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# A PRACTICAL PARALLEL DIFFERENCE SCHEME FOR PARABOLIC EQUATIONS *

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**Abstract**

A practical parallel difference scheme for parabolic equations is constructed as follows: to decompose the domain $\Omega$ into some overlapping subdomains, take flux of the last time layer as Neumann boundary conditions for the time layer on inner boundary points of subdomains, solve it with the fully implicit scheme on each subdomain, then take correspondent values of its neighbor subdomains as its values for inner boundary points of each subdomain and mean of its neighbor subdomain and itself at overlapping points. The scheme is unconditionally convergent. Though its truncation error is $O(\tau + h)$, the convergent order for the solution can be improved to $O(\tau + h^2)$.

*Key words*: Parallel difference scheme, Parabolic equation, Segment, Explicit–implicit.

## 1. Introduction

As we know, the drawback of pure explicit schemes for solving parabolic problems is a very restrictive constraint on a time step. Fully implicit schemes are unconditional stable, but their drawback is that on each time level, linear or nonlinear algebraic systems have to be solved, and it is not easy to parallel implementation. Therefore people wish to find a kind of difference schemes such that it has a lower restrictive constraint on a time step, and at best is unconditionally stable and convergent; the other hand, it has higher parallel character, that is, the solution domain can be decomposed freely, and loaded balance is convenient to be implemented, and only communication between neighbor CPU can be developed with few time. In this paper, we study this problem, and construct a practical parallel difference scheme for the following problem

$$\begin{cases} u_t = u_{xx}, & (x,t) \in \Omega \quad (1) \\ u(x,0) = \phi(x), & 0 \le x \le L \quad (2) \\ u(0,t) = u(L,t) = 0, & 0 \le t \le T \quad (3) \end{cases}$$

where $\Omega = [0,L] \times [0,T]$. For solving this equations, Zhou[1,2] presented some difference schemes with intrinsic parallelism, and proved their stability and convergence[3]. Zhang[4,5] also provided the alternating segment explicit–implicit scheme. In addition, there were still other alternating explicit–implicit schemes etc[6,7]. In this paper, a practical parallel difference scheme is presented, and can be extended to more general practical problems, including variable coefficient equations, or low order items, and two or more dimensional equations etc.

This paper is outlined as follows. In the second section, a practical parallel difference scheme and its convergence is proved. Some numerical experiments are presented in the third section.

## 2. Parallel Difference Scheme

### 2.1. Construction of the Parallel Difference Scheme

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Divide the domain$[0,L]\times[0,T]$ into small grids with the space steplength $h$ and time steplength $\tau$, where $Jh=L, N\tau=T$, and $J$ and $N$ are positive integers. Denote by $u_j^n(j=0,1,\cdots,J;n=0,1,\cdots,N)$ the discrete function defined on the discrete rectangular domain $\{(x_j,t^n)|j=0,1,\cdots,J;n=0,1,\cdots,N\}$ of the grid points, $\lambda=\frac{\tau}{h^2}$, $\Delta_\tau u_j^{n+1}=\frac{1}{\tau}(u_j^{n+1}-u_j^n)$, and $\delta^2 u_j^{n+1}=\frac{1}{h^2}(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1})$. Assume that we have obtained the values of the discrete function $\{u_j^n|j=1,2,\cdots,J-1\}$ on the $n$th time layer, and will take values $\frac{u_{k+2}^n-u_{k+1}^n}{h}$, $\frac{u_{k-2}^n-u_{k-1}^n}{h}$ of the $n$th time layer as Newmann boundary values on these boundary points of subdomains. For convenience, we only decompose the domain $\Omega$ into two subdomains, which are $[0,x_{k+2}]$ and $[x_{k-2},L]$. Thus, we can use the implicit scheme inside the two subdomain, and take the values of the $(n+1)$th time layer at these points $x=x_{k-1}, x=x_k$ and $x=x_{k+1}$ as estimating values, Denoting them by $\bar{u}_j^{n+1}(j=k-1,k,k+1)$ and $\tilde{u}_j^{n+1}(j=k-1,k,k+1)$ on two subdomains $[0,x_{k+2}]$ and $[x_{k-2},L]$ respectively. The difference scheme is written as follows

