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THE LONG-TIME BEHAVIOR OF SPECTRAL APPROXIMATE FOR KLEIN-GORDON-SCHRÖDINGER EQUATIONS *1)

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Abstract

Klein-Gordon-Schrödinger (KGS) equations are very important in physics. Some papers studied their well-posedness and numerical solution [1-4], and another works investigated the existence of global attractor in R^n and $\Omega \subset R^n$ $(n \leq 3)$ [5-6,11-12]. In this paper, we discuss the dynamical behavior when we apply spectral method to find numerical approximation for periodic initial value problem of KGS equations. It includes the existence of approximate attractor $\mathcal{A}_N$, the upper semi-continuity on $\mathcal{A}$ which is a global attractor of initial problem and the upper bounds of Hausdorff and fractal dimensions for $\mathcal{A}$ and $\mathcal{A}_N$, etc.

Key words: Klein-Gordon-Schrödinger equation, Spectral approximate, Global attractor, Hausdorff dimension, Fractal dimension.

1. Introduction

In this paper, we consider the following periodic initial value problem of dissipative KGS equations

$$\begin{cases} i\psi_t + \triangle\psi + i\nu\psi + \phi\psi = f(x), & x \in I, I = [0, 2\pi]^3, \\ \phi_{tt} + \gamma\phi_t - \triangle\phi + \phi - |\psi|^2 = g(x), & t > 0, \end{cases}$$

where ψ, ϕ are complex and real unknown functions, respectively, ν, γ are positive constants, f and g are given complex and real functions respectively, $i\nu\psi$ and $\gamma\phi_t$ are dissipative terms. We introduce the transformation $\theta = \phi_t + \delta\phi$, where δ is a small positive constant and then the above problem can be written as

$$\begin{cases} i\psi_t + \triangle\psi + i\nu\psi + \phi\psi = f(x), & (1.1) \\ \phi_t + \delta\phi = \theta, \qquad x \in I, t > 0, & (1.2) \\ \theta_t + (\gamma - \delta)\theta - \triangle\phi + (1 - \delta(\gamma - \delta))\phi - |\psi|^2 = g(x), & (1.3) \\ \psi(x + 2\pi e_i, t) = \psi(x, t), \phi(x + 2\pi e_i, t) = \phi(x, t), \theta(x + 2\pi e_i, t) = \theta(x, t), & (1.4) \\ \psi(x, 0) = \psi_0(x), \phi(x, 0) = \phi_0(x), \theta(x, 0) = \theta_0(x), & (1.5) \end{cases}$$

where e_i are unit vectors in the i-th direction.

Let $H_p^s(I)$ denote the periodic real or complex Sobolev space with the inner product $(\cdot, \cdot)_s$ and the norm $\|\cdot\|_s$. In particular, $H_p^0(I) = H(I)$, and its inner product and norm are $(\cdot, \cdot)$, $\|\cdot\|$, respectively. Denote $V = H_p^1(I) \times H_p^1(I) \times H(I)$, $\|(\psi, \phi, \theta)\|_V^2 = \|\psi\|_1^2 + \|\phi\|_1^2 + \|\theta\|^2$, $\widetilde{V} = H_p^2(I) \times H_p^2(I) \times H_p^1(I)$, $\|(\psi, \phi, \theta)\|_{\widetilde{V}}^2 = \|\psi\|_2^2 + \|\phi\|_2^2 + \|\theta\|_1^2$.

Assume that $S_N = \text{span}\{e^{ix\cdot j} | j \in Z^3, |j| \leq N\}$, P_N is an orthogonal projection from L^2 to S_N.

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Lemma 1[7]**.** *For any* $\sigma \geq 0$ *, if* $u \in H_p^\sigma(I)$ *then*

$$\|u - P_N u\|_j \leq c N^{-\sigma+j} \|u\|_\sigma, \qquad 0 \leq j \leq \sigma.$$

Lemma 2. *For* $u \in H_p^1(I)$*,* $n = 3$*, then*

$$\|u\|_{L^4} \leq c \|u\|_1^{3/4} \|u\|^{1/4}.$$

2. Some Results on the Problem (1.1)-(1.5)

Lemma 3. *Assume that* f *,* $g \in L^2(I)$ *and* $\|(\psi_0, \phi_0, \theta_0)\|_V \leq R$ *, then there exists a constant* $\delta_1 > 0$ *, such that if* $\delta \leq \delta_1$ *then solution of the problem (1.1)-(1.5) satisfies*

$$\|\psi(t)\|_1 + \|\phi(t)\|_1 + \|\theta(t)\| \leq D_1, t \geq t_1,$$

where D_1 *depends on* $\nu, \gamma, \delta, \|f\|, \|g\|$*;* t_1 *depends on* $\nu, \gamma, \delta, \|f\|, \|g\|$ *and* R*.*

Lemma 4. *Assume that* f *,* $g \in H_p^k(I)$ *,* $k \geq 0$ *and* $\|\psi_0\|_{k+2} + \|\phi_0\|_{k+2} + \|\theta_0\|_{k+1} \leq R$*, then the solution* $(\psi,\ \phi,\ \theta)$ *of the problem satisfies*

$$\|\psi(t)\|_{k+2} + \|\phi(t)\|_{k+2} + \|\theta(t)\|_{k+1} \leq D_{k+2}, \qquad t \geq t_{k+2},$$

where D_{k+2} *depends on* $\nu, \gamma, \delta, \|f\|_k, \|g\|_k$ *and* k *;* t_{k+2} *depends on* $\nu, \gamma, \delta, \|f\|_k, \|g\|_k$*,* k *and* R*.*

The proof of Lemma 3 and Lemma 4 are similar the paper[5] we don't represent.

Furthermore, we can prove

Lemma 5. *Under the conditions of Lemma 3, for* $0 \leq t \leq T$*, we have*

$$\|\psi(t)\|_1 + \|\phi(t)\|_1 + \|\theta(t)\| \leq L_1,$$

where L_1 *depends on* $\nu, \gamma, \delta, \|f\|, \|g\|$*, and* T*.*

Lemma 6. *Under the conditions of Lemma 4, for* $0 \leq t \leq T$*, we have*

$$\|\psi(t)\|_{k+2} + \|\phi(t)\|_{k+2} + \|\theta(t)\|_{k+1} \leq L_{k+2},$$

where L_{k+2} *depends on* $\nu, \gamma, \delta, \|f\|_k, \|g\|_k$*, and* T*.*

By the above Lemmas and the theory of partial differential equation we can obtain that the problem (1.1)-(1.5) defineds a continuous operator semigroup $\{S(t)\}_{t \geq 0}$, $S(t)(\psi_0, \phi_0, \theta_0) = (\psi(t), \phi(t), \theta(t))$. If we denote

$$B_1 = \{(\psi, \phi, \theta) \in V |\ \|\psi\|_1 + \|\phi\|_1 + \|\theta\| \leq M_1\}$$

and

$$B_k = \{(\psi, \phi, \theta) \in H_p^{k+2} \times H_p^{k+2} \times H_p^{k+1} |\ \|\psi\|_{k+2} + \|\phi\|_{k+2} + \|\theta\|_{k+1} \leq M_{k+2}\},$$

where $M_i,\ i = 1, 2, ..., k+2$ are proper large constants, then B_1 and B_k are bounded absorbing sets on V and $H_p^{k+2} \times H_p^{k+2} \times H_p^{k+1}$ respectively. We can also prove the following result using the technique introduced by Temam[8].

