

OPTIMAL DELAUNAY TRIANGULATIONS ^{*1)}

Long Chen Jin-chao Xu

(Mathematics Department, The Pennsylvania State University, University Park PA 16802, USA)

(E-mail: long_c@math.psu.edu xu@math.psu.edu)

Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

The Delaunay triangulation, in both classic and more generalized sense, is studied in this paper for minimizing the linear interpolation error (measure in L^p -norm) for a given function. The classic Delaunay triangulation can then be characterized as an optimal triangulation that minimizes the interpolation error for the isotropic function $\|\mathbf{x}\|^2$ among all the triangulations with a given set of vertices. For a more general function, a function-dependent Delaunay triangulation is then defined to be an optimal triangulation that minimizes the interpolation error for this function and its construction can be obtained by a simple lifting and projection procedure.

The optimal Delaunay triangulation is the one that minimizes the interpolation error among all triangulations with the same number of vertices, i.e. the distribution of vertices are optimized in order to minimize the interpolation error. Such a function-dependent optimal Delaunay triangulation is proved to exist for any given convex continuous function. On an optimal Delaunay triangulation associated with f , it is proved that ∇f at the interior vertices can be exactly recovered by the function values on its neighboring vertices. Since the optimal Delaunay triangulation is difficult to obtain in practice, the concept of nearly optimal triangulation is introduced and two sufficient conditions are presented for a triangulation to be nearly optimal.

Mathematics subject classification: 41A10, 41A25, 41A50, 41A60, 65M50, 65N50

Key words: Delaunay triangulation, Anisotropic mesh generation, N term approximation, Interpolation error, Mesh quality, Finite element.

1. Introduction

In this paper, we shall consider optimal triangulations from a function approximation point of view. Here "triangulation" is extended from the planar usage to arbitrary dimension: a triangulation \mathcal{T} decomposes a bounded domain $\Omega \subset \mathbb{R}^n$ into n -simplices such that the intersection of any two simplices in \mathcal{T} either consists of a common lower dimensional simplex or is empty.

The Delaunay triangulation (DT) of a finite point set V , the most commonly used unstructured triangulation, can be defined by the empty sphere property: no vertices in V are inside the circumsphere of any simplex in the triangulation. There are many optimality characterizations for Delaunay triangulation [7], in which the most well known is that in two dimensions it maximizes the minimum angle of triangles in the triangulation [15, 20]. We, however, would like to characterize the Delaunay triangulation from a function approximation point of view.

Let us denote $Q(\mathcal{T}, f, p) = \|f - f_{I,\mathcal{T}}\|_{L^p(\Omega)}$, where $f_{I,\mathcal{T}}(\mathbf{x})$ is the linear interpolation of f based the triangulation \mathcal{T} of a domain $\Omega \subset \mathbb{R}^n$. We shall prove that

$$Q(DT, \|\mathbf{x}\|^2, p) = \min_{\mathcal{T} \in \mathcal{P}_V} Q(\mathcal{T}, \|\mathbf{x}\|^2, p), \quad 1 \leq p \leq \infty, \quad (1.1)$$

* Received January 31, 2004.

¹⁾This work was supported in part by NSF DMS-0074299, NSF DMS-0209497, NSF DMS-0215392 and the Center for Computational Mathematics and Application at Penn State.

where \mathcal{P}_V is the set of all triangulations that have a given set V of vertices and Ω is chosen as the convex hull of V . This type of result was first proved in \mathbb{R}^2 by D'Azevedo and Simpson [4] for $p = \infty$ and then Rippa [19] for $1 \leq p < \infty$ in two dimensions. Our result is a generalization of their work to higher dimensions.

A Delaunay triangulation is therefore characterized as the optimal triangulation for piecewise linear interpolation to isotropic function $\|\mathbf{x}\|^2$ for a given point set in the sense of minimizing the interpolation error in L^p ($1 \leq p \leq \infty$) norm. Based on this characterization, we will introduce the concept of function-dependent Delaunay triangulation $(DT)_f$ for a given convex function with f in place of $\|\mathbf{x}\|^2$ in (1.1).

We further let the set V vary all sets of triangulations at most N points and consider the optimization problem corresponding to the error-based mesh quality $Q(\mathcal{T}, f, p)$. That is to find a triangulation \mathcal{T}^* such that

$$Q(\mathcal{T}^*, f, p) = \inf_{\mathcal{T} \in \mathcal{P}_N} Q(\mathcal{T}, f, p), \quad 1 \leq p \leq \infty, \quad (1.2)$$

where \mathcal{P}_N stands for the set of all triangulations with at most N vertices. Any minimizer of (1.2) is called an optimal Delaunay triangulation associated with f .

