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THE FINITE ELEMENT ANALYSIS OF THE CONTROLLED-SOURCE ELECTROMAGNETIC INDUCTION PROBLEMS BY FRACTIONAL-STEP PROJECTION METHOD *1)

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Abstract

This paper provides an convergence analysis of a fractional-step projection method for the controlled-source electromagnetic induction problems in heterogenous electrically conducting media by means of finite element approximations. Error estimates in finite time are given. And it is verified that provided the time step τ is sufficiently small, the proposed algorithm yields for finite time T an error of $\mathcal{O}(h^s+\tau)$ in the L^2-norm for the magnetic field $\boldsymbol{H}$, where h is the mesh size and $1/2 < s \leq 1$.

Mathematics subject classification: 65N30.
Key words: Controlled-source electromagnetic induction problems, Fractional-step projection method, Finite element, Error analysis.

1. Introduction

The numerical treatment of the controlled-source electromagnetic(CSEM) induction problems, which are widely applied in geophysical prospecting, have received much attention in the last decades (see [5, 11, 16, 17, ?]). So-called CSEM problems are actually that electric and magnetic fields at low frequencies (such that displacement currents can be neglected) satisfy the diffusive Maxwell's equations:

$$\nabla \times \boldsymbol{E} + \mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = \boldsymbol{0}, \tag{1.1}$$

$$\nabla \times \boldsymbol{H} - \sigma \boldsymbol{E} = \boldsymbol{J}_s, \tag{1.2}$$

where μ_0 is the magnetic permeability of free space, σ is the spatially varying electrical conductivity of the geological formation being studied with $0 < \bar{\sigma} \leq \sigma(\boldsymbol{x})$, and $\boldsymbol{J}_s(\boldsymbol{x}, t)$ is source electric current density.

By equations (1.1) and (1.2), eliminating electric field $\boldsymbol{E}$ we obtain

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times (\sigma^{-1} \nabla \times \boldsymbol{H}) = \nabla \times (\sigma^{-1} \boldsymbol{J}_s). \tag{1.3}$$

A constitutive equation $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ relates the magnetic induction and magnetic field vectors. The divergence-free condition

$$\nabla \cdot \boldsymbol{B} = 0 \quad \Longrightarrow \quad \nabla \cdot \boldsymbol{H} = 0 \tag{1.4}$$

is also imposed, indicating that no magnetic induction exists inside the solution domain Ω.

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For convenience of numerical treatment, we add a multiplier term $\nabla\phi$ to the left side of equation (1.3) to obtain

$$\mu_0\frac{\partial \boldsymbol{H}}{\partial t}+\nabla\times(\sigma^{-1}\nabla\times\boldsymbol{H})+\nabla\phi=\nabla\times(\sigma^{-1}\boldsymbol{J}_s). \tag{1.5}$$

By equation (1.4), we take the divergence in two side of above equation to get

$$\nabla^2\phi=0. \tag{1.6}$$

It can easily be founded that if we enforce a zero Dirichlet condition on ϕ along the boundary Γ of the solution domain Ω, then the multiplier function ϕ added in equation (1.5) is identically zero value on that domain. Thus, equation (1.5) is equivalent to equation (1.3).

In following context we shall concentrate our attention on the finite element approximation of the following initial-boundary value problem of equations (1.4) and (1.5):

$$\begin{cases} \mu_0\dfrac{\partial \boldsymbol{H}}{\partial t}+\nabla\times(\sigma^{-1}\nabla\times\boldsymbol{H})+\nabla\phi=\boldsymbol{F}, \quad \Omega\times(0,T), \\ \nabla\cdot\boldsymbol{H}=0, \quad \Omega\times[0,T], \\ \boldsymbol{H}\times\boldsymbol{n}=\boldsymbol{0}, \quad \partial\Omega\times[0,T], \\ \boldsymbol{H}(\cdot,0)=\boldsymbol{H}_0, \quad \Omega. \end{cases} \tag{1.7}$$

Here vector value function $\boldsymbol{F}=\nabla\times(\sigma^{-1}\boldsymbol{J}_s)$ and Ω is a bounded, simply-connected polyhedral domain with connected boundary $\Gamma=\partial\Omega$ and $\boldsymbol{n}$ the unit normal vector to Γ.

As is known to all, the fractional-step projection method of Chorin [6, 7] and Shen [22, 23] and Guermond [1, 12] has been successfully applied to solve the incompressible Navier-Stokes equations in primitive variables in recent years. This method is based on rather peculiar time-discretization of the Navier-Stokes equations, in which the convection-diffusion and the incompressibility are dealt with in two different substeps and therefore the original problem is converted into solving a convection-diffusion problem and a Poisson problem at each step. Thus, the projection method has advantage of much lower amount of computation in comparison with coupled techniques such as those that are based on the Uzawa operator (see [2, 3, 14] et al).

In view of above-mentioned virtues, the aim of this paper is to present and analyze a fractional-step projection algorithm for the controlled-source electromagnetic induction problem in heterogenous electrically conducting media by means of finite element approximations.

The remainder of this paper is organized as follows. Some preliminary results is stated and the fractional-step projection scheme is proposed in Section 2. Section 3 devotes to the error estimates with mild regularity assumptions on the solution of the continuous problems.