Theorem 1. *Suppose that* $f, g \in H$*, then the problem (1.1)-(1.5) has a global attractor* $\mathcal{A}$ *on* V*,which is a compact invariant subset of* V*,absorbs any bounded set of* V*.*

If f, $g \in H_p^k(I)$, $k \geq 0$, then (1.1)-(1.5) has a global attractor on $Q^k = H_p^{k+2} \times H_p^{k+2} \times H_p^{k+1}$ which is a compact invariant subset of Q^k and absorbs any bounded set of Q^k.

The proof of Theorem is similar to [5]. We can prove $S(t)$ is asymptotically compact in Q^k, that is if $(\psi_n, \phi_n, \theta_n)$ is bounded in Q^k and $t_n \longrightarrow \infty$, then $S(t_n)(\psi_n, \phi_n, \theta_n)$ is precompact in Q^k, thus from the Theorem I.1.1 of [8], we obtain the existence of global attractor $\mathcal{A}_k$ in Q^k.

3. Some Results on Spectral Approximation of (1.1)-(1.5)

For the problem (1.1)-(1.5), we consider the following Fourier spectral approximation: Find $(\psi_N,\phi_N,\theta_N)\in S_N^3$ satisfing

$$\begin{cases} i(\psi_{Nt},\chi)-(\nabla\psi_N,\nabla\chi)+i\nu(\psi_N,\chi)+(\phi_N\psi_N,\chi)=(f,\chi), & \forall\chi\in S_N, \quad (3.1)\\ (\phi_{Nt},\eta)+\delta(\phi_N,\eta)=(\theta_N,\eta), & \forall\eta\in S_N, \quad (3.2)\\ (\theta_{Nt},\zeta)+(\gamma-\delta)(\theta_N,\zeta)+(\nabla\phi_N,\nabla\zeta)+(1-\delta(\gamma-\delta))(\phi_N,\zeta) & \\ \qquad -(|\psi_N|^2,\zeta)=(g,\zeta), & \forall\zeta\in S_N, \quad (3.3)\\ \psi_N(0)=P_N\psi_0, \qquad \phi_N(0)=P_N\phi_0, \qquad \theta_N(0)=P_N\theta_0. & \quad (3.4)\end{cases}$$

Since the above problem is similar to (1.1)-(1.5) and for any u_N, $v_N\in S_N$,

$$\int_I u_N|v_N|^2dx=\int_I(P_Nu_N)|v_N|^2dx=\int_I u_NP_N(|v_N|^2)dx,$$

we can obtain

Lemma 7. *Assume that $f,g\in H$, then there exists a constant δ_1 such that for $\delta\le\delta_1$ and $\|(\psi_0,\phi_0,\theta_0)\|_V\le R$, the solution (ψ_N,ϕ_N,θ_N) of (3.1)-(3.4) satisfies*

$$\|(\psi_N(t),\phi_N(t),\theta_N(t))\|_V\le M, \qquad t\ge t_2,$$

where M depends on $\nu,\gamma,\delta,\|f\|,\|g\|$; t_2 depends on $\nu,\gamma,\delta,\|f\|,\|g\|$ and R.

Further assume that $f,g\in H_p^2$, $\psi_0,\phi_0\in H_p^2(I),\theta_0\in H_p^1(I)$ then we have

$$\sup_{t\in R^+}(\|\psi_N(t)\|_2^2+\|\phi_N(t)\|_2^2+\|\theta_N(t)\|_1^2)\le M_1$$

where M_1 depends on $\nu,\gamma,\delta,\|f\|_2,\|g\|_2$ only.

(3.1)-(3.4) is a system of ordinary differential equations.According to the theory of ordinary differential equations we get that for any $T\ge0$, (3.1)-(3.4) has a unique solution on the interval $[0,T]$. Hence (3.1)-(3.4) define a continuous operator semigroup $\{S_N(t)\}_{t\ge0}$ $S_N(t)(\psi_N(0),\phi_N(0),\theta_N(0))=(\psi_N(t),\phi_N(t),\theta_N(t))$ and we have

Theorem 2. *Assume that $f,g\in H$,$(\psi_0,\phi_0,\theta_0)\in\widetilde{V}$, then the operator semigroup $\{S_N(t)\}_{t\ge0}$ which is defined by (3.1)-(3.4) has a global attractor $\mathcal{A}_N$. It is a compact invariant subset of S_N^3 and absorbs any bounded set in S_N^3.*

4. The Error Estimate

Set

$$\psi-\psi_N=\psi-\widetilde{\psi}-(\psi_N-\widetilde{\psi})=a-b, \tag{4.1}$$

$$\phi-\phi_N=\phi-\widetilde{\phi}-(\phi_N-\widetilde{\phi})=c-d, \tag{4.2}$$

$$\theta-\theta_N=\theta-\widetilde{\theta}-(\theta_N-\widetilde{\theta})=e-h, \tag{4.3}$$

where $\widetilde{\psi}=P_N\psi,\widetilde{\phi}=P_N\phi,\widetilde{\theta}=P_N\theta$, hence they satisfy

$$(\nabla(\psi-\widetilde{\psi}),\nabla\chi)=0, \quad (\nabla(\phi-\widetilde{\phi}),\nabla\chi)=0, \quad (\nabla(\theta-\widetilde{\theta}),\nabla\chi)=0, \quad \forall\chi\in S_N \tag{4.4}$$

and by orthogonality b,d,h satisfy

$$\begin{cases} i(b_t,\chi)-(\nabla b,\nabla\chi)+i\nu(b,\chi)+(\phi_N\psi_N-\phi\psi,\chi) \\ =0, \qquad \forall\chi\in S_N, & (4.5)\\ (d_t,\eta)+\delta(d,\eta)-(h,\eta)=0, \qquad \forall\eta\in S_N, & (4.6)\\ (h_t,\zeta)+(\gamma-\delta)(h,\zeta)+(\nabla d,\nabla\zeta)+(1-\delta(\gamma-\delta))(d,\zeta)-(|\psi_N|^2-|\psi|^2,\zeta) \\ =0. & (4.7)\end{cases}$$

