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# THREE DIMENSION QUASI-WILSON ELEMENT FOR FLAT HEXAHEDRON MESHES *1)

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**Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday**

**Abstract**

The well known Wilson's brick is only convergent for regular cuboid meshes. In this paper a quasi-Wilson element of three dimension is presented which is convergent for any hexahedron meshes. Meanwhile the element is anisotropic, that is it can be used to any flat hexahedron meshes for which the regular condition is unnecessary.

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*Key words*: Nonconforming element, Three dimension Quasi-Wilson element, Anisotropic convergence.

## 1. Introduction

The classical finite element approximation relies on the regular [5] or nondegenerate [4] condition,ie. there exists a constant c such that

$$h_k/e_k \le c, \quad \forall K \tag{1.1}$$

where $h_K$ is diameter of $K$ and $\rho_K$ is diameter of the biggest ball contained in $K$. But recently some researches [2,3,7,18] show that the condition (1.1) is not necessary for the convergence of some finite elements, i.e, these elements can be well used in narrow meshes.

The well known Wilson's elements are nonconforming elements for the problems of order two. However the two dimension Wilson's element is only convergent for rectangular and parallelogram meshes. In order to extend this element to arbitrary quadrilateral meshes, various improved methods have been developed, see [6,7,10,11,12,13,14,15,17,18]. But seldom papers consider the three dimension Wilson's element. In the same way the three dimension Wilson's element is only convergent for regular cuboid meshes. In this paper a quasi-Wilson element of three dimension is presented. We prove that this element is convergent for any flat hexahedron meshes, this means its convergence is independent of regular(1.1).

## 2. Three Dimension Quasi-Wilson Element

Let $\hat{K} = [-1,1]^3$ be the reference element with vertices $\hat{A}_i(\hat{a}_{1i}, \hat{a}_{2i}, \hat{a}_{3i}), 1 \le i \le 8$, where $(\hat{a}_{11} \cdots \hat{a}_{18}) = (-1,1,1,-1,-1,1,1,-1)$ $(\hat{a}_{21}, \cdots, \hat{a}_{28}) = (-1,-1,1,1,-1,-1,1,1)$, $(\hat{a}_{31}, \cdots, \hat{a}_{38}) = (-1,-1,-1,-1,1,1,1,1)$. We define on $\hat{K}$ the finite element $(\hat{K}, \hat{p}, \hat{\sum})$ as following:

$$\hat{P} = span\{\hat{N}_1, \cdots, \hat{N}_8, \hat{\phi}(\hat{x}_1), \hat{\phi}(\hat{x}_2), \hat{\phi}(\hat{x}_3)\} \tag{2.1}$$

where

$$\hat{N}_i = \frac{1}{8}(1+\hat{a}_{1i}\hat{x}_1)(1+\hat{a}_{2i}\hat{x}_2)(1+\hat{a}_{3i}\hat{x}_3), 1 \le i \le 8, \hat{\phi}(t) = -\frac{3}{32}(t^2-1) + \frac{5}{64}(t^4-1)$$

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When $\phi(t) = \frac{1}{16}(t^2-1)$, it is Wilson's brick. Obviously $Q_1(\hat{k}) = span\{\hat{N}_1, \cdots, \hat{N}_8\}$, and $\hat{N}_i(\hat{A}_j) = \delta_{ij}, 1 \le i, j \le 8$

$$\hat{\sum} = \{\hat{v_1}, \cdots, \hat{v_8}, g_1(\hat{v}), g_2(\hat{v}), g_3(\hat{v})\} \tag{2.2}$$

where $\hat{v_i} = \hat{v}(\hat{A}_i), \quad 1 \le i \le 8, \quad g_j(\hat{v}) = \int_{\hat{K}} \frac{\partial^2 \hat{v}}{\partial \hat{x}_j^2} d\hat{x}, \quad 1 \le j \le 3$. It is easy to see that

$$\forall \quad \hat{v} \in \hat{P}, \quad \hat{v} = \hat{v}^0 + \hat{v}^1 \tag{2.3}$$

where

$$\hat{v}^0 = \sum_{i=1}^{8} \hat{v_i}\hat{N}_i(\hat{x}), \quad \hat{v}^1 = \sum_{i=1}^{3} \hat{g}_i(\hat{v})\hat{\phi}(\hat{x_i}), \tag{2.4}$$

Let $K$ be a convex hexahedron with vertices $A_i(a_{1i}, a_{2i}, a_{3i}), 1 \le i \le 8$. The mapping

$$F_K \in (\hat{Q_1}^3), F_K(\hat{x}) = (x_1^K(\hat{x}), x_2^K(\hat{x}), x_3^K(\hat{x})), x_i^K(\hat{x}) = \sum_{j=1}^{8} \hat{N}_j(\hat{x})a_{ij}, \quad 1 \le i \le 8 \tag{2.5}$$

makes $F_K(\hat{K}) = K, F_K(\hat{A}_i) = A_i, 1 \le i \le 8$

For any function $v(x)$ defined on K, we define $\hat{v}(\hat{x})$ by

$$\hat{v}(\hat{x}) = \hat{v}(x^K(\hat{x})), \quad or \quad \hat{v} = v \circ F_K$$

On the hexahedron element $K$, we define the shape function space $P_K$,

$$P_K = \{p = \hat{p} \circ F_K^{-1}; \hat{p} \in \hat{P}\}$$

Given a convex polyhedron domain $\Omega$, let $\bar{\Omega} = \bigcup_{K \in T_h} K$ be a decomposition $T_h$ of $\bar{\Omega}$. The finite-element space is defined by $V_h = \{v; v|_K \in P_K, \forall K \in T_h$; $v$ is continuous at the vertices of elements and vanishing at the vertices on the boundary of $\Omega\}$.