2. Fractional-step Projection Scheme

Firstly, we state some preliminary knowledge which will be frequently cited in the sequel. Throughout this paper we assume that $\Omega\subset\Re^3$ is a sufficiently smooth bounded, simply connected polyhedral domain with connected boundary $\Gamma=\partial\Omega$ and $\boldsymbol{n}$ is the unit normal vector to Γ. As usual, $W^{s,\,p}(\Omega)$ denotes the real Sobolev space, $0\le s<\infty,\ 0\le p\le\infty$, equipped with the norm $\|\cdot\|_{s,\,p}$ and semi-norm $|\cdot|_{s,\,p}$. The space $W_0^{s,\,p}$ is the completion of the space of smooth functions compactly supported in Ω with respect to the $\|\cdot\|_{s,\,p}$ norm (see [8, 13, 20]). For $p=2$,we denote the Hilbert spaces $W^{s,\,2}(\Omega)$ (resp., $W_0^{s,\,2}(\Omega)$) by $H^s(\Omega)$ (resp., $H_0^s(\Omega)$). The related norm is denoted by $\|\cdot\|_s$. The dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. For a fixed positive real number T, and a Banach space X, we denote by $L^p(X)$, $H^s(X)$ and $C(X)$ the space $L^p(0,T;X)$, $H^s(0,T;X)$ and $C(0,T;X)$, respectively.

Furthermore, we introduce the following Hilbert spaces

$$\begin{aligned} H(\mathbf{curl};\Omega) &= \{\boldsymbol{v}\in L^2(\Omega)^3:\ \nabla\times\boldsymbol{v}\in L^2(\Omega)^3\},\\ H_0(\mathbf{curl};\Omega) &= \{\boldsymbol{v}\in H(\mathbf{curl};\Omega):\ \boldsymbol{v}\times\boldsymbol{n}|_\Gamma=\mathbf{0}\},\\ H^\alpha(\mathbf{curl};\Omega) &= \{\boldsymbol{v}\in H^\alpha(\Omega)^3:\ \nabla\times\boldsymbol{v}\in H^\alpha(\Omega)^3,\alpha\geq 0\},\end{aligned}$$

where spaces $H(\mathbf{curl};\Omega)$ and $H^\alpha(\mathbf{curl};\Omega)$ are equipped with the following norms (see [9, 10]):

$$\|\boldsymbol{v}\|_{0,\mathbf{curl}}^2=\|\boldsymbol{v}\|_0^2+\|\nabla\times\boldsymbol{v}\|_0^2,\quad \|\boldsymbol{v}\|_{\alpha,\mathbf{curl}}^2=\|\boldsymbol{v}\|_\alpha^2+\|\nabla\times\boldsymbol{v}\|_\alpha^2.$$

For the sake of simplicity, we denote

$$\begin{aligned} &X=H_0(\mathbf{curl};\Omega),\quad V=\{\boldsymbol{v}\in X:\nabla\cdot\boldsymbol{v}=0\},\\ &M=H_0^1(\Omega),\quad H=\{\boldsymbol{v}\in L^2(\Omega)^3:\nabla\cdot\boldsymbol{v}=0,\boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma=0\}.\end{aligned}$$

Then we have the following orthogonal decomposition of $L^2(\Omega)^3$ (see [15]):

$$L^2(\Omega)^3=H\oplus\nabla(H_0^1(\Omega)). \tag{2.1}$$

Based on above discuss, the variational formulation of problem (1.7) reads as follows: Seek $(\boldsymbol{H},\phi)$ in the following spaces,

$$\begin{aligned} &\boldsymbol{H}\in L^\infty(0,T;L^2(\Omega)^3)\cap L^2(0,T;X),\ \boldsymbol{H}_t\in L^2(0,T;X^{-1}),\quad \text{for all } T>0,\\ &\phi\in L^2(0,T;M),\quad \text{for all } T>0.\end{aligned}$$

such that it satisfies the initial condition

$$\boldsymbol{H}(\boldsymbol{x},0)=\boldsymbol{H}_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega \tag{2.2}$$

and the following equations

$$\begin{cases} (\mu_0\dfrac{\partial\boldsymbol{H}}{\partial t},\ \boldsymbol{v})+(\sigma^{-1}\nabla\times\boldsymbol{H},\ \nabla\times\boldsymbol{v})+(\nabla\phi,\ \boldsymbol{v})=(\boldsymbol{F},\ \boldsymbol{v}),\ \forall\boldsymbol{v}\in X,\\ (\boldsymbol{H},\ \nabla\psi)=0,\ \forall\psi\in M\end{cases} \tag{2.3}$$

for $a.e.\ t\in(0,\ T)$. Problem (2.2)-(2.3) has a unique solution $(\boldsymbol{H},\ \phi)\in X\times M$ (see [10]).

Now we describe the finite element discretization of (2.2)-(2.3). Let $\mathcal{T}_h$ be a regular partition of Ω into tetrahedrons with mesh h (see [19]). An element of $\mathcal{T}_h$ is denoted by K. Let S_h be the standard continuous piecewise linear finite element space and V_h be the Nédélec $H(\text{curl};\Omega)$-conforming edge element space defined by

$$V_h=\{\boldsymbol{v}_h\in H(\text{curl};\Omega);\ \boldsymbol{v}_h=\boldsymbol{a}_K\times\boldsymbol{x}+\boldsymbol{b}_K \text{ on } K\in\mathcal{T}_h\}, \tag{2.4}$$

where $\boldsymbol{a}_K$ and b_K are two constant vectors. It was proved in Nédélec [18] that any function $\boldsymbol{v}$ in V_h can be uniquely determined by the degree of freedom in the the moment set $M_E(\boldsymbol{v}$ on each element $K\in\mathcal{T}_h$. Here $M_E(\boldsymbol{v}$ is defined as follows:

$$M_E=\{\int_e\boldsymbol{v}\cdot\tau ds;\ e \text{ is an edge of } K\},$$

where τ is the unit vector along the edge e. We know that the integrals required in the definition of $M_E(\boldsymbol{v})$ make sense for any $\boldsymbol{v}\in H^s(K)^3$, with $\alpha>1/2$. Thus we can define an interpolation $\Pi_h\boldsymbol{v}$ of any $\boldsymbol{v}\in H^s(K)^3$ such that $\Pi_h\boldsymbol{v}\in V_h$ and $\Pi_h\boldsymbol{v}$ has the same degree of freedom as $\boldsymbol{v}$ on each K in $\mathcal{T}_h$.

