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FROM ENERGY IMPROVEMENT TO ACCURACY ENHANCEMENT: IMPROVEMENT OF PLATE BENDING ELEMENTS BY THE COMBINED HYBRID METHOD *1)

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Abstract

By following the geometric point of view in mechanics, a novel expression of the combined hybrid method for plate bending problems is introduced to clarify its intrinsic mechanism of enhancing coarse-mesh accuracy of conforming or nonconforming plate elements. By adjusting the combination parameter $\alpha \in (0,1)$ and adopting appropriate bending moments modes, reduction of energy error for the discretized displacement model leads to enhanced numerical accuracy. As an application, improvement of Adini's rectangle is discussed. Numerical experiments show that the combined hybrid counterpart of Adini's element is capable of attaining high accuracy at coarse meshes.

Mathematics subject classification: 65N12, 65N30.
Key words: Finite element, Combined hybrid, Energy error.

1. Introduction

The combined hybrid finite element method [6,7,8,9] is capable of remarkably enhancing coarse-mesh accuracy of conventional lower order elements for linear elasticity problems. The 4-node plane quadrilateral CH(0-1) proposed in [9] is a successful example.

By following the geometric point of view in mechanics, a novel expression of the combined hybrid method was introduced in [10] to clarify its intrinsic mechanism of enhancing coarse-mesh accuracy and stability of lower order displacement schemes for linear elasticity problems. For a fixed coarse mesh and a given stress mode, e.g. the piecewise constant stress mode, one can adjust the energy of the finite element model such that the energy error reduces to zero by optimizing the combined parameter α and by adding energy compatible bubble displacements to the given conforming displacements. It was shown by numerical experiments that the smaller the energy error is, the higher numerical accuracy will be, and that combined hybrid schemes without energy error are of high accuracy at coarse meshes. This accuracy criterion of schemes at coarse meshes is different from the gradual convergence of the $h-$version and the $p-$version, i.e. it does not require the mesh size h being smaller or the degree p of elements being bigger for the combined hybrid method to achieve higher accuracy.

In the reference [11], the combined hybrid finite element method was applied to 4th-order plate bending problems. It was shown that the resultant schemes are stabilized, i.e., the convergence of the schemes is independent of inf-sup conditions and any other patch test. Then the deflection interpolant and the bending moments approximation can be chosen independently, which provides possibility of optimizing bending moments modes so as to obtain accurate plate elements.

Based on [11], the present paper is devoted to a further analysis of the mechanism of enhancing coarse-mesh accuracy of conventional plate elements of the combined hybrid method.

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By adopting rational bending moments modes and adjusting the combination parameter $\alpha \in (0,1)$, energy error of the discretized scheme can be reduced, and then an enhanced numerical accuracy at coarse meshes can be acquired. As an application, improvement of Adini's rectangle is discussed and numerical experiments show that the combined hybrid counterpart of Adini's element is capable of attaining high coarse-mesh accuracy.

In what follows the letter C will represent a constant which is independent of the mesh size $h = \max_K \{h_K\}$ and may be different at its each occurrence.

2. Combined Hybrid Variational Principle

Considering the following plate bending problem:

$$\left\{ \begin{array}{ll} \mathbf{divdiv}\sigma = f, & in\ \Omega, \\ \sigma = \mathbf{m}(\mathbf{D}_2 u), & in\ \Omega, \\ u = \nabla u \cdot \mathbf{n} = 0, & on\ \Gamma = \partial\Omega. \end{array} \right. \tag{2.1}$$

where $\Omega \subset \Re^2$ is a bounded open set, u represents vertical deflection, σ the bending moments, and $\mathbf{n}$ the outer normal unit vector along Γ. The operators $\mathbf{divdiv}, \mathbf{D}_2$ and $\mathbf{m}$ are defined respectively as follows:

$$\mathbf{divdiv}\tau = \partial_{11}\tau_{11} + 2\partial_{12}\tau_{12} + \partial_{22}\tau_{22},$$

$$\mathbf{D}_2 v = \begin{pmatrix} \partial_{11} v & \partial_{12} v \\ \partial_{12} v & \partial_{22} v \end{pmatrix},$$

$$\mathbf{m}(\tau) = \begin{pmatrix} \tau_{11} + \nu\tau_{22} & (1-\nu)\tau_{12} \\ (1-\nu)\tau_{12} & \nu\tau_{11} + \tau_{22} \end{pmatrix}$$

for any symmetric tensor $\tau = (\tau_{ij}), i,j = 1,2$, and $\nu \in (0, 0.5)$ denotes the Poisson's coefficient, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j},\ i,j = 1,2.$

As shown in the reference [11], the combined hybrid variational principle equivalent to the problem (2.1) reads as:

$$\inf_{(v,v_c)\in U\times U_c} \sup_{\tau\in\mathbf{V}} \{\tfrac{1-\alpha}{2} d(v,v) - f(v) - b_1(\tau, v - v_c) + \alpha[b_2(\tau,v) - \tfrac{1}{2}a(\tau,\tau)]\} \tag{2.2}$$

where

$$U := \{v \in \prod_{K\in T_h} H^2(K); u = \nabla u \cdot \mathbf{n} = 0, on\ \Gamma\},$$

$$\mathbf{V} := \prod_{K\in T_h} H(\mathbf{divdiv}; K) = \prod_{K\in T_h} \{\tau \in (L^2(K))^4_s; \mathbf{divdiv}\tau \in L^2(K)\}$$

and

$$U_c := H^2_0(\Omega) / \prod_{K\in T_h} H^2_0(K)$$

are respectively the deflection space, the symmetric bending moments vector space and the interelemental boundary deflection space, $T_h = \{K\}$ denotes a regular subdivision of Ω, with mesh diameter h_K for any $K \in T_h$, $(L^2(K))^4_s$ the space of square integrable 2×2 symmetric tensors, and

