon^h-u_\epsilon^H\|_0^{\frac12}|u_\epsilon^h-u_\epsilon^H|_1^{\frac32}|Q_Hv-v|_1\\
&\leq c(\epsilon^2+\epsilon^{\frac12}H^{\frac12}+H^{\frac52})|v|_1,\\
N(u_\epsilon^h-u_\epsilon^H,u_\epsilon^h-u_{\epsilon h},v) &\leq c|u_\epsilon^h-u_\epsilon^H|_1|u_\epsilon^h-u_{\epsilon h}|_1|v|_1 \leq c(\epsilon^3+H^{\frac72}+\epsilon H)|v|_1,
\end{aligned}$$

and

$$N(u_\epsilon^h-u_{\epsilon h},u_{\epsilon h}-u_\epsilon^H,v) \leq c(\epsilon^3+H^{\frac72}+\epsilon H)|v|_1.$$

Thus

$$|E|_1 \leq c(\epsilon^2+H^{3-s}+\epsilon^{\frac12}H^{\frac12}) \quad (s=0(n=2); \quad s=\frac12(n=3)).$$

Combining Theorem 2.2 with Theorem 3.1 and using the triangle inequality yield the following result.

Theorem 3.2. *If the assumptions in Theorem 3.1 are satisfied, then there holds the following estimate:*

$$|u-u^*|_1 \leq c(\epsilon+h+H^{3-s}+\epsilon^{\frac12}H^{\frac12}) \quad (s=0(n=2); \quad s=\frac12(n=3)).$$

Remark. From Theorem 2.3 and Theorem 3.2, it follows that if $\epsilon = O(H^{5-2s})$ and $h = O(H^{3-s})$ $(s=0(n=2); \quad s=\frac12(n=3))$ are chosen, the convergence order of the two-grid method is the same as that of the usual finite element method for H^1-error. We can also get the L^2-error estimate about $\|u-u^*\|_0$ by introducing a linear dual problem to (1.3).

4. Numerical Test

This section gives two examples to verify above analysis. We will numerically solve these two problems by the usual finite element method (FEM) and the two-grid method (TGM), respectively. In this part, we assume that $\Omega=(0,1)\times(0,1)$ and each unit is square. Let Ω be subdivided into coarse mesh units k with its diameter $H = \frac{1}{3}$ given in Fig 1.1, then we make any coarse mesh unit $k \in \tau_H$ into 3^3 same quadrilateral units T given in Fig 1.2. So we can get the fine mesh subdivision τ_h $(h = \frac{1}{27})$ of Ω.

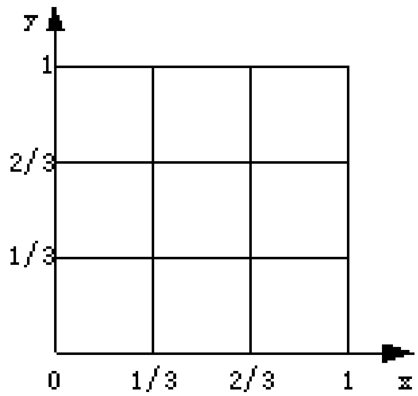


Fig1.1 Coarse mesh subdivision

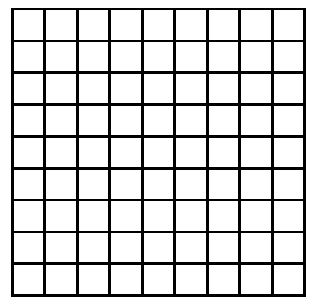


Fig1.2 Each coarse mesh unit subdivision

To easily check the accuracy of numerical tests, we choose two kinds of right-hand side f to get the corresponding exact solutions $u(x,y) = (u_1(x,y), u_2(x,y))$, $p(x,y)$ to problem (1.1) for n=2. The first solution is

$$\left\{ \begin{array}{l} u_1(x,y) = x^2(1-x)^2y(1-y)(1-2y), \\ u_2(x,y) = -y^2(1-y)^2x(1-x)(1-2x), \\ p(x,y) = \frac{1}{2}(x-y), \end{array} \right.$$

and the second solution is

$$\left\{ \begin{array}{l} u_1(x,y) = \frac{3\pi}{25} sin^2(3\pi x) sin(6\pi y), \\ u_2(x,y) = -\frac{3\pi}{25} cos(6\pi y) cos^2(3\pi x), \\ p = \frac{1}{18} cos(6\pi x). \end{array} \right.$$