The following interpolation error estimates can be seen in [9], [10], [15] and [19]:

Lemma 2.1. *There exists a constant $C>0$ such that*

$$\|\boldsymbol{u}-\Pi_h\boldsymbol{u}\|_0+\|\nabla\times(\boldsymbol{u}-\Pi_h\boldsymbol{u})\|_0\leq Ch^\alpha\|u\|_{\alpha,curl} \tag{2.5}$$

for all $\boldsymbol{u}\in H^\alpha(\boldsymbol{curl};\Omega)$ with $1/2<\alpha\leq 1$.

Furthermore, the interpolation operator Π_h has the following property (see [10]):

Lemma 2.2. Let $\boldsymbol{v} = \nabla\phi$ with $\phi \in H_0^1(\Omega)$. Then, if $\boldsymbol{v}$ is regular enough to ensure the existence of $\Pi_h \boldsymbol{v}$, we have $\Pi_h \boldsymbol{v} = \nabla\phi_h$ for some $\phi_h \in S_h \cap H_0^1(\Omega)$.

We now define two finite element subspaces of V_h and S_h which will be used in the sequel:

$$X_h = V_h \cap H_0(\mathbf{curl};\Omega), \quad M_h = S_h \cap H_0^1(\Omega).$$

Introduce the linear operator $B_h : X_h \to M_h$ and its transpose $B_h^T : M_h \to X_h$ so that for every couple $(\boldsymbol{v}_h, q_h) \in X_h \times M_h$ we have

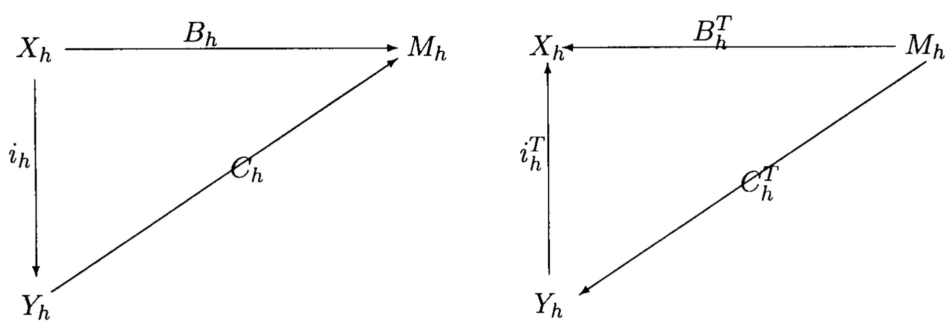
$$(B_h \boldsymbol{v}_h, q_h) = (\boldsymbol{v}_h, \nabla q_h) = (\boldsymbol{v}_h, B_h^T q_h). \tag{2.6}$$

It is not difficult to verify that B_h is surjective, that is to say, the mixed approximation satisfies the LBB condition (see [3, 4, 10, 18, 19]):

$$\inf_{q_h \in M_h} \sup_{\boldsymbol{v}_h \in X_h} \frac{(B_h \boldsymbol{v}_h,\ q_h)}{\|\boldsymbol{v}_h\|_{0,\mathbf{curl}} \|q_h\|_1} \geq c, \quad c > 0. \tag{2.7}$$

To build a discrete version of the Helmholtz decomposition of (2.1), we introduce an additional discrete setting. Define Y_h a finite dimensional subspace of $L^2(\Omega)^3$ and endow Y_h with the norm of $L^2(\Omega)^3$. For the sake of simplicity we assume that $X_h \subset Y_h$. Denote by i_h the continuous injection of X_h into Y_h; the transpose i_h^T is the L^2 projection of Y_h onto X_h. Furthermore, we assume that can build an operator $C_h : Y_h \to M_h$ such that we have the following.

(A1) The operator C_h is an extension of B_h and $i_h^T C_h^T = B_h^T$, that is to say, the following commutative diagrams hold:



Remark 2.3. Noticing that M_h is composed of continuous piecewise polynomial functions, so we have $M_h \subset H^1(\Omega)$. As a result, if we set $Y_h = X_h + \nabla M_h$, then $Y_h \subset L^2(\Omega)^3$. Furthermore, it can easily be verified that C_h defined by

$$(C_h \boldsymbol{v}_h,\ q_h) = (\boldsymbol{v}_h,\ \nabla q_h), \quad \forall (\boldsymbol{v}_h,\ q_h) \in Y_h \times M_h, \tag{2.8}$$

is an extension of B_h, and C_h^T is the restriction of ∇ to M_h.

Recall that B_h is assumed to be surjective, as a consequence, C_h is also surjective for C_h is an extension of B_h. The null space of C_h playing an important role in the sequel we set $H_h = \ker C_h$. This definition enables us to build a discrete counterpart of the aforementioned orthogonal decomposition $L^2(\Omega)^3 = H \oplus \nabla(H_0^1(\Omega))$.