We shall prove the existence of the optimal Delaunay triangulation for a convex function. With the formulation of $Q(\mathcal{T}, f, 1)$, we obtain a necessary condition. More precisely, if triangulation is optimal in the sense of minimizing $Q(\mathcal{T}, f, 1)$ for a convex function f in $\mathcal{C}^1(\Omega)$, then for an interior vertex \mathbf{x}_i , we have

$$\nabla f(\mathbf{x}_i) = -\frac{1}{|\Omega_i|} \sum_{\tau_j \in \Omega_i} [\nabla |\tau_j|(\mathbf{x}) \sum_{\mathbf{x}_k \in \tau_j, \mathbf{x}_k \neq \mathbf{x}_i} f(\mathbf{x}_k)]. \quad (1.3)$$

Here Ω_i is the patch of \mathbf{x}_i which consists of all simplices using \mathbf{x}_i as a vertex and $|A|$ is the Lebesgue measure of set A in \mathbb{R}^n . We free the vertex \mathbf{x}_i to be a variable $\mathbf{x} \in \Omega_i$ and treat $|\tau_j|(\mathbf{x})$ as a linear function of \mathbf{x} whose gradient is a combination of other vertices in τ_j ; See Fig. 2.

The identity (1.3) states that the gradient of f can be recovered exactly at a grid point on an optimal Delaunay triangulation by taking a special linear combination of function values on its neighboring nodes. If the triangulation is not optimal, (1.3) will guide us to move the vertex \mathbf{x}_i in its local patch to optimize the interpolation error and thus can be used as a mesh smoothing scheme.

While an optimal Delaunay triangulation is desired, but it is difficult to obtain in practice. We therefore introduce the concept of nearly optimal triangulation. We call a triangulation is nearly optimal if $Q(\mathcal{T}, f, p) \leq CQ(\mathcal{T}^*, f, p)$ with a constant C independent of the number of vertices N . For the practical propose, we present two sufficient conditions for a triangulation to be nearly optimal. One such a condition is that the triangulation should be quasi-uniform under a new metric obtained by a modification of the Hessian matrix of object function f .

By choosing f of interest, we develop an unified approach to generalize the main concepts and techniques used in isotropic mesh generation, for example the Delaunay triangulation and edge swapping algorithm, to anisotropic and high dimensional cases which become a challenging and active research in the last decade [19, 21, 10, 1, 9, 5, 13, 16]

The rest of this paper is organized as follows. In Section 2, we discuss the Delaunay triangulation and present the characterization in terms of linear interpolation error. In Section 3, we introduce optimal Delaunay triangulations, prove the existence and present a necessary condition for gradient recovery. In Section 4, sufficient conditions for a nearly optimal Delaunay are presented. The last section is the concluding remark.

2. Function-Dependent Delaunay Triangulation

The Delaunay triangulation (DT) is the most commonly used triangulation for the gen-

eration of unstructured meshes used in finite element method for solving partial differential equation. It is often defined as the dual of Voronoi diagram (see for example [6]). In this paper, we use an equivalent definition [17, 8] which only involves the triangulation itself.

For a given point set V , the convex hull of V is the smallest convex set which contains these points and we denote it by $CH(V)$.

Definition 2.1. *Let V be a finite set of points in \mathbb{R}^n , the Delaunay triangulation of V is the triangulation of $CH(V)$ so that it satisfies empty sphere condition for any simplex: there are no points in V inside the circumsphere of any simplex.*

We will soon discuss the existence of DT, which will follow easily after we characterize it as an optimal triangulation. Here we just point out that, in general DT is not unique since $n + 2$ (or more) points may lie on a common sphere and any triangulation of those points will be a DT. Fortunately this is the only possibility [6]. If we assume that points are affinely independent and no $n + 2$ points lie on a common sphere, a Delaunay triangulation is then uniquely determined by these points.

There are many optimality characterizations for the Delaunay triangulation in two dimensions. Lawson [15] and Sibson [20] observed that a DT maximizes the minimum angle of any triangle. Lambert [14] showed that a DT maximizes the arithmetic mean of the radius of inscribed circles of the triangles. Rippa [18] showed that a DT minimizes the integral of the squared gradients.