Consider the model problem,

$$\begin{cases} -\Delta u = f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{2.6}$$

where $\Omega$ is a bounded, convex polyhedron in three dimension.

Its weak form is find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^1(\Omega) \tag{2.7}$$

where $a(u, v) = \int_\Omega \nabla u . \nabla v dx, \quad f(v) = \int_\Omega f v dx$. By theorem 1.8 of [8] (see [9]), when $f \in L^p(\Omega), p > 2$,

$$u \in W^{2,p}(\Omega) \tag{2.8}$$

The quasi-Wilson element approximation of (2.7) is defined by find $u_h \in v_h$ such that

$$a_h(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h \tag{2.9}$$

where $a_h(u_h, v_h) = \sum_k \int_k \nabla u_h . \nabla v_n dx$. Since$V_h$ is not contained in $H_0^1(\Omega), V_h$ is a nonconforming approximation of $H_0^1(\Omega)$

For every $v_h \in V_h$, we define

$$|v_n|_{1,h}^2 = a_h(v_h, v_h)$$

It is easy to check that $|\cdot|_{1,h}$ is a norm over $V_h$. Every $v_h \in V_h$ can be written as

$$v_h = v_h^0 + v_h^1 \tag{2.10}$$

where $\forall K \in T_h, F_K : \hat{K} \to K$, with

$$v_h^0|_K = \sum_{i=1}^{8} \hat{N}_i(\hat{x})v_h(A_i) = \hat{v}^0 \circ F_K^{-1}, \quad v_h^1|_K = \sum_{i=1}^{3} \hat{\phi}_i(\hat{x_i})g_i(\hat{v}) = \hat{v}^1 \circ F_K^{-1}$$

Thus trilinear function $v_n^0$ is the conforming part of $v_n$ and $v_n^1$ is the nonconforming part of $v_n$.

From Strang's lemma [5], we have

$$|u-u_h|_{1,h} \leq C(inf_{v_h \in V_h}|u-v_h|_{1,h} + sup_{w_n \in V_h}\frac{|a_h(v,w_h)-f(w_h)|}{|w_h|_{1,h}}) \tag{2.11}$$

where C is independent of $T_h$.

Let $\hat{L}$ be the trilinear interpolation operator on $\hat{K}$ and $L_K v = (\hat{L}\hat{v}) \circ F_K^{-1} \in P_K$, then

$$\hat{L}\hat{v} = \sum_{i=1}^{8} \hat{N}_i(\hat{x})\hat{v}_i, \qquad L_K v = \sum_{i=1}^{8} \hat{N}_i(\hat{x}) \circ F_K^{-1} v_i$$

Let $L^0 v$ be the piecewise isoparametric trilinear interpolation of $v$ on $\hat{\Omega}$, such that

$$L^0 v|_K = L_K v, \forall K \in T_h$$

Taking $v_h \in V_h$ with $v_h^1 = 0$ and $v_h^0 = L^0 u$, *i.e.* $v_h = L^0 u$, we have

$$inf_{v_h \in V_h}|u-v_h|_{1,h} \leq |u-L^0 u|_{1,h} \tag{2.12}$$

In the next section we estimate $|u-L^0 u|_{1,h}$

## 3. The Anisotropy of the Trilinear Interpolation

First we briefly present the anisotropic interpolation theory [7].

Suppose $\hat{K}$ is the reference element, $W^{m,p}\hat{(}K)$ and $H^m(\hat{K})$ are the usual Sobolev spaces and $P_l(\hat{K})$ is the polynomial space of degree $\leq l$ on $\hat{K}$.

**Lemma 3.1 ([5],Theorem 3.14).** *For some integers $l \geq 0$ and $m \geq 0$, suppose $W^{l+1,p}(\hat{K}) \hookrightarrow W^{m,q}(\hat{K})$ and $\hat{I} \in \pounds(W^{l+1,p}(\hat{K}); W^{m,q}(\hat{K}))$–the space of continuous linear mappings from $W^{l+1,p}(\hat{K})$ into $W^{m,q}(\hat{K})$–be a mapping such that*

$$\hat{I}\hat{P} = \hat{p} \quad \forall \hat{p} \in P_l(\hat{K})$$

*Then there exists a constant $C(\hat{I},\hat{K})$ such that*

$$|\hat{v}-\hat{I}\hat{v}|_{m,q,\hat{K}} \leq c(\hat{I},\hat{K})|\hat{v}|_{l+1,p,\hat{K}}$$

Let the shape function space $\hat{P}$ be a polynomial space of dimension $m$. Assume, further, that $\hat{p}_1, \cdots, \hat{p}_m \subset \hat{P}$ and $\hat{d}_1, \cdots, \hat{d}_m \subset \hat{P}'$ be a pair of dual bases for $\hat{P}$ and $\hat{P}'$ respectively ,i.e.

$$\hat{d}_i(\hat{p}_j) = \delta_{ij}, 1 \leq i,j \leq m$$

where $\hat{d}_1, \cdots, \hat{d}_m$ are called the degree of freedom of the element.