Corollary 2.4. *We have the following orthogonal decomposition*

$$Y_h = H_h \oplus C_h^T(M_h). \tag{2.9}$$

We also assume that C_h^T satisfies the following hypothesis.

(A2) There exists $C > 0$ such that $\forall q_h$ in M_h,

$$\|C_h^T q_h\|_0 \leq C \|q_h\|_1.$$

Remark 2.5. Note that the assumption (A2) is automatically satisfied if we choose $Y_h = X_h + \nabla M_h$ and $C_h^T = \nabla$ (see [1, 12] for details).

We next introduce a partition of the time interval $[0,\ T]$: $t_n = n\tau$ for $0 \le n \le N$ where $\tau = T/N$. This section is concerned with the time scheme for computing approximations to the magnetic field $\boldsymbol{H}$ and the multiplier function ϕ at each time step t_n.

To avoid the technical difficulty of the blowing up of the error estimates at initial time induced by possible lack of regularity of the solution, we assume that the solution is as smooth as needed at $t = 0$.

Hereafter we denote by $\hat{\boldsymbol{H}}_h^0 \in X_h$ and $\hat{\phi}_h^0 \in M_h$ an approximation to $\boldsymbol{H}_0$ and $\phi(t=0)$ such that

$$\|\boldsymbol{H}_0 - \hat{\boldsymbol{H}}^0\|_0 + h(\|\nabla \times (\boldsymbol{H}_0 - \hat{\boldsymbol{H}}^0)\|_0 + \|\nabla(\phi(0) - \hat{\phi}_h^0)\|_0) \le Ch^2. \tag{2.10}$$

We are now interested in defining a fractional-step projection scheme for $1 \le n \le N$. We define two sequences of approximate magnetic fields $\{\tilde{\boldsymbol{H}}_h^n \in X_h\}$ and $\{\boldsymbol{H}_h^n\} \in Y_h$ and one sequence of approximate multiplier $\{\phi_h^n \in M_h\}$ as follows:

- **The fractional-step projection method**

Step 1 (*Initialization.*) The sequences $\{\boldsymbol{H}_h^n \in Y_h\}$ and $\{\tilde{\boldsymbol{H}}_h^n \in X_h\}$ are initialized by $\boldsymbol{H}_h^0 = \tilde{\boldsymbol{H}}_h^0 = \hat{\boldsymbol{H}}_h^0$ and the sequence $\{\phi_h^n \in M_h\}$ is initialized by $\phi_h^0 = \hat{\phi}_h^0$.

Step 2 (*Time loop.*) For $0 \le n < N$, seek $\{\tilde{\boldsymbol{H}}_h^n \in X_h\}$ such that

$$\begin{aligned} &(\mu_0 \frac{\tilde{\boldsymbol{H}}_h^n - i_h^T \boldsymbol{H}_h^{n-1}}{\tau},\ \boldsymbol{v}_h) + (\sigma^{-1}\nabla \times \tilde{\boldsymbol{H}}_h^n,\ \nabla \times \boldsymbol{v}_h) \\ &+(B_h^T \phi_h^{n-1},\ \boldsymbol{v}_h) = (\boldsymbol{F}^n,\ \boldsymbol{v}_h),\ \forall \boldsymbol{v}_h \in X_h \end{aligned} \tag{2.11}$$

and find $(\boldsymbol{H}_h^n,\ \phi_h^n) \in Y_h \times M_h$ such that

$$\begin{cases} \mu_0 \dfrac{\boldsymbol{H}_h^n - i_h \tilde{\boldsymbol{H}}_h^n}{\tau} + C_h^T(\phi_h^n - \phi_h^{n-1}) = 0, \\ C_h \boldsymbol{H}_h^n = 0. \end{cases} \tag{2.12}$$

Remark 2.6. The problem (2.11) clearly has a unique solution and problem (2.12) is also well posed thanks to Corollary 2.4. In addition, in practice we can set $Y_h = X_h + \nabla M_h$. Thus ,by setting $\phi_h^{-1} = \hat{\phi}_h^0$, the algorithm that is implemented read as follows for $n > 0$,

$$\begin{aligned} &(\mu_0 \frac{\tilde{\boldsymbol{H}}_h^n - \tilde{\boldsymbol{H}}_h^{n-1}}{\tau},\ \boldsymbol{v}_h) + (\sigma^{-1}\nabla \times \tilde{\boldsymbol{H}}_h^n,\ \nabla \times \boldsymbol{v}_h) \\ &= (\boldsymbol{F}^n,\ \boldsymbol{v}_h) + (2\phi_h^n - \phi_h^{n-1},\ \nabla \cdot \boldsymbol{v}_h),\ \forall \boldsymbol{v}_h \in X_h \end{aligned} \tag{2.13}$$

and the projection step (2.12) takes the following form:

$$\begin{cases} \mu_0 \dfrac{\boldsymbol{H}_h^n - \tilde{\boldsymbol{H}}_h^n}{\tau} + \nabla(\phi_h^n - \phi_h^{n-1}) = 0, \\ \nabla \cdot \boldsymbol{H}_h^n = 0. \end{cases} \tag{2.14}$$

Remark 2.7. If we choose $Y_h = X_h + \nabla M_h$, then the projection step (2.12) is actually to solve a discrete Poisson problem with zero Dirichlet boundary condition: find $\phi_h^n \in M_h$ such that

$$(\nabla(\phi_h^n - \phi_h^{n-1}),\ \nabla\psi_h) = -\frac{\mu_0}{\tau}(\nabla \cdot \tilde{\boldsymbol{H}}_h^n,\ \psi_h),\quad \forall \psi_h \in M_h. \tag{2.15}$$

If needed, the magnetic field $\boldsymbol{H}_h^n$ is given by

$$\boldsymbol{H}_h^n = \tilde{\boldsymbol{H}}_h^n - \frac{\tau}{\mu_0}\nabla(\phi_h^n - \phi_h^{n-1}). \tag{2.16}$$

Next we list a discrete Gronwall's inequality and some standard estimates on finite difference schemes, which shall be used our subsequent analysis.