Another interesting characterization of Delaunay triangulation is related to a convex hull in one dimensional higher space. To make it clear, let us introduce some notation first. We will identify \mathbb{R}^{n+1} as $\mathbb{R}^n \times \mathbb{R}$. A point in \mathbb{R}^{n+1} can be written as (\mathbf{x}, x_{n+1}) , where $\mathbf{x} \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}$. For a point \mathbf{x} in \mathbb{R}^n , we can lift it to the paraboloid $(\mathbf{x}, \|\mathbf{x}\|^2)$ living in \mathbb{R}^{n+1} and denote this lifting operator as $'$, namely $\mathbf{x}' = (\mathbf{x}, \|\mathbf{x}\|^2)$. For a given point set V in \mathbb{R}^n , we then have a set of points V' in \mathbb{R}^{n+1} by lifting point in V to the paraboloid. The convex hull $CH(V')$ can be divided into *lower* and *upper* parts; a facet belongs to the lower convex hull if it is supported by a hyperplane that separates V' from $(\mathbf{0}, -\infty)$. We may assume the facets of the lower convex hull are simplices since if $n + 2$ more vertices forms a facet, we can choose any triangulation of this facet; See Fig. 1. Brown [2] discovered that the projection of a lower convex hull of V' in \mathbb{R}^{n+1} is a DT of V in \mathbb{R}^n ; See also Edelsbrunner and Seidel [11].

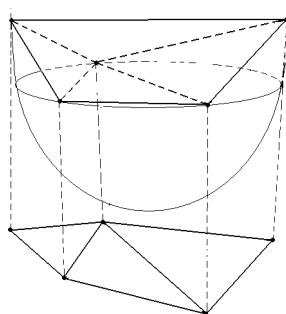


Figure 1. Projection of a lower convex hull

We, however, would like to characterize the Delaunay triangulation from a function approximation point of view. Let $\Omega \subset \mathbb{R}^n$ be a convex bounded domain, \mathcal{T} a triangulation of Ω , and $f_{I,\mathcal{T}}$ the piecewise linear finite element interpolation of a given continuous function f defined on $\bar{\Omega}$.

Definition 2.2. We define an error-based mesh quality $Q(\mathcal{T}, f, p)$ as

$$Q(\mathcal{T}, f, p) = \|f - f_{I,\mathcal{T}}\|_{L^p,\Omega} = \left(\int_{\Omega} |f(\mathbf{x}) - f_{I,\mathcal{T}}(\mathbf{x})|^p d\mathbf{x} \right)^{1/p}.$$

By choosing a special function $f(\mathbf{x}) = \|\mathbf{x}\|^2$, we will characterize the Delaunay triangulation as an optimal triangulation in the sense of minimizing this error-based mesh quality.

Theorem 2.3.

$$Q(DT, \|\mathbf{x}\|^2, p) = \min_{\mathcal{T} \in \mathcal{P}_V} Q(\mathcal{T}, \|\mathbf{x}\|^2, p), \quad \forall 1 \leq p \leq \infty,$$

where we choose $\Omega = CH(V)$ and denote \mathcal{P}_V all possible triangulations of Ω by using the points in V .

Proof. Let CS_{τ} be the circumsphere of τ and \mathbf{x}_o , R_{τ} be the CS_{τ} 's center and radius respectively. For a simplex τ , we consider the hyperplane in \mathbb{R}^{n+1}

$$F_{\tau} : x_{n+1} - 2\mathbf{x} \cdot \mathbf{x}_o + \|\mathbf{x}_o\|^2 - R_{\tau}^2 = 0.$$

It is easy to check that: (1) the lifting of vertices of τ lie on F_{τ} and they make up an n -simplex on F_{τ} , denoted by τ' , and (2) a point \mathbf{x} in \mathbb{R}^n is out of CS_{τ} if and only if its lifting point $(\mathbf{x}, \|\mathbf{x}\|^2)$ in \mathbb{R}^{n+1} lies above F_{τ} .

For any $\mathbf{x} \in \Omega$, let τ_1, τ_2 be the simplices containing \mathbf{x} in DT and \mathcal{T} respectively. By our definition of DT and (2), vertices of τ_2 can not be enclosed by the circumsphere of τ_1 . It means τ_2' lie above τ_1' . Note that

$$(\tau_1, f_{I,DT}(\tau_1)) = \tau_1', (\tau_2, f_{I,\mathcal{T}}(\tau_2)) = \tau_2'$$

and $f(\mathbf{x})$ is convex, we get $f(\mathbf{x}) \leq f_{I,DT}(\mathbf{x}) \leq f_{I,\mathcal{T}}(\mathbf{x})$. The desired result then follows.

This type of result was first proved in \mathbb{R}^2 by D'Azevedo and Simpson [4] for $p = \infty$ and Rippa [19] for $1 \leq p < \infty$. But their geometry approach is not easy to extend to higher dimensions. Theorem 2.3 is a generalization of their work to high dimensions.