Suppose a finite-element interpolation operator $\hat{I}$ satisfies.

$$\hat{d}_i(\hat{I}\hat{v}) = \hat{d}_i(\hat{v}), \quad 1 \leq i \leq m \tag{3.1}$$

Obviously

$$\hat{I}\hat{v} = \sum_{i=1}^{m} \hat{d}_i(\hat{v})\hat{p}_i \tag{3.2}$$

and

$$\hat{I}\hat{v} = \hat{v}, \quad \forall \hat{v} \in \hat{p} \tag{3.3}$$

Suppose $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index; the $\hat{D}^\alpha \hat{p}$ is also a polynomial space on $\hat{K}$. Let

$$dim\hat{D}^\alpha \hat{P} = r$$

and let $\{\hat{q}_i\}_{i=1}^r$ be a set of basis functions of $\hat{D}^\alpha \hat{P}$. Suppose that

$$\hat{D}^\alpha \hat{p}_i = \sum_{j=1}^r C_{ij}\hat{q}_j, \quad 1 \le i \le m.$$

then

$$\hat{D}^\alpha \hat{I}\hat{v} \overset{(3.2)}{=} \sum_{i=1}^m \hat{d}_i(\hat{v})\hat{D}^\alpha \hat{p}_i = \sum_{j=1}^r (\sum_{i=1}^m C_{ij}\hat{d}_i(\hat{v}))\hat{q}_i \triangleq \sum_{j=1}^r \beta_j(\hat{v})\hat{q}_i \tag{3.4}$$

where

$$\beta_j(\hat{v}) = \sum_{i=1}^m C_{ij}\hat{d}_i(\hat{v}) \tag{3.5}$$

From (3.5) and (3.1) we have

$$\beta_j(\hat{v}) = \sum_{i=1}^m C_{ij}\hat{d}_i(\hat{v}) = \sum_{i=1}^m C_{ij}\hat{d}_i(\hat{I}\hat{v})) = \beta_j(\hat{I}\hat{v}) \tag{3.6}$$

**Theorem 3.1**[7]**.** *Let $\alpha$ be a multi-index, $P_l(\hat{K}) \subset \hat{D}^\alpha \hat{P}$ and let $\hat{I} : W^{|\alpha|+l+1,p}(\hat{K}) \to \hat{P}$ be the above finite-element interpolation operator satisfying $\hat{I} \in \pounds(W^{|\alpha|+l+1,p}(\hat{K}); W^{|\alpha|+m,q}(\hat{K}))$, with $W^{l+1,p}(\hat{K}) \hookrightarrow w^{m,q}(\hat{K})$. If there exists an interpolation operator $\hat{T} : W^{l+1,p}(\hat{K}) \to \hat{D}^\alpha \hat{p}$ with $\hat{T} \in \pounds(w^{l+1,p}(\hat{K}); w^{m,q}(\hat{K}))$, and*

$$\hat{D}^\alpha \hat{I}\hat{v} = \hat{T}\hat{D}^\alpha \hat{v}, \qquad \forall \hat{v} \in W^{|\alpha|+l+1,p}(\hat{K}) \tag{3.7}$$

*Then, there exists a constant $C(\hat{I}, \hat{K})$ such that*

$$|\hat{D}^\alpha(\hat{v} - \hat{I}\hat{v})|_{m,q,\hat{K}} \le C(\hat{I}, \hat{K})|\hat{D}^\alpha(\hat{v})|_{l+1,p,\hat{K}} \quad \forall \hat{v} \in W^{|\alpha|+l+1,p}(\hat{K}) \tag{3.8}$$

*Proof.* Clearly, $|\hat{D}^\alpha(\hat{v} - \hat{I}\hat{v})|_{m,q,\hat{K}} = |\hat{D}^\alpha \hat{v} - \hat{T}\hat{D}^\alpha \hat{v}|_{m,q,\hat{K}}$

From $P_l(\hat{K}) \subset \hat{D}^\alpha \hat{p}$, then $\forall \hat{p} \in P_l(\hat{K})$, there exists $\hat{q} \in P_{l+|\alpha|} \subset P$ such that $\hat{p} = \hat{D}^\alpha \hat{q}$. We have

$$\hat{T}\hat{p} = \hat{T}\hat{D}^\alpha \hat{q} \overset{(3.7)}{=} \hat{D}^\alpha \hat{I}\hat{q} \overset{(3.3)}{=} \hat{D}^\alpha \hat{q} = \hat{p}$$

Then (3.8) follows from Lemma 3.1.

