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AN ALGORITHM FOR FINDING GLOBAL MINIMUM OF NONLINEAR INTEGER PROGRAMMING*

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Abstract

A filled function is proposed by R.Ge[2] for finding a global minimizer of a function of several continuous variables. In [4], an approach for finding a global integer minimizer of nonlinear function using the above filled function is given. Meanwhile a major obstacle is met: if $\rho > 0$ is small, and $\|x_I - \overset{*}{x}_I\|$ is large, where x_I - an integer point, $\overset{*}{x}_I$ - a current local integer minimizer, then the value of the filled function almost equals zero. Thus it is difficult to recognize the size of the value of the filled function and can not to find the global integer minimizer of nonlinear function. In this paper, two new filled functions are proposed for finding global integer minimizer of nonlinear function, the new filled function improves some properties of the filled function proposed by R. Ge [2].

Some numerical results are given, which indicate the new filled function (4.1) to find global integer minimizer of nonlinear function is efficient.

Key words: Local integer minimizer, Global integer minimizer, Filled function.

1. Introduction

It is well known, the problem for linear integer programming is an NP hard problem [3]. Certainly, nonlinear integer programming is more difficult than linear integer programming. There is no any algorithm for finding global minimizer of integer programming with quadratic constraints and linear objective function [1]. Therefore, generally assume that the domain of function is bounded when nonlinear integer programming is concerned. Even for this case, there also is lack of the algorithms to solve it.

In [4], we proposed an approach for finding global minimum of nonlinear integer programming (P) using the filled function method [2].

$$\text{(P)} \qquad \min_{x_I \in X_I} f(x_I)$$

where $X \subset R^n$-a bounded box, X_I-an integer point set in X, R_I^n- all integer points in R^n.

Redefined function $f(x_I)$,

$$f(x_I) = \begin{cases} f(x_I), & x_I \in X_I, \\ +\infty, & x_I \in R_I^n \setminus X_I. \end{cases}$$

Thus problem (P) is equivalent to the following problem $(\text{P})_I$

$$(\text{P})_I \qquad \min_{x_I \in R_I^n} f(x_I) \tag{1.1}$$

The filled function is as follows

$$p(x_I, \overset{*}{x}{}_I^1, r, \rho) = \begin{cases} \dfrac{1}{r + f(x_I)} \exp\Big(-\dfrac{\|x_I - \overset{*}{x}{}_I^1\|^2}{\rho}\Big), & x_I \in X_I, \\ +\infty & x_I \in R_I^n \setminus X_I \end{cases} \tag{1.2}$$

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Where $r + f(\overset{*1}{x}_I) > 0, \rho > 0, \overset{*1}{x}_I$ is a discrete local minimizer of $(P)_I$. We say that an integer point $\overset{*1}{x}_I$ is a (strictly) discrete local minimizer of $f(x_I)$, if $f(\overset{*1}{x}_I) \leq (<) f(\overset{*1}{x}_I \pm e_i), i = 1, \cdots, n$. Here $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$.

In fact, the filled function $p(x_I, \overset{*1}{x}_I, r, \rho)$ of (1.1) is

$$p(x_I, \overset{*1}{x}_I, r, \rho) = \begin{cases} \dfrac{1}{r + f(x_I)} \exp\Big(- \dfrac{\|x_I - \overset{*1}{x}_I\|^2}{\rho} \Big), & x_I \in X_I, \\ 0 & x_I \in R_I^n \setminus X_I \end{cases} \tag{1.3}$$

But from the view point of computition, it is unnecessary to search in $R_I^n \setminus X_I$, thus we use (1.2) as the filled function.

For the filled function (1.2), in theory, we prove that if we suitably choose $r, \rho > 0$, then by the procedure of finding local minimum of (1.2) we can find a point $\overset{*1}{\bar{x}}_I$ satisfying $f(\overset{*1}{\bar{x}}_I) \leq f(\overset{*1}{x}_I)$, thus by using $\overset{*1}{\bar{x}}_I$ as an initial point, we can find another discrete local minimizer $\overset{*2}{x}_I$ of (1.1) satisfying $f(\overset{*2}{x}_I) < f(\overset{*1}{x}_I)$. Using $\overset{*2}{x}_I$, we can construct a new filled function (1.2) $p(x_I, \overset{*2}{x}_I, r, \rho)$. Repeating the above procedure, finally we can find the global minimizer x_I^* of (1.1).

When we test this approach on some functions, one major disadvantage appears: If $\rho > 0$ is small, and $\|x_I - \overset{*1}{x}_I\|$ is large, then the value of $\exp\big(- \frac{\|x_I - \overset{*1}{x}_I\|^2}{\rho} \big)$ almost equals zero, thus it is difficult to recognize the size of $\exp\big(- \frac{\|x_I - \overset{*1}{x}_I\|^2}{\rho} \big)$, and hence the discrete local minimum of $p(x_I, \overset{*1}{x}_I, r, \rho)$.

In order to overcome it, we will propose the following filled function with one parameter:

$$p(x, x^*, \rho) = f(x^*) - f(x) - \rho \|x - x^*\|^2 \tag{1.4}$$

2. Some Definitions and an Algorithm of Finding the Local Discrete Minimum

Definition 2.1. *For any $x_I \in R_I^n$, the neighbour of x_I is defined by $N(x_I) = \{x_I, x_I \pm e_i, i = 1, \cdots, n\}$, where e_i denotes the unit vector whose i-th component equals 1, and other components equal 0.*

Definition 2.2. *It is said that the integer point $x_I^0 \in R_I^n$ is a (strictly) discrete local minimizer of $f(x_I)$, if $f(x_I^0) \leq (<) f(x_I)$ for all $x_I \in N(x_I^0)$.*

Definition 2.3. *It is said that the integer point $x_I^0 \in R_I^n$ is a (strictly) discrete global minimizer of $f(x_I)$, if $f(x_I^0) \leq (<) f(x_I)$ for all $x_I \in R_I^n$.*

Obviously, a discrete global minimum of $f(x_I)$ must also be a discrete local minimum of $f(x_I)$.

Similarly, a discrete local maximum and discrete global maximum of $f(x_I)$ can be defined.

Let $D = \{e_i, -e_i, i = 1, \cdots, n\}$.