$$\Delta_\tau u_j^{n+1}=\delta^2 u_j^{n+1},\quad j=1,2,\cdots,k-3,k+3,\cdots,J-1, \tag{4}$$

$$\Delta_\tau u_{k-2}^{n+1}=\frac{1}{h^2}(\bar{u}_{k-1}^{n+1}-2u_{k-2}^{n+1}+u_{k-3}^{n+1}), \tag{5}$$

$$\Delta_\tau \bar{u}_{k-1}^{n+1}=\frac{1}{h^2}(u_{k-2}^{n+1}-2\bar{u}_{k-1}^{n+1}+\bar{u}_k^{n+1}), \tag{6}$$

$$\Delta_\tau \bar{u}_k^{n+1}=\frac{1}{h^2}(\bar{u}_{k+1}^{n+1}-2\bar{u}_k^{n+1}+\bar{u}_{k-1}^{n+1}), \tag{7}$$

$$\Delta_\tau \bar{u}_{k+1}^{n+1}=\frac{1}{h^2}(u_{k+2}^{n}-u_{k+1}^{n}-\bar{u}_{k+1}^{n+1}+\bar{u}_k^{n+1}), \tag{8}$$

$$\Delta_\tau u_{k+2}^{n+1}=\frac{1}{h^2}(u_{k+3}^{n+1}-2u_{k+2}^{n+1}+\tilde{u}_{k+1}^{n+1}), \tag{9}$$

$$\Delta_\tau \tilde{u}_{k+1}^{n+1}=\frac{1}{h^2}(u_{k+2}^{n+1}-2\tilde{u}_{k+1}^{n+1}+\tilde{u}_k^{n+1}), \tag{10}$$

$$\Delta_\tau \tilde{u}_k^{n+1}=\frac{1}{h^2}(\tilde{u}_{k-1}^{n+1}-2\tilde{u}_k^{n+1}+\tilde{u}_{k+1}^{n+1}), \tag{11}$$

$$\Delta_\tau \tilde{u}_{k-1}^{n+1}=\frac{1}{h^2}(u_{k-2}^{n}-u_{k-1}^{n}-\tilde{u}_{k-1}^{n+1}+\tilde{u}_k^{n+1}), \tag{12}$$

and then we take

$$u_{k-1}^{n+1}=\bar{u}_{k-1}^{n+1},\ u_{k+1}^{n+1}=\tilde{u}_{k+1}^{n+1},\ u_k^{n+1}=\frac{1}{2}(\bar{u}_k^{n+1}+\tilde{u}_k^{n+1}). \tag{13}$$

Obviously this difference scheme is solved segmentally. Eqs.(4)$\sim$(8) is one segment, and Eqs.(4), (9) $\sim$ (12) is another segment. Both Eqs.(4)$\sim$(8) and Eqs.(4),(9)$\sim$(12) are solved by the fully implicit scheme, and Eqs.(8) and (12) are obtained by taking $\bar{u}_{k+2}^{n+1}-\bar{u}_{k+1}^{n+1}$ and $\tilde{u}_{k-2}^{n+1}-\tilde{u}_{k-1}^{n+1}$ of the fully implicit scheme as $u_{k+2}^n-u_{k+1}^n$ and $u_{k-2}^n-u_{k-1}^n$ respectively.

Using Eqs.(13), (4)$\sim$(12) can be rewritten as

$$\Delta_\tau u_j^{n+1}=\delta^2 u_j^{n+1},\quad j=1,2,\cdots,k-2,k+2,\cdots,J-1, \tag{14}$$

