Journal of Computational Mathematics, Vol.22, No.4, 2004, 567–580.

ON THE GENERAL ALGEBRAIC INVERSE EIGENVALUE PROBLEMS *

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Abstract

A number of new results on sufficient conditions for the solvability and numerical algorithms of the following general algebraic inverse eigenvalue problem are obtained: Given $n+1$ real $n\times n$ matrices $A=(a_{ij}),\ A_k=(a_{ij}^{(k)})(k=1,2,\dots,n)$ and n distinct real numbers $\lambda_1,\lambda_2,\dots,\lambda_n$, find n real numbers $c_1,c_2,\dots,c_n$ such that the matrix $A(c)=A+\sum\limits_{k=1}^{n}c_kA_k$ has eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$.

Mathematics subject classification: 15A18, 34A55.
Key words: Linear algebra, Eigenvalue problem, Inverse problem.

1. Introduction

We are interested in solving the following inverse eigenvalue problems:

Problem A(Additive inverse eigenvalue problem). Given an $n\times n$ real matrix $A=(a_{ij})$, and n distinct real numbers $\lambda_1,\lambda_2,\dots,\lambda_n$, find a real $n\times n$ diagonal matrix $D=diag(c_1,c_2,\dots,c_n)$ such that the matrix $A+D$ has eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$.

Problem M(Multiplicative inverse eigenvalue problem). Given an $n\times n$ real matrix $A=(a_{ij})$, and n distinct real numbers $\lambda_1,\lambda_2,\dots,\lambda_n$, find a real $n\times n$ diagonal matrix $D=diag(c_1,c_2,\dots,c_n)$ such that the matrix DA has eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$.

Problem G(General inverse eigenvalue problem). Given $n+1$ real $n\times n$ matrices $A=(a_{ij}),\ A_k=(a_{ij}^{(k)})(k=1,2,\dots,n)$ and n distinct real numbers $\lambda_1,\lambda_2,\dots,\lambda_n$, find n real numbers $c_1,c_2,\dots,c_n$ such that the matrix $A(c)=A+\sum\limits_{k=1}^{n}c_kA_k$ has eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$.

Evidently Problem **A** and **M** are special cases of Problem **G**. The solutions of Problem **G** are complicated. A number of results on sufficient conditions for the solvability, stability analysis of solution and numerical algorithms of Problem **G** with real symmetric matrices can be found in [1,3,11,12,14,16,19,20,21,22]. These results are all obtained by studying the following nonlinear system

$$\lambda_i(A(c))=\lambda_i,\ \ i=1,2,\dots,n \tag{1}$$

where $\lambda_i(A(c))$ is the ith eigenvalue of $A(c)$, or

$$det(A(c)-\lambda_iI)=0,\quad i=1,2,\dots,n. \tag{2}$$

Most numerical algorithms depend heavily on the fact that the eigenvalues of real symmetric matrix are real valued and, hence, can be totally ordered[13]. But non-symmetric matrices have not the fact. Less results on non-symmetric problems can be found. In this paper, we

* Received April 17, 2002.

use another approach to investigate Problem **G**. The main idea is to treat Problem **G** as the following equivalent problem.

$$A(c)T = T\Lambda \tag{3}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and T is a non-singular matrix. We see that the columns of T are the eigenvectors of $A(c)$. (3) is equivalent to a polynomial system(see Section 2). It is not necessary to consider ordering eigenvalues to solve the polynomial system.

In Section 2 it is proved that problem **G** is equivalent to a polynomial system. In Section 3 by studying the system with the help of Brouwer's fixed point theorem we obtain some new sufficient conditions on the solvability , which improve the results in[1,3,5,8,9]. In Section 4, we propose a linearly convergent iterative algorithm and a quadratically convergent iterative algorithm. Several examples are given in this paper.

Throughout this paper we use the following notation. Let $R^{n\times n}$ be the set of all $n \times n$ real matrices. $R^n = R^{n\times 1}$. Let

$$h_i^{(k)} = \sum_{j=1,\neq i}^{n} |a_{ij}^{(k)}|, \quad h_i = \sum_{k=1}^{n} h_i^{(k)}, \quad H = (h_i^{(k)}) \in R^{n\times n}.$$

Obviously, H is a nonnegative matrix. Let $\rho(H)$ be the spectral radius of H.

For a permutation π of the n items $\{1, \ldots, n\}$, let

$$s_{ij} = a_{ij} + \sum_{k=1}^{n} (\lambda_{\pi(k)} - a_{\pi(k),\pi(k)}) a_{ij}^{(k)}, \quad l_{ij} = |s_{ij}|, \quad i, j = 1, 2, \ldots, n, \ i \neq j \tag{4}$$

$$l_i = \sum_{j=1,\neq i}^{n} l_{ij}, \quad i = 1, 2, \ldots, n \tag{5}$$

2. Equivalent Polynomial System

Without loss of generality we can suppose that[1,3,8,9] $a_{ii} = 0 (i = 1, \ldots, n)$ in Problem **A**, $a_{ii} = 1 (i = 1, \ldots, n)$ in Problem **M**, and $a_{ii}^{(k)} = \delta_{ik} (i, k = 1, \ldots, n)$ in Problem **G**.

Theorem 1. *Problem* **G** *has a solution* $c_1, c_2, \ldots, c_n \in R$ *if and only if there exists a permutation* π *of the* n *items* $\{1, \ldots, n\}$ *such that the following polynomial system*

$$\begin{cases} (\lambda_{\pi(j)} - a_{ii} - c_i) t_{ij} = (a_{ij} + \sum\limits_{k=1}^{n} c_k a_{ij}^{(k)}) + \sum\limits_{l=1,\neq i,j}^{n} (a_{il} + \sum\limits_{k=1}^{n} c_k a_{il}^{(k)}) t_{lj}, \quad i, j = 1, \ldots, n, i \neq j \\ \lambda_{\pi(i)} - a_{ii} - c_i = \sum\limits_{l=1,\neq i}^{n} (a_{il} + \sum\limits_{k=1}^{n} c_k a_{il}^{(k)}) t_{li}, \quad i = 1, \ldots, n \end{cases} \tag{6}$$

has a solution $c_i \in R$, $t_{ij} \in R$ $(i, j = 1, \ldots, n, \ i \neq j)$.