Definition 2.4. *It is said that $d_0 \in D$ is a descent direction of $f(x_I)$ at $x_I^0 \in R_I^n$, if $f(x_I^0 + d_0) < f(x_I^0)$.*

A discrete local minimum of $(P)_I$ can be found by the following algorithm.

Algorithm 2.1.

Step 1. Select any initial point $x_I^0 \in X_I \subseteq R_I^n$.

Step 2. If x_I^0 is a discrete local minimizer of $(P)_I$, then stop. Otherwise, a descent direction d_0 of $f(x_I)$ at x_I^0 can be found.

Step 3. Let $x_I^0 := x_I^0 + d_0$, go to step 2.

3. Some Properties of Filled Function (1.4) and the Algorithm for Finding Discrete Global Minimum

Assume that $\overset{*}{x}{}_I^1$ is a discrete local minimizer of $f(x_I)$. Consider the following filled function that is based on (1.4):

$$p(x_I,\overset{*}{x}{}_I^1,\rho)=\begin{cases} f(\overset{*}{x}{}_I^1)-f(x_I)-\rho\|x_I-\overset{*}{x}{}_I^1\|^2, \rho>0, & x_I\in X_I,\\ +\infty, & x_I\notin X_I. \end{cases} \tag{3.1}$$

Theorem 3.1. *If $\rho>0$ and $\overset{*}{x}{}_I^1$ is a discrete local minimizer of $(P)_I$, then $\overset{*}{x}{}_I^1$ is a discrete strictly local maximizer of $p(x_I,\overset{*}{x}{}_I^1,\rho)$.*

Proof. Since $\overset{*}{x}{}_I^1$ is a discrete local minimizer of $(P)_I$, we have

$$f(\overset{*}{x}{}_I^1)\le f(\overset{*}{x}{}_I^1+d),\qquad \text{for all}\quad d\in D. \tag{3.2}$$

Thus

$$\begin{aligned} p(\overset{*}{x}{}_I^1,\overset{*}{x}{}_I^1,\rho) &= f(\overset{*}{x}{}_I^1)-f(\overset{*}{x}{}_I^1)-\rho\|\overset{*}{x}{}_I^1-\overset{*}{x}{}_I^1\|^2=0\\ &> f(\overset{*}{x}{}_I^1)-f(\overset{*}{x}{}_I^1+d)-\rho\|\overset{*}{x}{}_I^1+d-\overset{*}{x}{}_I^1\|^2\\ &= p(\overset{*}{x}{}_I^1+d,\overset{*}{x}{}_I^1,\rho). \end{aligned} \tag{3.3}$$

Theorem 3.2. *If $\rho>0$, $x_I^1,x_I^2\in R_I^n, f(x_I^2)\ge f(x_I^1)\ge f(\overset{*}{x}{}_I^1)$, $\|x_I^2-\overset{*}{x}{}_I^1\|>\|x_I^1-\overset{*}{x}{}_I^1\|>0$, then $p(x_I^2,\overset{*}{x}{}_I^1,\rho)<p(x_I^1,\overset{*}{x}{}_I^1,\rho)<0=p(\overset{*}{x}{}_I^1,\overset{*}{x}{}_I^1,\rho)$.*

Proof. From the conditions of the theorem, we have

$$\begin{aligned} f(\overset{*}{x}{}_I^1)-f(x_I^2)-\rho\|x_I^2-\overset{*}{x}{}_I^1\|^2 &= p(x_I^2,\overset{*}{x}{}_I^1,\rho)\\ &< f(\overset{*}{x}{}_I^1)-f(x_I^1)-\rho\|x_I^1-\overset{*}{x}{}_I^1\|^2\\ &= p(x_I^1,\overset{*}{x}{}_I^1,\rho)<0=p(\overset{*}{x}{}_I^1,\overset{*}{x}{}_I^1,\rho). \end{aligned} \tag{3.4}$$

Theorem 3.3. *If $x_I^1,x_I^2\in R_I^n, f(x_I^1)\ge f(x_I^2)\ge f(\overset{*}{x}{}_I^1),\|x_I^2-\overset{*}{x}{}_I^1\|>\|x_I^1-\overset{*}{x}{}_I^1\|>0$, then when $\rho>\dfrac{f(x_I^1)-f(x_I^2)}{\|x_I^2-\overset{*}{x}{}_I^1\|^2-\|x_I^1-\overset{*}{x}{}_I^1\|^2}\ge 0$, it holds*

$$p(x_I^2,\overset{*}{x}{}_I^1,\rho)<p(x_I^1,\overset{*}{x}{}_I^1,\rho)<0=p(\overset{*}{x}{}_I^1,\overset{*}{x}{}_I^1,\rho). \tag{3.5}$$

Proof. Obviously, $p(x_I^1,\overset{*}{x}{}_I^1,\rho)<0$, and $\|x_I^2-\overset{*}{x}{}_I^1\|^2-\|x_I^1-\overset{*}{x}{}_I^1\|^2\ge 1$, $f(x_I^1)\ge f(x_I^2)\ge f(\overset{*}{x}{}_I^1)$. Therefore, we only need discuss that if $\rho>\frac{f(x_I^1)-f(x_I^2)}{\|x_I^2-\overset{*}{x}{}_I^1\|^2-\|x_I^1-\overset{*}{x}{}_I^1\|^2}$, then $p(x_I^2,\overset{*}{x}{}_I^1,\rho)<p(x_I^1,\overset{*}{x}{}_I^1,\rho)<0$. Because

$$\rho>\frac{f(x_I^1)-f(x_I^2)}{\|x_I^2-\overset{*}{x}{}_I^1\|^2-\|x_I^1-\overset{*}{x}{}_I^1\|^2}=\frac{f(x_I^1)-f(\overset{*}{x}{}_I^1)-(f(x_I^2)-f(\overset{*}{x}{}_I^1))}{\|x_I^2-\overset{*}{x}{}_I^1\|^2-\|x_I^1-\overset{*}{x}{}_I^1\|^2}, \tag{3.6}$$

thus we obtain

$$\begin{aligned} f(\overset{*}{x}{}_I^1)-f(x_I^2)-\rho\|x_I^2-\overset{*}{x}{}_I^1\|^2 &= p(x_I^2,\overset{*}{x}{}_I^1,\rho)\\ &< f(\overset{*}{x}{}_I^1)-f(x_I^1)-\rho\|x_I^1-\overset{*}{x}{}_I^1\|^2\\ &= p(x_I^1,\overset{*}{x}{}_I^1,\rho)<0=p(\overset{*}{x}{}_I^1,\overset{*}{x}{}_I^1,\rho). \end{aligned} \tag{3.7}$$