We present numerical results generated by FEM and TGM in Figs2-5, and use CPU_I, I=1,2 to stand for the CPU time used by TGM and FEM, respectively. We only present results for the exact u_1 and interpolated numerical solutions generated by FEM and TGM along y-axis at x=0.8. We give the detail results of the first solution for the different λ in Table 1 and Table 2. From Figs2-5, we know the efficiency of two kinds of algorithms for the first solution is better than that for the second one. From Figs2-5, Table 1 and Table 2, we also know, TGM can save a lot of computation time; the finer the grid is, the more the saved CPU time is and the bigger λ is, the better the convergence order is. This results well coincide with the theoretic analysis in this article.

Fig2: $u_1(x,y)=2x^2(1-x)^2y(x-y)(1-2y)$ (for $\lambda=0.01$)

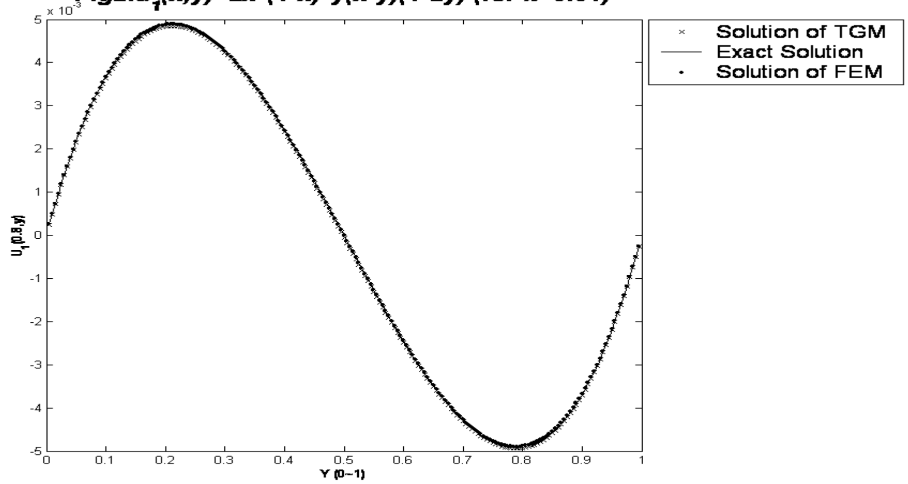


Fig3: $u_1(x,y)=3\pi/25\sin(3\pi x)^2\sin(6\pi y)$ (for $\lambda=0.1$)