Proof. Suppose Problem **G** has a solution $c = (c_1, c_2, \ldots, c_n)^T \in R^n$. Since the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(c)$ are all different , the Jordan canonical form of $A(c)$ is $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and therefore there exists a nonsingular matrix $S = (s_{ij}) \in C^{n\times n}$ such that

$$A(c) = S\Lambda S^{-1},$$

that is

$$A(c)S = S\Lambda. \tag{7}$$

Noting that $A(c)$ is a real matrix only with real eigenvalues, then the similarity matrix S can be taken to be real. Notice that $S \in R^{n\times n}$ is nonsingular, hence $detS \neq 0$, then there exists a permutation π of the n items $\{1,\ldots,n\}$ such that $\prod_{i=1}^{n} s_{i,\pi(i)} \neq 0$. Without loss of generality we can suppose that $s_{i,\pi(i)} = 1$ $(i = 1,2,\ldots,n)$. Let

$$P = (p_{ij}) \in R^{n\times n}$$

where

$$p_{\pi(i),j} = \delta_{ij}, \ \ i,j = 1,\ldots,n.$$

P is a permutation matrix. Let

$$T = (t_{ij}) = SP, \ \ \Lambda_\pi = diag(\lambda_{\pi(1)}, \ \lambda_{\pi(2)}, \ \ldots, \lambda_{\pi(n)}).$$

Clearly, $t_{ij} = s_{i,\pi(j)}, \ t_{ii} = 1$ $(i,j = 1,2,\ldots,n)$, $\Lambda_\pi = P^T \Lambda P$. Hence,

$$A(c)T = T\Lambda_\pi. \tag{8}$$

It is easy to show that (6) and (8) are equivalent.

Conversely, there exists a permutation π such that the system (6) has a solution $c_i \in R, \ t_{ij} \in R, \ i,j = 1,2,\ldots,n, \ i \neq j$. Let $t_{ii} = 1$ and $\Lambda_\pi = diag(\lambda_{\pi(1)}, \ \lambda_{\pi(2)},\ldots,\lambda_{\pi(n)})$. Then it is easy to show that $T = (t_{ij}) \in R^{n\times n}$, $c = (c_1,c_2,\ldots,c_n)^T \in R^n$ satisfy (8), that is, λ_i $(i = 1,\ldots,n)$ are all the eigenvalues of $A(c)$. Hence $c_1,c_2,\ldots,c_n$ is a solution to Problem **G**.

Remark 1. Let

$$x_i = \lambda_{\pi(i)} - a_{ii} - c_i, \ \ i = 1,2,\ldots,n. \tag{9}$$

Then (4) can be written as

$$(\lambda_{\pi(j)} - \lambda_{\pi(i)})t_{ij} - \sum_{l=1,\neq i,j}^{n} s_{il}t_{lj} = s_{ij} - \sum_{k=1}^{n} x_k a_{ij}^{(k)} - x_i t_{ij} - \sum_{k=1}^{n} x_k \sum_{l=1,\neq i}^{n} a_{il}^{(k)} t_{lj}, \quad i,j = 1,2,\ldots,n, \ i \neq j \tag{10}$$

$$x_i + \sum_{k=1}^{n} x_k \sum_{l=1,\neq i}^{n} a_{il}^{(k)} t_{li} = \sum_{l=1,\neq i}^{n} s_{il}t_{li}, \quad i = 1,2,\ldots,n. \tag{11}$$

Applying Theorem 1 to the additive and multiplicative inverse eigenvalue problems, we get the following corollaries.

Corollary 1. *Problem* **A** *has a solution* $c_1,c_2,\ldots,c_n \in R$ *if and only if there exists a permutation* π *of* $\{1,2,\ldots,n\}$ *such that the following polynomial system*

$$\begin{cases} (\lambda_{\pi(i)} - \lambda_{\pi(j)})t_{ij} + \sum\limits_{k=1,\neq i,j}^{n} a_{ik}t_{kj} = -a_{ij} + (\sum\limits_{l=1,\neq i}^{n} a_{il}t_{li})t_{ij}, \ \ i,j = 1,\ldots,n, \ i \neq j \\ c_i = \lambda_{\pi(i)} - \sum\limits_{j=1,\neq i}^{n} a_{ij}t_{ji}, \ i = 1,\ldots,n \end{cases} \tag{12}$$

has a solution $c_i \in R, \ t_{ij} \in R, \ i,j = 1,2,\ldots,n, \ i \neq j$.