Theorem 3.4. *If $\rho>0$ satisfies the condition of Theorem 3.3, and $x_I^1\ne\overset{*}{x}{}_I^1, x_I^1,\overset{*}{x}{}_I^1\in R_I^n, f(x_I^1)\ge f(\overset{*}{x}{}_I^1)$, then we can define $d_0\in D$, such that $\|x_I^1+d_0-\overset{*}{x}{}_I^1\|>\|x_I^1-\overset{*}{x}{}_I^1\|$, and it holds either*

$$f(x_I^1+d_0)<f(\overset{*}{x}{}_I^1) \tag{3.8}$$

or

$$p(x_I^1 + d_0, \overset{*}{x}{}_I^1, \rho) < p(x_I^1, \overset{*}{x}{}_I^1, \rho) < p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho) = 0. \tag{3.9}$$

Proof. From $x_I^1 \neq \overset{*}{x}{}_I^1, x_I^1, \overset{*}{x}{}_I^1 \in R_I^n, f(x_I^1) \geq f(\overset{*}{x}{}_I^1)$, and $0 = \| \overset{*}{x}{}_I^1 - \overset{*}{x}{}_I^1 \| < \|x_I^1 - \overset{*}{x}{}_I^1 \|$, by Theorem 3.2, we have

$$p(x_I^1, \overset{*}{x}{}_I^1, \rho) < p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho) = 0. \tag{3.10}$$

Furthermore, let $|x_I^{1j_0} - \overset{*}{x}{}_I^{1j_0}| = \max_{1 \leq i \leq n} |x_I^{1i} - \overset{*}{x}{}_I^{1i}| > 0$. If $x_I^{1j_0} - \overset{*}{x}{}_I^{1j_0} > 0$, then let $d_0^{j_0} = 1$. If $x_I^{1j_0} - \overset{*}{x}{}_I^{1j_0} < 0$, then let $d_0^{j_0} = -1$, and $d_0^i = 0$ for other $i \neq j_0$. Thus $d_0 \in D$, and $\|x_I^1 + d_0 - \overset{*}{x}{}_I^1 \| > \|x_I^1 - \overset{*}{x}{}_I^1 \|$. For this d_0, there are two cases:

(1) $f(x_I^1 + d_0) < f(\overset{*}{x}{}_I^1)$,

(2) $f(x_I^1 + d_0) \geq f(\overset{*}{x}{}_I^1)$.

If case (1) holds, then the theorem is true. If case (2) holds, due to $f(x_I^1 + d_0) \geq f(\overset{*}{x}{}_I^1)$, $f(x_I^1) \geq f(\overset{*}{x}{}_I^1)$, and $\|x_I^1 + d_0 - \overset{*}{x}{}_I^1 \| > \|x_I^1 - \overset{*}{x}{}_I^1 \|$, by Theorem 3.2 and 3.3 that we have

$$p(x_I^1 + d_0, \overset{*}{x}{}_I^1, \rho) < p(x_I^1, \overset{*}{x}{}_I^1, \rho). \tag{3.11}$$

Thus the theorem is true.

Remark. This theorem indicates either d_0 is a descent direction of $f(x_I)$ at x_I^1, or d_0 is a descent direction of $p(x_I, \overset{*}{x}{}_I^1, \rho)$ at x_I^1, when $f(x_I^1) \geq f(\overset{*}{x}{}_I^1)$.

Theorem 3.5. *If $\rho > 0$ satisfies the condition of Theorem 3.3, then for the discrete local minimizer $\overset{*}{x}{}_I^0$ of $p(x_I, \overset{*}{x}{}_I^1, \rho)$, it holds either $f(\overset{*}{x}{}_I^0) < f(\overset{*}{x}{}_I^1)$ or there exists $d_0 \in D$ such that $f(\overset{*}{x}{}_I^0 + d_0) < f(\overset{*}{x}{}_I^1)$.*

Proof. If it is not true, then there exists a discrete local minimizer $\overset{*}{\overline{x}}{}_I^0$ of $p(x_I, \overset{*}{x}{}_I^1, \rho)$ such that

$$f(\overset{*}{\overline{x}}{}_I^0) \geq f(\overset{*}{x}{}_I^1), \; f(\overset{*}{\overline{x}}{}_I^0 + d) \geq f(\overset{*}{x}{}_I^1) \tag{3.12}$$

Furthermore, $\overset{*}{x}{}_I^1$ is a strictly discrete local maximizer of $p(x_I, \overset{*}{x}{}_I^1, \rho)$, thus $\overset{*}{x}{}_I^1 \neq \overset{*}{\overline{x}}{}_I^0$. By Theorem 3.4, there exists $d_0 \in D$ such that $\| \overset{*}{\overline{x}}{}_I^0 + d_0 - \overset{*}{x}{}_I^1 \| > \| \overset{*}{\overline{x}}{}_I^0 - \overset{*}{x}{}_I^1 \|$, and it holds either

$$f(\overset{*}{\overline{x}}{}_I^0 + d_0) < f(\overset{*}{x}{}_I^1) \tag{3.13}$$

or

$$p(\overset{*}{\overline{x}}{}_I^0 + d_0, \overset{*}{x}{}_I^1, \rho) < p(\overset{*}{\overline{x}}{}_I^0, \overset{*}{x}{}_I^1, \rho) \tag{3.14}$$

it contradicts with the condition that $\overset{*}{\overline{x}}{}_I^0$ is a discrete local minimizer of $p(x_I, \overset{*}{x}{}_I^1, \rho)$ and (3.12). Thus the theorem is true.

Remark. This theorem indicates either $\overset{*}{x}{}_I^0$ or $\overset{*}{x}{}_I^0 + d_0$ can be used as an initial point for finding another discrete local minimizer $\overset{*}{x}{}_I^2$ of $f(x_I)$, which satisfies $f(\overset{*}{x}{}_I^2) < f(\overset{*}{x}{}_I^1)$.