$a(\sigma,\tau)=\int_\Omega \mathbf{m}^{-1}(\sigma):\tau d\mathbf{x},$
$b_1(\tau,v-v_c)=\sum\oint_{\partial K}[M_{nn}(\tau)\nabla(v-v_c)\cdot\mathbf{n}+M_{ns}(\tau)\nabla(v-v_c)\cdot\mathbf{s}-Q_n(\tau)(v-v_c)]ds,$
$b_2(\tau,v)=\sum\int_K \tau:\mathbf{D}_2 v d\mathbf{x},$
$d(u,v)=\sum\int_K \mathbf{m}(\mathbf{D}_2 u):\mathbf{D}_2 v d\mathbf{x},$
$f(v)=\int_\Omega f v d\mathbf{x},$
$M_{nn}(\tau)=(\tau\mathbf{n})\cdot\mathbf{n},\ M_{ns}(\tau)=(\tau\mathbf{n})\cdot\mathbf{s},\ Q_n(\tau)=\nabla(tr(\tau))\cdot\mathbf{n},$
$\mathbf{n}=$ *unit outer normal vector along* ∂K,
$\mathbf{s}=$ *unit tangent vector along* ∂K.

According to optimality conditions of saddle point problems, the combined hybrid variational principle (2.2)is equivalent to:

Find $(\sigma,u,u_c)\in\mathbf{V}\times U\times U_c$ such that

$$\alpha a(\sigma,\tau)-\alpha\, b_2(\tau,u)+b_1(\tau,u-u_c)=0,\qquad \forall\tau\in\mathbf{V} \tag{2.3}$$

$$\alpha b_2(\sigma,v)-b_1(\sigma,v-v_c)+(1-\alpha)d(u,v)=f(v),\quad \forall(v,v_c)\in U\times U_c \tag{2.4}$$

where the combination parameter $\alpha\in(0,1)$.

To discuss finite element discretizations of the problem (2.2) or its equivalent problem (2.3)(2.4), the weakly compatible finite dimensional deflection subspace $U^h\subset U$. is introduced (see [11]), i.e. U^h satisfies:

(D1) $\forall v\in U^h, d(v,v)=0$ implies $v=0$;

(D2) A coupling linear mapping $T_c: v\in U^h\to v_c=T_c(v)\in U_c$ can be determined by the nodal parameters of $v\in U^h$, i.e. $v\in U^h$ has a corresponding elemental boundary conforming component $T_c(v)\in U_c^h$.

As pointed out in [11], all the conventional plate elements with C^1-continuous vertices are weakly compatible. In fact, the weakly compatible subspace U^h is of either one of the following two characteristics:

C1) The set of nodal parameters of $v\in U^h$ on each side K' of element K (a triangle or a quadrilateral) is

$$\coprod_{K'}(v)=\{v(Q_i),\partial_1 v(Q_i),\partial_2 v(Q_i),i=1,2\},$$

where Q_1 and Q_2 are the endpoints of K'. And then T_c can be constructed as

$$\forall v\in U^h, T_c(v)|_{K'}\in P_3(K'),\nabla T_c(v)\cdot\mathbf{n}|_{K'}\in P_1(K') \tag{2.5}$$

such that for $i=1,2$,

$$T_c(v)(Q_i)=v(Q_i),\quad \nabla T_c(v)(Q_i)\cdot\mathbf{s}=\nabla v(Q_i)\cdot\mathbf{s}, \tag{2.6}$$

$$\nabla T_c(v)(Q_i)\cdot\mathbf{n}=\nabla v(Q_i)\cdot\mathbf{n}, \tag{2.7}$$

where $P_t(K)$ denote the set of polynomials of degree $\le t$ for an integer $t\ge 0$;

C2) The set of nodal parameters of $v\in U^h$ on each side K' of element K is

$$\sum_{K'}=\{v(Q_i),\partial_1 v(Q_i),\partial_2 v(Q_i),i=1,2;\nabla v(Q_3)\cdot\mathbf{n}\},$$

where $Q_3 = Q_{12}$ is the midpoint of K'. Then T_c can be constructed as

$$\forall v \in U^h, T_c(v)|_{K'} \in P_3(K'), \nabla T_c(v) \cdot \mathbf{n}|_{K'} \in P_2(K') \tag{2.8}$$

such that for $i = 1, 2$,

$$T_c(v)(Q_i) = v(Q_i), \quad \nabla T_c(v)(Q_i) \cdot \mathbf{s} = \nabla v(Q_i) \cdot \mathbf{s}, \tag{2.9}$$

and for $i = 1, 2, 3$,

$$\nabla T_c(v)(Q_i) \cdot \mathbf{n} = \nabla v(Q_i) \cdot \mathbf{n}. \tag{2.10}$$

Let $\mathbf{V}^h \subset \mathbf{V}$ be a finite dimensional subspace of piecewise-independent bending moments approximation.