For a general quadratic function $f(\mathbf{x}) = \mathbf{x}^t H \mathbf{x}$ with a symmetric positive definite matrix H , we may define a Delaunay triangulation denoted by $(DT)_H$ under metric H by changing the circumsphere used in Definition 2.1 to the circum ellipse $(\mathbf{x} - \mathbf{x}_o)^t H (\mathbf{x} - \mathbf{x}_o) = R_{\tau,H}$. Similar to Theorem 2.3, we have

Theorem 2.4.

$$Q((DT)_H, \mathbf{x}^t H \mathbf{x}, p) = \min_{\mathcal{T} \in \mathcal{P}_V} Q(\mathcal{T}, \mathbf{x}^t H \mathbf{x}, p), \quad \forall 1 \leq p \leq \infty.$$

If the eigenvalues of H are in different scales, say $\lambda_{max} = 1, \lambda_{min} = \epsilon, \epsilon \ll 1$, the optimal triangulation for a given point set V is the Delaunay triangulation under metric H which means the edges should be stretched according to the metric and thus, it results long thin elements. Therefore, from an approximation point of view, long thin elements may be good for the finite element method.

For a general convex (or concave) function f , we can generalize the definition of DT by this optimality characterization.

Definition 2.5. For a given convex (or concave) function $f \in \mathcal{C}(\bar{\Omega})$ and a given point set V , $(DT)_f$ is a Delaunay triangulation of $CH(V)$ respect to f if it satisfies

$$Q((DT)_f, f, p) = \min_{\mathcal{T} \in \mathcal{P}_V} Q(\mathcal{T}, f, p).$$

We will now use the lifting and projection to construct our optimal triangulation $(DT)_f$.

Theorem 2.6. *Let \mathcal{T}' be the triangulation obtained by the projection of the lower convex hull of the lifting points. For any convex function f , $\mathcal{T}' = (DT)_f$, i.e.*

$$Q(\mathcal{T}', f, p) = \min_{\mathcal{T} \in \mathcal{P}_V} Q(\mathcal{T}, f, p).$$

This theorem follows from the following simple lemma.

Lemma 2.7. *For any $\mathcal{T} \in \mathcal{P}_V$,*

$$f_{I, \mathcal{T}'}(\mathbf{x}) \leq f_{I, \mathcal{T}}(\mathbf{x}), \quad \forall x \in \Omega.$$

Proof. For any $\mathbf{x} \in \Omega$, let τ_1, τ_2 be the simplices containing \mathbf{x} in \mathcal{T}' and \mathcal{T} , respectively. By the definition of \mathcal{T}' , the lifting of vertices of τ_2 lie above $F_{\tau_1'}$. It means τ_2' lies above $F_{\tau_1'}$. Note that

$$(\tau_1, f_{I, \mathcal{T}'}(\tau_1)) = \tau_1', (\tau_2, f_{I, \mathcal{T}}(\tau_2)) = \tau_2',$$

the desired result then follows.

Thus a global algorithm to construct a function dependent Delaunay triangulation can be obtained by using the standard algorithm of constructing convex hull for a given point set. A local algorithm is to replace a small subsets of elements by other such sets while preserving the position of the points. In two dimensions, it is called edge swapping, which is one of the most popular algorithm used to generate a Delaunay triangulation. By Theorem 2.6, we can use the interpolation error as a criterion of choosing the right triangulation locally. For special function $f(\mathbf{x}) = \|\mathbf{x}\|^2$, in views of Theorem 2.3 this criterion is equivalent to the most commonly used empty sphere criterion. We will discuss further these issues in a future paper.

3. Optimal Delaunay Triangulation

We have shown that when points are fixed, a Delaunay triangulation optimizes the connectivity for a finite point set. Now we free the points to find the optimal triangulation.

Definition 3.1. *Let \mathcal{P}_N stand for the set of all triangulations with at most N vertices. Given a continuous convex function f on $\bar{\Omega}$ and $1 \leq p \leq \infty$, a triangulation $\mathcal{T}^* \in \mathcal{P}_N$ is optimal if*

$$Q(\mathcal{T}^*, f, p) = \inf_{\mathcal{T} \in \mathcal{P}_N} Q(\mathcal{T}, f, p).$$

By results from previous section, an optimal triangulation must be a Delaunay triangulation. Thus we call it an optimal Delaunay triangulation. The following theorem concerns the existence of the optimal Delaunay triangulation.

Theorem 3.2. *Given $1 \leq p < \infty$ and a convex function f , an optimal triangulation in \mathcal{P}_N exists.*

Proof. By the lifting method, we note that adding a new point into a simplex τ the new triangulation obtained by connecting it to the vertices of τ is not worse than the original one since f is convex. Therefore, we may prove the result for the triangulation with exactly N vertices.