Now, we have to point out two problems:

1. From the condition of theorem 3.3, if $\rho > \overline{f} - \underline{f}$, where $\overline{f} > \max_{x_I \in X_I} f(x_I), \underline{f} < \min_{x_I \in X_I} f(x_I)$, then for any $x_I^1, x_I^2 \in R_I^n$, $\|x_I^2 - \overset{*}{\overline{x}}{}_I^1 \| > \|x_I^1 - \overset{*}{\overline{x}}{}_I^1 \|$, it always holds

$$p(x_I^2, \overset{*}{\overline{x}}{}_I^1, \rho) < p(x_I^1, \overset{*}{\overline{x}}{}_I^1, \rho). \tag{3.15}$$

Because if $\rho > \overline{f} - \underline{f}$, then $\rho > \overline{f} - \underline{f} \geq \frac{f(x_I^1)-f(x_I^2)}{\|x_I^2-\overset{*}{\overline{x}}{}_I^1\|^2-\|x_I^1-\overset{*}{\overline{x}}{}_I^1\|^2}$, then $f(\overset{*}{\overline{x}}{}_I^1) - f(x_I^2) - \rho\|x_I^2 - \overset{*}{\overline{x}}{}_I^1\|^2 = p(x_I^2, \overset{*}{\overline{x}}{}_I^1, \rho) < f(\overset{*}{\overline{x}}{}_I^1) - f(x_I^1) - \rho\|x_I^1 - \overset{*}{\overline{x}}{}_I^1\|^2 = p(x_I^1, \overset{*}{\overline{x}}{}_I^1, \rho)$, in spite of $f(x_I^1), f(x_I^2) > f(\overset{*}{\overline{x}}{}_I^1)$ or $f(x_I^1), f(x_I^2) < f(\overset{*}{\overline{x}}{}_I^1)$. It is a disadvantage.

In order to overcome this disadvantage, we propose a strategy: If we have found a discrete local minimizer $\overset{*}{\overline{x}}{}_I^1$ of $f(x_I)$, then let $f(x_I) := \min(f(x_I), f(\overset{*}{\overline{x}}{}_I^1))$, and $p(x_I, \overset{*}{\overline{x}}{}_I^1, \rho) = f(\overset{*}{\overline{x}}{}_I^1) - \min(f(x_I), f(\overset{*}{\overline{x}}{}_I^1)) - \rho\|x_I - \overset{*}{\overline{x}}{}_I^1\|^2$. For this filled function $p(x_I, \overset{*}{\overline{x}}{}_I^1, \rho)$, the condition $\rho > \frac{f(x_I^1)-f(x_I^2)}{\|x_I^2-\overset{*}{\overline{x}}{}_I^1\|^2-\|x_I^1-\overset{*}{\overline{x}}{}_I^1\|^2}$ is unnecessary for theorem 3.3.

2. During the procedure for minimizing $p(x_I, \overset{*}{x}{}_I^1, \rho)$, we only need find a point $\overset{*}{x}{}_I^0$ such that $f(\overset{*}{x}{}_I^0) \leq f(\overset{*}{x}{}_I^1)$ or $f(\overset{*}{x}{}_I^0 + d_0) \leq f(\overset{*}{x}{}_I^1)$, $d_0 \in D$. It is equivalent to $\overline{(P)_{I\mu}}$:

$$\overline{(P)_I} \qquad \begin{array}{ll} \min & p(x_I, \overset{*}{x}{}_I^0, \rho) \\ \text{s.t.} & f(x_I) \leq f(\overset{*}{x}{}_I^0), \quad x_I \in R_I^n. \end{array}$$

The corresponding exact penalty function is

$$\overline{(P)_{I\mu}} \qquad \begin{aligned} &\min_{x_I \in R_I^n} p(x_I, \overset{*}{x}{}_I^0, \rho) + \mu(\max(0, f(x_I) - f(\overset{*}{x}{}_I^0)))^2 \\ &= f(\overset{*}{x}{}_I^0) - \min(f(\overset{*}{x}{}_I^0), f(x_I)) - \rho\|x_I - \overset{*}{x}{}_I^0\|^2 \\ &\quad + \mu(\max(0, f(x_I) - f(\overset{*}{x}{}_I^0)))^2 \\ &= p(x_I, \overset{*}{x}{}_I^0, \rho, \mu). \end{aligned}$$

4. Some Properties of $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$

Now we prove

$$p(x_I, \overset{*}{x}{}_I^1, \rho, \mu) = f(\overset{*}{x}{}_I^1) - \min(f(\overset{*}{x}{}_I^1), f(x_I)) - \rho\|x_I - \overset{*}{x}{}_I^1\|^2 + \mu(\max(0, f(x_I) - f(\overset{*}{x}{}_I^1)))^2 \quad (4.1)$$

also is a filled function, where $\overset{*}{x}{}_I^1$ is a discrete local minimizer of $f(x_I)$.

Theorem 4.1. *If* $0 < \mu < \frac{\rho}{L^2}$, $\overset{*}{x}{}_I^1$ *is a discrete local minimizer of* $f(x_I)$, *then* $\overset{*}{x}{}_I^1$ *is a strictly discrete local maximizer of* $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$, *where L is Lipschitzian.*

Proof. Obviously, $p(\overset{*}{x}{}_I^1 \overset{*}{x}{}_I^1, \rho, \mu) = 0$, and for all $d \in D$, $f(\overset{*}{x}{}_I^1) \leq f(\overset{*}{x}{}_I^1 + d)$. So

$$\begin{aligned} p(\overset{*}{x}{}_I^1 + d, \overset{*}{x}{}_I^1, \rho, \mu) &= f(\overset{*}{x}{}_I^1) - \min(f(\overset{*}{x}{}_I^1), f(\overset{*}{x}{}_I^1 + d)) - \rho\|\overset{*}{x}{}_I^1 + d - \overset{*}{x}{}_I^1\|^2 \\ &\quad + \mu(\max(0, f(\overset{*}{x}{}_I^1 + d) - f(\overset{*}{x}{}_I^1)))^2 \\ &= -\rho\|d\|^2 + \mu(f(\overset{*}{x}{}_I^1 + d) - f(\overset{*}{x}{}_I^1))^2 \leq -\rho\|d\|^2 + \mu L^2\|d\|^2 \\ &< 0 = p(\overset{*}{x}{}_I^1 \overset{*}{x}{}_I^1, \rho, \mu). \end{aligned} \quad (4.2)$$