By virtue of the coupling operator T_c, we take $U_c^h = T_c(U^h)$ as the approximation of the interelemental boundary deflection subspace U_c so that the three variables in the continuous problem (2.3)(2.4) will reduce to two in the discretized problem.

The subspaces $\mathbf{V}^h \subset \mathbf{V}$ and $U^h \subset U$ are equipped with the following norms:

$$||\tau||_{\mathbf{V}} := [\int_\Omega \mathbf{m}^{-1}(\tau) : \tau d\mathbf{x} + \sum_K h_K^4 |divdiv\tau|_{0,K}^2]^{\frac{1}{2}}, \ \ \forall \tau \in \mathbf{V}^h,$$

$$||v||_U := (\sum \int_K \mathbf{m}(\mathbf{D}_2 v) : \mathbf{D}_2 v)^{\frac{1}{2}}, \ \ \forall v \in U^h.$$

The problem (2.3)(2.4) is discretized as:

Find $(\sigma_h, u_h) \in \mathbf{V}^h \times U^h$ such that

$$\alpha a(\sigma_h, \tau) - \alpha b_2(\tau, u_h) + b_1(\tau, u_h - T_c(u_h)) = 0, \qquad \forall \tau \in \mathbf{V}^h \tag{2.11}$$

$$\alpha b_2(\sigma_h, v) - b_1(\sigma_h, v - T_c(v)) + (1-\alpha) d(u_h, v) = f(v), \quad \forall v \in U^h. \tag{2.12}$$

And the corresponding discretized variational problem is

$$\Pi_{CH}(\sigma_h, u_h; \alpha) = \inf_{v \in U^h} \sup_{\tau \in \mathbf{V}^h} \Pi_{CH}(\tau, v; \alpha) \tag{2.13}$$

where the energy functional

$$\Pi_{CH}(\tau, v; \alpha) := \frac{1-\alpha}{2} d(v, v) - f(v) - b_1(\tau, v - T_c(v)) + \alpha[b_2(\tau, v) - \frac{1}{2} a(\tau, \tau)].$$

Remark 2.1. If the weakly compatible space $U^h \subset C^0(\bar{\Omega})$, then by the construction of T_c one has $T_c(v)|_{\partial K} = v|_{\partial K}$, and the following relation holds:

$$b_1(\tau, v - T_c(v)) = \sum \oint_{\partial K} M_{nn}(\tau) \nabla (v - T_c(v)) \cdot \mathbf{n} ds, \ \forall (\tau, v) \in \mathbf{V}^h \times U^h. \tag{2.14}$$

Moreover, if $U^h \subset C^1(\bar{\Omega})$, then

$$b_1(\tau, v - T_c(v)) = 0, \ \forall (\tau, v) \in \mathbf{V}^h \times U^h. \tag{2.15}$$

Remark 2.2. Since $\tau \in \mathbf{V}^h$ is piecewise independent, the parameters of bending moments can then be eliminated at elemental level. The derivation of element stiffness matrix of the combined hybrid method (2.11)(2.12) is in a similar way to [9, Appendix].

Remark 2.3. For the extreme case $\alpha = 1$, the combined hybrid scheme (2.11)(2.12) reduces to a hybrid scheme, i.e.:

Find $(\sigma_h, u_h) \in \mathbf{V}^h \times U^h$ such that

$$a(\sigma_h, \tau) - b_2(\tau, u_h) + b_1(\tau, u_h - T_c(u_h)) = 0, \quad \forall \tau \in \mathbf{V}^h \tag{2.16}$$

$$b_2(\sigma_h, v) - b_1(\sigma_h, v - T_c(v)) = f(v), \qquad \forall v \in U^h. \tag{2.17}$$

And the following inf-sup condition is required for the finite dimensional subspace $\mathbf{V}^h$ and U^h:

$$\sup_{\tau \in \mathbf{V}^h} \frac{b_2(\tau, v) - b_1(\tau, v - T_c(v))}{||\tau||_V} \geq C||v||_U, \quad \forall v \in U^h. \tag{2.18}$$

For the combined hybrid scheme (2.11)(2.12) it is not necessary to impose any inf-sup condition a priori on finite element subspace $\mathbf{V}^h \times U^h$. In fact there holds the following gradual convergence theorem (see [11]):

Lemma 2.1. *Assume that (σ, u) is the exact solution to the problem (2.1). Then for $\forall \alpha \in (0,1)$ and for an arbitrary $\mathbf{V}^h$ which contains at least piecewise-constant bending moments mode, i.e.*

$$\mathbf{V}^h \supset \mathbf{V}_0^h := \{\tau \in \mathbf{V}; \tau_{ij}|_K = const., \tau_{ij} = \tau_{ji}, i, j = 1, 2, \forall K \in T_h\},$$

there exists a unique combined hybrid finite element solution $(\sigma_h, u_h) \in \mathbf{V}^h \times U^h$ to the problem (2.11)(2.12) such that

$$\begin{aligned} &||\sigma - \sigma_h||_{0,\Omega} + ||u - u_h||_U \\ &\leq C\{\inf_{\tau \in \mathbf{V}^h} ||\sigma - \tau||_{\mathbf{V}} + \inf_{v \in U_h} [||u - v||_U + \sup_{\tau \in \mathbf{V}^h} \tfrac{b_1(\tau, v - T_c(v))}{||\tau||_{\mathbf{V}}}]\}. \end{aligned} \tag{2.19}$$

The Lemma 2.1 provides general reliability with the gradual convergence error estimates at finer meshes. In the following section we will clarify the mechanism of enhancing coarse-mesh accuracy of the combined hybrid method (2.11)(2.12) and discuss how to improve accuracy of conventional plate elements at coarse meshes.