Setting $\chi = b$ in (4.5) and taking the imaginary part,we get

$$\frac{1}{2}\frac{d}{dt}\|b\|^2 + \nu\|b\|^2 \leq [\|\phi_N\|_\infty(\|a\| + \|b\|) + \|\psi\|_\infty(\|c\| + \|d\|)]\|b\| \\ \leq c_1(\|b\|^2 + \|a\|^2 + \|c\|^2 + \|d\|^2). \tag{4.8}$$

Taking $\eta = d$ in (4.6), we have

$$\frac{1}{2}\frac{d}{dt}\|d\|^2 + \delta\|d\|^2 \leq \frac{\delta}{2}\|d\|^2 + c_2\|h\|^2. \tag{4.9}$$

Setting $\zeta = h$ in (4.7), we get

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 + (\nu - \delta)\|h\|^2 - (h, \triangle d) + (1 - (\delta(\gamma - \delta))(d, h) + ((\bar{a} - \bar{b})\psi_N + (a - b)\bar{\psi}, h) \\ = 0. \tag{4.10}$$

In order to eliminate the term $(h, \triangle d)$, we set $\eta = \triangle d$ in (4.6) again, and then we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla d\|^2 + \delta\|\nabla d\|^2 = -(h, \triangle d). \tag{4.11}$$

By (4.10) and (4.11)

$$\frac{1}{2}\frac{d}{dt}(\|h\|^2 + \|\nabla d\|^2) + \delta\|\nabla d\|^2 + (\gamma - \delta)\|h\|^2 + (1 - (\delta(\gamma - \delta))(d, h) \\ + ((\bar{a} - \bar{b})\psi_N + (a - b)\bar{\psi}, h) = 0.$$

Using the ε-inequality we have

$$\frac{1}{2}\frac{d}{dt}(\|h\|^2 + \|\nabla d\|^2) + \frac{1}{2}(\gamma - \delta)\|h\|^2 + \delta\|\nabla d\|^2 \\ \leq c_3(\|d\|^2 + \|a\|^2 + \|b\|^2). \tag{4.12}$$

Finally setting $\chi = -\triangle b$ in (4.5) and taking the imagenary part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla b\|^2 + \nu\|\nabla b\|^2 = Im((d - c)\psi_N + (b - a)\phi, \triangle b),$$

here we have used $(a_t, \triangle b) = (a, \triangle b) = 0$. Using Lemma 7, $\|\psi_N\|_\infty \leq C$, and

$$\int_I |a||\nabla\psi_N||\nabla b|dx \leq \|\nabla b\|\|\nabla\psi_N\|_{L^3}\|a\|_{L^6} \leq c(\|b\|_1^2 + \|\nabla\psi_N\|_{L^3}^2\|a\|_{L^6}^2)$$

by the interpolation inequality

$$\|\nabla\psi_N\|_{L^3} \leq c\|\psi_N\|_2^{\frac{1}{2}}\|\nabla\psi_N\|^{\frac{1}{2}} \leq C,$$

embedding theorem $\|a\|_{L^6} \leq c\|a\|_1$, thus

$$\int_I |a||\nabla\psi_N||\nabla b|dx \leq c_4(\|b\|_1^2 + \|a\|_1^2).$$

Similarly estimate for the another terms, hence we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla b\|^2 \leq c_5(\|b\|_1^2 + \|d\|_1^2 + \|a\|_1^2 + \|c\|_1^2). \tag{4.13}$$

Summing up (4.8), (4.9), (4.12) and (4.13) we obtain

$$\frac{d}{dt}(\|b\|_1^2+\|d\|_1^2+\|h\|^2)+2\nu\|b\|^2+\delta\|d\|^2+2(\gamma-\delta)\|h\|^2+2\delta\|\nabla d\|^2$$
$$\leq c_6(\|b\|_1^2+\|d\|_1^2+\|h\|^2)+c_7(\|a\|_1^2+\|c\|_1^2).$$

By the Gronwall inequality for $t\in[0,T]$ we have

$$\|b(t)\|_1^2+\|d(t)\|_1^2+\|h(t)\|^2\leq C_T N^{-2\sigma}(\|\psi\|_{\sigma+1}^2+\|\phi\|_{\sigma+1}^2).$$

Finally using the trigonometric inequality and Lemma 1, we get

$$\|\psi(t)-\psi_N(t)\|_1^2+\|\phi(t)-\phi_N(t)\|_1^2+\|\theta(t)-\theta_N(t)\|^2$$
$$\leq C_T N^{-2\sigma}(\|\psi\|_{\sigma+1}^2+\|\phi\|_{\sigma+1}^2+\|\theta\|_\sigma^2). \tag{4.14}$$

Theorem 3. *Assume that the problem (1.1)-(1.5) have a unique solution and it satisfies* ψ, $\phi\in L^\infty(R^+;H_p^{\sigma+1}(I))$, $\theta\in L^\infty(R^+;H_p^\sigma(I))$, $\sigma\geq 1$, *then we have the error estimate (4.14) for the solution* $(\psi_N(t),\phi_N(t),\theta_N(t))$ *of the Fourier spectral method (3.1)-(3.4).*

Using the results of [8]: Let $H_\eta\subset H$ is a class subspace,$0<\eta\leq\eta_0$, $\bigcup_{0<\eta\leq\eta_0} H_\eta$ is dense in H. For every $\eta>0$, the semigroup operator $S_\eta(t)$ maps H_η into itself and is continuous for $t\geq 0$,and for every compact interval $\widetilde{I}\subset R^+$,

$$\delta_\eta(\widetilde{I})=\sup_{\substack{u_0\in H_\eta\\ \|u_0\|\leq R}}\sup_{t\in\widetilde{I}} dist(S_\eta(t)u_0,S(t)u_0)\to 0,\quad \eta\to 0.$$

Assume that for every $\eta>0$, $\{S_\eta(t)\}_{t\geq 0}$ has a global attractor which absorbs any bounded open neighourhood of $\mathcal{A}_\eta\bigcup\mathcal{A}$.