Lemma 2.8. ([**21**]) *Let $\{\omega_j\}$ be a nonnegative sequence, and $\{\beta_j\}$ be a nonnegative and monotone non-decreasing sequence. If $\{\eta_n\}$ is a nonnegative sequence such that*

$$\eta_0 \le \beta_0, \quad \eta_n \le \beta_n + \sum_{j=0}^{n-1}\omega_j\eta_j, \quad n \ge 1,$$

then we have the following estimate

$$\eta_n \le \beta_n \exp(\sum_{j=0}^{n-1}\omega_j).$$

Lemma 2.9. ([**9**]) *For $B = H^1(\boldsymbol{curl};\Omega)$ or $B = H^s(\Omega)^3,\ s \ge 0$, we have*

$$\begin{aligned}
&\|\partial_\tau \boldsymbol{u}^n\|_B^2 \le \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\boldsymbol{u}'(t)\|_B^2 dt,\ \boldsymbol{u} \in H^1(0,T;B),\\
&\|\partial_\tau^2 \boldsymbol{u}^n\|_B^2 \le \frac{1}{\tau}\int_{t_{n-2}}^{t_n}\|\boldsymbol{u}''(t)\|_B^2 dt,\ \boldsymbol{u} \in H^2(0,T;B),\\
&\|\boldsymbol{u}_t^n - \partial_\tau \boldsymbol{u}^n\|_B^2 \le \tau\int_{t_{n-1}}^{t_n}\|\boldsymbol{u}''(t)\|_B^2 dt,\ \boldsymbol{u} \in H^2(0,T;B),
\end{aligned} \tag{2.17}$$

where $\partial_\tau \boldsymbol{u}^n = (\boldsymbol{u}^n - \boldsymbol{u}^{n-1})/\tau$.

3. Error Estimates

In this section, we shall derive error bounds on the magnetic field. Without loss of the generality, we assume that μ_0 and σ ia a constant in the sequel. (It is straightforward to extend the analysis to the non-constant or elementwise constant case by simply this coefficient inside the integrals or norm and bounding it by taking its maximum or minimum value if necessary).

For convenience, we introduce the notations:

$$\|\boldsymbol{w}\|_{\mu_0}^2 = (\mu_0\boldsymbol{w},\ \boldsymbol{w}), \quad \|\boldsymbol{w}\|_\sigma^2 = (\sigma^{-1}\boldsymbol{w},\ \boldsymbol{w}), \quad \forall \boldsymbol{w} \in L^2(\Omega)^3,$$

We define projection operator $(P_h,\ Q_h):\ H_0(\mathbf{curl};\Omega) \times H_0^1(\Omega) \to X_h \times M_h$ as follows:

$$\begin{aligned}
a(P_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, Q_h q - q) = 0,\ \forall \boldsymbol{v}_h \in X_h,\\
b(P_h\boldsymbol{u} - \boldsymbol{u}, r_h) = 0,\ \forall r_h \in M_h
\end{aligned} \tag{3.1}$$

where bilinear forms $a(\boldsymbol{u},\boldsymbol{v})$ and $b(\boldsymbol{u},r)$ defined by

$$a(\boldsymbol{u},\boldsymbol{v}) = (\sigma^{-1}\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{u}), \quad b(\boldsymbol{u},r) = (\boldsymbol{u},\nabla r).$$

It is clear that (P_h, Q_h) is well defined in $H_0(\mathbf{curl};\Omega)\times H_0^1(\Omega)$. And it is easy to see that ([10])

$$\|P_h\boldsymbol{u} - \boldsymbol{u}\|_{0,\text{curl}} + \|Q_h r - r\|_1 \le Ch^s(\|\boldsymbol{u}\|_{s,\mathbf{curl}} + \|r\|_{1+s}), \tag{3.2}$$

where $1/2 < s \le 1$.

In addition, for the sake of conciseness we set

$$\boldsymbol{e}_h^n = P_h\boldsymbol{H}(t_n) - \boldsymbol{H}_h^n,\ \ \tilde{\boldsymbol{e}}_h^n = P_h\boldsymbol{H}(t_n) - \tilde{\boldsymbol{H}}_h^n,\ \ \varepsilon_h^n = Q_h\phi(t_n) - \phi_h^n,$$

We now turn our attention to the error analysis of the fractional-step projection scheme. We have the following error estimates theorem.