3. Mechanism of Enhancing Coarse-mesh Accuracy for Plate Elements

Let $u_1^h \in U^h$ be the finite element solution of the conforming or nonconforming displacement model, i.e.

$$\Pi_P(u_1^h) = \inf_{v \in U^h} \Pi_P(v) \tag{3.1}$$

where $\Pi_P(v) := \frac{1}{2}d(v, v) - f(v)$ is the potential energy functional. Then the variational energy functional form (2.13) can be rewritten as:

$$\begin{aligned} &\Pi_{CH}(\sigma_h, u_h; \alpha) = \inf_{v \in U^h} \sup_{\tau \in \mathbf{V}^h} \Pi_{CH}(\tau, v; \alpha) \\ &= \inf_{v \in U^h} \{\Pi_P(v) + \sup_{\tau \in \mathbf{V}^h} [-\tfrac{\alpha}{2}d(v, v) + \alpha b_2(\tau, v) - \tfrac{\alpha}{2}a(\tau, \tau) - b_1(\tau, v - T_c(v))]\} \\ &= \inf_{v \in U^h} \{\Pi_P(v) - \inf_{\tau \in \mathbf{V}^h} [\tfrac{\alpha}{2}a(\tau - \mathbf{m}(\mathbf{D}_2 v), \tau - \mathbf{m}(\mathbf{D}_2 v)) + b_1(\tau, v - T_c(v))]\}. \end{aligned} \tag{3.2}$$

From this novel expression of the combined hybrid energy functional, we easily obtain the following two conclusions:

Lemma 3.1. *Let α and U^h be given. Then a bending moments subspace $\bar{\mathbf{V}}^h$ larger than $\mathbf{V}^h$ leads to a bigger energy, i.e. $\bar{\mathbf{V}}^h \supset \mathbf{V}^h$ implies*

$$\Pi_{CH}(\bar{\sigma}_h, \bar{u}_h; \alpha) = \inf_{v \in U^h} \sup_{\tau \in \bar{\mathbf{V}}^h} \Pi_{CH}(\tau, v; \alpha) > \inf_{v \in U^h} \sup_{\tau \in \mathbf{V}^h} \Pi_{CH}(\tau, v; \alpha) = \Pi_{CH}(\sigma_h, u_h; \alpha).$$

Lemma 3.2. *Let U^h and $\mathbf{V}^h$ be given. Then the energy function $\Pi_{CH}(\sigma_h, u_h; \alpha)$ is monotone decreasing with respect to α, i.e. $\bar{\alpha} > \alpha$ implies*

$$\Pi_{CH}(\sigma_h, u_h; \bar{\alpha}) = \inf_{v \in U^h} \sup_{\tau \in \mathbf{V}^h} \Pi_{CH}(\tau, v; \bar{\alpha}) < \inf_{v \in U^h} \sup_{\tau \in \mathbf{V}^h} \Pi_{CH}(\tau, v; \alpha) = \Pi_{CH}(\sigma_h, u_h; \alpha).$$

For the weakly compatible subspace U^h, there holds one of the following three cases:
Case 1: $U^h \subset C^1(\bar{\Omega})$, which indicates that the corresponding displacement scheme (3.1) is conforming, and that the potential energy relation

$$\Pi_P(u_1^h) > \Pi_P(u) = \inf_{v \in H_0^2(\Omega)} \Pi_P(v) \tag{3.3}$$

holds, where $u \in H_0^2(\Omega)$ is the exact deflection of the plate bending problem (2.1) and $\Pi_P(u)$ is the exact potential energy value;
Case 2: $U^h \not\subset C^1(\bar{\Omega})$ together with the potential energy relation (3.3) holds;
Case 3: $U^h \not\subset C^1(\bar{\Omega})$ together with the relation

$$\Pi_P(u_1^h) < \Pi_P(u) \tag{3.4}$$

holds.

As pointed out in [10], the combined hybrid method is of an intrinsic mechanism of enhancing energy-accuracy of conforming and nonconforming displacement elements at coarse meshes for linear elasticity problems. Numerical experiments in [6,10] and in Section 5 below also show that higher energy-accuracy schemes always enjoy higher numerical accuracy. As far as the fourth-order problem (2.1) is concerned, there holds the following proposition:

Proposition 3.1. *By two ways: adopting rational bending moments mode $\mathbf{V}^h$ and choosing appropriate combination parameter α, the conventional displacement scheme (3.1) can be improved by the combined hybrid method (2.11)(2.12) in a sense that the energy-error inequality*

$$|\Pi_{CH}(\sigma_h, u_h; \alpha) - \Pi_P(u)| < |\Pi_P(u_1^h) - \Pi_P(u)| \tag{3.5}$$

holds at coarse meshes, i.e. the corresponding combined hybrid scheme is of higher energy-accuracy than the displacement scheme (3.1).