**Theorem 3.2**[7]**.** *The assumptions are the same as Theorem 3.1. If*

$$\beta_i(\hat{v}) = F_j(\hat{D}^\alpha \hat{v}), 1 \le j \le r \tag{3.9}$$

*where*

$$F_j \in (W^{l+1,p}(\hat{K}))', \quad 1 \le j \le r \tag{3.10}$$

*then (3.8) holds.*

*Proof.* Define the interpolation operator $\hat{T} : W^{l+1,p}(\hat{K}) \to \hat{D}^\alpha \hat{p}$ by

$$\hat{T}\hat{w} = \sum_{i=1}^r F_i(\hat{w})\hat{q}_i$$

Then, for all $\hat{w} \in W^{l+1,p}(\hat{K})$,

$$||\hat{T}\hat{w}||_{m,q,\hat{K}} \le \sum_{i=1}^r |F_i(\hat{v})| ||\hat{q}_i||_{m,q,\hat{K}} \overset{(3.10)}{\le} \hat{C}||\hat{w}||_{l+1,p,\hat{k}}$$

Thus $\hat{T} \in \pounds(W^{l+1,p}(\hat{K}); W^{m,q}(\hat{K})) \quad \forall \hat{v} \in W^{|\alpha+l+1,p}(\hat{K})$, we have

$$\hat{T}\hat{D}^\alpha \hat{v} = \sum_{i=1}^r F_i(\hat{D}^\alpha \hat{v})\hat{q}_i \overset{(3.9)}{=} \sum_{i=1}^r \beta_i(\hat{v})\hat{q}_i \overset{(3.4)}{=} \hat{D}^\alpha \hat{I}\hat{v}$$

Hence (3.8) follows from Theorem 3.1.

**Remark 3.1.** Apel etc. have presented other conditions for getting (3.8), see Lemma 3 and 4 of [2], however our result , i.e, Theorem 3.2 is easier to use than Apel's.

For trilinear interpolation operator $\hat{L}$ on $\hat{K}$,

$$\hat{P} = Q_1(\hat{k}), \quad \hat{L}\hat{v} = \sum_{i=1}^{8} \hat{N}_i(\hat{x})\hat{v}_i$$

Let $\alpha = (1,0,0)$,then $\hat{D}^\alpha \hat{P} = Q_1(\hat{x_2}, \hat{x_3}) = span\{1, \hat{x_2}, \hat{x_3}, \hat{x_2}\hat{x_3}\}$. By simple computation, we have

$$\hat{D}^\alpha \hat{L}\hat{v} = \sum_{i=1}^{8} \frac{\partial \hat{N}_i(\hat{x})}{\partial \hat{x}_1}\hat{v}_i = \sum_{i=1}^{4} \beta_i^{(1)}(\hat{v})M_i^{(1)}$$

where $M_i^{(1)} = \frac{1}{8}(1+b_i^{(1)}\hat{x}_2)(1+c_i^{(1)}\hat{x}_3), 1 \le i \le 4, (b_1^{(1)}, \cdots, b_4^{(1)}) = (-1,1,-1,1), (c_1^{(1)}, \cdots, c_4^{(1)}) = (-1,-1,1,1), span\{M_1^{(1)}, \cdots, M_4^{(1)}\} = Q_1(\hat{x}_2\hat{x}_3)$, and

$$\beta_1^{(1)}(\hat{v}) = \hat{v}_2 - \hat{v}_1 = \int_{\hat{l}_{12}} \frac{\partial \hat{v}}{\partial \hat{x}_1} ds, \quad \beta_2^{(1)}(\hat{v}) = \hat{v}_3 - \hat{v}_4 = \int_{\hat{l}_{43}} \frac{\partial \hat{v}}{\partial \hat{x}_1} ds,$$

$$\beta_3^{(1)}(\hat{v}) = \hat{v}_6 - \hat{v}_5 = \int_{\hat{l}_{56}} \frac{\partial \hat{v}}{\partial \hat{x}_1} ds, \quad \beta_4^{(1)}(\hat{v}) = \hat{v}_7 - \hat{v}_8 = \int_{\hat{l}_{87}} \frac{\partial \hat{v}}{\partial \hat{x}_1} ds,$$

where $\hat{l}_{ij} = \overrightarrow{\hat{A}_i\hat{A}_j}$. Thus

$$\beta_i^{(1)}(\hat{v}) = F_i^{(1)}(\frac{\partial \hat{v}}{\partial \hat{x_1}}), \quad 1 \le i \le 4. \tag{3.11$_1$}$$

and

$$|F_i^{(1)}(\hat{w})| = |\int_{\hat{l_{jk}}} \hat{w} ds| \le \hat{c}||\hat{w}||_{0,p,\hat{l_{jk}}}$$

Where $p = 2 + \varepsilon, 0 < \varepsilon << 1$.