Corollary 2. *Problem* **M** *has a solution* $c_1,c_2,\ldots,c_n \in R$ *if and only if there exists a permutation* π *of* $\{1,2,\ldots,n\}$ *such that the following polynomial system*

$$\begin{cases} (\lambda_{\pi(j)} - c_i)t_{ij} = c_i(a_{ij} + \sum\limits_{l=1,\neq i,j}^{n} a_{il}t_{lj}), \ \ i,j = 1,\ldots,n, \ i \neq j \\ \lambda_{\pi(i)} = c_i(1 + \sum\limits_{l=1,\neq i}^{n} a_{il}t_{li}), \ \ i = 1,\ldots,n \end{cases} \tag{13}$$

has a solution $c_i \in R,\ t_{ij} \in R,\ i,j = 1,2,\ldots,n,\ i \neq j.$

3. Sufficient Conditions for the Existence of Real Solutions

Theorem 2. *For Problem* **G***, suppose that*

$$a_{ii}^{(k)} = \delta_{ik},\ i,k = 1,2,\ldots,n \tag{14}$$

and there exist a constant $K > 0$ *and a permutation* π *of* $\{1,2,\ldots,n\}$ *such that*

$$\rho(H) < 1/K, \tag{15}$$

$$|\lambda_{\pi(i)} - \lambda_{\pi(j)}| \geq (\frac{1}{K}+1)\sigma_i + (\frac{1}{K}-1)\left[l_{ij} + \sum_{k=1}^{n}\sigma_k a_{ij}^{(k)}\right] \tag{16}$$

$$i,j = 1,2,\ldots,n, i \neq j$$

where $\sigma_i,\ i = 1,2,\ldots,n$ *satisfy*

$$\sigma_i = Kl_i + K\sum_{k=1}^{n}\sigma_k h_i^{(k)},\ \ i = 1,2,\ldots,n \tag{17}$$

Then there exists $c = (c_1, c_2, \ldots, c_n)^T \in R^n$ *with*

$$\left|c_i - (\lambda_{\pi(i)} - a_{ii})\right| \leq \sigma_i,\ \ i = 1,2,\ldots,n \tag{18}$$

such that the eigenvalues of $A(c)$ *are* $\lambda_1, \lambda_2, \ldots, \lambda_n$*.*

The proofs of Theorem 2 will be based on the following lemmas.

Lemma 1. *Under the conditions of Theorem 2 there exists only one nonnegative vector* $(\sigma_1, \sigma_2, \ldots, \sigma_n)^T \in R^n$ *satisfying (17).*

Proof. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)^T \in R^n$, $l = (l_1, l_2, \ldots, l_n)^T \in R^n$. Then (17) is equivalent to

$$\sigma = Kl + KH\sigma$$

that is

$$(I - KH)\sigma = Kl$$

If $\rho(H) < 1/K, \rho(KH) < 1$, then $I - KH$ is invertible and

$$(I - KH)^{-1} = \sum_{n=1}^{\infty} K^n H^n$$

Hence,

$$\sigma = \sum_{n=1}^{\infty} K^{n+1} H^n l \geq 0.$$

Proof of Theorem 2. Let

$$t = (t_{12}, t_{13}, \ldots, t_{1n}, t_{21}, t_{23}, \ldots, t_{2n}, \ldots, t_{n1}, t_{n2}, \ldots, t_{n,n-1})^T \in R^{n^2-n},$$

$$c = (c_1, c_2, \ldots, c_n)^T \in R^n,\ \ x = (x_1, x_2, \ldots, x_n)^T \in R^n.$$

Define

$$\Omega = \left\{t \in R^{n^2-n} : |t_{ij}| \leq K,\ i,j = 1,2,\ldots,n,\ i \neq j\right\}. \tag{19}$$

Obviously, Ω is a nonempty convex closed set in R^{n^2-n}. Now let f be the map with

$$f : \Omega \to R^{n^2-n}$$

and

$$\begin{aligned} f(t) &= (F_{12}(t), F_{13}(t), \ldots, F_{1n}(t), F_{21}(t), F_{23}(t), \ldots, F_{2n}(t), \\ &\quad \ldots, F_{n1}(t), F_{n2}(t), \ldots, F_{n,n-1}(t))^T \end{aligned}$$

with

$$(\lambda_{\pi(j)} - \lambda_{\pi(i)} + x_i)F_{ij} - \sum_{l=1,\neq i,j}^{n} (s_{il} - \sum_{k=1}^{n} x_k a_{il}^{(k)})F_{lj} = s_{ij} - \sum_{k=1}^{n} x_k a_{ij}^{(k)}, \tag{20}$$

$$i, j = 1, 2, \ldots, n,\ i \neq j$$

where $x_i (i = 1, 2, \ldots, n)$ satisfy

$$x_i + \sum_{k=1}^{n} x_k \sum_{l=1,\neq i}^{n} a_{il}^{(k)} t_{li} = \sum_{l=1,\neq i}^{n} s_{il} t_{li}, \quad i = 1, 2, \ldots, n. \tag{21}$$

We show that $f(\Omega) \subseteq \Omega$ and continuous. Let $t \in \Omega$, that is, $|t_{ij}| \leq K,\ i, j = 1, 2, \ldots, n,\ i \neq j$. By (15) and (16), we have

$$\begin{aligned} |x_i| &= \left| \sum_{l=1,\neq i}^{n} s_{il} t_{li} + \sum_{k=1}^{n} x_k \sum_{l=1,\neq i}^{n} a_{il}^{(k)} t_{li} \right| \\ &\leq K \sum_{l=1,\neq i}^{n} l_{il} + K \sum_{k=1}^{n} |x_k| h_i^{(k)}. \end{aligned} \tag{22}$$

Then we have

$$|x| \leq Kl + KH|x| \tag{23}$$

where

$$|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T.$$

Noting that $(I - KH)^{-1} > 0$, hence $|x| \leq K(I - KH)^{-1} l = \sigma$. Suppose that p, q satisfy

$$|t_{pq}| = \|t\|_\infty = \max_{i \neq j} |t_{ij}|.$$

Then we have

$$\begin{aligned} |\lambda_{\pi(q)} - \lambda_{\pi(p)}||F_{pq}| &= \left| s_{pq} - \sum_{k=1}^{n} x_k a_{pq}^{(k)} - x_p F_{pq} + \sum_{l=1,\neq p,q}^{n} (s_{pl} - \sum_{k=1}^{n} x_k a_{pl}^{(k)}) F_{lq} \right| \\ &\leq l_{pq} + \sum_{k=1}^{n} |x_k||a_{pq}^{(k)}| + |x_p||F_{pq}| + \sum_{l=1,\neq p,q}^{n} (l_{pl} + \sum_{k=1}^{n} |x_k||a_{pl}^{(k)}|)|F_{lq}| \\ &\leq l_{pq} + \sum_{k=1}^{n} \sigma_k |a_{pq}^{(k)}| + \sigma_p |F_{pq}| + \sum_{l=1,\neq p,q}^{n} (l_{pl} + \sum_{k=1}^{n} \sigma_k |a_{pl}^{(k)}|)|F_{pq}|. \end{aligned}$$