Proof. For Case 1, the relation $U^h \subset C^1(\bar{\Omega})$, as pointed out in Remark 2.1, implies the equality (2.15), and then (3.2) reduces to

$$\Pi_{CH}(\sigma_h, u_h; \alpha) = \inf_{v \in U^h} [\Pi_P(v) - \inf_{\tau \in \mathbf{V}^h} \frac{\alpha}{2} a(\tau - \mathbf{m}(\mathbf{D}_2 v), \tau - \mathbf{m}(\mathbf{D}_2 v))]. \tag{3.6}$$

If $\mathbf{V}^h$ is such that $\mathbf{V}^h|_K \not\supset \mathbf{m}(\mathbf{D}_2(U^h|_K)$, the inequality $a(\tau, \tau) > 0$ will yield

$$\Pi_{CH}(\sigma_h, u_h; \alpha) < \Pi_{CH}(\sigma_h, u_h; 0) = \Pi_P(u_1^h). \tag{3.7}$$

This inequality implies that the potential energy of the conforming model decreases by subtracting the term $\inf_{\tau \in \mathbf{V}^h} \frac{\alpha}{2} a(\tau - \mathbf{m}(\mathbf{D}_2 v), \tau - \mathbf{m}(\mathbf{D}_2 v))$.

Thus by virtue of the continuity of the energy $\Pi_{CH}(\sigma_h, u_h; \alpha)$ with respect to α, we know that by adjusting the combination parameter α one can obtain (3.5). Further more, if we take the piecewise-constant mode as the approximation of bending moments, i.e. $\mathbf{V}^h = \mathbf{V}_0^h$, we can conclude that there exists $\alpha^* \in (0, 1)$ such that

$$\Pi_{CH}(\sigma_h, u_h; \alpha^*) = \Pi_P(u), \tag{3.8}$$

that is to say, the scheme (3.6) with $\alpha = \alpha^*$ is of zero energy-error. In fact, when $\alpha = 1$, the hybrid scheme (2.16)(2.17) is divergent due to not satisfying the inf-sup condition (2.18), then the energy $\Pi_{CH}(\sigma_h, u_h; \alpha)$ will far exceeding the exact energy $\Pi_P(u)$ when α approaches to 1, i.e. there exists $\alpha_0 \in (0.5, 1)$ such that

$$\Pi_{CH}(\sigma_h, u_h; \alpha_0)| < \Pi_P(u) < \Pi_P(u_1^h). \tag{3.9}$$

Note that when $\alpha = 0$,

$$\Pi_{CH}(\sigma_h, u_h; 0) = \Pi_P(u_1^h),$$

then by the continuity of the energy $\Pi_{CH}(\sigma_h, u_h; \alpha)$ with respect to α, there exists $\alpha^* \in (0, \alpha_0)$ such that (3.8) holds.

For Case 2, one can construct $\mathbf{V}^h$ such that (2.15) holds. Thus by an argument similar to Case 1 the energy relation (3.5) can be obtained with an appropriate combination parameter α. In fact, the piecewise-constant-bending-moments mode $\mathbf{V}^h = \mathbf{V}_0^h$ can also do if

$$a(\tau - \mathbf{m}(\mathbf{D}_2 v), \tau - \mathbf{m}(\mathbf{D}_2 v)) \gg |b_1(\tau, v - T_c(v))|, \;\; \forall(\tau, v) \in \mathbf{V}_0^h \times U^h. \tag{3.10}$$

For Case 3, one can use a bending moments approximation subspace $\mathbf{V}^h$ large enough such that

$$\inf_{\tau \in \mathbf{V}^h} [\frac{\alpha}{2} a(\tau - \mathbf{m}(\mathbf{D}_2 v), \tau - \mathbf{m}(\mathbf{D}_2 v)) + b_1(\tau, v - T_c(v))] < 0. \tag{3.11}$$

Thus by (3.2) there holds

$$\Pi_{CH}(\sigma_h, u_h; \alpha) > \Pi_P(u_1^h),$$

and (3.5) can also be attained by adjusting α. Note that if only $\mathbf{V}^h$ is large enough, the inequality (3.11) will always hold.

Remark 3.1. Adini's rectangular element falls into Case 3, as will be discussed in next section.

From the proof of Proposition 3.1 we easily get the following corollary:

Corollary 3.1. *Assume that $U^h \subset C^1(\bar{\Omega})$ and $\mathbf{V}^h = \mathbf{V}_0^h$. Then there exists a parameter $\alpha \in (0, 1)$ such that the combined hybrid scheme (2.11)(2.12) is of zero energy-error, i.e. the energy relation*

$$\Pi_{CH}(\sigma_h, u_h; \alpha) = \Pi_P(u)$$

holds.

4. Application: Improvement of Adini's Rectangular Element

As an application of the mechanism of enhancing coarse-mesh accuracy of the combined hybrid method, this section is devoted to improvement of Adini's rectangular element. For the sake of simplicity, we assume that Ω is a polygonal domain and $K \in T_h$ is an arbitrary rectangle with vertices $Q_i(x_i, y_i), i = 1, 2, 3, 4$, central point (x_0, y_0), and side lengthes h_x and h_y.