Theorem 4. *Under the above assumptions, we have*

$$d_H(\mathcal{A}_\eta,\mathcal{A})\to 0,\qquad \eta\to 0.$$

where

$$d_H(\mathcal{A}_\eta,\mathcal{A})=\sup_{x\in\mathcal{A}_\eta}\inf_{y\in\mathcal{A}} dist_H(x,y).$$

In order to use Theorem 4, we take $H=V$,$H_\eta=H_N=S_N^3$. Obviously, $S_N(t)$ maps S_N^3 into S_N^3 for all $t\geq 0$, and it is continous for every N. From Theorem 3

$$\delta_N(\widetilde{I})=\sup_{\substack{(\psi_0,\phi_0,\theta_0)\in S_N^3\\ \|(\psi_0,\phi_0,\theta_0)\|_V\leq R}}\sup_{t\in\widetilde{I}}\|(S_N(t)(\psi_0,\phi_0,\theta_0)-S(t)(\psi_0,\phi_0,\theta_0)\|_V\to 0,\qquad N\to\infty.$$

Hence we have

Theorem 5. *Under the assumptions of Theorems 3* ($\sigma\geq 1$), *for* $\mathcal{A}$ *and* $\mathcal{A}_N$,*we have*

$$d_V(\mathcal{A}_N,\mathcal{A})\to 0,\qquad N\to+\infty.$$

5. Upper Bounds of Hausdorff and Fractal Dimension for $\mathcal{A}$ and $\mathcal{A}_N$

In this section we give an estimate of the upper bounds of Hausdorff and fractal dimension for $\mathcal{A}$ and $\mathcal{A}_N$. First of all we introduce some definitions and lemmas.

Let H is a Hilbert space, $X\subset H$ is a compact set, $S(t)$ maps X into H which is a continuous map and satisfies

(i) $S(t)X = X, \quad t\geq 0$;

(ii) $S(t)$ is uniformly differentiable on X, i.e. for every $u\in X$, where exists a linear operator $L(t,v)\in\mathcal{L}(H,H)$ and

$$\sup_{\substack{v\in X\\ 0<|u-v|_H\leq\varepsilon}} \frac{|S(t)u-S(t)v-L(t;v)(u-v)|_H}{|u-v|_H}\to 0, \qquad \varepsilon\to 0;$$

(iii) $\sup|L(t;v)|_{\mathcal{L}(H,H)}<+\infty$.

Note that

$$|\zeta_1\wedge\zeta_2\wedge\cdots\wedge\zeta_m|_H^2 = \det_{1\leq i,j\leq m}(\zeta_i,\zeta_j), \qquad \forall\zeta_1,\cdots,\zeta_m\in H, \tag{5.1}$$

$$\overline{\omega}_m(t) = \sup_{v_0\in X}\omega_m(L(t,v_0)), \tag{5.2}$$

where

$$\omega_m(L(t,v_0)) = \sup_{\substack{\zeta_i\in H\\ |\zeta_i|_H\leq 1, i=1,2,\cdots,m}} |L(t;v_0)\zeta_1\wedge L(t;v_0)\zeta_2\wedge\cdots\wedge L(t;v_0)\zeta_m|_H. \tag{5.3}$$

then we have

Lemma 8[8]**.** *Assume that the conditions (i)-(iii) hold. If for some $d>0$, $t_0>0$ such that $\sup_{u\in X}\omega_d(L(t_0,u))<1$, then the Hausdorff dimension $d_H(X)$ of X is finite and less than or equal to d, where $\omega_d(L)=\omega_m^{1-s}(L)\omega_{m+1}^s(L)$, $d=m+s$,and its fractal dimension $d_F(x)$ is finite too.*

In order to apply the Lemma to the above, we need to verify (i)-(iii).The condition (i) is obviously satisfied for $\mathcal{A}$ and $S(t)$ in $X=\widetilde{V}$. Now we examine the conditions (ii) and (iii).

The linearization of problem (1.1)-(1.5) is

$$\begin{cases} iU_t+\triangle U+i\nu U+\phi U+\psi V=0, & (5.4)\\ V_t+\delta V=W, & (5.5)\\ W_t+(\gamma-\delta)W-\triangle V+(1-\delta(\gamma-\delta))V-(\psi\bar{U}+U\bar{\psi})=0, & (5.6)\\ U(x+2\pi e_i,t)=U(x,t), V(x+2\pi e_i,t)=V(x,t), W(x+2\pi e_i,t)=W(x,t), & (5.7)\\ U(0)=\psi_1, V(0)=\phi_1, W(0)=\theta_1. & (5.8)\end{cases}$$

Using the priori estimate and the theory of linear partial differential equation, we obtain

Theorem 6. *Assume that $(\psi_1,\phi_1,\theta_1)\in V$, then there exists a unique solution for (5.4)-(5.8) $(U(t),V(t),W(t))\in C(R^+,V)$ and the following estimate holds*

$$\|U(t)\|_1^2+\|V(t)\|_1^2+\|W(t)\|^2\leq C_T(\|\psi_1\|_1^2+\|\phi_1\|_1^2+\|\theta_1\|^2), \quad 0\leq t\leq T, \ \forall T>0.$$

Now we prove the solution operator $S(t)$ of (1.1)-(1.5) is uniformly differentiable on $\widetilde{V}$. First of all we study the difference of solutions corresponding to different initial values.

Suppose that the problem (1.1)-(1.5) has solutions $(\psi_1^*,\phi_1^*,\theta_1^*)$ and $(\psi_0^*,\phi_0^*,\theta_0^*)$ which corresponding to $(\psi_0+\psi_1,\phi_0+\phi_1,\theta_0+\theta_1)$ and (ψ_0,ϕ_0,θ_0), respectively, then their difference (Ψ,Φ,Θ) satisfies $(\Psi(0),\Phi(0),\Theta(0))=(\psi_1,\phi_1,\theta_1)$ and the equation

$$\begin{cases} i\Psi_t+\triangle\Psi+i\nu\Psi+\phi_1^*\Psi+\Phi\psi_0^*=0, & (5.9)\\ \Phi_t+\delta\Phi=\Theta, & (5.10)\\ \Theta_t+(\gamma-\delta)\Theta-\triangle\Phi+(1-\delta(\gamma-\delta))\Phi-(\psi_1^*\bar{\Psi}+\bar{\psi_0^*}\Psi)=0, & (5.11)\end{cases}$$

Since the above equation is similar to (5.4)-(5.6),we can obtian

$$\frac{d}{dt}(\|\Psi(t)\|_1^2+\|\Phi(t)\|_1^2+\|\Theta(t)\|^2)\leq C(\|\Psi(t)\|_1^2+\|\Phi(t)\|_1^2+\|\Theta(t)\|^2)$$

i.e

$$\|\Psi(t)\|_1^2+\|\Phi(t)\|_1^2+\|\Theta(t)\|^2\le e^{Ct}(\|\psi_1\|_1^2+\|\phi_1\|_1^2+\|\theta_1\|^2). \tag{5.12}$$