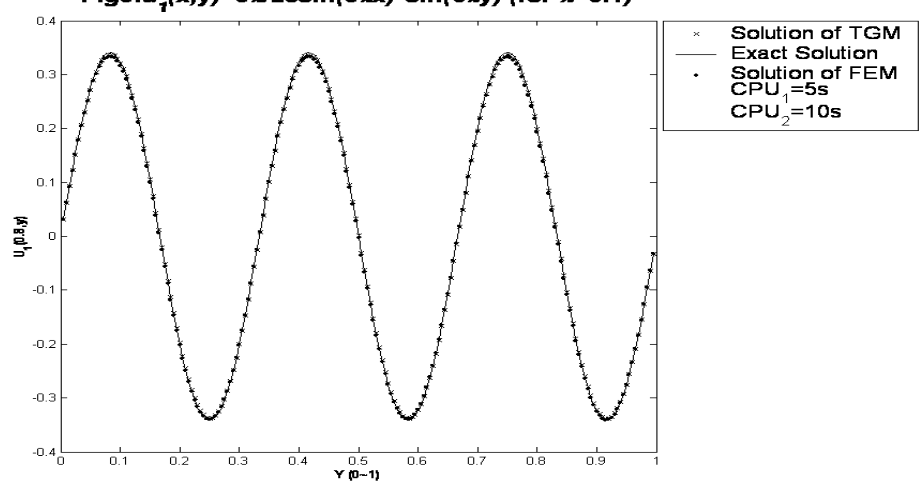
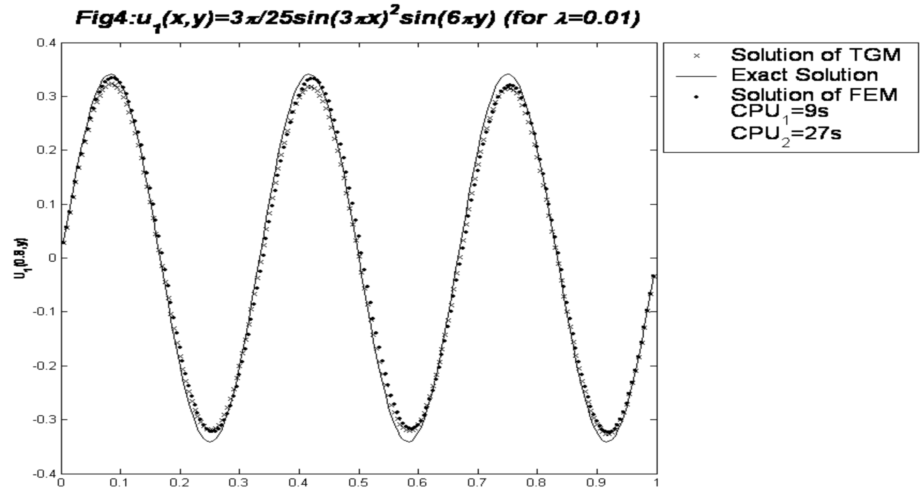
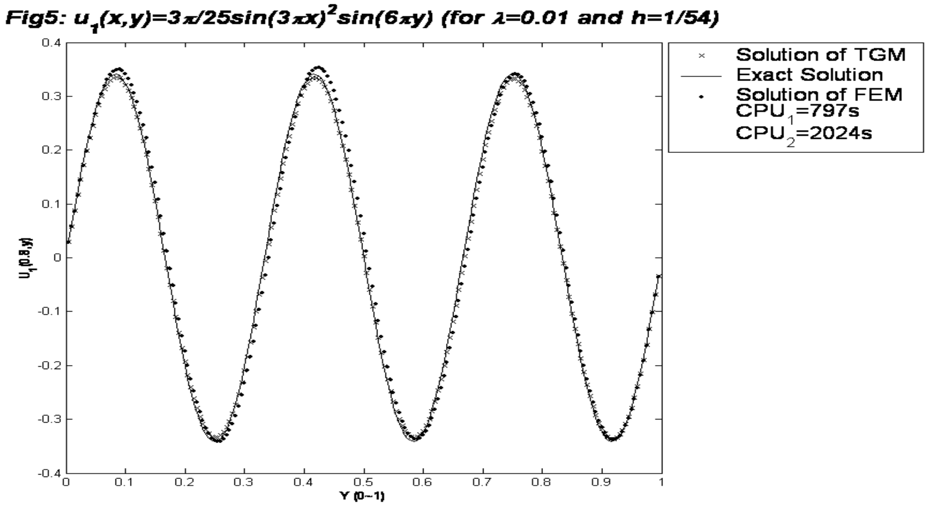


Fig4: $u_1(x,y)=3\pi/25\sin(3\pi x)^2\sin(6\pi y)$ (for $\lambda=0.01$)





λ	two-grid method		finite element method	
	H^1-error	Time(s)	H^1-error	Time(s)
λ=0.001	1.492615e-3	6	1.262735e-3	16
λ=0.005	1.167979e-3	5	1.131656e-3	15
λ=0.01	1.132879e-3	4	1.116366e-3	14
λ=0.05	1.108092e-3	6	1.107803e-3	14
λ=0.1	1.106157e-3	6	1.107159e-3	13
λ=0.5	1.104389e-3	6	1.104572e-3	13
λ=1	1.104262e-3	6	1.104311e-3	13

Table1: Comparisons of H^1- Error and CPU time

λ	two-grid method		finite element method	
	H^1-error	Time(s)	H^1-error	Time(s)
λ=0.001	7.370373e-4	496	6.231852e-4	1661
λ=0.005	6.110039e-4	458	6.060208e-4	1446
λ=0.01	6.060601e-4	436	6.043017e-4	1455
λ=0.05	6.030500e-4	429	6.054160e-4	1402
λ=0.1	6.028151e-4	438	6.053308e-4	1592
λ=0.5	6.026011e-4	435	6.030067e-4	1308
λ=1	6.025856e-4	435	6.025862e-4	1104

Table2: Comparisons of H^1- Error and CPU time with h=1/54

Remark. Above numerical examples is performed on IBM pentium 4 whose memory and mobile CPU are 256MB and 1.80 GHz, respectively.

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