The deflection subspace of Adini's C^0- interpolants is defined as

$$U_A^h := \{v \in U \bigcap C^0(\bar{\Omega}); v|_K \in P_3(K) \bigoplus \bigvee\{xy^3, x^3y\}, \forall K \in T_h\}$$

with the nodal parameters set of $v \in U^h$ on K

$$\coprod\nolimits_K^A(v) = \{v(Q_i), \partial_1 v(Q_i), \partial_2 v(Q_i), i = 1, 2, 3, 4\}.$$

Consider the piecewise-incomplete-quadratic mode $\mathbf{V}_{0-2}^h$ defined by: $\forall \tau \in \mathbf{V}_{0-2}^h$,

$$\tau|_K = \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \eta & 0 & 0 & 0 & \xi^2 & 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & \xi & 0 & 0 & 0 & \xi^2 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \eta & \xi & 0 & 0 & \xi^2 & 0 & 0 & \eta^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{13} \end{pmatrix}, \tag{4.1}$$

where $\beta = (\beta_i) \in \Re^{13}$ are the parameters of bending moments, ξ and η are the local co-ordinates defined as:

$$\xi = (x - x_0)/h_x, \quad \eta = (y - y_0)/h_y$$

for $\forall (x, y) \in K$.

By (2.5), (2.6), (2.7) and (2.14), there holds

Lemma 4.1. *For* $\forall (\tau, v) \in \mathbf{V}_{0-2}^h \times U_A^h$,

$$b_1(\tau, v - T_c(v)) = 0. \tag{4.2}$$

Proof. Let us argue on the referential square $\hat{K} = [-1, 1]^2$ with four vertices

$$\hat{Q}_1 = (\xi_1, \eta_1) = (-1, -1), \hat{Q}_2 = (\xi_2, \eta_2) = (1, -1),$$

$$\hat{Q}_3 = (\xi_3, \eta_3) = (1, 1), \hat{Q}_4 = (\xi_4, \eta_4) = (-1, 1).$$

Then for $\forall \hat{v} \in P_{\hat{K}} = P_3(\hat{K}) \bigoplus \bigvee \{\xi\eta^3, \xi^3\eta\}$, we can write

$$\hat{v} = \sum_{i=1}^{4} (\hat{v}(i) p_i + \hat{v}_\xi(i)\phi_i + \hat{v}_\eta(i)\psi_i$$

with

$$\begin{cases} p_i = \frac{(1+\xi_i\xi)(1+\eta_i\eta)}{4}(1 + \frac{\xi_i\xi + \eta_i\eta}{2} - \frac{\xi^2+\eta^2}{2}), \\ \phi_i = -\frac{(1+\eta_i\eta)(1+\xi_i\xi)^2(1-\xi_i\xi)}{8}\xi_i, \\ \psi_i = -\frac{(1+\xi_i\xi)(1+\eta_i\eta)^2(1-\eta_i\eta)}{8}\eta_i, \end{cases}$$

where $\hat{v}(i), \hat{v}_\xi(i), \hat{v}_\eta(i)$ $(i = 1, 2, 3, 4)$ denote the twelve degrees of freedom of $\hat{K}$.

In what follows we will prove that for $\forall \tau \in \mathbf{V}_{0-2}^h$ and $\hat{v}$,

$$\begin{aligned} &\oint_{\partial \hat{K}} M_{nn}(\tau)(\nabla \hat{v} - \nabla T_c(\hat{v})) \cdot \mathbf{n} ds \\ = &\int_{e_{12}+e_{34}} \tau_{22}(\nabla \hat{v} - \nabla T_c(\hat{v})) \cdot \mathbf{n} ds + \int_{e_{23}+e_{41}} \tau_{11}(\nabla \hat{v} - \nabla T_c(\hat{v})) \cdot \mathbf{n} ds \\ := &(I) + (II) = 0, \end{aligned}$$

where e_{ij} denote the edge of rectangle $\hat{K}$ with endpoints $\hat{Q}_i$ and $\hat{Q}_j$.

In fact, for the term (I), some trivial calculations show that

$$\nabla T_c(\hat{v}) \cdot \mathbf{n}|_{\eta=\pm 1} = \sum_{i=1..4} \hat{v}_\eta(i) \frac{\partial \psi_i}{\partial \eta}|_{\eta=\pm 1},$$

$$\frac{\partial p_1}{\partial \eta}|_{\eta=\pm 1} = \frac{(1-\xi)\xi(1+\xi)}{8} = \frac{\partial p_3}{\partial \eta}|_{\eta=\pm 1},$$

$$\frac{\partial p_2}{\partial \eta}|_{\eta=\pm 1} = -\frac{(1-\xi)\xi(1+\xi)}{8} = \frac{\partial p_4}{\partial \eta}|_{\eta=\pm 1},$$

$$\frac{\partial \phi_1}{\partial \eta}|_{\eta=\pm 1} = -\frac{(1-\xi)^2(1+\xi)}{8} = -\frac{\partial \phi_4}{\partial \eta}|_{\eta=\pm 1},$$

$$\frac{\partial \phi_2}{\partial \eta}|_{\eta=\pm 1} = \frac{(1+\xi)^2(1-\xi)}{8} = -\frac{\partial \phi_3}{\partial \eta}|_{\eta=\pm 1}.$$

Then we have

$$(I) = \int_{e_{12}} -\tau_{22} \sum(\hat{v}(i)\tfrac{\partial p_i}{\partial \eta} + \hat{v}_\xi(i)\tfrac{\partial \phi_i}{\partial \eta})ds + \int_{e_{34}} \tau_{22} \sum(\hat{v}(i)\tfrac{\partial p_i}{\partial \eta} + \hat{v}_\xi(i)\tfrac{\partial \phi_i}{\partial \eta})ds$$
$$= \int_{-1}^{1}[(\tau_{22}|_{\eta=1} - \tau_{22}|_{\eta=-1}) \sum(\hat{v}(i)\tfrac{\partial p_i}{\partial \eta} + \hat{v}_\xi(i)\tfrac{\partial \phi_i}{\partial \eta})|_{\eta=\pm 1}]d\xi \quad = 0.$$

Similarly, we can prove $(II) = 0$.