$$\Delta_\tau u_{k-1}^{n+1}=\delta^2 u_{k-1}^{n+1}+r_{k-1}^{n+1}, \tag{15}$$

$$\Delta_\tau u_k^{n+1}=\delta^2 u_k^{n+1}+r_k^{n+1}, \tag{16}$$

$$\Delta_\tau u_{k+1}^{n+1}=\delta^2 u_{k+1}^{n+1}+r_{k+1}^{n+1}, \tag{17}$$

where

$$
\begin{aligned}
r_{k-1}^{n+1} =& \frac{1}{(\lambda^2+3\lambda+1)h^2}\Big\{(1+\lambda)u_k^n+(1+\lambda)\lambda u_{k-1}^{n+1}+(1-\lambda)\lambda u_{k+1}^n+\lambda^2 u_{k+2}^n \\
& -(\lambda^2+3\lambda+1)u_k^{n+1}\Big\}, \\
r_k^{n+1} =& \frac{1}{\lambda(\lambda^2+3\lambda+1)h^2}\Big\{(1+\lambda)u_k^n+\frac{(1+\lambda)\lambda}{2}(u_{k-1}^{n+1}+u_{k+1}^{n+1})+\frac{(1-\lambda)\lambda}{2} \\
& (u_{k-1}^n+u_{k+1}^n)+\frac{\lambda^2}{2}(u_{k+2}^n+u_{k-2}^n)\Big\}-\frac{1}{\lambda h^2}u_k^n-\delta^2 u_k^{n+1}, \\
r_{k+1}^{n+1} =& \frac{1}{(\lambda^2+3\lambda+1)h^2}\Big\{(1+\lambda)u_k^n+(1+\lambda)\lambda u_{k+1}^{n+1}+(1-\lambda)\lambda u_{k-1}^n+\lambda^2 u_{k-2}^n \\
& -(\lambda^2+3\lambda+1)u_k^{n+1}\Big\}.
\end{aligned}
$$

Obviously, the parallel difference scheme (14)$\sim$ (17) is a correct scheme to the fully implicit scheme, which adds correspondent correct items $r_{k-1}^{n+1}, r_k^{n+1}, r_{k+1}^{n+1}$ at the points $k-1, k$ and $k+1$ respectively. By calculation, we know that the truncation error of the equation (14) and (16) are $O(\tau+h^2)$, however, the truncation error of the equations (15) and (17) are both $O(\tau+h)$.

The boundary condition for above difference scheme (14)$\sim$ (17) is

$$u_0^n = u_J^n = 0,\ \forall\ n=0,1,\cdots,N,$$

and the initial condition is

$$u_j^0=\phi_j^0,\ j=0,1,\cdots,J-1.$$

### 2.2. Convergence for the Difference Scheme

**Lemma 1.** *Denote $a^+ = max(a,0)$.*

*(i) Assume that $\{w_j\}$ satisfies $-\delta^2 w_j \le 0,\ j=1,2,\cdots,J-1,$ and $w_0\le 0, W_J \le 0,$ we have $w_j \le 0,\quad j=1,2,\cdots,J-1.$*

*(ii) Assume that $\{z_j^{n+1}\}$ satisfies*

$$
\begin{aligned}
&\Delta_\tau z_j^{n+1}-\delta^2 z_j^{n+1}\le 0,\quad j=1,2,\cdots,J-1,\\
&z_0^{n+1}\le b_0^{n+1},\ z_J^{n+1}\le b_J^{n+1},
\end{aligned}
$$

*we have*

$$\max_{1\le j\le J-1}(z_j^{n+1})^+ \le max\Big((b_0)^+,(b_J)^+,\max_{1\le j\le J-1}(z_j^n)^+\Big).$$

**Theorem 2.** *Denote that $u(x_j,t^n), u_j^n$ are the solution of Eqs. (1) $\sim$(3) and (14)$\sim$(17) respectively, $e_j^n = u(x_j,t^n)-u_j^n$. Then for $\forall\lambda>0$, the parallel difference scheme (14)$\sim$(17)is unconditionally convergent, that is , for $\forall\lambda>0$, we have*

$$\max_{0\le j\le J}|e_j^{n+1}| = O(\tau+h^2),\ \forall n=0,1,\cdots,N-1.$$

*Proof.* Assume that $u(x_j,t^n), u_j^n$ are the solution of Eqs.(1)$\sim$(3) and (14)$\sim$(17) respectively, $e_j^n=u(x_j,t^n)-u_j^n$. Then $e_j^n$ satisfies