Theorem 4.2. *If* $\rho, \mu > 0, \mu < \frac{\rho}{L^2}$, $\|x_I^1 - \overset{*}{x}{}_I^1\| > \|x_I^2 - \overset{*}{x}{}_I^1\| > 0, f(x_I^2) \geq f(x_I^1) \geq f(\overset{*}{x}{}_I^1)$, *then* $p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) < p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) < 0 = p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho, \mu)$, *where L is Lipschitzian.*

Proof. Obviously,

$$\begin{aligned} p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) - p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) &= -\rho(\|x_I^1 - \overset{*}{x}{}_I^1\|^2 - \|x_I^2 - \overset{*}{x}{}_I^1\|^2) \\ &\quad + \mu((f(x_I^1) - f(\overset{*}{x}{}_I^1))^2 - (f(x_I^2) - f(\overset{*}{x}{}_I^1))^2) < 0. \end{aligned} \quad (4.3)$$

Furthermore,

$$\begin{aligned} p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) &= f(\overset{*}{x}{}_I^1) - \min(f(\overset{*}{x}{}_I^1), f(x_I^2)) - \rho\|x_I^2 - \overset{*}{x}{}_I^1\|^2 \\ &\quad + \mu(f(x_I^2) - f(\overset{*}{x}{}_I^1))^2 \\ &\leq (-\rho + L^2\mu)\|x_I^2 - \overset{*}{x}{}_I^1\|^2 \\ &< 0 = p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho, \mu). \end{aligned} \tag{4.4}$$

Therefore, theorem is true.

Theorem 4.3. *If* $0 < \mu < \frac{\rho}{8M^2}, \|x_I^1 - \overset{*}{x}{}_I^1\| > \|x_I^2 - \overset{*}{x}{}_I^1\| > 0, f(x_I^1) \geq f(x_I^2) \geq f(\overset{*}{x}{}_I^1)$, *then* $p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) < p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) < 0 = p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho, \mu)$, where $M \geq| f(x_I) |$, for all $x_I \in X_I$.

Proof. Obviously,

$$\begin{aligned} &p(x_I^1, \overset{*}{x}{}_I^1, p, \mu) - p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) \\ &= -\rho(\|x_I^1 - \overset{*}{x}{}_I^1\|^2 - \|x_I^2 - \overset{*}{x}{}_I^1\|^2) + \mu((f(x_I^1) - f(\overset{*}{x}{}_I^1))^2 - (f(x_I^2) - f(\overset{*}{x}{}_I^1))^2) \\ &= (\|x_I^1 - \overset{*}{x}{}_I^1\|^2 - \|x_I^2 - \overset{*}{x}{}_I^1\|^2)\Big(-\rho + \mu\frac{(f(x_I^1) - f(\overset{*}{x}{}_I^1))^2 - (f(x_I^2) - f(\overset{*}{x}{}_I^1))^2}{\|x_I^1 - \overset{*}{x}{}_I^1\|^2 - \|x_I^2 - \overset{*}{x}{}_I^1\|^2}\Big) \\ &\leq (\|x_I^1 - \overset{*}{x}{}_I^1\|^2 - \|x_I^2 - \overset{*}{x}{}_I^1\|^2)(-\rho + 8\mu M^2 \\ &< 0, \end{aligned} \tag{4.5}$$

when $0 < \mu < \frac{\rho}{8M^2}$, where $M \geq| f(x_I) |$, for all $x_I \in X_I$ and $\|x_I^1 - \overset{*}{x}{}_I^1\|^2 - \|x_I^2 - \overset{*}{x}{}_I^1\|^2 \geq 1$.

Furthermore,

$$p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) < 0 = p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho, \mu) = 0. \tag{4.6}$$

Therefore, theorem is true.

Theorem 4.4. *If* $\mu > 0$ *satisfies the conditions of the above theorems, and* $x_I^1 \neq \overset{*}{x}{}_I^1, f(x_I^1) \geq f(\overset{*}{x}{}_I^1)$, *then we can define* $d_0 \in D$, *such that* $\|x_I^1 + d_0 - \overset{*}{x}{}_I^1\| > \|x_I^1 - \overset{*}{x}{}_I^1\|$, *and either*

$$f(x_I^1 + d_0) < f(\overset{*}{x}{}_I^1). \tag{4.7}$$

or $p(x_I^1 + d_0, \overset{*}{x}{}_I^1, \rho, \mu) < p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) < p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho, \mu) = 0$.

Proof. Similarly to the one to Theorem 3.4.

Theorem 4.5. *If* $\mu > 0$ *satisfies the condition of theorem 4.4, then the discrete local minimizer* $\overset{*}{x}{}_I^0$ *of* $p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu)$ *holds either*

$$f(\overset{*}{x}{}_I^0) < f(\overset{*}{x}{}_I^1) \tag{4.8}$$

or there exists $d_0 \in D$, *such that* $f(\overset{*}{x}{}_I^0 + d_0) < f(\overset{*}{x}{}_I^1)$.

Proof. Similarly to the one to Theorem 3.5.

Theorem 4.6. *If* $\overset{*}{x}{}_I^0$ *is a global minimizer of* $f(x_I)$, *then for all* $x_I \neq \overset{*}{x}{}_I^0$,

$$p(x_I, \overset{*}{x}{}_I^0, \rho, \mu) < 0 = p(x_I, \overset{*}{x}{}_I^0, \rho, \mu), \qquad 0 < \mu < \frac{\rho}{L^2}. \tag{4.9}$$

where L *is Lipschitzian.*

Proof. Since $\overset{*}{x}{}_I^0$ is a global minimizer of $f(x_I)$, then for all $x_I, f(\overset{*}{x}{}_I^0) \leq f(x_I)$. Thus

$$\begin{aligned} p(x_I, \overset{*}{x}{}_I^0, \rho, \mu) &= f(\overset{*}{x}{}_I^0) - \min(f(\overset{*}{x}{}_I^0), f(x_I)) - \rho\|x_I - \overset{*}{x}{}_I^0\|^2 + \mu(\max(0, f(x_I) - f(\overset{*}{x}{}_I^0))^2 \\ &\leq (-\rho + \mu L^2)\|x_I - \overset{*}{x}{}_I^0\|^2 < 0. \end{aligned} \tag{4.10}$$

Theorem 4.7. *If* $0 < \mu < \frac{\rho}{L^2}$, $\rho > h_1^* = \max_{f(\overset{*}{x}{}_I^1) > f(\overset{*}{x}{}_I^2)} f(\overset{*}{x}{}_I^1) - f(\overset{*}{x}{}_I^2) > 0, \overset{*}{x}{}_I^1, \overset{*}{x}{}_I^2$ *are the minimizers of* $f(x_I)$, $\overset{*}{x}{}_I^0$ *is a local discrete minimizer of* $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$, *then* $p(\overset{*}{x}{}_I^0, \overset{*}{x}{}_I^1, \rho, \mu) < 0$.