From imbedding theorem (Theorem 5.4 of [1]),

$$w^{1,p}(\hat{k}) \hookrightarrow w^{0,p}(\hat{l}_{ij})$$

Hence.

$$|F_i^{(1)}(\hat{w})| \le \hat{c}||\hat{w}||_{0,p,\hat{l_{jk}}} \le \hat{c}||\hat{w}||_{1,p,\hat{k}}, \quad 1 \le i \le 4 \tag{3.11$_2$}$$

Let $\alpha = (0,1,0)$, similarly

$$\hat{D}^\alpha \hat{L}\hat{v} = \sum_{i=1}^{4} \beta_i^{(2)}(\hat{v})M_i^{(2)}$$

Where $M_i^{(2)} = \frac{1}{8}(1 + b_i^{(2)}\hat{x_1})(1 + c_i^{(2)}\hat{x_3}), \quad 1 \le i \le 4, (b_1^{(2)}, \cdots, b_4^{(2)}) = (-1,1,-1,1), (c_1^{(2)}, \cdots, c_4^{(2)}) = (-1,-1,1,1), span\{M_1^{(2)}, \cdots, M_4^{(2)}\} = Q_1(\hat{x_1}, \hat{x_3})$, and

$$\beta_1^{(2)} = \int_{\hat{l}_{14}} \frac{\partial \hat{v}}{\partial \hat{x_2}} ds, \quad \beta_2^{(2)}(\hat{v}) = \int_{\hat{l}_{23}} \frac{\partial \hat{v}}{\partial \hat{x_2}} ds, \quad \beta_3^{(2)}(\hat{v}) = \int_{\hat{l}_{58}} \frac{\partial \hat{v}}{\partial \hat{x_2}} ds, \quad \beta_4^{(2)}(\hat{v}) = \int_{\hat{l}_{67}} \frac{\partial \hat{v}}{\partial \hat{x_2}} ds$$

In the same way we have

$$\beta_i^{(2)}(\hat{v}) = F_i^{(2)}(\frac{\partial \hat{v}}{\partial \hat{x}_2}), \quad |F_i^{(2)}(\hat{w})| \le \hat{c}||\hat{w}||_{1,p,\hat{K}} \tag{3.12}$$

The same results hold for $\alpha = (0,0,1)$. Thus the conditions of Theorem 3,2 are satisfied for $|\alpha| = 1, m = l = 0, p = 2 + \varepsilon, q = 2$. Form(3,8) we obtain

$$||\hat{D}^\alpha(\hat{v} - \hat{L}\hat{v})||_{0,\hat{K}} \le \hat{c}|\hat{D}^\alpha \hat{v}|_{1,p,\hat{K}}, \quad \forall \hat{v} \in W^{2,p}(\hat{K}) \tag{3.13}$$

A general hexahedron element $K$ can always be regarded as a perturbation of a cuboid $\tilde{K}$, since under translation and rotation of the coordinate system, the $L^2$-norm, the 1-seminorm and 2- seminorm are unchanged,we simply suppose that the faces of $K$ are parallel to the coordinate planes. Let the vertices of $\tilde{K}$ be $\tilde{A}_i(\tilde{a}_{1i}, \tilde{a}_{2i}, \tilde{a}_{3i}), \quad 1 \le i \le 8, \quad \tilde{A}_1\tilde{A}_2//\tilde{x}_1 axis, \quad \tilde{A}_1\tilde{A}_4//\tilde{x}_2 axis,$ $\tilde{A}_1\tilde{A}_5//\tilde{x}_3 axis.$

$$\tilde{a}_{12} - \tilde{a}_{11} = h_1, \quad \tilde{a}_{24} - \tilde{a}_{21} = h_2, \quad \tilde{a}_{35} - \tilde{a}_{31} = h_3$$

We suppose that

$$h_1 \sim h_2 >> h_3 \tag{3.14}$$

That is we consider flat hexahedron element for which the regular condition(1.1) is unnecessary. Let the vertices of $K$ be $A_i(a_{1i}, a_{2i}, a_{3i}), \quad i \le i \le 8.$ we assume that the perturbation of $a_{ji}$ to $\tilde{a}_{ji}$ is $O(h_j)$ along $x_j$-direction and is $O(h_3)$ along other direction, $1 \le j \le 3, 1 \le i \le 8.$ Precisely we assume that

$$a_{ki} - a_{kj} = \alpha_{ij}^{(K)} h_K, \quad (i,j) \in X_K$$

$$|a_{ki} - a_{kj}| = \tilde{\alpha_{ij}}^{(K)} h_3, (i,j) \in X/X_K, \quad 1 \le k \le 3 \tag{3.15}$$

where $X_1 = \{(2,1), (3,4), (6,5), (7,8)\}$, $X_2 = \{(4,1), (3,2), (8,5), (7,6)\}$, $X_3 = \{(5,1), (6,2),$ $(7,3), (8,4)\}$, $X = X_1 \bigcup X_2 \bigcup X_3$

$$0 < \sigma_o \le \alpha_{ij}^{(K)} \le \sigma, \quad 0 \le \tilde{\alpha}_{ij}^{(K)} \le \sigma_0, \quad 1 \le k \le 3, \quad (i,j) \in X$$

Let $\hat{K} = [-1,1]^3$ be the reference element described in section 2, then mapping $F_K$ of (2.5): $\hat{K} \to K$. Let $J_K$ be the Jacobian matrix of $F_K$, then

$$J_K = (\frac{\partial x_i}{\partial \hat{x}_j})_{3\times 3}, \quad J_K^{-1} = (\frac{\partial \hat{x}_i}{\partial x_j})_{3\times 3}$$

From the expressions of $\hat{N}_i (1 \le i \le 8)$ and (3.15), it can be obtained that

$$|\frac{\partial x_i}{\partial \hat{x}_i}| \le \hat{c} h_i, 1 \le i \le 3; \quad |\frac{\partial x_i}{\partial \hat{x}_j}| \le \hat{c} h_3, i \ne j,$$