Thus the relation (4.2) is obtained.

Numerical experiments in Section 5 show that for Adini's displacement element U_A^h, the energy inequality (3.4) holds, i.e.

$$\Pi_{PA} := \Pi_P(u_1^h) = \inf_{v \in U_A^h} \Pi_P(v) < \Pi_P(u). \tag{4.3}$$

So according to (3.11) and the relation (4.2), a bending moments subspace larger than $\mathbf{V}_{0-2}^h$ is needed for improving Adini's element U_A^h by the combined hybrid method. To this end we introduce

$$\mathbf{V}_2^h := \{\tau \in \mathbf{V}; \tau_{ij}|_K \in \bigvee\{1, \xi, \eta, \xi^2, \eta^2\}, \tau_{ij} = \tau_{ji}, i,j = 1,2, \forall K \in T_h\}.$$

Take $U^h = U_A^h$ in the problem (2.11)(2.12) or in its equivalent form (2.13), and let $\mathbf{V}^h$ be taken respectively as $\mathbf{V}_0^h$, $\mathbf{V}_{0-2}^h$ and $\mathbf{V}_2^h$, we then get the combined hybrid plate elements $\mathbf{V}_0^h \times \mathbf{U}_A^h$, $\mathbf{V}_{0-2}^h \times \mathbf{U}_A^h$ and $\mathbf{V}_2^h \times \mathbf{U}_A^h$ which are denoted respectively by CHA0(α), CHA1(α) and CHA2(α). These combined hybrid elements are all with 12 parameters on K due to the elimination of the bending moments parameters (see Remark 2.2), as same as Adini's element.

For convenience we also denote the energies of the combined hybrid elements CHA0(α), CHA1(α) and CHA2(α) respectively by $\Pi_{CHA0}(\alpha)$, $\Pi_{CHA1}(\alpha)$ and $\Pi_{CHA2}(\alpha)$.

Since

$$\mathbf{V}_0^h \subset \mathbf{V}_{0-2}^h \subset \mathbf{V}_2^h,$$

by Lemma 3.1 there hold

$$\Pi_{CHA0}(\alpha) < \Pi_{CHA1}(\alpha) < \Pi_{CHA2}(\alpha). \tag{4.4}$$

And by (4.2) and (4.3) there hold

$$\Pi_{CHA1}(\alpha) < \Pi_{PA} < \Pi_P(u). \tag{4.5}$$

The inequalities (4.4) and (4.5) are also confirmed by numerical experiments in Section 5. From Table 1-4 one can see that $\Pi_{CHA2}(0.5) > \Pi_P(u) > \Pi_{PA}$, which indicate by Lemma 3.2 that an appropriate parameter α bigger than 0.5 can lead to a scheme of more accurate energy (see, e.g. CHA2(0.8) and CHA2(0.9) in Table 1-4).

5. Numerical Experiments

Some test problems are now calculated for the case of a thin isotropic square plate of side length $L = 1$ and Poison's ratio $\nu = 0.3$ which is modelled by (4×4), (8×8) and (16×16) finite element meshes respectively.

Two types of boundary conditions are considered: simply-supported boundary conditions and clamped conditions. The applied transverse loading is in the form of a unit uniform load or a unit center concentrated load. Numerical results of energy and central displacement of the square plate are given in Table 1-4.

The numerical results of the combined hybrid elements CHA0(0.5), CHA1(0.5) and CHA2(0.5) show that

$$\Pi_{CHA0}(0.5) < \Pi_{CHA1}(0.5) < \Pi_{PA}$$

which is conformable to (4.4), that

$$\Pi_{PA} < \Pi_P(u) < \Pi_{CHA2}(0.5),$$

and that the energy results of CHA2(0.5) are more accurate than those of Adini's element, so do the results of central displacement.

The numerical results of CHA2(α) with $\alpha = 0.8,\ 0.9$ are also calculated. From Table 1-4 one can see that CHA2(0.8) and CHA2(0.9) are of high accuracy for energy and displacement at the coarse mesh (4×4). Comparisons are also made with the conforming Bogner-Fox-Schmit (BFS) element [3] with 16 parameters, the 12-parameter and 16-parameter rectangular elements proposed by Prof. Shi and his coauthor[5].

Note that the hybrid element CHA2(1) (see Remark 2.3) also gives uniformly good results. In fact, one can test the inf-sup condition (2.18) for $\mathbf{V}_2^h \times U_A^h$ by some trivial calculations.