Let us take a sequence of triangulations $\{\mathcal{T}^k\}_{k=1}^{\infty} \subset \mathcal{P}_N$ with vertices $V^k = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \Omega^N$ such that

$$\lim_{k \rightarrow \infty} Q(\mathcal{T}^k, f, p) = \inf_{\mathcal{T} \in \mathcal{P}_N} Q(\mathcal{T}, f, p).$$

By the compactness of $\bar{\Omega}$, we may suppose that there exists $V^* \in \bar{\Omega}^N$ such that $\lim_{k \rightarrow \infty} V^k = V^*$, namely

$$\lim_{k \rightarrow \infty} \mathbf{x}_i^k = \mathbf{x}_i^*, \quad \forall \mathbf{x}_i^k \in V^k, i = 1, \dots, N.$$

Because of the finite possible connectivity for N vertices, we may also assume $\{\mathcal{T}^k\}_{k=1}^\infty$ yield the same connectivity. We index the simplices as $\tau_1^k, \dots, \tau_{N_T}^k$. The signed volume $\text{vol}(\tau)$ of a simplex τ with vertices $\mathbf{x}_1, \dots, \mathbf{x}_{n+1}$ is defined as

$$\text{vol}(\tau) = \det |\mathbf{x}_2 - \mathbf{x}_1 \ \mathbf{x}_3 - \mathbf{x}_1 \ \dots \ \mathbf{x}_{n+1} - \mathbf{x}_1|,$$

which is obviously a continuous function with respect to its vertices. Since $\text{vol}(\tau_j^k) > 0, k = 1, \dots, n, \dots, j = 1, \dots, N_T$, we conclude that $\text{vol}(\tau_j^\infty) \geq 0, j = 1, \dots, N_T$ and thus V^* with the same connectivity yields a triangulation \mathcal{T}^* . It might be nonconforming in the sense that there exist vertices lie in the interior of the boundary of some simplices. For a non-conforming triangulation, the linear interpolation is well defined except on the boundary of each simplex which is a measure zero set. Thus the interpolation error is well defined. Furthermore, a careful check of the proof of Lemma 2.7 tells us that $Q(\mathcal{T}', \|\mathbf{x}\|^2, p) = \min_{\mathcal{T} \in \tilde{\mathcal{P}}_V} Q(\mathcal{T}, \|\mathbf{x}\|^2, p)$, where $\tilde{\mathcal{P}}_V$ includes the non-conforming triangulation, namely we allow hanging points. Since $Q(\mathcal{T}^k, f, p) = \sum_j Q(\tau_j^k, f, p)$, the quality $Q(\mathcal{T}^k, f, p)$ is continuous with respect to k , and thus

$$Q(DT^*, f, p) \leq Q(\mathcal{T}^*, f, p) = \lim_{k \rightarrow \infty} Q(\mathcal{T}^k, f, p) = \inf_{\mathcal{T} \in \mathcal{P}_N} Q(\mathcal{T}, f, p).$$

Where DT^* , the Delaunay triangulation of points V^* , is our optimal triangulation.

Remark 3.3. We can also define the optimal triangulation in the class \mathcal{P}_{N_T} , the set of triangulations with at most N_T simplices. It is slightly different with the optimal triangulation in \mathcal{P}_N since with the same vertices, we can have two triangulations with different number of simplices. The existence of such optimal triangulation can be proved in the similar way.

The optimal triangulation will benefit for the approximation since it contains more information than linear approximant. We first present an explicit formula for $Q(\mathcal{T}, f, 1)$. We denote \mathbf{x}_i the vertices of the triangulation, Ω_i the union of all simplices using \mathbf{x}_i , and $|A|$ the Lebesgue measure of set A in \mathbb{R}^n .

Lemma 3.4. For a convex function f ,

$$Q(\mathcal{T}, f, 1) = \frac{1}{n+1} \sum_{\mathbf{x}_i \in \mathcal{T}} f(\mathbf{x}_i) |\Omega_i| - \int_{\Omega} f(\mathbf{x}) d\mathbf{x}.$$

Proof. Because $f_{I, \mathcal{T}}(\mathbf{x}) \geq f(\mathbf{x})$ in Ω , we get

$$\begin{aligned} Q(\mathcal{T}, f, 1) &= \sum_{\tau \in \mathcal{T}} \int_{\tau} f_{I, \mathcal{T}}(\mathbf{x}) d\mathbf{x} - \int_{\Omega} f(\mathbf{x}) d\mathbf{x} \\ &= \frac{1}{n+1} \sum_{\tau \in \mathcal{T}} \left(|\tau| \sum_{k=1}^{n+1} f(\mathbf{x}_{\tau, k}) \right) - \int_{\Omega} f(\mathbf{x}) d\mathbf{x} \\ &= \frac{1}{n+1} \sum_{\mathbf{x}_i \in \mathcal{T}} f(\mathbf{x}_i) |\Omega_i| - \int_{\Omega} f(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Since $\sum_{i=1}^N |\Omega_i| = (n+1)|\Omega|$, the first term is indeed a numerical quadrature scheme of the integral of f . The quality is just to measure the error of this scheme. We are going to find the optimal one.