Hence by (17) and (16) we can get

$$|F_{pq}| \leq \frac{l_{pq}+\sum_{k=1}^{n}\sigma_k|a_{pq}^{(k)}|}{|\lambda_{\pi(q)}-\lambda_{\pi(p)}|-\sigma_p-\sum_{l=1,\neq p,q}^{n}(l_{pl}+\sum_{k=1}^{n}\sigma_k|a_{pl}^{(k)}|)} \quad (24)$$

$$= \frac{l_{pq}+\sum_{k=1}^{n}\sigma_k|a_{pq}^{(k)}|}{|\lambda_{\pi(q)}-\lambda_{\pi(p)}|-(1+1/K)\sigma_p+(l_{pq}+\sum_{k=1}^{n}\sigma_k|a_{pq}^{(k)}|)} \quad (25)$$

$$\leq K.$$

Let $t \in \Omega$. By (15), (21) and the implicit function theorem, the vector-valued function $x : \Omega \mapsto \Gamma = \{x \in R^n : \ |x| \leq \sigma\}$ is analytic. By (20) (16) and the implicit function theorem, the vector-valued function $f : \Omega \times \Gamma \mapsto \Omega$ is analytic. By the chain rule, $f : \Omega \mapsto \Omega$ is analytic.

Then we have $f(\Omega) \subseteq \Omega$ and continuous. By Brouwer's fixed point theorem, f has a fixed point in Ω. Hence, by Remark 1 and Theorem 1, we can get Theorem 2.

Applying Theorem 2 to the additive and multiplicative inverse eigenvalue problems, we get the following corollaries.

Corollary 3. *For Problem* **A**, *suppose that*

$$a_{ii} = 0, \ \ i = 1, 2, \ldots, n \quad (26)$$

and there exist a constant $K > 0$ and a permutation π of $\{1, 2, \ldots, n\}$ such that

$$|\lambda_{\pi(i)} - \lambda_{\pi(j)}| \geq (K+1)\sum_{l=1,\neq i}^{n}|a_{il}| + (\frac{1}{K}-1)|a_{ij}|, \ \ i,j = 1,2,\ldots,n, \ i \neq j. \quad (27)$$

Then there exists $D = diag(c_1, c_2, \ldots, c_n) \in R^{n\times n}$ with

$$|c_i - \lambda_{\pi(i)}| \leq K\sum_{j=1,\neq i}^{n}|a_{ij}|, \ \ i = 1,2,\ldots,n \quad (28)$$

such that the eigenvalues of $A + D$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Corollary 4. *For Problem* **M**, *suppose that*

$$a_{ii} = 1, \ \ i = 1, 2, \ldots, n \quad (29)$$

and there exist a constant $K > 0$ and a permutation π of $\{1, 2, \ldots, n\}$ such that

$$g_i = \sum_{j=1,\neq i}^{n}|a_{ij}| < \frac{1}{K}, \ \ i = 1,2,\ldots,n \quad (30)$$

and

$$|\lambda_{\pi(i)} - \lambda_{\pi(j)}| \geq \frac{|\lambda_{\pi(i)}|}{1-Kg_i}\left[(K+1)\sum_{l=1,\neq i}^{n}|a_{il}| + (\frac{1}{K}-1)|a_{ij}|\right], \ \ i,j = 1,2,\ldots,n, \ i \neq j \quad (31)$$

Then there exists $D = diag(c_1, c_2, \ldots, c_n) \in R^{n\times n}$ with

$$|c_i - \lambda_{\pi(i)}| \leq \frac{K|\lambda_{\pi(i)}|g_i}{1-Kg_i}, \ \ i = 1,2,\ldots,n \quad (32)$$

such that the eigenvalues of DA are $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Remark 2. In fact, K is the bound of the normalized eigenvectors in Theorem 2, Corollary 3 and Corollary 4. We can get many sufficient conditions on the solvability by choosing different values of K. Especially, letting $K = 1$, we can obtain the results in [1,3,5,8].

Example 1. For $\lambda_1 = 4,\ \lambda_2 = -8,\ A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0.2 \\ 8.1 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0.1 \\ 0.2 & 1 \end{pmatrix}$, consider Problem **G**. It can be verified that if $\pi(1) = 1,\ \pi(2) = 2,\ K = 0.8$ then $\sigma_1 = 0.4275,\ \sigma_2 = 3.4872$. Applying Theorem 2, we know that Problem G in this example is solvable. In fact $c_1 = -0.001787,\ c_2 = -10.998213$.

We can't infer the solvability of Example 1 from the results in [1,3,5].

4. Numerical Methods

4.1. A Linearly Convergent Iterative Algorithm

Algorithm L.