When $h_1 \ge h_2 \ge max(4, 3\sigma_1/\sigma_0) h_3, \ \frac{1}{32}\sigma_0^2\sigma_1 h_1 h_2 h_3 \le det J_K \le \frac{3}{4}\sigma_1^3 h_1 h_2 h_3$
and

$$|\frac{\partial \hat{x}_i}{\partial x_j}| \le \hat{C} \cdot min(h_i^{-1}, h_j^{-1})$$

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \quad h^\alpha = h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3}$, then the above inequalities can be expressed as

$$|\frac{\partial x^\alpha}{\partial \hat{x}^\beta}| \le \hat{C} \cdot min(h^\alpha, h^\beta), |\alpha| = |\beta| = 1 \tag{3.16}$$

$$|\frac{\partial \hat{x^\alpha}}{\partial x^\beta}| \le \hat{C} \cdot min(h^{-\alpha}, h^{-\beta}), |\alpha| = |\beta| = 1 \tag{3.17}$$

$$c_0 H \le det J_k \le c_1, H \tag{3.18}$$

where $H = h_1 h_2 h_3$. Thus

$$|D^\beta v| \le \hat{c} \sum_{|\alpha|=1} h^{-\alpha} |\hat{D}^\alpha \hat{v}|, |\beta| = 1, \quad |\hat{D}^\beta \hat{v}| \le \hat{c} \sum_{|\alpha|=1} h^\alpha |D^\alpha v|, |\beta| = 1 \tag{3.19}$$

$$|D^{\alpha+\beta} \hat{v}| \le \hat{c} h^\alpha \sum_{|\nu|=1} \sum_{|\mu|=1} h^\mu |D^{\mu+\nu} v|, |\alpha| = |\beta| = 1 \tag{3.20}$$

Hence

$$|u - Lu|_{1,K} = \sum_{|\beta|=1} ||D^\beta (u - Lu)||_{0,K}$$

$$\overset{(3.19)(3.18)}{\leq} \hat{c}\sum_{|\beta|=1}\sum_{|\alpha|=1} h^{-\alpha}H^{\frac{1}{2}}||D^{\alpha}(\hat{u}-\hat{L}\hat{u})||_{0,\hat{K}}$$

$$\overset{(3.13)}{\leq} \hat{c}\sum_{|\alpha|=1} h^{-\alpha}H^{\frac{1}{2}}|D^{\alpha}\hat{u}|_{1,p,\hat{K}}$$

$$= \hat{c}\sum_{|\alpha|=1} h^{-\alpha}H^{\frac{1}{2}}\sum_{|\beta|=1}||D^{\alpha+\beta}\hat{u}||_{0,p,\hat{K}}$$

$$\overset{(3.20)(3.18)}{\leq} \hat{c}H^{\frac{1}{2}-\frac{1}{p}}\sum_{|\nu|=1}\sum_{|\mu|=1} h^{\mu}||D^{\mu+\nu}u||_{0,p,K}$$

$$= \hat{c}H^{\frac{1}{2}-\frac{1}{p}}\sum_{|\mu|=1} h^{\mu}|D^{\mu}u|_{1,p,k} \tag{3.21}$$

Where $\hat{c}$ is independent of (1.1).

## 4. Anisotropic Convergence of Three Dimension Quasi-Wilson Element

From the results of section 3, we immediately get interpolation error as following:

$$inf_{v_n\in V_n}|u-v_n|_{1,h} \overset{(2.12)}{\leq} |u-L^0u|_{1,h} = (\sum_{K\in T_n}|u-Lu|^2_{1,K})^{\frac{1}{2}}$$

$$\overset{(3.21)}{\leq} c(\sum_{K\in T_n} H^{1-\frac{2}{p}}\sum_{\mu=1} h^{2\mu}|D^{\mu}u|^2_{1,p,K})^{\frac{1}{2}} \tag{4.1}$$

Now we estimate the consistence error, i.e. the second term of (2.11).

For all $v_h\in V_h, v_h$ can be expressed as

$$v_h = v_h^0 + v_h^1 \tag{4.2}$$

Where $v_h^0$ is the piecewise trilinear function, $v_K^0 = v_h^0|_K = \sum_{i=1}^{8}\hat{N}_i(\hat{x})v_h(A_i^K)$, obviously $v_h^0 \in C_0^0(\bar{\Omega}) = \{v\in C^0(\bar{\Omega}); v|_{\partial\Omega} = 0\}$, $v_h^0$ is the conforming part of $v_h$, $v_K^1 \overset{\Delta}{=} v_h^1|_K = \sum_{j=1}^{3}\hat{\phi}(\hat{x}_j)g_j(\hat{v}_h)$, $v_h^1$ is the nonconforming part of $v_h$. For any $K\in T_h$, let $F_K:\hat{K}\to K, \hat{v} = v_K\circ F_K$, then $\hat{v} = \hat{v}^0+\hat{v}^1, \hat{v}^0 = v_k^0\circ F_K, \hat{v}^1 = v_K^1\circ F_K$.