Let us look at the interpolation error in the patch Ω_i . We replace the vertex \mathbf{x}_i by any $\mathbf{x} \in \Omega_i$, keeping the connectivity, and try to minimize the error locally as a function of \mathbf{x} ; See Fig. 2.

Theorem 3.5. If triangulation is optimal in the sense of minimizing $Q(\mathcal{T}, f, 1)$ for a convex function f in $C^1(\Omega)$, then for an interior vertex \mathbf{x}_i , we have

$$\nabla f(\mathbf{x}_i) = -\frac{1}{|\Omega_i|} \sum_{\tau_j \in \Omega_i} (\nabla |\tau_j|(\mathbf{x}) \sum_{\mathbf{x}_k \in \tau_j, \mathbf{x}_k \neq \mathbf{x}_i} f(\mathbf{x}_k)). \quad (3.1)$$

Proof. By Lemma 3.4,

$$Q(\Omega_i, f, 1) = \frac{1}{n+1} \sum_{\tau_j \in \Omega_i} (|\tau_j(\mathbf{x})| \sum_{\mathbf{x}_k \in \tau_j, \mathbf{x}_k \neq \mathbf{x}} f(\mathbf{x}_k)) + \frac{|\Omega_i|}{n+1} f(\mathbf{x}_i) - \int_{\Omega_i} f(\mathbf{x}) d\mathbf{x}.$$

Since we only adjust the location of \mathbf{x}_i , $\int_{\Omega_i} f(\mathbf{x}) d\mathbf{x}$ is a constant. We only need to minimize

$$E(\mathbf{x}) \equiv \frac{1}{n+1} \sum_{\tau_j \in \Omega_i} (|\tau_j(\mathbf{x})| \sum_{\mathbf{x}_k \in \tau_j, \mathbf{x}_k \neq \mathbf{x}} f(\mathbf{x}_k)) + \frac{|\Omega_i|}{n+1} f(\mathbf{x}). \tag{3.2}$$

The domain of $E(\mathbf{x})$ is the biggest convex set A contained in Ω_i such that $\mathbf{x} \in A$ will not result in overlapping simplices. Since there exists a small neighborhood of \mathbf{x}_i in A , A is not empty and \mathbf{x}_i is an interior point of A . By the optimality of the triangulation, we conclude that \mathbf{x}_i is a critical point of $E(\mathbf{x})$.

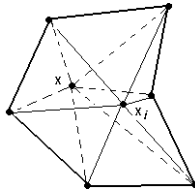


Figure 2. Local minimization

Remark 3.6. The theorem is also true if the triangulation is only a local minimizer. It is useful in practice since most algorithms only give us local minimizers.

Note that the right hand side of (3.1) only involves the nodal values of function, it can be used to recovery the gradient from its nodal value if the triangulation is optimal. If the triangulation is not optimal, (3.1) will guide us to move the vertex \mathbf{x}_i in its local patch to optimize the interpolation error and thus can be used as a mesh smoothing.

4. Nearly Optimal Triangulation

In previous section, we have given a necessary condition for an optimal Delaunay triangulation for $p = 1$ and a convex function f . In general, it is difficult, if it is not impossible, to find useful sufficient conditions for the optimal Delaunay triangulation. Instead we try to find a triangulation \mathcal{T}_N such that $Q(\mathcal{T}_N, f, p) \leq CQ(\mathcal{T}_N^*, f, p)$, where C is a constant independent of N and \mathcal{T}_N^* is the optimal Delaunay triangulation in \mathcal{P}_N . We call such a \mathcal{T}_N a nearly optimal triangulation. In this section, we will give two practical sufficient conditions based on the error estimate in our recent work [3].

Given a function $f \in \mathcal{C}^2(\Omega)$, we call a symmetric positive definite matrix $H \in \mathbb{R}^{n \times n}$ to be a majorizing Hessian of f if

$$|\xi^t (\nabla^2 f)(\mathbf{x}) \xi| \leq c_0 \xi^t H(\mathbf{x}) \xi, \quad \xi \in \mathbb{R}^n, \mathbf{x} \in \Omega$$

for some positive constant c_0 .