Further we consider the difference

$$(\widetilde{\psi},\widetilde{\phi},\widetilde{\theta})=(\Psi,\Phi,\Theta)-(U,V,W) \tag{5.13}$$

where (U,V,W) is the solution of (5.4)-(5.8), then $(\widetilde{\psi},\widetilde{\phi},\widetilde{\theta})$ satisfies $(\widetilde{\psi}(0),\widetilde{\phi}(0),\widetilde{\theta}(0))=0$, and the equation

$$\begin{cases} i\widetilde{\psi_t}+\triangle\widetilde{\psi}+i\nu\widetilde{\psi}+\phi_1^*\Psi+\psi_0^*\Phi-\phi_0^*U-\psi_0^*V=0, & (5.14)\\ \widetilde{\phi}_t+\delta\widetilde{\phi}=\widetilde{\theta}, & (5.15)\\ \widetilde{\theta}_t+(\gamma-\delta)\widetilde{\theta}-\triangle\widetilde{\phi}+(1-\delta(\gamma-\delta))\widetilde{\phi} & \\ \qquad -(\psi_1^*\overline{\Psi}+\bar{\psi}_0^*\Psi-\psi_0^*\overline{U}-\bar{\psi}_0^*U)=0. & (5.16)\end{cases}$$

From (5.14)-(5.16),it follows that

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\psi}\|^2+\nu\|\widetilde{\psi}\|^2\le\int_I|\psi_0^*\widetilde{\phi}\widetilde{\psi}|dx+\int_I|U\Phi\widetilde{\psi}|dx, \tag{5.17}$$

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\phi}\|^2+\delta\|\widetilde{\phi}\|^2=(\widetilde{\theta},\widetilde{\phi}), \tag{5.18}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{\psi}\|^2+\nu\|\nabla\widetilde{\psi}\|^2-Im(\phi_1^*\Psi+\psi_0^*\Phi-\phi_0^*U-\psi_0^*V,\triangle\widetilde{\psi})=0. \tag{5.19}$$

Using (5.15) again

$$(-\triangle\widetilde{\phi},\widetilde{\theta})=\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{\phi}\|^2+\delta\|\nabla\widetilde{\phi}\|^2,$$

we have

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\widetilde{\phi}\|^2+\|\widetilde{\theta}\|^2)+(\gamma-\delta)\|\widetilde{\theta}\|^2+\delta\|\nabla\widetilde{\phi}\|^2+(1-\delta(\gamma-\delta))(\widetilde{\phi},\widetilde{\theta})$$
$$-(\psi_1^*\overline{\Psi}+\bar{\psi}_0^*\Psi-\psi_0^*\overline{U}-\bar{\psi}_0^*U,\widetilde{\theta})=0. \tag{5.20}$$

Summing up (5.17)-(5.20) and noting that

$$\phi_1^*\Psi+\psi_0^*\Phi-\phi_0^*U-\psi_0^*V=\phi_1^*\widetilde{\psi}+\psi_0^*\widetilde{\phi}+U\Phi,$$

$$\psi_1^*\overline{\Psi}+\bar{\psi_0}^*\Psi-\psi_0^*\overline{U}-\bar{\psi_0}^*U=\psi_1^*\overline{\widetilde{\psi}}+\bar{\psi}_0^*\widetilde{\psi}+\bar{U}\Psi,$$

we have

$$\frac{1}{2}\frac{d}{dt}(\|\widetilde{\psi}\|_1^2+\|\widetilde{\phi}\|_1^2+\|\widetilde{\theta}\|^2)+\nu\|\widetilde{\psi}\|_1^2+\delta\|\widetilde{\phi}\|_1^2+(\gamma-\delta)\|\widetilde{\theta}\|^2$$
$$\le\int_I|\psi_0^*\widetilde{\phi}\widetilde{\psi}|dx+\int_I|U\Phi\widetilde{\psi}|dx+\int_I|\widetilde{\theta}\widetilde{\phi}|dx+\delta(\gamma-\delta)\int_I|\widetilde{\theta}\widetilde{\phi}|dx$$
$$+\int_I|(\psi_1^*\overline{\widetilde{\psi}}+\bar{\psi}_0^*\widetilde{\psi}+\bar{U}\Psi)\widetilde{\theta}|dx+\int_I[|\widetilde{\psi}\nabla\phi_1^*\nabla\widetilde{\psi}|+|\nabla(\psi_0^*\widetilde{\phi}+U\Phi)\nabla\widetilde{\psi}|]dx. \tag{5.21}$$

For the terms on the right side of (5.21) we have following estimations

$$\int_I|\widetilde{\theta}\widetilde{\phi}|dx\le\|\widetilde{\theta}\|^2+\|\widetilde{\phi}\|^2,\qquad\int_I|\psi_0^*\widetilde{\phi}\widetilde{\psi}|dx\le\|\psi_0^*\|_\infty(\|\widetilde{\phi}\|^2+\|\widetilde{\psi}\|^2),$$

$$\int_I|(\psi_1^*\overline{\widetilde{\psi}}+\bar{\psi}_0^*\widetilde{\psi})\widetilde{\theta}|dx\le(\|\psi_1^*\|_\infty+\|\psi_0^*\|_\infty)(\|\widetilde{\psi}\|^2+\|\widetilde{\theta}\|^2),$$

$$\int_I |U\Phi\widetilde{\psi}|dx \le c\|\widetilde{\psi}\|\|\Phi\|_{L^4}\|U\|_{L^4} \le c\|\widetilde{\psi}\|\|\Phi\|_1^{3/4}\|\Phi\|^{1/4}\|U\|_1^{3/4}\|U\|^{1/4}$$

$$\le c\|\widetilde{\psi}\|^2 + c(\|\Phi\|_1^3\|\Phi\| + \|U\|_1^3\|U\|) \le c\|\widetilde{\psi}\|^2 + c(\|\Phi\|_1^4 + \|U\|_1^4),$$

here we used Lemma 2 to obtain the last inequality. Similarly

$$\int_I |\bar{U}\Psi\widetilde{\theta}|dx \le c\|\widetilde{\theta}\|^2 + c(\|\Psi\|_1^4 + \|U\|_1^4).$$