Proof. By theorem 4.1, $\overset{*}{x}{}_I^1$ is a strictly local discrete maximizer of $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$. If $\overset{*}{x}{}_I^0$ is a local discrete minimizer of $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$, so $\overset{*}{x}{}_I^0 \neq \overset{*}{x}{}_I^1$,

(i) $f(\overset{*}{x}{}_I^0) < f(\overset{*}{x}{}_I^1)$, $\| \overset{*}{x}{}_I^0 - \overset{*}{x}{}_I^1 \|^2 \geq 1$,

$$p(\overset{*}{x}{}_I^0, \overset{*}{x}{}_I^1, \rho, \mu) = f(\overset{*}{x}{}_I^1) - f(\overset{*}{x}{}_I^0) - \rho \| \overset{*}{x}{}_I^0 - \overset{*}{x}{}_I^1 \|^2 \leq h_1^* - \rho < 0, \tag{4.11}$$

(ii) $f(\overset{*}{x}{}_I^0) \geq f(\overset{*}{x}{}_I^1)$,

$$p(\overset{*}{x}{}_I^0, \overset{*}{x}{}_I^1, \rho, \mu) = -\rho \| \overset{*}{x}{}_I^0 - \overset{*}{x}{}_I^1 \|^2 + \mu (f(\overset{*}{x}{}_I^0) - f(\overset{*}{x}{}_I^1))^2 \leq (-\rho + \mu L^2) \| \overset{*}{x}{}_I^0 - \overset{*}{x}{}_I^1 \|^2 < 0. \tag{4.12}$$

Furthermore, we have the following theorem.

Theorem 4.8. *If $f(x_I^1) < f(\overset{*}{x}{}_I^1), x_I^1 \neq \overset{*}{x}{}_I^1$, $\rho > h_1^* = \max\limits_{f(\overset{*}{x}{}_I^1) > f(\overset{*}{x}{}_I^2)} f(\overset{*}{x}{}_I^1) - f(\overset{*}{x}{}_I^2) > 0$, $\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^2$ are local discrete minimizers of $f(x_I)$, then $p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) < 0$.*

Proof. From $f(x_I^1) < f(\overset{*}{x}{}_I^1)$, $\rho > h_1^*$, then there exists another minimizer $\overset{*}{x}{}_I^2$ of $f(x_I)$, by x_I^1 as an initial point, $f(\overset{*}{x}{}_I^2) < f(\overset{*}{x}{}_I^1)$, and $\|x_I^1 - \overset{*}{x}{}_I^1 \|^2 \geq 1$, thus

$$\begin{aligned} p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) &= f(\overset{*}{x}{}_I^1) - f(x_I^1) - \rho \|x_I^1 - \overset{*}{x}{}_I^1 \|^2 \\ &\leq f(\overset{*}{x}{}_I^1) - f(\overset{*}{x}{}_I^2) - \rho \\ &\leq h_1^* - \rho < 0. \end{aligned} \tag{4.13}$$

Theorem 4.9. *If $f(x_I^2) > f(\overset{*}{x}{}_I^1), f(x_I^1) \leq f(\overset{*}{x}{}_I^1), \|x_I^1 - \overset{*}{x}{}_I^1 \| > \|x_I^2 - \overset{*}{x}{}_I^1 \| > 0$, $h_1^* = \max\limits_{f(\overset{*}{x}{}_I^1) > f(\overset{*}{x}{}_I^2)} f(\overset{*}{x}{}_I^1) - f(\overset{*}{x}{}_I^2)$, $\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^2$ are local discrete minimizers of $f(x_I), \rho > h_1^*$, then*

$$p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) < p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) < 0 = p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho, \mu). \tag{4.14}$$

Proof. Because

$$\begin{aligned} p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) &= f(\overset{*}{x}{}_I^1) - \min(f(\overset{*}{x}{}_I^1), f(x_I^2)) - \rho \|x_I^2 - \overset{*}{x}{}_I^1 \|^2 \\ &\quad + \mu(\max(0, f(x_I^2) - f(\overset{*}{x}{}_I^1)))^2 \\ &= -\rho \|x_I^2 - \overset{*}{x}{}_I^1 \|^2 + \mu (f(x_I^2) - f(\overset{*}{x}{}_I^1))^2, \end{aligned} \tag{4.15}$$

$$\begin{aligned} p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) &= f(\overset{*}{x}{}_I^1) - \min(f(\overset{*}{x}{}_I^1), f(x_I^1)) - \rho \|x_I^1 - \overset{*}{x}{}_I^1 \|^2 \\ &\quad + \mu(\max(0, f(x_I^1) - f(\overset{*}{x}{}_I^1)))^2 \\ &= f(\overset{*}{x}{}_I^1) - f(x_I^1) - \rho \|x_I^1 - \overset{*}{x}{}_I^1 \|^2 \end{aligned} \tag{4.16}$$

Then by the condition of theorem, we have

$$\begin{aligned} & p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) - p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) \\ &= f(\overset{*}{x}{}_I^1) - f(x_I^1) - \rho \|x_I^1 - \overset{*}{x}{}_I^1 \|^2 + \rho \|x_I^2 - \overset{*}{x}{}_I^1 \|^2 - \mu (f(x_I^2) - f(\overset{*}{x}{}_I^1))^2 \\ &< f(\overset{*}{x}{}_I^1) - f(\overset{*}{x}{}_I^2) - \rho \\ &\leq h_1^* - \rho < 0. \end{aligned} \tag{4.17}$$

where $\overset{*}{x}{}_I^2$ is another local minimizer of $f(x_I)$, by x_I^1 as an initial point, $f(\overset{*}{x}{}_I^2) < f(\overset{*}{x}{}_I^1)$. thus

$$p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) < p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) < 0 = p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho, \mu). \tag{4.18}$$

Remark.