**Lemma 4.1.**

*i)* $$\int_K \frac{\partial v_K^1}{\partial x_j}dx = 0, 1\leq i\leq 3 \tag{4.3}$$

*ii)* $$||\hat{v}^1||_{0,\hat{K}} \leq |\hat{v}^1|_{1,\hat{K}} \tag{4.4}$$

*iii)* $$||\frac{\partial\hat{v}}{\partial\hat{x_i}}||^2_{0,\hat{K}} = ||\frac{\partial\hat{v}^0}{\partial\hat{x_i}}||^2_{0,\hat{K}} + ||\frac{\partial\hat{v}^1}{\partial\hat{x_i}}||^2_{0,\hat{K}} \tag{4.5}$$

*iv)* $$|v_K^1|_{1,K}\leq\hat{c}|v_K|_{1,K} \quad ||v_K^1||_{0,K}\leq\hat{c}h_K|v_K|_{1,K} \tag{4.6}$$

*Proof.* It is easy to check that

$$\int_{-1}^{1}\phi(t)p(t)dt = 0, \forall p\in P_1;\quad \int_{-1}^{1}\phi'(t)q(t)dt = 0, \forall q\in P_2 \tag{4.7}$$

From

$$I = J_K.J_K^{-1} = (\frac{\partial x_i}{\partial\hat{x}_j})_{3\times3}\cdot(\frac{\partial\hat{x}_i}{\partial x_j})_{3\times3}$$

We have

$$(detJ_K)\frac{\partial \hat{x}_i}{\partial x_j} = Y_{ij} \in Q_2(\hat{k}), 1 \leq i, j \leq 3 \tag{4.8}$$

For example, $Y_{11} = \frac{\partial x_2}{\partial \hat{x_2}}\frac{\partial x_3}{\partial \hat{x_3}} - \frac{\partial x_2}{\partial \hat{x_3}}\frac{\partial x_3}{\partial \hat{x_2}} \in Q_2(\hat{K})$, etc. $Y_{ij}$ for $\hat{x_k}(1 \leq k \leq 3)$ is a polynomial of degree two.

i)

$$\int_K \frac{\partial v_2^1}{\partial x_j} = \int_{\hat{K}} \sum_{i=1}^{3} \frac{\partial \hat{v}^1}{\partial \hat{x}_i}\frac{\partial \hat{x}_i}{\partial x_j} detJ_K d\hat{x} \overset{(4.8)}{=} \sum_{i=1}^{3} g_i(\hat{v}) \int_{\hat{K}} \phi'(\hat{x}_i) Y_{ij} d\hat{x} \overset{(4.7)}{=} 0$$

ii) By simple computations we have

$$||\hat{v}||_{0,\hat{k}}^2 = \int_{\hat{K}} (\sum_{i=1}^{3} g_i(\hat{v})\hat{\phi}(x_i))^2 d\hat{x} \overset{(4.7)}{=} \sum_{i=1}^{3} g_i^2(\hat{v}) \int_{\hat{K}} \hat{\phi}(\hat{x}_i)^2 d\hat{x}$$

$$= 4\int_{-1}^{1} \hat{\phi}(\hat{x}_i)^2 d\hat{x} \sum_{i=1}^{3} g_i^2(\hat{v}) = \frac{2}{45}\frac{1}{56}\sum_{i=1}^{3} g_i^2(\hat{v})$$

$$|\hat{v}^1|_{1,\hat{K}}^2 = \sum_{i=1}^{3} ||\frac{\partial \hat{v}^1}{\partial \hat{x}_i}||_{0,\hat{K}}^2 = \sum_{i=1}^{3} g_i^2(\hat{v}) \int_{\hat{K}} \phi'(\hat{x}_i)^2 d\hat{x} = \frac{1}{56}\sum_{i=1}^{3} g_i^2(\hat{v})$$

Hence

$$||\hat{v}||_{0,\hat{K}} = \sqrt{\frac{2}{45}} |\hat{v}|_{1,\hat{K}} \leq |\hat{v}|_{1,\hat{K}}$$

iii)

$$||\frac{\partial \hat{v}}{\partial \hat{x}_i}||_{0,\hat{K}}^2 = ||\frac{\partial \hat{v}^0}{\partial \hat{x}_i}||_{0,\hat{K}}^2 + ||\frac{\partial \hat{v}^1}{\partial \hat{x}_i}||_{0,\hat{K}}^2 + 2\int_{\hat{K}} \frac{\partial \hat{v}^0}{\partial \hat{x}_i}\frac{\partial \hat{v}^1}{\partial \hat{x}_i} d\hat{x}$$

$\frac{\partial \hat{v}^0}{\partial \hat{x}_i}$ is a constant for $\hat{x}_i$, $\frac{\partial \hat{v}^1}{\partial \hat{x}_i} = g_i(\hat{v})\phi'(\hat{x_i})$, hence, $\int_{\hat{K}} \frac{\partial \hat{v}^0}{\partial \hat{x}_i}\frac{\partial \hat{v}^1}{\partial \hat{x}_i} d\hat{x} = 0$ by (4.7), and (4.5) holds.