For a triangulation \mathcal{T}_N , where N is the number of simplices, we will define a new metric

$$H_p = (\det H)^{-\frac{1}{2p+n}} H.$$

There are two conditions for a triangulation \mathcal{T}_N to be an nearly optimal triangulation. The first assumption asks the mesh to capture the high oscillation of the Hessian metric, namely H does not change very much on each element.

(A1) There exist two positive constants α_0 and α_1 such that

$$\alpha_0 \xi^t H_\tau \xi \leq \xi^t H(\mathbf{x}) \xi \leq \alpha_1 \xi^t H_\tau \xi, \quad \xi \in \mathbb{R}^n,$$

where H_τ is the average of H over τ .

The second condition asks that \mathcal{T} is quasi-uniform under the new metric induced by H_p .

(A2) There exists two positive constants β_0 and β_1 such that

$$\frac{\sum_i \tilde{d}_{\tau,i}^2}{|\tilde{\tau}|^{2/n}} \leq \beta_0, \quad \forall \tau \in \mathcal{T} \text{ and } \frac{\max_{\tau \in \mathcal{T}} |\tilde{\tau}|}{\min_{\tau \in \mathcal{T}} |\tilde{\tau}|} \leq \beta_1, \tag{4.1}$$

where $|\tilde{\tau}|$ is the volume of τ and $\tilde{d}_{\tau,i}$ is the length of the i -th edge of τ under the new metric H_p , respectively.

Theorem 4.1. *Let $f \in C^2(\bar{\Omega})$ and \mathcal{T}_N satisfy assumptions (A1) and (A2). The following error estimate holds:*

$$Q(\mathcal{T}, f, p) \leq C \beta_0 \beta_1^{(2p+n)/n} N^{-2/n} \|\sqrt{\det(H)}\|_{L^{\frac{pn}{2p+n}}(\bar{\Omega})}$$

for some constant $C = C(n, p, c_0, \alpha_0, \alpha_1)$.

For a convex function, we also have a lower bound [3].

Theorem 4.2. *Suppose there is a family of triangulations $\{\mathcal{T}_N\}$ of Ω satisfying*

$$\lim_{N \rightarrow \infty} \max_{\tau \in \mathcal{T}_N} \text{diam}(\tau) = 0,$$

where N is the element number of the triangulation, then for a strictly convex (or concave) function $f \in C^2(\bar{\Omega})$, we have:

$$\liminf_{N \rightarrow \infty} N^{2/n} Q(\mathcal{T}_N, f, p) \geq LC_{n,p} \|\sqrt{\det(\nabla^2 f)}\|_{L^{\frac{pn}{2p+n}}(\bar{\Omega})}.$$

The equality holds if and only if all edges are asymptotic equal under the metric H_p .

For a sequence of optimal Delaunay triangulations \mathcal{T}_N^* , it is easy to show that

$$\lim_{N \rightarrow \infty} Q(\mathcal{T}_N^*, f, p) = 0.$$

With the fact that f is strictly convex, we know $\{\mathcal{T}_N^*\}$ satisfying

$$\lim_{N \rightarrow \infty} \max_{\tau \in \mathcal{T}_N^*} \text{diam}(\tau) = 0.$$

Note that the number of vertices, and the number of simplices are in the same order when one of them goes to infinity. Thus for N large enough, a triangulation \mathcal{T}_N which satisfies condition (A1) and (A2) will be a nearly optimal triangulation, namely

$$Q(\mathcal{T}_N, f, p) \leq CN^{-2/n} \|\sqrt{\det(H)}\|_{L^{\frac{pn}{2p+n}}(\bar{\Omega})} \leq CQ(\mathcal{T}_N^*, f, p).$$

We summarize it as a theorem.

Theorem 4.3. *For a strictly convex function f and sufficient large N , the triangulation \mathcal{T}_N which satisfies assumptions (A1) and (A2) is a nearly optimal triangulation.*

(A1) is easy to be satisfied if $f \in C^2$ and N is large enough. (A2) will direct our construction of the nearly optimal triangulation. The first inequality in (4.1) means that each τ is shape-regular under the metric H_p . The second inequality (4.1) means that all elements τ are of comparable size (under the new metric), which is a global condition. It means that the mesh will concentrate at the region where $\det H_p(\mathbf{x})$ is large. A nearly optimal mesh should satisfy both local and global conditions. The dependence of β_0 and β_1 appeared in the estimate

suggests that the constants β_0 and β_1 can be used as a measure of the mesh quality. For a related discussion and application of these mesh qualities to variational mesh adaptation, we refer to Huang [12].