Furthermore

$$\int_I |\psi_0^*\nabla\widetilde{\phi}\nabla\widetilde{\psi}|dx \le \|\psi_0^*\|_\infty(\|\nabla\widetilde{\phi}\|^2 + \|\nabla\widetilde{\psi}\|^2),$$

$$|\int_I U\nabla\widetilde{\psi}\nabla\Phi dx| \le \|\nabla\widetilde{\psi}\|\|U\|_{L^p}\|\nabla\Phi\|_{L^q} \le c(\|\nabla\widetilde{\psi}\|^2 + \|U\|_{L^p}^2\|\nabla\Phi\|_{L^q}^2),$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.Using the interpolation inequality

$$\|U\|_{L^p} \le c\|U\|_1^\alpha\|U\|^{1-\alpha} \le c\|U\|_1$$

$$\|\nabla\Phi\|_{L_q} \le C(\|\triangle\Phi\| + \|\nabla\Phi\|)^\beta\|\nabla\Phi\|^{1-\beta},$$

where $\alpha = \frac{3p-6}{2p},\ \beta = 3(\frac{1}{2} - \frac{1}{q}) = \frac{3}{p},\ 3 < p < 6$. Thus by for given $(\psi_0, \phi_0, \theta_0) \in \widetilde{V}$, the solution $(\psi(t), \phi(t), \theta(t)) \in \widetilde{V}$

$$|\int_I U\nabla\widetilde{\psi}\nabla\Phi dx| \le c\|\nabla\widetilde{\psi}\|^2 + c\|U\|_1^2(\|\nabla\Phi\| + \|\triangle\Phi\|)^{2\beta}\|\nabla\Phi\|^{2(1-\beta)}$$

$$\le c(\|\nabla\widetilde{\psi}\|^2 + \|U\|_1^2\|\nabla\Phi\|^2 + \|U\|_1^2\|\nabla\Phi\|^{2(1-\frac{3}{p})})$$

$$\le c(\|\nabla\widetilde{\psi}\|^2 + \|\Phi\|_1^3 + \|U\|_1^{\frac{12}{5}} + \|\Phi\|_1^4),$$

where we choose $p = q = 4,\ \alpha = \beta = \dfrac{3}{4},\ s = \dfrac{6}{5},\ r = 6.$

$$\int_I |\Phi\nabla U\nabla\widetilde{\psi}|dx \le \|\Phi\|_\infty\|\nabla U\|\|\nabla\widetilde{\psi}\| \le c\|\Phi\|_2^{\frac{3}{4}}\|\Phi\|^{\frac{1}{4}}\|\nabla U\|\|\nabla\widetilde{\psi}\|$$

$$\le c\|\Phi\|^{\frac{1}{4}}\|\nabla U\|\|\nabla\widetilde{\psi}\| \le c(\|\widetilde{\psi}\|_1^2 + \|\Phi\|_1^{\frac{9}{4}} + \|U\|_1^{\frac{18}{7}}),$$

Similarly, $\int_I |\widetilde{\psi}\nabla\phi_1^*\nabla\widetilde{\psi}|dx,\ \int_I |\widetilde{\phi}\nabla\psi_0^*\nabla\widetilde{\psi}|dx$ can be estimated.

Substituting above estimations to (5.21), we get

$$\frac{d}{dt}(\|\widetilde{\psi}\|_1^2 + \|\widetilde{\phi}\|_1^2 + \|\widetilde{\theta}\|^2)$$

$$\le c(\|\widetilde{\psi}\|_1^2 + \|\widetilde{\phi}\|_1^2 + \|\widetilde{\theta}\|^2) + c(\|\Psi\|_1^4 + \|\Phi\|_1^4 + \|U\|_1^4 + \|V\|_1^4 + \|\Phi\|_1^3 + \|U\|_1^{\frac{12}{5}} + \|\Phi\|_1^{\frac{9}{4}} + \|U\|_1^{\frac{18}{7}})$$

and using Gronwall's inequality

$$\sup_{t\in[0,T]}(\|\widetilde{\psi}(t)\|_1^2 + \|\widetilde{\phi}(t)\|_1^2 + \|\widetilde{\theta}(t)\|^2)$$

$$\le C_T \sup_{t\in[0,T]}(\|\Psi(t)\|_1^4 + \|\Phi(t)\|_1^4 + \|U(t)\|_1^4 + \|V(t)\|_1^4 + \|\Phi\|_1^3 + \|U\|_1^{\frac{12}{5}} + \|\Phi\|_1^{\frac{9}{4}} + \|U\|_1^{\frac{18}{7}})$$

$$\le C_T(\|\psi_1\|_1^4 + \|\phi_1\|_1^4 + \|\theta_1\|^4 + \|\phi_1\|_1^3 + \|\psi_1\|_1^{\frac{12}{5}} + \|\phi_1\|_1^{\frac{9}{4}} + \|\psi_1\|_1^{\frac{18}{7}}).$$

Finally we obtain

$$\sup_{0<\|(\psi_1,\phi_1,\theta_1)\|_V\leq\varepsilon\ (\psi_1,\phi_1,\theta_1)\in\widetilde{V}}$$

$$\frac{\|S(t)(\psi_0+\psi_1,\phi_0+\phi_1,\theta_0+\theta_1)-S(t)(\psi_0,\phi_0,\theta_0)-L(t;u_0)(\psi_1,\phi_1,\theta_1)\|_V^2}{\|(\psi_1,\phi_1,\theta_1)\|_V^2}$$

$$\longrightarrow 0, \qquad \varepsilon\to 0,$$

where $u_0=(\psi_0,\phi_0,\theta_0)$.

Lemma 9. *Assume that the semigroup operator of (1.1)-(1.5) and (5.4)-(5.8) are $S(t)$ and $L(t;u_0)$ respectively, then they satisfy the conditions (i)-(iii) of Lemma 8, i.e, $S(t)$ is uniformly differentiable on $\widetilde{V}$.*

Now we prove that there exist m,t_0 such that $d=m+s$, $\sup_{u\in\mathcal{A}}\omega_d(L(t_0;u))<1$.

Let $Q^1(t),\cdots,Q^m(t)$ are solutions of (5.4)-(5.8) with initial values $\xi_1,\cdots,\xi_m\in V$, respectively. Note that $q(t)=e^{\delta t}Q(t)$ then $q(t)=(e^{\delta t}U(t),e^{\delta t}V(t),e^{\delta t}W(t))=(u(t),v(t),w(t))$ and

$$\begin{aligned}\det_{1\leq i,j\leq m}(Q^i(t),Q^j(t))_V&=|Q^1(t)\wedge Q^2(t)\wedge\cdots\wedge Q^m(t)|_V^2\\&=e^{-2m\delta t}|q^1(t)\wedge q^2(t)\wedge\cdots\wedge q^m(t)|_V^2.\end{aligned}\tag{5.22}$$

We shall prove this is exponential decalying if $t\to+\infty$.