iv)

$$|v_K^1|_{1,K}^2 = \sum_{|\alpha|=1} ||D^\alpha v_K^1||_{0,K}^2 \overset{(3.19)(3.18)}{\leq} \hat{c} \sum_{|\beta|=1} Hh^{-2\beta} ||\hat{D}^\beta \hat{v}^1||_{0,\hat{K}}^2$$

$$\overset{(4.5)}{\leq} \hat{c} \sum_{|\beta|=1} Hh^{-2\beta} ||\hat{D}^\beta \hat{v}||_{0,\hat{K}}^2 \leq \hat{c} \sum_{|\beta|=1} ||D^\beta v_K||_{0,K}^2 = \hat{c}|v_K|_{1,K}^2 \tag{4.9}$$

In the similar way

$$||v_K^1||_{0,K} \overset{(3.18)}{\leq} \hat{c}H^{\frac{1}{2}} ||\hat{v}^1||_{0,\hat{K}} \overset{(4.4)}{\leq} \hat{c}H^{\frac{1}{2}} |\hat{v}^1|_{1,\hat{K}}$$

$$\overset{(3.19)}{\leq} \hat{c} \sum_{|\beta|=1} h^\beta ||D^\beta v_K^1||_{0,K}$$

$$\leq \hat{c}h_K |v_K^1|_{1,K} \overset{(4.9)}{\leq} \hat{c}h_K |v_K|_{1,K}$$

since $w_h^0$ is the conforming part of $w_h$,

$$|a_h(u, w_h) - f(w_h)| = |a_h(u, w_h^1) - f(w_h^1)| \leq |a_h u, w_h^1)| + |f(w_h^1)| \tag{4.10}$$

$$\forall \alpha \in R^3, |a_h(u, w_h^1)| = \sum_{K \in T_n} \int_K \nabla u . \nabla w_h^1 dx \overset{(4.3)}{=} \sum_{k \in T_h} \int_K (\nabla u - \alpha) . \nabla w_h^1 dx$$

Hence

$$|a_h(u,w_h^1)| \leq \sum_{k\in T_n} inf_{\alpha\in R^3}||\nabla u-\alpha||_{0,K}||\nabla w_h^1||_{0,K}$$

$$\stackrel{(3.18)}{\leq} \hat{c}\sum_{K\in T_n} inf_{\alpha\in R^3} H^{\frac{1}{2}}||\hat{\nabla} u-\alpha||_{0,\hat{K}}|w_h|_{1,K} \stackrel{Lemma3.1}{\leq} \hat{c}\sum_{K\in T_n} H^{\frac{1}{2}}|\hat{\nabla} u|_{1,\hat{K}}|w_h|_{1,K}$$

$$= \hat{c}\sum_{K\in T_n} H^{\frac{1}{2}} \sum_{|\beta|=1} ||\hat{D}^\beta \hat{\nabla} u||_{0,\hat{K}}|w_h|_{1,K} \leq \hat{c}\sum_{K\in T_n} H^{\frac{1}{2}} \sum_{|\beta|=1} ||\hat{D}^\beta \hat{\nabla} u||_{0,p,\hat{K}}|w_h|_{1,K}$$

$$\stackrel{(3.19)(3.18)}{\leq} \hat{c}\sum_{K\in T_n} H^{\frac{1}{2}-\frac{1}{p}} \sum_{|\beta|=1} h^\beta||D^\beta \nabla u||_{0,p,K}|w_h|_{1,K}$$

$$\leq \hat{c}\sum_{K\in T_n} H^{\frac{1}{2}-\frac{2}{p}} \sum_{|\beta|=1} h^\beta|D^\beta u|_{1,p,K}|w_h|_{1,K} \leq \hat{c}(\sum_{K\in T_n} H^{1-\frac{2}{p}} \sum_{|\beta|=1} h^{2\beta}|D^\beta u|^2_{1,p,K})^{\frac{1}{2}}|w_h|_{1,h} \quad (4.11)$$

$$|f(w_h^1)| \leq \sum_{K\in T_h} ||f||_{0,k}||w_h^1||_{0,k} \stackrel{(4.6)}{\leq} \sum_{k\in T_h} \hat{c}h_k||f||_{0,k}|w_h|_{1,k}$$

$$\leq \hat{c}(\sum_{K\in T_h} h_K^2||f||^2_{0,K})^{\frac{1}{2}}|w_h|_{1,h} \quad (4.12)$$

From (4.10)(4.11)(4.12) we obtain

$$sup_{w_h\in V_h}\frac{|a_h(u,w_h)-f(w_h)|}{|w_h|_{1,h}} \leq \hat{c}[\sum_{K\in T_n} H^{1-\frac{2}{p}} \sum_{|\beta|=1} h^{2\beta}|D^\beta u|^2_{1,p,K}) + h_K^2||f||^2_{0,K}]^{\frac{1}{2}} \quad (4.13)$$

From (2.11)(4.1)(4.13) we have

Theorem 4.1 suppose $u$ and $u_h$ are the solution of (2.7) and (2.9), then

$$|u-u_h|_{1,h} \leq \hat{c}[\sum_{K\in T_n} (H^{1-\frac{2}{p}} \sum_{|\beta|=1} h^{2\beta}|D^\beta u|^2_{1,p,K} + h_K^2||f||^2_{0,K})]^{\frac{1}{2}} \quad (4.14)$$

where $\hat{c}$ is independent of $h_K/\rho_K, \forall K \in T_h$.

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