$$
\begin{aligned}
&\Delta_\tau e_j^{n+1}-\delta^2 e_j^{n+1}=R_j^{n+1},\quad j=1,2,\cdots,k-2,k+2,\cdots,J-1,\\
&\Delta_\tau e_{k-1}^{n+1}-\delta^2 e_{k-1}^{n+1}-\tilde r_{k-1}^{n+1}=R_{k-1}^{n+1},\\
&\Delta_\tau e_k^{n+1}-\delta^2 e_k^{n+1}-\tilde r_k^{n+1}=R_k^{n+1},\\
&\Delta_\tau e_{k+1}^{n+1}-\delta^2 e_{k+1}^{n+1}-\tilde r_{k+1}^{n+1}=R_{k+1}^{n+1},\\
&e_0^n=e_J^n=0,\ n=1,2,\cdots,N,\\
&e_j^0=0,\ j=0,1,\cdots,J,
\end{aligned}
$$

where

$$\begin{aligned}
\tilde{r}_{k-1}^{n+1} =&\frac{1}{(\lambda^2+3\lambda+1)h^2}\Big\{(1+\lambda)e_k^n+(1+\lambda)\lambda e_{k-1}^{n+1}+(1-\lambda)\lambda e_{k+1}^n+\lambda^2 e_{k+2}^n \\
&-(\lambda^2+3\lambda+1)e_k^{n+1}\Big\}, \\
\tilde{r}_k^{n+1} =&\frac{1}{\lambda(\lambda^2+3\lambda+1)h^2}\Big\{(1+\lambda)e_k^n+\frac{(1+\lambda)\lambda}{2}(e_{k-1}^{n+1}+e_{k+1}^{n+1})+\frac{(1-\lambda)\lambda}{2}(e_{k-1}^n+e_{k+1}^n) \\
&+\frac{\lambda^2}{2}(e_{k+2}^n+e_{k-2}^n)\Big\}-\frac{1}{\lambda h^2}e_k^n-\delta^2 e_k^{n+1}, \\
\tilde{r}_{k+1}^{n+1} =&\frac{1}{(\lambda^2+3\lambda+1)h^2}\Big\{(1+\lambda)e_k^n+(1+\lambda)\lambda e_{k+1}^{n+1}+(1-\lambda)\lambda e_{k-1}^n+\lambda^2 e_{k-2}^n \\
&-(\lambda^2+3\lambda+1)e_k^{n+1}\Big\},
\end{aligned}$$

and

$$\begin{aligned}
R_j^{n+1} &= O(\tau+h^2), j=1,2,\cdots,k-2,k+2,\cdots,J-1, \\
R_{k-1}^{n+1} &= O(\tau+h^2)+\frac{\lambda^3 h}{\lambda^2+3\lambda+1}\frac{\partial^3 u}{\partial x^3}(x_k,t^{n+1}), \\
R_k^{n+1} &= O(\tau+h^2), \\
R_{k+1}^{n+1} &= O(\tau+h^2)-\frac{\lambda^3 h}{\lambda^2+3\lambda+1}\frac{\partial^3 u}{\partial x^3}(x_k,t^{n+1}).
\end{aligned}$$

It is obvious that for $j=1,2,\cdots,k-2,k+2,\cdots,J-1$, we have $|R_j^{n+1}|\leq C_1(\tau+h^2)$, where $C_1$ is a positive constant only dependent on $u(x,t)$. At the same time, $|R_{k-1}^{n+1}|\leq C_2, |R_k^{n+1}|\leq C_3(\tau+h^2), |R_{k+1}^{n+1}|\leq C_2$, where $C_2$ and $C_3$ are both positive constants only dependent on $u(x,t)$ and $\lambda$.