Table 1. Energy Π and central displacement u of square plate under simply-supported boundary conditions and a unit uniform load

	Elements	4×4	8×8	16×16	Exact
	Adini	-9.053e-4	-8.653e-4	-8.548e-4	
	CHA0(0.5)	-9.619e-4	-8.812e-4	-8.589e-4	
	CHA1(0.5)	-9.360e-4	-8.748e-4	-8.573e-4	
Π	CHA2(0.5)	-8.423e-4	-8.462e-4	-8.499e-4	-8.512e-4
	CHA2(0.8)	-8.586e-4	-8.503e-4	-8.509e-4	
	CHA2(0.9)	-8.628e-4	-8.511e-4	-8.511e-4	
	CHA2(1)	-8.673e-4	-8.518e-4	-8.513e-4	
	Adini	0.004330	0.004129	0.004079	
	CHA0(0.5)	0.004592	0.004195	0.004096	
	CHA1(0.5)	0.004468	0.004165	0.004088	
	CHA2(0.5)	0.003977	0.004037	0.004056	
u	CHA2(0.8)	0.004043	0.004056	0.004061	0.004062
	CHA2(0.9)	0.004054	0.004060	0.004062	
	CHA2(1)	0.004061	0.004063	0.004062	
	12-parameter	0.004052	0.004062	0.004062	
	16-parameter	0.004052	0.004062	0.004062	
	BFS	0.004065	0.004063	0.004062	

Table 2. Energy Π and central displacement u of square plate under simply-supported boundary conditions and a unit center concentrated load

	Elements	4×4	8×8	16×16	Exact
	Adini	-6.166e-3	-5.914e-3	-5.835e-3	
	CHA0(0.5)	-6.957e-3	-6.176e-3	-5.916e-3	
	CHA1(0.5)	-6.759e-3	-6.116e-3	-5.898e-3	
Π	CHA2(0.5)	-5.647e-3	-5.752e-3	-5.786e-3	-5.801e-3
	CHA2(0.8)	-5.778e-3	-5.795e-3	-5.799e-3	
	CHA2(0.9)	-5.815e-3	-5.807e-3	-5.802e-3	
	CHA2(1)	-5.854e-3	-5.819e-3	-5.805e-3	
	Adini	0.01233	0.01183	0.01167	
	CHA0(0.5)	0.01392	0.01235	0.01183	
	CHA1(0.5)	0.01352	0.01223	0.01180	
	CHA2(0.5)	0.01129	0.01150	0.01157	
u	CHA2(0.8)	0.01156	0.01159	0.01160	0.01160
	CHA2(0.9)	0.01163	0.01161	0.01160	
	CHA2(1)	0.01171	0.01164	0.01161	
	12-parameter	0.01136	0.01155	0.01159	
	16-parameter	0.01140	0.01155	0.01159	
	BFS	0.01147	0.01157	0.01159	

Table 3. Energy Π and central displacement u of square plate under clamped boundary conditions and a unit uniform load

	Elements	4×4	8×8	16×16	Exact
Π	Adini	-2.114e-4	-2.002e-4	-1.960e-4	-1.946e-4
	CHA0(0.5)	-2.840e-4	-2.225e-4	-2.020e-4	
	CHA1(0.5)	-2.754e-4	-2.200e-4	-2.013e-4	
	CHA2(0.5)	-1.882e-4	-1.919e-4	-1.939e-4	
	CHA2(0.8)	-1.930e-4	-1.935e-4	-1.943e-4	
	CHA2(0.9)	-1.942e-4	-1.939e-4	-1.944e-4	
	CHA2(1)	-1.952e-4	-1.942e-4	-1.945e-4	
u	Adini	0.001403	0.001304	0.001275	0.001265
	CHA0(0.5)	0.001741	0.001398	0.001299	
	CHA1(0.5)	0.001680	0.001381	0.001295	
	CHA2(0.5)	0.001211	0.001248	0.001261	
	CHA2(0.8)	0.001239	0.001259	0.001264	
	CHA2(0.9)	0.001244	0.001261	0.001265	
	CHA2(1)	0.001248	0.001263	0.001265	
	12-parameter	0.001236	0.001260	0.001265	
	16-parameter	0.001249	0.001263	0.001265	
	BFS	0.001321	0.001272	0.001266	

Table 4. Energy Π and central displacement u of square plate under clamped boundary conditions and a unit center concentrated load

	Elements	4×4	8×8	16×16	Exact
Π	Adini	-3.067e-3	-2.901e-3	-2.836e-3	-2.806e-3
	CHA0(0.5)	-3.944e-3	-3.192e-3	-2.925e-3	
	CHA1(0.5)	-3.789e-3	-3.141e-3	-2.910e-3	
	CHA2(0.5)	-2.679e-3	-2.760e-3	-2.793e-3	
	CHA2(0.8)	-2.774e-3	-2.798e-3	-2.804e-3	
	CHA2(0.9)	-2.802e-3	-2.809e-3	-2.807e-3	
	CHA2(1)	-2.832e-3	-2.820e-3	-2.810e-3	
u	Adini	0.006135	0.005803	0.005672	0.005612
	CHA0(0.5)	0.007887	0.006384	0.005851	
	CHA1(0.5)	0.007577	0.006282	0.005819	
	CHA2(0.5)	0.005357	0.005520	0.005585	
	CHA2(0.8)	0.005547	0.005597	0.005608	
	CHA2(0.9)	0.005604	0.005618	0.005614	
	CHA2(1)	0.005664	0.005640	0.005620	
	12-parameter	0.005324	0.005544	0.005597	
	16-parameter	0.005387	0.005561	0.005600	
	BFS	0.005622	0.005597	0.005606	

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