1. From theorem 4.8, we know that if $\overset{*}{x}{}_I^1$ is not a discrete global minimizer of $f(x)$, then there exists $x_I^1, f(x_I^1) < f(\overset{*}{x}{}_I^1)$, and starting $\overset{*}{x}{}_I^1$ to minimize $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$, it is possible to find a point x_I^1, such that $p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) < 0 = p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho, \mu)$, and $f(x_I^1) < f(\overset{*}{x}{}_I^1)$.

2. From theorem 4.9, we know that if $\overset{*}{x}{}_I^1$ is not a discrete global minimizer of $f(x)$, and there exist $x_I^1, x_I^2 \neq \overset{*}{x}{}_I^1$, $f(x_I^2) > f(\overset{*}{x}{}_I^1), f(x_I^1) \leq f(\overset{*}{x}{}_I^1)$, $\|x_I^1 - \overset{*}{x}{}_I^1\| > \|x_I^2 - \overset{*}{x}{}_I^1\|$, then when $\rho > h_1^*$, and starting $\overset{*}{x}{}_I^1$ to minimize $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$, it is possible to find a point x_I^1 such that $p(x_I^1, \overset{*}{x}{}_I^1, \rho, \mu) < p(x_I^2, \overset{*}{x}{}_I^1, \rho, \mu) < 0 = p(\overset{*}{x}{}_I^1, \overset{*}{x}{}_I^1, \rho, \mu)$, and $f(x_I^1) \leq f(\overset{*}{x}{}_I^1)$.

3. If we do not concern the theorem 4.7 - theorem 4.9, then from theorem 4.1 - theorem 4.6, we can take $\rho = 1$. Thus the filled function $p(x_I, \overset{*}{x}_I, \rho, \mu)$ can be $p(x_I, \overset{*}{x}_I, \mu) = f(\overset{*}{x}_I) - \min(f(\overset{*}{x}_I), f(x_I)) - \|x_I - \overset{*}{x}_I\|^2 + \mu(\max(0, f(x_I) - f(\overset{*}{x}_I)))^2$.

5. Algorithm

Step 0. Choose $\mu, \rho > 0$, initial point $x_I^0, \mu_0 \in (0,1), m_0 := 0$, positive integer number M (estimated number of the local minimizers of $f(x)$).

Step 1. $\min_{x_I \in X_I} f(x_I) = f(\overset{*}{x}{}_I^1)$, constructing the filled function $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$.

(1) If $m_0 < M, m_0 := m_0 + 1$, then go to step 2;

(2) If $m_0 = M$, then goto step 5.

Step 2. Choose some $x_I^1 \neq \overset{*}{x}{}_I^1$;

Step 3. If $f(x_I^1) > f(\overset{*}{x}{}_I^1)$, then determine a search direction $d_0 \neq 0$, minimize $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$, goto step 4.

Step 4. 1. If during minimizing $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$, we find a point $\bar{x}_I^1 \neq \overset{*}{x}{}_I^1$, such that $f(\bar{x}_I^1) \leq f(\overset{*}{x}{}_I^1)$, then let $x_I^0 := \bar{x}_I^1$, goto step 1;

2. If during minimizing $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$, the boundary of X_I is attained, then restart, and goto step 3;

3. If during minimizing $p(x_I, \overset{*}{x}{}_I^1, \rho, \mu)$, we find a minimizer $\overset{*}{\bar{x}}{}_I^1 \neq \overset{*}{x}{}_I^1$,

(i) If $f(\overset{*}{\bar{x}}{}_I^1) \leq f(\overset{*}{x}{}_I^1)$, then let $x_I^0 := \overset{*}{\bar{x}}{}_I^1$, goto step 1;

(ii) If $f(\overset{*}{\bar{x}}{}_I^1) > f(\overset{*}{x}{}_I^1)$, it implies that $\mu > 0$ too large, then let $\mu := \mu_0 \mu$, goto step 3.

Step 5. End. $\overset{*}{x}{}_I^1, f(\overset{*}{x}{}_I^1)$ respectively are the global point and the global value of $f(x_I)$.

6. Numerical Results

We tested the algorithm on the following problems.

Problem 1. Rosenbrock function:

$$\begin{array}{ll} \min & f(x_I) = (x_{1I}^2 - x_{2I})^2 + 2(x_{1I} - 1)^2, \\ \text{s.t.} & 0 \leq x_{1I},\ x_{2I} \leq 10, \\ & x_{1I}, x_{2I} - \text{integer points.} \end{array}$$

Problem 2[5].

$$\begin{array}{ll} \min & f(x_I) = [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)] \\ & \qquad * [30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)] \\ \text{s.t.} & x_{1I},\ x_{2I} = 0.001i, i = -2000, -1999, \ldots, 1999, 2000. \\ & x_{1I}, x_{2I} - \text{integer points.} \end{array}$$

Problem 3[5]**.**

$$\begin{array}{ll} \min & f(x_I) = \dfrac{33.7539}{x_{1I}} + \dfrac{1.4430}{x_{2I}} + \dfrac{1.3885}{x_{3I}} \\ \text{s.t.} & x_{1I} + x_{2I} + x_{3I} = 24 \\ & 1 \le x_{1I} \le 16 \\ & 1 \le x_{2I} \le 20 \\ & 1 \le x_{3I} \le 28 \\ & x_{iI}, i = 1,2,3 - \text{integer points.} \end{array}$$

The penalty function is $p(x_I) = \mid x_{1I} + x_{2I} + x_{3I} - 24 \mid$, the penalty parameter $\bar{\mu} = 35$.