We consider the following stationary discrete problem

$$\begin{aligned}
&-\delta^2 E_j = C_1(\tau+h^2), \quad j=1,2,\cdots,k-2,k+2,\cdots,J-1, \\
&-\delta^2 E_{k-1} = C_2 h, \\
&-\delta^2 E_k = C_3(\tau+h^2) \\
&-\delta^2 E_{k+1} = C_2 h, \\
&E_0 = E_J = 0.
\end{aligned}$$

Assume that $E_j=E_j^{(1)}+E_j^{(2)}$, where $E_j^{(1)}$ and $E_j^{(2)}$ are the solution of the following problem respectively

$$\begin{aligned}
&-\delta^2 E_j^{(1)} = C_1(\tau+h^2), \quad j=1,2,\cdots,k-2,k+2,\cdots,J-1, \\
&-\delta^2 E_{k-1}^{(1)} = 0, \\
&-\delta^2 E_k^{(1)} = C_3(\tau+h^2), \\
&-\delta^2 E_{k+1}^{(1)} = 0, \\
&E_0^{(1)} = E_J^{(1)} = 0.
\end{aligned}$$

and

$$-\delta^2 E_j^{(2)} = 0, \quad j = 1, 2, \cdots, k-2, k+2, \cdots, J-1, \tag{18}$$

$$-\delta^2 E_{k-1}^{(2)} = C_2 h, \tag{19}$$

$$-\delta^2 E_k^{(2)} = 0, \tag{20}$$

$$-\delta^2 E_{k+1}^{(2)} = C_2 h, \tag{21}$$

$$E_0^{(2)} = E_J^{(2)} = 0. \tag{22}$$

Then for $E_j^{(1)} - \frac{1}{2}C_4(\tau + h^2)x_j(1 - x_j)$ , where $C_4 = max(C_1, C_3)$, using the lemma 1(i), we can obtain

$$E_j^{(1)} \leq \frac{1}{8}C_4(\tau + h^2), \quad j = 1, 2, \cdots, J-1.$$

We compute the solution of the problem (18)$\sim$ (22) directly, and obtain

$$E_j^{(2)} \leq 2C_2 h^2 L, \quad j = 1, 2, \cdots, J-1.$$

when $n = 0$, $e_j^0 \leq E_j, j = 1, 2, \cdots, J-1$. We use induction on $n$, and assume that $e_j^n \leq E_j, j = 1, 2, \cdots, J-1$. For $e_j^{n+1} - E_j$, Using the lemma 1(ii) leads to $e_j^{n+1} \leq E_j, j = 1, 2, \cdots, J-1$. At the same time, we can obtain that $e_j^{n+1} \geq -E_j, j = 1, 2, \cdots, J-1$. Finally we have

$$e_j^n = O(\tau + h^2), \quad j = 1, 2, \cdots, J-1; n = 1, 2, \cdots, N.$$

Thus, we complete the proof of the theorem 2.

## 3. Numerical Experiments

In this section, we test the validity of the parallel difference scheme, comparing it with AGE[8] and ASE-I[4]. First, we consider the following problem

$$\begin{cases} u_t = u_{xx}, & (x,t) \in \Omega \\ u(x,0) = \sin(\pi x), & 0 \leq x \leq 1 \\ u(0,t) = u(1,t) = 0, & 0 \leq t \leq T \end{cases}$$

where $\Omega = [0,1] \times [0,T]$. Denote our scheme as GS and precise solution as PS. Table 1 and 2 show the solution of GS with AGE and ASE-I under different time steplength, taking $x = 0.1, 0.2, \cdots, 0.9$ as examples.

Table 1. $h = 0.002, \tau = 16.0 \times 10^{-6}, T = 0.1$

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| PS($10^{-1}$) | 1.1517 | 2.1907 | 3.0152 | 3.5446 | 3.7270 | 3.5446 | 3.0152 | 2.1907 | 1.1517 |
| GS($10^{-1}$) | 1.1518 | 2.1909 | 3.0155 | 3.5449 | 3.7273 | 3.5449 | 3.0155 | 2.1909 | 1.1518 |
| AGE($10^{-1}$) | 1.1512 | 2.1897 | 3.0138 | 3.5430 | 3.7253 | 3.5430 | 3.0138 | 2.1897 | 1.1512 |
| ASE-I($10^{-1}$) | 1.1518 | 2.1909 | 3.0155 | 3.5448 | 3.7273 | 3.5448 | 3.0155 | 2.1909 | 1.1518 |