1) Choose a starting value $t_{ij}^{(0)} = 0$ for all $i, j = 1, 2, \ldots, n,\ \ i \neq j$. For $m = 1, 2, 3, \ldots,$

i) compute $c_i^{(m)},\ \ i = 1, 2, \ldots, n$ by solving the linear system

$$x_i^{(m)} + \sum_{k=1}^{n} x_k^{(m)} \sum_{l=1,\neq i}^{n} a_{il}^{(k)} t_{li}^{(m-1)} = \sum_{l=1,\neq i}^{n} s_{il} t_{li}^{(m-1)}, \quad i = 1, 2, \ldots, n. \tag{33}$$

ii)compute $t_{ij}^{(m)},\ i, j = 1, 2, \ldots, n,\ i \neq j$ by solving n linear systems

$$(\lambda_{\pi(j)} - \lambda_{\pi(i)} + x_i^{(m)}) t_{ij}^{(m)} - \sum_{l=1,\neq i,j}^{n} (s_{il} - \sum_{k=1}^{n} x_k^{(m)} a_{il}^{(k)}) t_{lj}^{(m)} = s_{ij} - \sum_{k=1}^{n} x_k^{(m)} a_{ij}^{(k)}, \quad i = 1, 2, \ldots, n,\ \ i \neq j \tag{34}$$

for j=1,2,...,n.

2) Compute $c_i = \lambda_{\pi(i)} - a_{ii} - x_i^{(m)},\ i = 1, 2, \ldots, n.$

The following theorem is the main result of this section.

Theorem 3. *For Problem* **G***, suppose that*

$$a_{ii}^{(k)} = \delta_{ik},\ i, k = 1, 2, \ldots, n \tag{35}$$

and there exist a constant $K > 0$ and a permutation π of $\{1, 2, \ldots, n\}$ such that

$$\rho(H) < 1/K, \tag{36}$$

and

$$|\lambda_{\pi(i)} - \lambda_{\pi(j)}| \geq \max\left\{ (\frac{1}{K} + 1)\sigma_i + (\frac{1}{K} - 1)\left[l_{ij} + \sum_{k=1}^{n} \sigma_k a_{ij}^{(k)} \right], \right. \left. \sigma_i - (l_{ij} + \sum_{k=1}^{n} \sigma_k a_{ij}^{(k)}) + (K+1)\tau_i + (K-1)\sum_{k=1}^{n} \tau_k a_{ij}^{(k)} \right\} \tag{37}$$

$$i, j = 1, 2, \ldots, n,\ i \neq j$$

where σ_i, τ_i $i = 1, 2, \ldots, n$ satisfy

$$\sigma_i = K l_i + K \sum_{k=1}^{n} \sigma_k h_i^{(k)},\ \ i = 1, 2, \ldots, n, \tag{38}$$

and

$$\tau_i = \sigma_i/K + K\sum_{k=1}^{n}\tau_k h_i^{(k)}, \;\; i = 1,2,\ldots,n, \tag{39}$$

Then (i) there exists $c = (c_1, c_2, \ldots, c_n)^T \in R^n$ *with*

$$\left|c_i - (\lambda_{\pi(i)} - a_{ii})\right| \le \sigma_i, \;\; i = 1,2,\ldots,n \tag{40}$$

such that the eigenvalues of $A(c)$ *are* $\lambda_1, \lambda_2, \ldots, \lambda_n$, *and (ii)* $c_1, c_2, \ldots, c_n$ *can be obtained with Algorithm L. The iterates* $\{t_{ij}^{(m)}\}_{i\neq j}$ *generated by Algorithm L converge linearly to the unique solution* $\{t_{ij}\}_{i\neq j}$ *of the eqution (6) with*

$$\max_{i\neq j}|t_{ij} - t_{ij}^{(m)}| \le \frac{\max_{i\neq j}|t_{ij}^{(m+1)} - t_{ij}^{(m)}|}{1-\varrho} \le \frac{\varrho^{m-k}}{1-\varrho}\max_{i\neq j}|t_{ij}^{(k+1)} - t_{ij}^{(k)}|, \tag{41}$$

$$k \le m \tag{42}$$

where

$$\varrho = \max_{i\neq j}\frac{(K+1)\tau_i + (K-1)\sum_{k=1}^{n}\tau_k a_{ij}^{(k)} - \sigma_i/K}{|\lambda_{\pi(i)} - \lambda_{\pi(j)}| - (1+1/K)\sigma_i - (l_{ij} + \sum_{k=1}^{n}\sigma_k|a_{ik}|)} \tag{43}$$

Proof. We use the notations of the proof of Theorem 2. From the proof of Theorem 2, we know that $F(\Omega) \subseteq \Omega$. It is sufficient to show that F is a contraction operator mapping Ω into itself.

Let $t^{(1)}, t^{(2)} \in \Omega$. We have

$$(\lambda_{\pi(j)} - \lambda_{\pi(i)} + x_i^{(1)})F_{ij}(t^{(1)}) - \sum_{l=1,\neq i,j}^{n}(s_{il} - \sum_{k=1}^{n}x_k^{(1)}a_{il}^{(k)})F_{lj}(t^{(1)}) = s_{ij} - \sum_{k=1}^{n}x_k^{(1)}a_{ij}^{(k)} \tag{44}$$

$$i,j = 1,2,\ldots,n, \; i \neq j$$

where $x_i^{(1)} (i = 1,2,\ldots,n)$ satisfy

$$x_i^{(1)} + \sum_{k=1}^{n}x_k^{(1)}\sum_{l=1,\neq i}^{n}a_{il}^{(k)}t_{li}^{(1)} = \sum_{l=1,\neq i}^{n}s_{il}t_{li}^{(1)}, \;\; i = 1,2,\ldots,n \tag{45}$$

and

$$(\lambda_{\pi(j)} - \lambda_{\pi(i)} + x_i^{(2)})F_{ij}(t^{(2)}) - \sum_{l=1,\neq i,j}^{n}(s_{il} - \sum_{k=1}^{n}x_k^{(2)}a_{il}^{(k)})F_{lj}(t^{(2)}) = s_{ij} - \sum_{k=1}^{n}x_k^{(2)}a_{ij}^{(k)} \tag{46}$$

$$i,j = 1,2,\ldots,n, \; i \neq j$$

where $x_i^{(2)} (i = 1,2,\ldots,n)$ satisfy

$$x_i^{(2)} + \sum_{k=1}^{n}x_k^{(2)}\sum_{l=1,\neq i}^{n}a_{il}^{(k)}t_{li}^{(2)} = \sum_{l=1,\neq i}^{n}s_{il}t_{li}^{(2)}, \;\; i = 1,2,\ldots,n. \tag{47}$$