Theorem 3.1. *Assume that time step τ is sufficiently small and hypotheses (A1)-(A2) hold. Moreover, assume that for some $1/2 < s \le 1$, the solution of the continuous problem (2.3) has the following regularity:*

$$\boldsymbol{H} \in H^1(0,T; X \cap H^s(\boldsymbol{curl},\Omega)), \quad \boldsymbol{H} \in H^2(0,T; L^2(\Omega)^3), \quad \phi \in H^1(0,T; M \cap H^{1+s}(\Omega)).$$

Then the solution to the fractional-step scheme (2.11)-(2.12) satisfies:

$$\max_{1\le n\le N}\{\|\boldsymbol{H}(t_n) - \boldsymbol{H}_h^n\|_{\mu_0} + \|\boldsymbol{H}(t_n) - \tilde{\boldsymbol{H}}_h^n\|_{\mu_0}\} \le C(h^s + \tau). \tag{3.3}$$

Proof. The whole proof is divided into the following five steps.

Step 1. For conciseness we denote $\boldsymbol{H}^n = \boldsymbol{H}(t_n)$, $\phi^n = \phi(t_n)$. The accurate solution of (1.7) satisfies at time t_n:

$$\begin{cases} (\mu_0 \dfrac{\boldsymbol{H}^n - \boldsymbol{H}^{n-1}}{\tau},\ \boldsymbol{v}_h) + (\sigma^{-1}\nabla\times\boldsymbol{H}^n,\ \nabla\times\boldsymbol{v}_h) \\ \quad +(B_h^T\phi^n,\ \boldsymbol{v}_h) = (\boldsymbol{F}^n,\ \boldsymbol{v}_h) + (\mu_0\boldsymbol{R}^n,\ \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in X_h, \\ B_h P_h \boldsymbol{H}^n = 0, \end{cases} \tag{3.4}$$

where

$$\boldsymbol{R}^n = \frac{\boldsymbol{H}^n - \boldsymbol{H}^{n-1}}{\tau} - \frac{\partial}{\partial t}\boldsymbol{H}(t_n) = -\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(t - t_{n-1})\boldsymbol{H}_{tt}(t)dt.$$

Then by the Cauchy-Schwarz inequality we derive the following bound

$$\|\boldsymbol{R}^n\|_0^2 \le \tau\int_{t_{n-1}}^{t_n}\|\boldsymbol{H}''(t)\|_0^2 dt, \quad n = 1,\cdots,N. \tag{3.5}$$

By subtracting (2.11) from the first equation of (3.4) and using the definition of the projection operator (P_h, Q_h), we derive the following error equation:

$$\begin{aligned} &(\mu_0\frac{\tilde{\boldsymbol{e}}_h^n - i_h^T\boldsymbol{e}_h^{n-1}}{\tau},\ \boldsymbol{v}_h) + (\sigma^{-1}\nabla\times\tilde{\boldsymbol{e}}_h^n,\ \nabla\times\boldsymbol{v}_h) + (B_h^T\theta_h^{n-1},\ \boldsymbol{v}_h) \\ &= (\mu_0\partial_\tau(P_h\boldsymbol{H}^n - \boldsymbol{H}^n),\ \boldsymbol{v}_h) + (\boldsymbol{R}^n,\ \boldsymbol{v}_h), \end{aligned} \tag{3.6}$$

where

$$\theta_h^{n-1} = Q_h\phi^n - \phi_h^{n-1} = \tau\partial_\tau Q_h\phi^n + \varepsilon_h^{n-1}. \tag{3.7}$$

On the other hand, since $B_hP_h\boldsymbol{H}^n = 0$, $n = 1,\cdots,N$, and C_h is an extension of B_h, we can obtain the system of equations that controls $\boldsymbol{e}_h^n$ and ε_h^n from (2.12) and the second equation of (3.4):

$$\begin{cases} \mu_0\dfrac{\boldsymbol{e}_h^n - i_h\tilde{\boldsymbol{e}}_h^n}{\tau} + C_h^T(\varepsilon_h^n - \theta_h^{n-1}) = 0, \\ C_h\boldsymbol{e}_h^n = 0. \end{cases} \tag{3.8}$$

Step 2. To get a bound on $\tilde{\boldsymbol{e}}_h^n$, we take $\boldsymbol{v}_h = 2\tau\tilde{\boldsymbol{e}}_h^n \in X_h$ in (3.6). Using the relation $2(a,\ a-b) = |a|^2 + |a-b|^2 - |b|^2$, we have

$$\begin{aligned} &\|\tilde{\boldsymbol{e}}_h^n\|_{\mu_0}^2 + \|\tilde{\boldsymbol{e}}_h^n - i_h^T\boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 - \|i_h^T\boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 \\ &\quad +2\tau\|\nabla\times\tilde{\boldsymbol{e}}_h^n\|_\sigma^2 + 2\tau(B_h^T\theta_h^{n-1},\ \tilde{\boldsymbol{e}}_h^n) \\ &= 2\tau(\mu_0\partial_\tau(P_h\boldsymbol{H}^n - \boldsymbol{H}^n),\ \tilde{\boldsymbol{e}}_h^n) + 2\tau(\boldsymbol{R}^n,\ \tilde{\boldsymbol{e}}_h^n) \\ &=: (I_1) + (I_2). \end{aligned} \tag{3.9}$$