Table 2. $h = 0.002, \tau = 200.0 \times 10^{-6}, T = 0.1$

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| PS($10^{-1}$) | 1.1517 | 2.1907 | 3.0152 | 3.5446 | 3.7270 | 3.5446 | 3.0152 | 2.1907 | 1.1517 |
| GS($10^{-1}$) | 1.1528 | 2.1928 | 3.0181 | 3.5480 | 3.7306 | 3.5480 | 3.0181 | 2.1928 | 1.1528 |
| AGE($10^{-1}$) | 1.0391 | 1.9740 | 2.7158 | 3.1919 | 3.3556 | 3.1919 | 2.7158 | 1.9740 | 1.0391 |
| ABE-I($10^{-1}$) | 1.1515 | 2.1883 | 3.0089 | 3.4010 | 3.5818 | 3.4010 | 3.0089 | 2.1883 | 1.1515 |

From Table 1 and 2, we can see that GS has better precision than AGE and ASE-I. In Table 5. we take two CPU for examples, and give user time and communicative time under a time step. From Table 5, we can see that communicative time, compared with user time, is very smaller. This suggests that GS has higher parallel character.

Second, we consider the following two–dimensional parabolic equation

$$\begin{cases} u_t = u_{xx} + u_{yy}, & (x,y,t) \in \Omega \times [0,T] \\ u(x,y,0) = 3\sin x \sin y, & (x,y) \in \Omega \\ u(x,y,t) = 0, & (x,y,t) \in \partial\Omega \times [0,T] \end{cases}$$

where $\Omega = \{(x,y)|0 \le x \le 1, 0 \le y \le 1\}$, and $\partial\Omega$ the boundary of $\Omega$. Here ASE-I is developed as ABE-I[5]. At the same time, we compare GS with AGE and ABE-I, taking $x = 0.5(y = 0.1, 0.2, \cdots, 0.9)$ as examples (see Table 3 and 4),

Table 3. $h = 0.01, \tau = 1.0 \times 10^{-4}, T = 0.1$

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| PS($10^{-1}$) | 0.4292 | 0.8164 | 1.1238 | 1.3211 | 1.3891 | 1.3211 | 1.1238 | 0.8164 | 0.4292 |
| GS($10^{-1}$) | 0.4291 | 0.8163 | 1.1237 | 1.3210 | 1.3890 | 1.3210 | 1.1237 | 0.8163 | 0.4291 |
| AGE($10^{-1}$) | 0.4276 | 0.8134 | 1.1196 | 1.3162 | 1.3840 | 1.3162 | 1.1196 | 0.8134 | 0.4276 |
| ABE-I($10^{-1}$) | 0.4290 | 0.8160 | 1.1235 | 1.3209 | 1.3889 | 1.3209 | 1.1235 | 0.8160 | 0.4290 |

Table 4. $h = 0.01, \tau = 4.0 \times 10^{-4}, T = 0.1$

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| PS($10^{-1}$) | 0.4292 | 0.8164 | 1.1238 | 1.3211 | 1.3891 | 1.3211 | 1.1238 | 0.8164 | 0.4292 |
| GS($10^{-1}$) | 0.4227 | 0.8040 | 1.1067 | 1.3010 | 1.3679 | 1.3010 | 1.1067 | 0.8040 | 0.4227 |
| AGE($10^{-1}$) | 0.4023 | 0.7652 | 1.0534 | 1.2386 | 1.3026 | 1.2386 | 1.0534 | 0.7652 | 0.4023 |
| ASE-I($10^{-1}$) | 0.4198 | 0.7987 | 1.1034 | 1.2989 | 1.3643 | 1.2989 | 1.1034 | 0.7987 | 0.4198 |

and give user time and communicative time under a time step (see Table 6.). We also draw the same conclusion that our difference scheme has better parallel character.

Table 5. $h = 0.0001, \tau = 1.0 \times 10^{-8}$

| | First CPU | Second CPU |
|---|---|---|
| Comm.time | 0.000033 | 0.000023 |
| User time | 0.057685 | 0.078579 |

Table 6. $h = 0.01, \tau = 1.0 \times 10^{-4}$

| | First CPU | Second CPU |
|---|---|---|
| Comm.time | 0.000093 | 0.000073 |
| User time | 0.757685 | 0.778579 |

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