Subtracting (47) from (45), we can get

$$\begin{aligned} & x_i^{(1)} - x_i^{(2)} + \sum_{k=1}^{n}(x_k^{(1)} - x_k^{(2)}) \sum_{l=1,\neq i}^{n} a_{il}^{(k)} t_{li}^{(2)} \\ = & \sum_{l=1,\neq i}^{n} s_{il}(t_{li}^{(1)} - t_{li}^{(2)}) - \sum_{k=1}^{n} x_k^{(1)} \sum_{l=1,\neq i}^{n} a_{il}^{(k)}(t_{li}^{(1)} - t_{li}^{(2)}), \\ & i = 1, 2, \ldots, n. \end{aligned} \tag{48}$$

Then

$$\begin{aligned} |x_i^{(1)} - x_i^{(2)}| &\leq \sum_{k=1}^{n} |x_k^{(1)} - x_k^{(2)}| \sum_{l=1,\neq i}^{n} |a_{il}^{(k)}||t_{li}^{(2)}| + \sum_{l=1,\neq i}^{n} (|s_{il}| + \sum_{k=1}^{n} |x_k^{(1)}||a_{il}^{(k)}|)|t_{li}^{(1)} - t_{li}^{(2)}| \\ &\leq K \sum_{k=1}^{n} |x_k^{(1)} - x_k^{(2)}| h_i^{(k)} + \|t^{(1)} - t^{(2)}\|_\infty \sum_{l=1,\neq i}^{n} (|s_{il}| + \sum_{k=1}^{n} \sigma_k |a_{il}^{(k)}|) \\ &= K \sum_{k=1}^{n} |x_k^{(1)} - x_k^{(2)}| h_i^{(k)} + \frac{1}{K}\sigma_i \|t^{(1)} - t^{(2)}\|_\infty, \quad i = 1, 2, \ldots, n, \end{aligned}$$

that is, we have

$$|x_i^{(1)} - x_i^{(2)}| - K \sum_{k=1}^{n} |x_k^{(1)} - x_k^{(2)}| h_i^{(k)} \leq \frac{1}{K}\sigma_i \|t^{(1)} - t^{(2)}\|_\infty, \quad i = 1, 2, \ldots, n. \tag{49}$$

(49) is equivalent to

$$(I - KH)|x^{(1)} - x^{(2)}| \leq \frac{1}{K}\|t^{(1)} - t^{(2)}\|_\infty \sigma \tag{50}$$

where $|x^{(1)} - x^{(2)}| = (|x_1^{(1)} - x_1^{(2)}|, |x_2^{(1)} - x_2^{(2)}|, \cdots, |x_n^{(1)} - x_n^{(2)}|)^T$. Hence we have

$$|x^{(1)} - x^{(2)}| \leq \frac{1}{K}(I - KH)^{-1}\|t^{(1)} - t^{(2)}\|_\infty \sigma = \|t^{(1)} - t^{(2)}\|_\infty \tau, \tag{51}$$

that is ,

$$|x_i^{(1)} - x_i^{(2)}| \leq \|t^{(1)} - t^{(2)}\|_\infty \tau_i, \;\; i = 1, 2, \cdots, n. \tag{52}$$

Subtracting (44) from (46), we can get

$$\begin{aligned} & (\lambda_{\pi(j)} - \lambda_{\pi(i)} + x_i^{(1)})(F_{ij}(t^{(1)}) - F_{ij}(t^{(2)})) \\ & - \sum_{l=1,\neq i,j}^{n} (s_{il} - \sum_{k=1}^{n} x_k^{(1)} a_{il}^{(k)})(F_{lj}(t^{(1)}) - F_{lj}(t^{(2)})) \\ = & -(x_i^{(1)} - x_i^{(2)})F_{ij}(t^{(2)}) - \sum_{l=1,\neq i,j}^{n} \sum_{k=1}^{n} (x_k^{(1)} - x_k^{(2)}) a_{il}^{(k)} F_{lj}(t^{(2)}) - \sum_{k=1}^{n} (x_k^{(1)} - x_k^{(2)}) a_{ij}^{(k)} \end{aligned} \tag{53}$$

for all $i, j = 1, 2, \cdots, n, \;\; , i \neq j$. Suppose that p, q satisfy

$$|F_{pq}^{(1)} - F_{pq}^{(2)}| = \max_{i \neq j} |F_{ij}^{(1)} - F_{ij}^{(2)}| = \|F^{(1)} - F^{(2)}\|_\infty.$$

By (53) and (52), we have

$$\begin{aligned}&\left(|\lambda_{\pi(q)}-\lambda_{\pi(p)}|-\sum_{l=1,\neq p,q}^{n} l_{pl}-\sigma_p-\sum_{l=1,\neq p,q}^{n}\sum_{k=1}^{n}\sigma_k|a_{pl}^{(k)}|\right)|F_{pq}^{(1)}-F_{pq}^{(2)}|\\ \leq\ &(K\tau_p+K\sum_{l=1,\neq p,q}^{n}\sum_{k=1}^{n}\tau_k|a_{pl}^{(k)}|+\sum_{k=1}^{n}\tau_k|a_{pq}^{(k)}|)\|t^{(1)}-t^{(2)}\|_\infty. \end{aligned}\tag{54}$$