Note that, using (3.2) and (3.5),

$$
\begin{aligned}
(I_1) &\le C_\gamma \tau \|\partial_\tau (P_h \boldsymbol{H}^n - \boldsymbol{H}^n)\|_{\mu_0}^2 + \gamma\tau \|\tilde{\boldsymbol{e}}_h^n\|_{\mu_0}^2 \\
&\le \gamma_1 \tau \|\tilde{\boldsymbol{e}}_h^n - i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + \gamma_2 \tau \|i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + C_\gamma h^{2s} \int_{t_{n-1}}^{t_n} (\|\boldsymbol{H}'(t)\|_{s,\text{crul}}^2 + \|\nabla \phi'(t)\|_{1+s}^2) dt, \\
(I_2) &\le C_{\gamma'} \tau \|\boldsymbol{R}^n\|_0^2 + \gamma' \tau \|\tilde{\boldsymbol{e}}_h^n\|_{\mu_0}^2 \\
&\le \gamma_1' \tau \|\tilde{\boldsymbol{e}}_h^n - i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + \gamma_2' \tau \|i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + C_{\gamma'} \tau^2 \int_{t_{n-1}}^{t_n} \|\boldsymbol{H}''(t)\|_0^2 dt.
\end{aligned}
\tag{3.10}
$$

By (3.9) and (3.10) we obtain

$$
\begin{aligned}
&\|\tilde{\boldsymbol{e}}_h^n\|_{\mu_0}^2 + \|\tilde{\boldsymbol{e}}_h^n - i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 - \|i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 \\
&\quad + 2\tau \|\nabla \times \tilde{\boldsymbol{e}}_h^n\|_\sigma^2 + 2\tau (B_h^T \theta_h^{n-1},\ \tilde{\boldsymbol{e}}_h^n) \\
&\le (\gamma_1 + \gamma_1') \tau \|\tilde{\boldsymbol{e}}_h^n - i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + (\gamma_2 + \gamma_2') \tau \|i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 \\
&\quad + C_\gamma h^{2s} \int_{t_{n-1}}^{t_n} (\|\boldsymbol{H}'(t)\|_{s,\text{curl}}^2 + \|\nabla \phi'(t)\|_{1+s}^2) dt + C_{\gamma'} \tau^2 \int_{t_{n-1}}^{t_n} \|\boldsymbol{H}''(t)\|_0^2 dt.
\end{aligned}
\tag{3.11}
$$

Taking time stepsize τ such that

$$
\tau \le \frac{1}{2(\gamma_1 + \gamma_1')},
$$

then we have

$$
\begin{aligned}
&\|\tilde{\boldsymbol{e}}_h^n\|_{\mu_0}^2 + \frac{1}{2} \|\tilde{\boldsymbol{e}}_h^n - i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 - \|i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + 2\tau \|\nabla \times \tilde{\boldsymbol{e}}_h^n\|_\sigma^2 + 2\tau (B_h^T \theta_h^{n-1},\ \tilde{\boldsymbol{e}}_h^n) \\
&\le C\tau \|i_h^T \boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + C_\gamma h^{2s} \int_{t_{n-1}}^{t_n} (\|\boldsymbol{H}'(t)\|_{s,\text{curl}}^2 + \|\nabla \phi'(t)\|_{1+s}^2) dt + C_{\gamma'} \tau^2 \int_{t_{n-1}}^{t_n} \|\boldsymbol{H}''(t)\|_0^2 dt.
\end{aligned}
\tag{3.12}
$$

Step 3. To obtain some control on $2\tau(\tilde{\boldsymbol{e}}_h^n,\ B_h^T \theta_h^{n-1})$, using $C_h \boldsymbol{e}_h^n = 0$, we take the inner product of first equation of (3.8) by $2\mu_0^{-1} \tau^2 C_h^T \theta_h^{n-1}$,

$$
-2\tau(\tilde{\boldsymbol{e}}_h^n,\ B_h^T \theta_h^{n-1}) + 2\mu_0^{-1} \tau^2 (C_h^T \varepsilon_h^n - C_h^T \theta_h^{n-1},\ C_h^T \theta_h^{n-1}) = 0.
$$

By relation $2(a-b,\ b) = \|a\|^2 - \|a-b\|^2 - \|b\|^2$, we have

$$
-2\tau(\tilde{\boldsymbol{e}}_h^n,\ B_h^T \theta_h^{n-1}) + \mu_0^{-1} \tau^2 \|C_h^T \varepsilon_h^n\|_0^2 - \|\boldsymbol{e}_h^n - \tilde{\boldsymbol{e}}_h^n\|_{\mu_0}^2 = \mu_0^{-1} \tau^2 \|C_h^T \theta_h^{n-1}\|_0^2. \tag{3.13}
$$

By (3.7), we get

$$
\begin{aligned}
\|C_h^T \theta_h^{n-1}\|_0^2 &= \|\tau C_h^T \partial_\tau Q_h \phi^n + C_h^T \varepsilon_h^{n-1}\|_0^2 \\
&\le \|C_h^T \varepsilon_h^{n-1}\|_0^2 + C\tau \int_{t_{n-1}}^{t_n} \|\nabla Q_h \phi'(t)\|_0^2 dt.
\end{aligned}
\tag{3.14}
$$

Thus, combining (3.13) and (3.14) to obtain

$$
\begin{aligned}
&-2\tau(\tilde{\boldsymbol{e}}_h^n,\ B_h^T \theta_h^{n-1}) + \mu_0^{-1} \tau^2 \|C_h^T \varepsilon_h^n\|_0^2 - \|\boldsymbol{e}_h^n - \tilde{\boldsymbol{e}}_h^n\|_{\mu_0}^2 \\
&\le \mu_0^{-1} \tau^2 \|C_h^T \varepsilon_h^{n-1}\|_0^2 + C\tau^3 \int_{t_{n-1}}^{t_n} \|\nabla Q_h \phi'(t)\|_0^2 dt.
\end{aligned}
\tag{3.15}
$$