Problem 4[5]**.**

$$\begin{array}{ll} \min & f(x_I) = -x_{3I} - x_{4I} - x_{5I} \\ \text{s.t.} & 20x_{1I} + 30x_{2I} + x_{3I} + 2x_{4I} + 2x_{5I} \le 180 \\ & 30x_{1I} + 20x_{2I} + 2x_{3I} + x_{4I} + 2x_{5I} \le 150 \\ & -60x_{1I} + x_{3I} \le 0 \\ & -75x_{2I} + x_{4I} \le 0 \\ & 0 \le x_{iI} \le 1, \quad i = 1,2 \\ & 0 \le x_{iI} \le 75, \quad i = 3,4,5 \\ & x_{iI}, i = 1,2,3,4,5 - \text{integer points.} \end{array}$$

The penalty function is $p(x_I) = \max\{0, 20x_{1I} + 30x_{2I} + x_{3I} + 2x_{4I} + 2x_{5I} - 180, 30x_{1I} + 20x_{2I} + 3x_{3I} + x_{4I} + 2x_{5I} - 150, -60x_{1I} + x_{3I}, -75x_{2I} + x_{4I}\}$, the penalty parameter $\bar{\mu} = 225$.

Problem 5[5]**.**

$$\begin{array}{ll} \min & f(x_I) = x_{1I}x_{2I}x_{3I} + x_{1I}x_{4I}x_{5I} + x_{2I}x_{4I}x_{6I} + x_{6I}x_{7I}x_{8I} + x_{2I}x_{5I}x_{7I} \\ \text{s.t.} & 2x_{1I} + 2x_{4I} + 8x_{8I} \ge 12, \\ & 11x_{1I} + 7x_{4I} + 13x_{6I} \ge 41, \\ & 6x_{2I} + 9x_{4I}x_{6I} + 5x_{7I} \ge 60, \\ & 3x_{2I} + 5x_{5I} + 7x_{8I} \ge 42, \\ & 6x_{2I}x_{7I} + 9x_{3I} + 5x_{5I} \ge 53, \\ & 4x_{3I}x_{7I} + x_{5I} \ge 13, \\ & 2x_{1I} + 4x_{2I} + 7x_{4I} + 3x_{5I} + x_{7I} \le 69, \\ & 9x_{1I}x_{8I} + 6x_{3I}x_{3I} + 4x_{3I}x_{7x} \le 47, \\ & 12x_{2I} + 8x_{2I}x_{8I} + 2x_{3I}x_{6I} \le 73, \\ & 9x_{3I} + 4x_{5I} + 2x_{6I} + 9x_{8I} \le 31, \\ & 0 \le x_{iI} \le 7, \quad i = 1,3,4,6,8, \\ & 0 \le x_{iI} \le 15, \quad i = 2,5,7, \\ & x_{iI}, i = 1,2,\ldots,8 - \text{integer points.} \end{array}$$

The penalty function

$$\begin{aligned} p(x_I) = & \max(0, -2x_{1I} - 2x_{4I} - 8x_{8I} + 12, -11x_{1I} - 7x_{4I} - 13x_{6I} + 41, \\ & -6x_{2I} - 9x_{4I}x_{6I} - 5x_{7I} + 60, -2x_{2I} - 5x_{5I} - 7x_{8I} + 42, \\ & -6x_{2I}x_{7I} - 9x_{3I} - 5x_{5I} + 53, -4x_{3I}x_{7I} - x_{5I} + 13, \\ & 2x_{1I} + 4x_{2I} + 7x_{4I} + 3x_{5I} + x_{7I} - 69, 9x_{1I}x_{8I} + 6x_{3I}x_{5I} + 4x_{3I}x_{7I} - 47, \\ & 12x_{2I} + 8x_{2I}x_{8I} + 2x_{3I}x_{6I} - 73, x_{3I} + 4x_{5I} + 2x_{6I} + x_{8I} - 31), \end{aligned}$$

the penalty parameter $\bar{\mu} = 6500$.

Problem 6[6]**.**

$$\begin{array}{ll} \min & f(x_I) = \sum_{i=1}^{10} (x_{iI}^4 - 4.9x_{iI}^2) \\ \text{s.t.} & -5 \le x_{iI} \le 5, \quad i = 1,2,\ldots,10 \\ & x_{iI}, i = 1,2,\ldots,10 - \text{integer points.} \end{array}$$

Problem 7[6]**.**

$$\begin{array}{ll} \min & f(x_I) = x_{1I}^4 + x_{2I}^4 + 16[x_{1I}x_{2I} + (4 + x_{2I})^2] \\ \text{s.t.} & -10 \le x_{iI} \le 10, \quad i = 1,2 \\ & x_{iI}, i = 1,2 - \text{integer points.} \end{array}$$

Problem 8[7]**.**

$$\begin{array}{ll} \min & f(x_I) = \left(\dfrac{1}{6.931} - \dfrac{x_{1I}x_{2I}}{x_{3I}x_{4I}}\right)^2 \\ \text{s.t.} & 12 \le x_{iI} \le 60, \quad i = 1,2,3,4 \\ & x_{iI}, i = 1,2,3,4 - \text{integer points.} \end{array}$$

The computational results are shown in Table. The meanings of the symbols used in Table are as follows:

No: the order of the problems;
x^0: the initial point;
x^*: the global minimizer;
$f(x^*)$: the global minimum function value;
ρ, μ: parameters;
Nf: the number of function evaluations;
Np: the number of filled function evaluations;
NL: the number of finding local minima.

Table

No	x^0	x^*	$f(x^*)$	ρ	μ	Nf	Np	NL
1	$(10,10)^T$	$(1,1)^T$	0	1	10^{-4}	38	28	2
2	$(-2,-2)^T$	$(0,-1.0)^T$	3.0	-	-	6502	0	0
3	$(1,1,1)^T$	$(16,4,4)^T$	2.817494	-	-	85	0	0
4	$(0,0,0,0,0)^T$	$(1,1,24,52,0)^T$	-76	1	10^{-10}	402	264	1
5	$(3,3,0,0,3,3,0,0)^T$	$(5,4,1,1,6,3,2,0)^T$	110	1	10^{-10}	168	70	1
6	$(0,0,\ldots,0)^T$	$(1,1,\ldots,1)^T$	-39	-	-	202	0	0
7	$(0,0)^T$	$(2,-3)^T$	17	-	-	18	0	0
8	$(21,27,48,49)^T$	$(13,30,51,53)^T$	2.3×10^{-11}	1	10^4	124	1667	7

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