Hence

$$\begin{aligned}\|F^{(1)}-F^{(2)}\|_\infty &\leq \max_{i\neq j}\frac{K\tau_i+K\sum\limits_{l=1,\neq i,j}^{n}\sum\limits_{k=1}^{n}\tau_k|a_{il}^{(k)}|+\sum\limits_{k=1}^{n}\tau_k|a_{ij}^{(k)}|}{|\lambda_{\pi(j)}-\lambda_{\pi(i)}|-\sum\limits_{l=1,\neq i,j}^{n}l_{il}-\sigma_i-\sum\limits_{l=1,\neq i,j}^{n}\sum\limits_{k=1}^{n}\sigma_k|a_{il}^{(k)}|}\|t^{(1)}-t^{(2)}\|_\infty\\ &= \max_{i\neq j}\frac{(K+1)\tau_i+(K-1)\sum\limits_{k=1}^{n}\tau_k|a_{ij}^{(k)}|-\sigma_i/K}{|\lambda_{\pi(j)}-\lambda_{\pi(i)}|-(1+1/K)\sigma_i+(l_{ij}+\sum\limits_{k=1}^{n}\sigma_k|a_{ij}^{(k)}|)}\|t^{(1)}-t^{(2)}\|_\infty\\ &= \varrho\|t^{(1)}-t^{(2)}\|_\infty\end{aligned}$$

From (37) and (43), we know that $\varrho<1$, it follows that F is a contraction with contraction number ϱ. Now the statements of the theorem can be deduced from the Banach fixed point theorem.

4.2. Newton's Method

Algorithm N.

1) Choose a starting value $x_i^{(0)},\ t_{ij}^{(0)},\ i,j=1,2,\cdots,n,\ i\neq j$. For $m=1,2,\cdots,M$ compute $x_i^{(m)},\ t_{ij}^{(m)},\ i,j=1,2,\cdots,n,\ i\neq j$ by solving the linear system

$$\begin{cases}(\lambda_{\pi(j)}-\lambda_{\pi(i)}+x_i^{(m-1)})t_{ij}^{(m)}-\sum\limits_{l=1,\neq i,j}^{n}(s_{il}-\sum\limits_{k=1}^{n}x_k^{(m-1)}a_{il}^{(k)})t_{lj}^{(m)}+x_i^{(m)}t_{ij}^{(m-1)}\\ \qquad +\sum\limits_{k=1}^{n}x_k^{(m)}(a_{ij}^{(k)}+\sum\limits_{l=1,\neq i,j}^{n}a_{il}^{(k)}t_{lj}^{(m-1)})\\ =s_{ij}+x_i^{(m-1)}t_{ij}^{(m-1)}+\sum\limits_{k=1}^{n}x_k^{(m-1)}\sum\limits_{l=1,\neq i,j}^{n}a_{il}^{(k)}t_{lj}^{(m-1)},\\ \qquad x_i^{(m)}+\sum\limits_{k=1}^{n}x_k^{(m)}\sum\limits_{l=1,\neq i}^{n}a_{il}^{(k)}t_{li}^{(m-1)}-\sum\limits_{l=1,\neq i}^{n}(s_{il}-\sum\limits_{k=1}^{n}x_k^{(m-1)}a_{il}^{(k)})t_{li}^{(m)}\\ =\sum\limits_{k=1}^{n}\sum\limits_{l=1,\neq i}^{n}x_k^{(m-1)}a_{il}^{(k)}t_{li}^{(m-1)}\end{cases}$$

$$i,j=1,2,\cdots,n\ i\neq j.$$

2) Compute $c_i=\lambda_{\pi(i)}-a_{ii}-x_i^{(M)},\ \ i=1,2,\cdots,n.$

Remark 3. By a standard argument (see [18]), it follows that the iterates $\{c_i^{(m)}\}$ generated by Algorithm N converge quadratically to the solution $\{c_i^*\}$ when a starting value $\{t_{ij}^{(0)}\}_{i\neq j}$ is sufficiently close to the solution of (6).

4.3. Numerical Examples

We have tested Algorithms described in this paper with Matlab 5.3.

Examples 2[22]. This is a general inverse eigenvalue problem with symmetric matrices.

$$A=\begin{bmatrix}0&2&3&1\\2&0&2&2\\3&2&0&3\\1&2&3&0\end{bmatrix},\quad A_1=\begin{bmatrix}1&0.001&0.001&0\\0.001&0&0.001&0.001\\0.001&0.001&0&0.001\\0&0.001&0.001&0\end{bmatrix},$$

$$A_2=\begin{bmatrix}0&-0.001&0&0\\-0.001&1&-0.001&0\\0&-0.001&0&-0.001\\0&0&-0.001&0\end{bmatrix},\quad A_3=\begin{bmatrix}0&0.002&0.002&0.002\\0.002&0&0.002&0.002\\0.002&0.002&1&0.002\\0.002&0.002&0.002&0\end{bmatrix},$$

$$A_4=\begin{bmatrix}0&0.002&0.001&0\\0.002&0&0.002&0.001\\0.001&0.002&0&0.002\\0&0.001&0.002&1\end{bmatrix},\quad \lambda_i=-30,\ -10,\ 10,\ 30.$$

Let $\pi(i)=i,\ i=1,2,\cdots,n$. We have calculated this example with Algorithm L and Algorithm N. We choose the starting point with zeros. After 10 iterations with Algorithm L or after 4 iterations with Algorithm N we obtain the numerical results as follows

$$c_i=-29.58520425277408,\ -9.86261231114425,\ 10.10052215992911,\ 29.34729440398922$$

$$\lambda_i(A(c))=-29.99999999999999,\ -9.99999999999999,\ 10.00000000000003,\ 29.99999999999999$$

After 5 and 3 iterations with Algorithm L we obtain the numerical results as follows, respectively.