Obviously $q(t)$ satisfies

$$\begin{cases}iu_t+\triangle u+i(\nu-\delta)u+\phi u+\psi v=0,\\ v_t=w,\\ w_t+(\gamma-2\delta)w-\triangle v+(1-\delta(\gamma-\delta))v-(\psi\bar{u}+\bar{\psi}u)=0.\end{cases}$$

Since

$$\mu\frac{d}{dt}(\|u\|^2+\|v\|^2)=\mu[-2\nu\|u\|^2-2Im\int_I\psi\bar{u}vdx-2\delta\|v\|^2+2\int_I vwdx],$$

similarly, we can obtain

$$\frac{d}{dt}I_\mu(q(t))=J_\mu(q(t))$$

where

$$I_\mu(q(t))=\|u\|_1^2+\|v\|_1^2+\|w\|_1^2-\int_I\phi|u|^2dx-2Re\int_I\psi\bar{u}vdx+\mu(\|u\|^2+\|v\|^2),$$

$$\begin{aligned}J_\mu(q(t))=&-(\nu-\delta)\|u\|_1^2-(\gamma-2\delta)\|w\|^2-Im\int_I\psi\bar{u}vdx+(\nu-\delta)Re\int_I\bar{\psi}uvdx\\&+(\nu-\delta)\int_I\phi|u|^2dx-\frac{1}{2}\int_I(\theta-\delta\phi)|u|^2dx-Re\int_I\psi_t\bar{u}vdx\\&-Re\int_I\psi\bar{u}wdx+\delta(\gamma-\delta)\int_I vwdx+\int_I(\psi\bar{u}+\bar{\psi}u)wdx\\&+\mu[-2\nu\|u\|^2-2Im\int_I\psi\bar{u}vdx-2\delta\|v\|^2+2\int_I vwdx].\end{aligned}$$

We can take μ sufficiently large such that $(I_\mu(q(t)))^{1/2}$ induces an equivalent norm in V.

Note that $q(t)=(u(t),v(t),w(t))$, $\widetilde{q}(t)=(\widetilde{u}(t),\widetilde{v}(t),\widetilde{w}(t))$ and

$$\Psi_\mu(t,q(t),\widetilde{q}(t))=\int_I(u\overline{\widetilde{u}}+\nabla u\cdot\nabla\overline{\widetilde{u}}+v\overline{\widetilde{v}}+\nabla v\cdot\nabla\overline{\widetilde{v}}+w\overline{\widetilde{w}}-\phi u\overline{\widetilde{u}})dx$$

$$-\frac{1}{2}\int_I[(\psi\bar{u}+\bar{\psi}u)v+(\psi\bar{u}_1+\bar{\psi}u_1)v_1]dx+\mu\int_I(u\overline{\widetilde{u}}+v\overline{\widetilde{v}})dx.$$

Obviously $\Psi_\mu(t,q(t),\widetilde{q}(t))$ is a symmetrical positive definite bilinear functional in $V\times V$ and satisfies $\Psi_\mu(t,q(t),q(t))=I_\mu(q(t))$. Hence $\Psi_\mu(t,q(t),q(t))$ defines an inner product in $V\times V$.

Using the ε-inequality we have

$$|J_\mu(q(t))|\leq c^*(\|u\|^2+\|v\|^2).$$

Let

$$H_m(t)=\det_{1\leq i,j\leq m}\Psi_\mu(t,q^i(t),q^j(t))=\prod_{l=1}^m\max_{\substack{F\subset R^m\\ \dim F=l}}\min_{\substack{x\in F\\ \sum_1^m|x_j|^2=1}}\Psi_\mu(t,\sum_1^m x_jq^j,\sum_1^m x_jq^j)$$

$$\leq\prod_{l=1}^m\max_{\substack{F\subset R^m\\ \dim F=l}}\min_{\substack{x\in F\\ \sum_1^m|x_j|^2=1}}\widetilde{c_1}(\sum_1^m x_jq^j,\sum_1^m x_jq^j)_V=\widetilde{c_1}^{\,m}\det_{1\leq i,j\leq m}(q^i,q^j)_V$$

$$=\widetilde{c_1}^{\,m}|q^1(t)\wedge q^2(t)\wedge\cdots\wedge q^m(t)|_V^2.$$

Taking μ properly we have

$$|q^1(t)\wedge q^2(t)\wedge\cdots\wedge q^m(t)|_V^2=\prod_{l=1}^m\max_{\substack{F\subset R^m\\ \dim F=l}}\min_{\substack{x\in F\\ \sum_1^m|x_j|^2=1}}(\sum_1^m x_jq^j(t),\sum_1^m x_jq^j(t))_V$$

$$\leq\prod_{l=1}^m\max_{\substack{F\subset R^m\\ \dim F=l}}\min_{\substack{x\in F\\ \sum_1^m|x_j|^2=1}}\Psi_\mu(t,\sum_1^m x_jq^j(t),\sum_1^m x_jq^j(t))$$

$$=\det_{1\leq i,j\leq m}\Psi_\mu(t,\sum_1^m x_jq^j(t),\sum_1^m x_jq^j(t))=H_m(t)\tag{5.23}$$

According to the results of [9], we get

$$\frac{d}{dt}H_m(t)=H_m(t)\sum_{l=1}^{m}\max_{\substack{F\subset R^m\\ \dim F=l}}\min_{\substack{x\in F\\ \sum_1^m|x_j|^2=1}}\frac{J_\mu(\sum_1^m x_jq^j(t))}{I_\mu(\sum_1^m x_jq^j(t))}$$

$$\le H_m(t)\sum_{l=1}^{m}\max_{\substack{F\subset R^m\\ \dim F=l}}\min_{\substack{x\in F\\ \sum_1^m|x_j|^2=1}}\frac{c^*(\|\sum_1^m x_ju^j\|^2+\|\sum_1^m x_jv^j\|^2)}{c^{**}(\|\sum_1^m x_ju^j\|_1^2+\|\sum_1^m x_jv^j\|_1^2+\|\sum_1^m x_jw^j\|^2)}$$

$$\le \frac{c^*}{c^{**}}H_m(t)\sum_{l=1}^{m}\max_{\substack{F\subset R^m\\ \dim F=l}}\min_{\substack{x\in F\\ \sum_1^m|x_j|^2=1}}\Big(\frac{\|\sum_1^m x_ju^j\|^2}{\|\sum_1^m x_ju^j\|_1^2}+\frac{\|\sum_1^m x_jv^j\|^2}{\|\sum_1^m x_jv^j\|_1^2}\Big)$$

$$\le \frac{2c^*}{c^{**}}H_m(t)\sum_{l=1}^{m}\frac{1}{1+\lambda_l}$$

and $\lambda_l\sim\widehat{c}\,l^{\frac{2}{3}}$ is the l-th eigenvalue of $-\Delta$. Hence

$$\frac{d}{dt}H_m(t)\le\frac{2c^*}{c^{**}}H_m(t)\sum_{l=1}^{m}\frac{1}{1+\widehat{c}\,l^{\frac{2}{3}}}\le\widetilde{c}m^{\frac{1}{3}}\ H_m(t)$$

where $\widetilde{c}$ is a constant, and

$$H_m(t)\le\exp(\widetilde{c}m^{\frac{1}{3}}t)H_m(0). \tag{5.24}$$

Thus by (5.22)-(5.24) we have

$$|Q^1(t)\wedge Q^2(t)\wedge\cdots\wedge Q^m(t)|_V^2\le\exp\{t(\widetilde{c}m^{\frac{1}{3}}-2\delta m)\}H_m(0)$$

$$\le\ \exp\{t(\widetilde{c}m^{\frac{1}{3}}-2\delta m)\}\widetilde{c_1}^{\,m}|q^1(0)\wedge q^2(0)\wedge\cdots\wedge q^m(0)|_V^2$$

$$=\ \exp\{t(\widetilde{c}m^{\frac{1}{3}}-2\delta m)\}\widetilde{c_1}^{\,m}|\xi^1\wedge\xi^2\wedge\cdots\wedge\xi^m|_V^2. \tag{5.25}$$