5. Concluding Remarks

In this paper, we have shown that a Delaunay triangulation is an optimal triangulation for linear approximation to the isotropic function $\|\mathbf{x}\|^2$ when grid points are fixed. We further define an error-based quality and consider the optimal triangulation when grid points are free to move. The existence of the optimal triangulation is proved if the function is convex and a necessary condition for the optimal triangulation minimizing $Q(\mathcal{T}, f, 1)$ and two sufficient conditions for a nearly optimal triangulation are obtained.

By this unified approach, we can generalize DT and edge flipping to anisotropic case. Mesh smoothing can be also developed based on this approach. We will study the optimization problem for the optimal Delaunay triangulation and develop a multigrid-like algorithm in a further work.

References

- [1] Thomas Apel, Anisotropic finite elements: Local estimates and applications, Book Series: Advances in Numerical Mathematics, Stuttgart: Teubner, 1999.
- [2] K.Q. Brown, Voronoi diagrams from convex hulls, *Inform. Process. Lett.*, **9** (1979), 223-228.
- [3] Long Chen, Pengtao Sun, and Jinchao Xu, Optimal anisotropic simplicial meshes for minimizing interpolation errors in Lp-norm, *Submitted to Math. Comp.*, 2003.
- [4] E.F. D'Azevedo and R.B. Simpson, On optimal interpolation triangle incidences, *SIAM Journal on Scientific and Statistical Computing*, **6** (1989), 1063-1075.
- [5] Dari EA and Buscaglia GC, Mesh optimization: how to obtain good unstructured 3-d finite element meshes with not-so-good mesh generators, *Structural Optimization*, **8** (1994), 181-188.
- [6] H. Edelsbrunner, Triangulations and meshes in computational geometry, *Acta Numerica*, (2000), 1-81.
- [7] F.Aurenhammer and R.Klein, Handbook of Computational Geometry, Amsterdam, Netherlands: North-Holland, 2000.
- [8] Steven Fortune, Voronoi diagrams and delaunay triangulations, In Computing in Euclidean Geometry, Edited by Ding-Zhu Du and Frank Hwang, World Scientific, Lecture Notes Series on Computing - Vol. 1, 1992.
- [9] LA Freitag and C. Ollivier-Gooch, Tetrahedral mesh improvement using swapping and smoothing, *International Journal of Numerical Methods in Engineering*, **40** (1997), 3979-4002.
- [10] W. G. Habashi, M. Fortin, J. Dompierre, M.-G. Vallet, D. A t Ali-Yahia, Y. Bourgault, M. P. Robichaud, and A. Tam et S. Boivin, Anisotropic mesh optimization for structured and unstructured meshes, Dans 28th Computational Fluid Dynamics Lecture Series, Lecture Series 1997-02, von Karman Institute, von Karman Institute for Fluid Dynamics, mars 1997, 1997.
- [11] H.Edelsbrunner and R.Seidel. Voronoi diagrams and arrangements, *Disc. and Comp. Geom.*, **8**:1 (1986), 25-44.
- [12] Weizhang Huang, Measuring mesh quality and application to variational mesh adaptation, *SIAM J.Sci. Comput. (Submitted)*, 2003.
- [13] Shewchuk J.R. Tetrahedral mesh generation by delaunay refinement, In *14th Annual ACM Symposium on Computational Geometry*, (1998), 86-95.
- [14] T. Lamber, The delaunay triangulation maximize the mean inradius, In *Proc. 6th Canad. Conf. Comput. Geom.*, (1994), 201-206.
- [15] C.L. Lawson, Software for C1 surface interpolation, *Mathematical Software III*, pages 161-194 J.R. Rice, ed., Academic Press, 1977.
- [16] Xiangyang Li, Sliver-free three dimensional Delaunay mesh generation, PhD thesis, University of Illinois at Urbana Champaign, 2000.

- [17] Marshall Bern and David Eppstein, Mesh generation and optimal triangulation, In *Computing in Euclidean Geometry*, Edited by Ding-Zhu Du and Frank Hwang, World Scientific, Lecture Notes Series on Computing, **1** (1992).
- [18] Shmuel Rippa, Minimal roughness property of the delaunay triangulation, *Comput. Aided Geom. Design*, **7** (1990), 489-497.
- [19] Shmuel Rippa, Long and thin triangles can be good for linear interpolation, *SIAM J. Numer. Anal.*, **29** (1992), 257-270.
- [20] R. Sibson, Locally equiangular triangulations, *Computer Journal*, **21** (1978), 243-245.
- [21] R. Bruce Simpson, Anisotropic mesh transformations and optimal error control, *Applied Numerical Mathematics*, **14** (1994), 183-198.