$$c_i=-29.58520419590780,\ -9.86261215603080,\ 10.10052207002704,\ 29.34729477087801$$

$$c_i=-29.58516135604969,\ -9.86245726424444,\ 10.10038568489738,\ 29.34769336240272$$

With Algorithm N, after 3 and 2 iterations we obtain the numerical results as follows, respectively.

$$c_i=-29.58520425277407,\ -9.86261231114415,\ 10.10052215992869,\ 29.34729440398965$$

$$\lambda_i(A(c))=-30.00000000000000,\ -9.99999999999991,\ 9.99999999999963,\ 30.00000000000041$$

and

$$c_i=-29.58520420103676,\ -9.86261327912172,\ 10.10051855637134,\ 29.34729667155401$$

$$\lambda_i(A(c))=-29.99999997334557,\ -10.00000098388054,\ 9.99999660557243,\ 30.00000209942053$$

Example 3. This is a general inverse eigenvalue problem with nonsymmetric matrices.

$$A=\begin{bmatrix}0&2&3&1\\2&0&2&2\\3&2&0&3\\1&2&3&0\end{bmatrix},\quad A_1=\begin{bmatrix}1&0.1&0.1&0\\0.1&0&-0.1&-0.1\\0.1&0.1&0&-0.1\\0&0.1&0.1&0\end{bmatrix},$$

$$A_2=\begin{bmatrix}0&-0.1&0&0\\-0.1&1&-0.1&0\\0&-0.1&0&-0.1\\0&0&-0.1&0\end{bmatrix},\quad A_3=\begin{bmatrix}0&0.2&0.2&0.2\\0.2&0&0.2&0.2\\0.2&0.2&1&0.2\\0.2&0.2&0.2&0\end{bmatrix},$$

$$A_4=\begin{bmatrix}0&0.2&0.1&0\\-0.2&0&0.2&-0.1\\0.1&-0.2&0&0.2\\0&0.1&-0.2&1\end{bmatrix},\quad \lambda_i=-30,\ -10,\ 10,\ 30.$$

Let $\pi(i) = i,\ i = 1, 2, \cdots, n$. We have calculated this example with Algorithm L and Algorithm N. We choose the starting point with zeros. With Algorithm L, after 50 iterations we obtain the numerical results as follows

$$c_i = -31.52522503488440,\ -10.33136021413202,\ 11.83846051944945,\ 30.01812472956697$$

$$\lambda_i(A(c)) = -30.00000000000000,\ -10.00000000000000,\ 30.00000000000002,\ 10.00000000000000$$

and after 15 and 10 iterations we obtain the numerical results as follows, respectively.

$$c_i = -31.52522564150056,\ -10.33136115771607,\ 11.83845932001919,\ 30.01812525932049$$

$$\lambda_i(A(c)) = -30.00000041187057,\ -10.00000083332667,\ 30.00000007996793,\ 9.99999894535239$$

and

$$c_i = -31.52529877609995,\ -10.33147490525617,\ 11.83831472160383,\ 30.01818912765464$$

$$\lambda_i(A(c)) = -30.00005006932031,\ -10.00010128545978,\ 30.00000972334760,\ 9.99987179933484$$

With Algorithm N, after 4 ,3 and 2 iterations we obtain the numerical results as follows, respectively.

$$c_i = -31.52522503488441,\ -10.33136021413202,\ 11.83846051944943,\ 30.01812472956700,$$

$$\lambda_i(A(c)) = -29.99999999999999,\ -10.00000000000001,\ 30.00000000000002,\ 9.99999999999997,$$

$$c_i = -31.52522493156483,\ -10.33135987825058,\ 11.83845983228851,\ 30.01812500596253,$$

$$\lambda_i(A(c)) = -29.99999984320893,\ -9.99999942302649,\ 30.00000003698973,\ 9.99999925768129,$$

and

$$c_i = -31.52646043774289,\ -10.33591698793258,\ 11.83464175081756,\ 30.02176761727010,$$

$$\lambda_i(A(c)) = -30.00009759704869,\ -10.00488735039668,\ 30.00209172947121,\ 9.99692516038639.$$

Example 4[11]. Consider Problem A with symmetric matrices. Let

$$A = \begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}, \quad \lambda_i = 6,\ 3,\ 3.$$

It is easy to verify that $c_i = 2,\ -1,\ -1$ is an exact solution of this problem. Applying Algorithm N to this problem with the starting point with $x^{(0)} = (-7.8, 3.7, 4.2)^T$ and $t_{ij}^{(0)}$ being the (i, j) element of the eigenmatrix with diagonal elements being 1 of $A + diag(1.8, -0, 7, -1.2)$ we find

M	$\|c^{(M)} - c\|_2$	$\|\lambda - \lambda^{(M)}\|_2$
1	5.842469×10^{-2}	2.451207×10^{-2}
2	1.531832×10^{-3}	6.446797×10^{-4}
3	9.913302×10^{-7}	4.171958×10^{-7}
4	4.122296×10^{-13}	1.736661×10^{-13}

Here $\lambda^{(M)}$ denotes the vector of the eigenvalues of $A(c^{(M)})$. Observe that the speed of convergence is slightly faster than Algorithm 4.6.2 in [11] and it is the same as Algoithm 4.6.1 in [11]. In Alorithm 4.6.1 in [11] all the eigenvecors of $A(c^{(M)})$ have to be computed per step, which is very time consuming. In Algorithm N in this paper and in Algorithm 4.6.2 in [11] only some linear systems have to be solved per step, which is less time consuming.

From these examples we find that the convergent speed of Algorithm L is much slower than Algorithm N, but it requires less operations in each iteration.

Acknowledgments. The author would like to express his gratitude to Professor Zhu Benren and Professor Zhang Guanquan for helping to arrange his visits to Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences and for their consistent encouragement and support.

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