We have some control on $\boldsymbol{e}_h^n$ by taking the inner product of (3.8) by $2\tau\boldsymbol{e}_h^n$ and using $C_h\boldsymbol{e}_h^n = 0$ and relation $2(a-b,a) = \|a\|^2 + \|a-b\|^2 - \|b\|^2$,

$$\|\boldsymbol{e}_h^n\|_{\mu_0}^2 + \|\boldsymbol{e}_h^n - \tilde{\boldsymbol{e}}_h^n\|_{\mu_0}^2 - \|\tilde{\boldsymbol{e}}_h^n\|_{\mu_0}^2 = 0. \tag{3.16}$$

Step 4. After summing up (3.12)+(3.15)+(3.16) and noting that $\|i_h^T\boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 = \|\boldsymbol{e}_h^{n-1}\|_{\mu_0}^2$, we get

$$\begin{aligned}&\|\boldsymbol{e}_h^n\|_{\mu_0}^2 + \frac{1}{2}\|\tilde{\boldsymbol{e}}_h^n - i_h^T\boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + 2\tau\|\nabla\times\tilde{\boldsymbol{e}}_h^n\|_\sigma^2 + \mu_0^{-1}\tau^2\|C_h^T\varepsilon_h^n\|_0^2\\ &\le (1+C\tau)\|\boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + \mu_0^{-1}\tau^2\|C_h^T\varepsilon_h^{n-1}\|_0^2 + C_{\gamma'}\tau^2\int_{t_{n-1}}^{t_n}\|\boldsymbol{H}''(t)\|_0^2dt\\ &\quad +C_\gamma h^{2s}\int_{t_{n-1}}^{t_n}[\|\boldsymbol{H}'(t)\|_{s,\mathrm{curl}}^2 + \|\nabla\phi'(t)\|_{1+s}^2]dt + C\tau^3\int_{t_{n-1}}^{t_n}\|\nabla Q_h\phi'(t)\|_0^2dt.\end{aligned} \tag{3.17}$$

By taking the sum from $n=1$ to $m(\le N)$, we have

$$\begin{aligned}&\|\boldsymbol{e}_h^m\|_{\mu_0}^2 + \frac{1}{2}\sum_{n=1}^m\|\tilde{\boldsymbol{e}}_h^n - i_h^T\boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + 2\tau\sum_{n=1}^m\|\nabla\times\tilde{\boldsymbol{e}}_h^n\|_\sigma^2 + \mu_0^{-1}\tau^2\|C_h^T\varepsilon_h^m\|_0^2\\ &\le \|\boldsymbol{e}_h^0\|_{\mu_0}^2 + C\tau\sum_{n=1}^m\|\boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + \mu_0^{-1}\tau^2\|C_h^T\varepsilon_h^0\|_0^2 + C_{\gamma'}\tau^2\int_0^T\|\boldsymbol{H}''(t)\|_0^2dt\\ &\quad +C_\gamma h^{2s}\int_0^T[\|\boldsymbol{H}'(t)\|_{s,\mathrm{curl}}^2 + \|\nabla\phi'(t)\|_{1+s}^2]dt + C\tau^3\int_0^T\|\nabla Q_h\phi'(t)\|_0^2dt\end{aligned}$$

From the initialization hypothesis (2.10) we infer that the term $\|\boldsymbol{e}_h^0\|_{\mu_0}^2 + \mu_0^{-1}\tau^2\|C_h^T\varepsilon_h^0\|_0^2$ is bounded from above by $C(h^s+\tau)^2$. As a result, we can apply the discrete Gronwall Lemma 2.8, which leads to

$$\|\boldsymbol{e}_h^m\|_{\mu_0}^2 + \frac{1}{2}\sum_{n=1}^m\|\tilde{\boldsymbol{e}}_h^n - i_h^T\boldsymbol{e}_h^{n-1}\|_{\mu_0}^2 + 2\tau\sum_{n=1}^m\|\nabla\times\tilde{\boldsymbol{e}}_h^n\|_\sigma^2 \le C(h^s+\tau)^2. \tag{3.18}$$

From this bound we deduce for $n = 1,\cdots,N$

$$\|\boldsymbol{e}_h^n\|_{\mu_0} \le C(h^s+\tau) \tag{3.19}$$

and

$$\|\tilde{\boldsymbol{e}}_h^n - i_h^T\boldsymbol{e}_h^{n-1}\|_{\mu_0} \le C(h^s+\tau). \tag{3.20}$$

Furthermore, we get for $n = 1,\cdots,N$

$$\|\tilde{\boldsymbol{e}}_h^n\|_0 \le \|\tilde{\boldsymbol{e}}_h^n - i_h^T\boldsymbol{e}_h^{n-1}\|_0 + \|\boldsymbol{e}_h^{n-1}\|_0 \le C(h^s+\tau). \tag{3.21}$$

Hence we infer from (3.2) and (3.20)-(3.21) for $n = 1,2,\cdots,N$

$$\|\boldsymbol{H}^n - \boldsymbol{H}_h^n\|_0 \le \|\boldsymbol{H}^n - P_h\boldsymbol{H}^n\|_0 + \|\boldsymbol{e}_h^n\|_0 \le C(h^s+\tau)$$

and

$$\|\boldsymbol{H}^n - \tilde{\boldsymbol{H}}_h^n\|_0 \le \|\boldsymbol{H}^n - P_h\boldsymbol{H}^n\|_0 + \|\tilde{\boldsymbol{e}}_h^n\|_0 \le C(h^s+\tau).$$

So for we have obtained the desired result.

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