If $m>\frac{\widetilde{c}}{2\delta}m^{\frac{1}{3}}\ge m-1$, then we have

Theorem 7. *The global attractor $\mathcal{A}$ defined by Theorem 1 then the Hausdorff dimension of $\mathcal{A}$ is less than m and its fractal dimension is less than 2m in $\widetilde{V}$.*

By the result of paper [10]: Let H is a Hilbert space, $\{S(t)\}_{t\ge0}$ and $\{S_N(t)\}_{t\ge0}$ are C^1 operator semigroup and as $u_N(t)=S_N(t)u_{0N}$ approaches $u(t)=S(t)u_0$ as $N\ge N_0$,where $u_{0N}\in S_N$ is some approximation of u_0, i.e. for any $R>0,T>0,$ there exists a constant $C(R,T)$ such that $\forall\ \|u_0\|_H\le R$,

$$\|S_N(t)u_{0N}-S(t)u_0\|_H\le C(R,T)N^{-1}, \tag{5.26}$$

$$\|S'_N(t,u_{0N})-S'(t,u_0)\|_{\mathcal{L}(H,H)}\le C(R,T)N^{-1}. \tag{5.27}$$

Suppose that $\mathcal{A}$ and $\mathcal{A}_N$ are compact set which satisfy $S(t)\mathcal{A}=\mathcal{A}$, and $S_N(t)\mathcal{A}_N=\mathcal{A}_N,t\ge0,N\ge N_0$ and $d_H(\mathcal{A}_N,\mathcal{A})\to0,N\to+\infty$, then we have

Theorem 8[10]**.** *Suppose the above assumptions hold and there exists $T>0,1\le d\le N$, such that*

$$\omega_d(S'(T;u))<1,\qquad \forall u\in\mathcal{A},$$

then we have $d_H(\mathcal{A}) \le d$ and $d_H(\mathcal{A}_N) \le d$ when N is sufficiently large.

For our problem, $S^{'}(t, u_0)$ and $S_N^{'}(t, u_{0N})$ ara solution operators of (5.4)-(5.8) and

$$\begin{cases} i(U_{Nt},\chi) - (\nabla U_N, \nabla\chi) + i\nu(U_N,\chi) + (\psi_N V_N + U_N\phi_N,\chi) = 0, \quad \forall\chi \in S_N, & (5.28)\\ (V_{Nt},\eta) + \delta(V_N,\eta) = (W_N,\eta), \quad \forall\eta \in S_N, & (5.29)\\ (W_{Nt},\zeta) + (\gamma-\delta)(W_N,\zeta) + (\nabla V_N, \nabla\zeta) + (1-\delta(\gamma-\delta))(V_N,\zeta) & \\ \quad -(U_N\bar{\psi}_N + \bar{U}_N\psi_N,\zeta) = 0, \quad \forall\zeta \in S_N, & (5.30)\\ U_N(0) = P_N\psi_1,\ V_N(0) = P_N\phi_1,\ W_N(0) = P_N\theta_1, & (5.31)\\ U_N(x+2\pi e_i,t) = U_N(x,t),\ V_N(x+2\pi e_i,t) = V_N(x,t), & (5.32)\\ W_N(x+2\pi e_i,t) = W_N(x,t) & \end{cases}$$

respectively .By Theorem 3, the condition (5.26) is hold ($\sigma = 1$) and using similar error estimate between (5.4)-(5.8) and (5.28)-(5.32) we can also obtain the (5.27). Using Theorem 7 we obtain

Theorem 9. *The global attractor $\mathcal{A}_N$ which defined by semigroup $\{S_N(t)\}_{t\ge 0}$ of (4.5)-(4.7) has same upper bound of Hausdorff dimension with $\mathcal{A}$ for sufficiently large N.*

Similarly we can obtain that $\mathcal{A}$ and $\mathcal{A}_N$ have same fractal dimension when N is sufficiently large.

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References

[1] Guo Boling, The global solutions of some problems for a system of equations of Schrödinger-Klein-Gordon field., *Scientia Sinica, Ser. A,* **9** (1982), 897-910.

[2] I.Fukuda & M.Tsutsumi, On Klein-Gordon-Schrödinger equations III, *Math Japan,* **24** (1979), 307-321.

[3] Xiang Xinmin, Spectral method for solving the system of equations of Schrödinger-Klein-Gordon field., *J. Compu. & Appl. Math.*, **21** (1988), 161-171.

[4] Guo Boling & Mial Changxing, Asymptotic behavior of conpled Klein-Gordon-Schrödinger equations, *Sci. China, Ser A,* **7** (1995), 705-714.

[5] Kenming Lu & Bixiang Wang, Global attactors for the Klein-Gordon-Schrödinger equations in unbounded domains, *J. Diff. Equa.,* **170** (2001), 281-316.

[6] B.Guo & Y.Li, Attractors for Klein-Gordon-Schrödinger equations in R^3, *J. Diff. Equa.,* **136** (1997), 356-377.

[7] C.Canuto & A.Quarteroni, Approximation result for orthogonal polynomials in Sobolev spaces, *Math Comput.*, **157** (1982), 67-86.

[8] R.Temam, Infinite-dimensional dynamical systems in mechanics and physics, 2nd ed, Springer-Verlag, 1997.

[9] J.M Ghidaglia, Finite dimensional behavior for weakly damped driven Schrödinger equations, *Ann. Institut H.Poincaré, Analyse Nonlinéairer,* **5** (1988) 365-405.

[10] J.Lorenz, Numerics of invariant manifolds and attractors, *Comtemporaray Math.*, **172** (1994), 185-202.

[11] B.Wang & H.Lange, Attracrors for Klein-Gordon-Schrödinger equation, *J. Math. Phys.*, **40** (1999), 2445-2457.

[12] P.Biler, Attractors for the system of Schrödinger and Klein-Gordon equations with Yukawa coupling, *SIAM J. Math. Anal.*, **21** (1990), 1190-1212.